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Time-Like Conformal Homogeneous Hypersurfaces with Three Distinct Principal Curvatures*

Yanbin LIN[1] Ying LÜ[2] Changping WANG[3]

Abstract A hypersurface $x(M)$ in Lorentzian space R^4_1 is called conformal homogeneous, if for any two points p, q on M, there exists σ, a conformal transformation of R^4_1, such that $\sigma(x(M)) = x(M)$, $\sigma(x(p)) = x(q)$. In this paper, the authors give a complete classification for regular time-like conformal homogeneous hypersurfaces in R^4_1 with three distinct principal curvatures.

Keywords Lorentzian metric, Conformal metric, Conformal space form, Conformal homogeneous, Time-like hypersurface
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1 Introduction

Let $\{R^6_2, \langle\cdot,\cdot\rangle\}$ be a Lorentzian space form of dimension 6 and the inner product is defined as

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 - u_5v_5 - u_6v_6.$$

The conformal space Q^4_1 is defined in the light cone by

$$Q^4_1 = \{[u] \in RP^5 \mid u \in R^6_2,\ \langle u, u\rangle = 0\},$$

which is the conformal compactification of Lorentzian space forms R^4_1, S^4_1 and H^4_1. The conformal transformation group is therefore isomorphic to $O(4, 2)/\{\pm 1\}$. Since the hypersurfaces in three Lorentzian space forms are conformally equivalent to each other, we choose R^4_1 as the ambient space to study the conformal properties of hypersurfaces. More details on the conformal space Q^m_1 can be found in [3, 6].

Suppose that $x : M^3 \to (R^4_1, \langle\, ,\, \rangle_1)$ is a time-like hypersurface in Lorentzian space form, in which $\langle\, ,\, \rangle_1$ is a Lorentzian inner product with signatures $(+, +, +, -)$. If at every point p, $\{e_i\}$ is a basis of T_pM with dual basis $\{\omega^i\}$ and n is the space-like unit normal vector, then there is

Manuscript received July 11, 2018.
[1]School of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, Fujian, China.
E-mail: 1564863659@qq.com
[2]School of Mathematical Sciences, Xiamen University, Xiamen 361005, Fujian, China.
E-mail: lueying@xmu.edu.cn
[3]College of Mathematics and Informatics, Fujian Normal University, Fuzhou 350117, China.
E-mail: cpwang@fjnu.edu.cn
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a naturally induced Lorentzian metric on M,

$$g = \langle \mathrm{d}x, \mathrm{d}x \rangle_1 = \sum_{i,j} g_{ij} \omega^i \otimes \omega^j.$$

The structure equations are in form of

$$\mathrm{d}x = \sum_i \omega^i e_i(x), \quad \mathrm{d}e_i(x) = \sum_j \omega_i^j e_j(x) + h_{ij}\omega^j n, \quad \mathrm{d}n = -\sum_{i,j} S_i^j \omega^i e_j(x),$$

in which $h = \sum\limits_{i,j} h_{ij}\omega^i \otimes \omega^j$ is the second fundamental form and $S = \sum\limits_{i,j} S_i^j \omega^i \otimes e_j$ is the shape operator.

According to the algebraic lemma in [9], we have the following lemma.

Lemma 1.1 *There exists a basis* $\{e_i\}$ *such that the matrices of shape operator and induced metric are exactly in one of the following forms*:

(i) $S = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$, $g = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$, $\lambda_2 \leq \lambda_3$;

(ii) $S = \begin{bmatrix} \lambda_1 & \pm 1 & \\ & \lambda_1 & \\ & & \lambda_3 \end{bmatrix}$, $g = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$;

(iii) $S = \begin{bmatrix} \lambda_1 & & 1 \\ & \lambda_1 & \\ & 1 & \lambda_1 \end{bmatrix}$, $g = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$;

(iv) $S = \begin{bmatrix} a & b & \\ -b & a & \\ & & \lambda_3 \end{bmatrix}$, $g = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$, $b \neq 0$.

All principal curvatures are real in the first three cases. In the last one S has a pair of conjugate eigenvalues $\lambda_1 = \overline{\lambda_2} = a + \mathbf{i}b$ whose eigenvectors are respectively $e_1 + \mathbf{i}e_2$ and $\mathbf{i}e_1 + e_2$.

Since R_1^4 is embedded in Q_1^4 via

$$R_1^4 = Q_1^4 \backslash \pi, \quad \pi := \{[u] \mid u \in R_2^6,\ u_1 + u_6 = 0\}, \quad v \mapsto \Big[\Big(\frac{1 - \langle v, v\rangle_1}{2}, v, \frac{1 + \langle v, v\rangle_1}{2}\Big)^{\mathrm{T}}\Big],$$

every transformation $T \in O(4,2)$ will induce a conformal transformation σ on R_1^4. Therefore, the hypersurface $x(M)$ will be conformally transformed into another hypersurface $\sigma(x(M)) = \widetilde{x}(M)$. If we define a natural lift as

$$y : M \to R_2^6, \quad p \mapsto \Big(\frac{1 - \langle x(p), x(p)\rangle_1}{2}, x(p), \frac{1 + \langle x(p), x(p)\rangle_1}{2}\Big)^{\mathrm{T}},$$

then x relates to $\widetilde{x}$ by $[T \cdot y] = [\widetilde{y}]$.

It is well known that the principal directions are invariant to the conformal transformations, i.e., $\mathrm{d}\sigma(e)$ is still a principal direction of $\widetilde{x}$ with respect to curvature $\sigma(\lambda)$, provided that e is a principal direction of x with respect to λ. So the type of shape operator will not be changed by σ, while the values of principal curvatures could. However, no matter the eigenvalues are real or complex, $M_{ij,k} = \frac{\lambda_i - \lambda_k}{\lambda_j - \lambda_k}$ are conformal invariant functions and

$$g_0 := \frac{1}{2}((\lambda_1 - \lambda_2)^2 + (\lambda_2 - \lambda_3)^2 + (\lambda_1 - \lambda_3)^2)g$$

is a conformal invariant real 2-form. This means

$$\frac{\lambda_i - \lambda_k}{\lambda_j - \lambda_k} = \frac{\sigma(\lambda_i) - \sigma(\lambda_k)}{\sigma(\lambda_j) - \sigma(\lambda_k)}, \quad \sigma^* \circ (\widetilde{x}^{-1})^* \circ \widetilde{g}_0 = (x^{-1})^* \circ g_0.$$

It is easy to see that g_0 is non-degenerate if and only if x has distinct principal curvatures in types (i)–(iii) or $(\lambda_3 - a)^2 \neq 3b^2$ in type (iv). We call the hypersurface is conformal regular if g_0 is non-degenerate.

Definition 1.1 *A hypersurface $x(M)$ is called conformal homogeneous, if for any two points p, q on M, there exists a conformal transformation of R^4_1, say $\sigma_{p,q}$, such that*

$$\sigma_{p,q}(x(M)) = x(M), \quad \sigma_{p,q}(x(p)) = x(q).$$

Basically, the hypersurface is generated by and thus invariant to the conformal transformations induced from G, a subgroup of $O(4,2)$. In other words, there is $T_{p,q} \in G$ such that

$$[G \cdot y(p)] = [y(M)], \quad [T_{p,q} \cdot y(p)] = [y(q)].$$

Following the idea of Wang's work in [10], many authors studied the conformal structure of space-like hypersurfaces in Lorentzian space form in [3–6, 8]. As for the conformal homogeneous hypersurfaces, the cases in Riemannian space form can be found in [1–2, 11]. In Lorentzian space form, we have already classified space-like conformal homogeneous hypersurfaces in R^4_1 (see [7]). So the study of time-like case becomes our main interest. In this paper we assume that the hypersurfaces have three distinct principal curvatures. This means that types (ii) and (iii) do not occur. We call it "real case" for type (i) and "complex case" for type (iv). The main theorem is the following.

Theorem 1.1 *Suppose that $x(M)$ is a time-like hypersurface in R^4_1 with three distinct principal curvatures. If it is conformal regular and homogeneous, then it must be conformally equivalent to a hypersurfce generated by one of the following subgroup of $O(4,2)$:*

$$(1) \quad \begin{bmatrix} \cosh\psi & \sinh\psi & & & & \\ \sinh\psi & \cosh\psi & & & & \\ & & \cos\theta & -\sin\theta & & \\ & & \sin\theta & \cos\theta & & \\ & & & & \cosh\phi & \sinh\phi \\ & & & & \sinh\phi & \cosh\phi \end{bmatrix},$$

$$(2) \quad \begin{bmatrix} \cos\psi & & & & & -\sin\psi \\ & \cos\theta & & & -\sin\theta & \\ & & \cos\phi & -\sin\phi & & \\ & & \sin\phi & \cos\phi & & \\ & \sin\theta & & & \cos\theta & \\ \sin\psi & & & & & \cos\psi \end{bmatrix},$$

$$(3)\quad \begin{bmatrix} \mathrm{e}^{\psi} & \frac{\theta^2}{2}\mathrm{e}^{-\psi} & \theta & & -\psi\mathrm{e}^{\psi} & \\ & \mathrm{e}^{-\psi} & & & & \\ & \theta\mathrm{e}^{-\psi} & 1 & & & \\ & & & 1 & \phi\mathrm{e}^{\psi} & \\ & & & & \mathrm{e}^{\psi} & \\ & \psi\mathrm{e}^{-\psi} & & \phi & \frac{\phi^2}{2}\mathrm{e}^{\psi} & \mathrm{e}^{-\psi} \end{bmatrix},$$

$$(4)\quad \begin{bmatrix} \mathrm{e}^{-\psi} & \frac{\theta^2}{2}\mathrm{e}^{\psi} & \theta & & & \\ & \mathrm{e}^{\psi} & & & & \\ & \theta\mathrm{e}^{\psi} & 1 & & & \\ & & & 1 & \phi\mathrm{e}^{A\psi} & \\ & & & & \mathrm{e}^{A\psi} & \\ & & & \phi & \frac{\phi^2}{2}\mathrm{e}^{A\psi} & \mathrm{e}^{-A\psi} \end{bmatrix}, \quad A \neq 0, \pm 1,$$

$$(5)\quad \begin{bmatrix} \mathrm{e}^{-2\psi} & \mathrm{e}^{2\psi}\left(\frac{\theta^2}{2} - \frac{\phi^4}{6}\right) & \theta & \phi^2 & \frac{\phi^3}{3}\mathrm{e}^{\psi} & 2\phi\mathrm{e}^{-\psi} \\ & \mathrm{e}^{2\psi} & & & & \\ & \theta\mathrm{e}^{2\psi} & 1 & & & \\ & -\phi^2\mathrm{e}^{2\psi} & & 1 & \phi\mathrm{e}^{\psi} & \\ & -2\phi\mathrm{e}^{2\psi} & & & \mathrm{e}^{\psi} & \\ & -\frac{\phi^3}{3}\mathrm{e}^{2\psi} & & \phi & \frac{\phi^2}{2}\mathrm{e}^{\psi} & \mathrm{e}^{-\psi} \end{bmatrix},$$

$$(6)\quad \begin{bmatrix} e^{\psi}\cos\phi & & & & & -e^{\psi}\sin\phi \\ & e^{-\psi}\cos\phi & & & -e^{-\psi}\sin\phi & \\ & & \cos\theta & -\sin\theta & & \\ & & \sin\theta & \cos\theta & & \\ & e^{-\psi}\sin\phi & & & e^{-\psi}\cos\phi & \\ e^{\psi}\sin\phi & & & & & e^{\psi}\cos\phi \end{bmatrix}.$$

We organize this paper as follows. In Section 2 we set up a conformal structure for hypersurfaces. The integrable conditions are provided. We prove a theorem that helps us to identify two hypersurfaces easily in the sense of conformal equivalency. We also prove, with our setup, the coefficients in structure equations for homogeneous hypersurfaces are constant. In Section 3 we list several examples of homogeneous hypersurfaces. The classification is done in Section 4 by analysing the coefficients and using the theorem to identify them to the examples.

2 The Conformal Frame for Hypersurfaces

In real case the principal curvatures will turn from λ_i into $-\lambda_i$ when we choose $-n$ as a new normal vector field. So we can always assume $\lambda_1 < \lambda_3$ and then we obtain a conformal invariant 2-form

$$g_c = (\lambda_3 - \lambda_1)^2 g = 2(M_{23,1}^2 + M_{21,3}^2 + 1)^{-1} g_0.$$

We choose a new tangent frame on M as $E_i = \frac{e_i}{\lambda_1 - \lambda_3}$ so that $g_c(E_i, E_j) = g(e_i, e_j) = g_{ij}$.

As for the complex case, we similarly have the following conformal invariant 2-form provided that g_0 is non-degenerate,

$$g_c = b^2 g = -\frac{1}{2}(1 + M_{31,2}^2 + M_{32,1}^2)^{-1} g_0.$$

We set the following complex-valued tangent frame on M: $E_1 = \frac{e_1 + \mathbf{i}e_2}{\sqrt{2}b}$, $E_2 = \frac{\mathbf{i}e_1 + e_2}{\sqrt{2}b}$, $E_3 = \frac{e_3}{b}$. So E_i are all principal directions satisfying $g_c(E_i, E_j) = g_{ij}$.

Definition 2.1 *Given a regular hypersurface $x(M)$, we define the following terms*:

(1) *Conformal metric* g_c;

(2) *conformal orthonormal basis* $\{E_i\}$;

(3) *canonical lift* $Y=\rho_c y$, *where* ρ_c *equals* $\lambda_1-\lambda_3$ (*real case*) *or* b (*complex case*) *so that*

$$g_c=\langle \mathrm{d}Y,\mathrm{d}Y\rangle;$$

(4) *conformal normal vector* $\xi=\lambda_\alpha y+\xi_n$, *where*

$$\alpha=\begin{cases}1 & (\textit{real case})\\ 3 & (\textit{complex case})\end{cases},\qquad \xi_n:=(-\langle x,n\rangle_1,n,\langle x,n\rangle_1)^{\mathrm{T}}.$$

Let $Y_i=E_i(Y)$. Then $\{Y,\widehat{Y},\xi,Y_1,Y_2,Y_3\}$ is a moving frame of R^6_2 defined on M, where $\widehat{Y}$ is uniquely determined by $\langle\widehat{Y},\widehat{Y}\rangle=0$, $\langle\widehat{Y},Y\rangle=1$ and orthogonal to the others.

The structure equations with respect to this frame are given below:

$$\begin{aligned}
\mathrm{d}Y&=\sum_i\theta^iY_i,\\
\mathrm{d}\xi&=\boldsymbol{\Omega}Y+\sum_i\epsilon_i\boldsymbol{\Omega}^iY_i,\\
\mathrm{d}\widehat{Y}&=-\boldsymbol{\Omega}\xi-\sum_i\epsilon_i\boldsymbol{\Theta}^iY_i,\\
\mathrm{d}Y_i&=\boldsymbol{\Theta}^iY-\boldsymbol{\Omega}^i\xi-\epsilon_i\theta^i\widehat{Y}+\sum_j\epsilon_j\boldsymbol{\Omega}^{ij}Y_j,
\end{aligned}$$

in which θ^i is the dual basis of E_i, $\epsilon_i=g_{ii}$ and

$$\boldsymbol{\Omega}=\sum_i\Omega_i\theta^i,\quad \boldsymbol{\Omega}^i=\sum_j\Omega^i_j\theta^j,\quad \boldsymbol{\Theta}^i=\sum_j\Theta^i_j\theta^j,\quad \boldsymbol{\Omega}^{ij}=\sum_k\Omega^{ij}_k\theta^k=-\boldsymbol{\Omega}^{ji}.$$

Due to its definition we have $E_i(\xi_n)=-\lambda_iE_i(y)$, then $E_i(\lambda_iy+\xi_n)=\frac{E_i(\lambda_i)}{\rho_c}Y$. Meanwhile,

$$E_i(\lambda_iy+\xi_n)=E_i\Big(\xi+\frac{\lambda_i-\lambda_\alpha}{\rho_c}Y\Big)=E_i(\xi)+E_i\Big(\frac{\lambda_i-\lambda_\alpha}{\rho_c}\Big)Y+\frac{\lambda_i-\lambda_\alpha}{\rho_c}Y_i.$$

From the structure equations, $E_i(\xi)=\Omega_iY+\sum\limits_j\epsilon_j\Omega^j_iY_j$. Therefore,

$$\boldsymbol{\Omega}^i=\epsilon_ic_i\theta^i,\quad c_i=\frac{\lambda_\alpha-\lambda_i}{\rho_c}.$$

Lemma 2.1 *The integrable conditions with respect to* $\{Y,\widehat{Y},\xi,Y_1,Y_2,Y_3\}$ *are*

$$\Theta^i_j=\Theta^j_i,\quad \mathrm{d}\theta^i=\sum_{p,q}\epsilon_i\Omega^{iq}_p\theta^p\wedge\theta^q,$$

and

$$\sum_{p,q}\Big(E_p(\Omega_q)+\sum_i \epsilon_i\Omega_i\Omega_p^{iq}\Big)\theta^p\wedge\theta^q=\sum_{p,q}c_p\Theta_q^p\theta^p\wedge\theta^q, \tag{2.1}$$

$$\sum_p E_p(\epsilon_i c_i)\theta^p\wedge\theta^i+\sum_{p,q}c_i\Omega_p^{iq}\theta^p\wedge\theta^q=\sum_p\epsilon_i\Omega_p\theta^p\wedge\theta^i+\sum_{p,q}c_q\Omega_p^{iq}\theta^p\wedge\theta^q, \tag{2.2}$$

$$\begin{aligned}&\sum_{p,q}\Big(E_p(\Omega_q^{ij})+\sum_h\epsilon_h\Omega_h^{ij}\Omega_p^{hq}\Big)\theta^p\wedge\theta^q\\ &=\sum_p\epsilon_j\Theta_p^i\theta^p\wedge\theta^j+\sum_p\epsilon_i\Theta_p^j\theta^i\wedge\theta^p-\epsilon_i\epsilon_jc_ic_j\theta^i\wedge\theta^j+\sum_{p,q,h}\epsilon_h\Omega_p^{ih}\Omega_q^{hj}\theta^p\wedge\theta^q,\end{aligned} \tag{2.3}$$

$$\sum_{p,q}\Big(E_p(\Theta_q^i)+\sum_j\epsilon_j\Theta_j^i\Omega_p^{jq}\Big)\theta^p\wedge\theta^q=-\sum_p\epsilon_ic_i\Omega_p\theta^i\wedge\theta^p+\sum_{p,q,h}\epsilon_h\Omega_p^{ih}\Theta_q^h\theta^p\wedge\theta^q. \tag{2.4}$$

We define W, the conformal curvature of M, as below:

$$W=\begin{cases}M_{23,1}=\dfrac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1} & \text{(real case)},\\ \mathbf{i}(1-2M_{32,1})=\dfrac{\lambda_3-a}{b} & \text{(complex case)}.\end{cases}$$

Then W is conformal invariant and completely determines $\mathbf{\Omega}^i$ by

$$\begin{aligned}c_1=0,\quad c_2=W,\quad c_3=1\quad&\text{in real case},\\ c_1=W-\mathbf{i},\quad c_2=\overline{c}_1,\quad c_3=0\quad&\text{in complex case}.\end{aligned}$$

We claim that the whole system can be derived only from the conformal orthonormal basis and the conformal curvature.

Theorem 2.1 *$\{W;E_1,E_2,E_3\}$ is a complete system for M.*

Proof From the structure equations we have the Lie brackets of basis

$$[E_i,E_j]=\epsilon_i\Omega_i^{ji}E_i-\epsilon_j\Omega_j^{ij}E_j+\epsilon_k(\Omega_i^{jk}-\Omega_j^{ik})E_k. \tag{2.5}$$

One can see that

$$\begin{aligned}\Omega_i^{ij}&=-\Omega_i^{ji}=\langle[E_j,E_i],E_i\rangle,\\ \Omega_j^{ik}&=\frac{1}{2}(\langle[E_j,E_i],E_k\rangle+\langle[E_k,E_j],E_i\rangle+\langle[E_k,E_i],E_j\rangle),\end{aligned}$$

which means that $\mathbf{\Omega}^{ij}$ is determined by $[E_i,E_j]$.

By (2.2) we get $\epsilon_i\Omega_j-E_j(\epsilon_ic_i)=(c_j-c_i)\Omega_i^{ij}$. So $\mathbf{\Omega}$ is determined by W and E_i.

Moreover, both sides of (2.3) acting on $E_i\otimes E_j$ $(i\neq j)$ yields

$$\epsilon_j\Theta_i^i+\epsilon_i\Theta_j^j=E_i(\Omega_j^{ij})-E_j(\Omega_i^{ij})+\sum_h\epsilon_h(\Omega_i^{ih}\Omega_j^{jh}+\Omega_h^{ij}\Omega_i^{hj}+\Omega_i^{hj}\Omega_j^{ih}+\Omega_j^{ih}\Omega_h^{ij})+\epsilon_i\epsilon_jc_ic_j.$$

Acting on $E_k\otimes E_j$ $(j,k\neq i)$ yields

$$\epsilon_j\Theta_k^i=E_k(\Omega_j^{ij})-E_j(\Omega_k^{ij})+\sum_h\epsilon_h(\Omega_h^{ij}\Omega_k^{hj}+\Omega_j^{ih}\Omega_k^{hj}-\Omega_h^{ij}\Omega_j^{hk}-\Omega_k^{ih}\Omega_j^{hj}).$$

Obviously, $\mathbf{\Theta}^i$ is also determined by W and E_i.

Suppose that there are two immersed hypersurfaces in R^4_1 having same conformal curvature W and conformal orthonormal basis $\{E_i\}$ with same Lie brackets, then they must have same $\mathbf{\Omega}$, $\mathbf{\Omega}^i$, $\mathbf{\Omega}^{ij}$ and $\mathbf{\Theta}^i$. As a consequence, they are equivalent to each other up to a conformal transformation of R^4_1 induced by some $T \in O(4,2)$.

Remark 2.1 The above theorem can help us quickly finding out if two immersed hypersurfaces are conformally equivalent. All we have to do is calculating the principal curvatures with corresponding principal directions, then W, E_i and their Lie brackets can be easily obtained and compared with each other.

Now we suppose that $x(M)$ is a conformal homogeneous hypersurface in R^4_1, i.e., for any two point p and q, there is a conformal transformation $\sigma_{p,q}$ induced from $T_{p,q} \in O(4,2)$ such that $\sigma_{p,q}(x(p)) = x(q)$ and $\sigma_{p,q}(x(M)) = x(M)$. We will show that the frame defined in this section is conformal invariant, as stated in the following lemma.

Lemma 2.2 $T_{p,q} \cdot (Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_p = (Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_q$.

Proof In real case, we assume that $\gamma(t)$ is a curve on M satisfying $\gamma(0) = p$, $\gamma(1) = q$. Since $x(M)$ is generated by transformations induced from $G \subset O(4,2)$, there is a curve $T(t)$ in G such that $T(0) = \mathrm{id}$, $T(1) = T_{p,q}$ and $[T(t) \cdot Y(p)] = [Y(\gamma(t))]$. So $Y(\gamma(t)) = \rho_t T(t) \cdot Y(p)$ for some non-vanishing function ρ_t. This is equivalent to saying that $T(t)$ induce a family of conformal transformations σ_t such that $\sigma_t(x(p)) = x(\gamma(t))$. Because $g_c = \langle \mathrm{d}Y, \mathrm{d}Y \rangle$ is conformal invariant, for any fixed t we have

$$\langle \mathrm{d}Y, \mathrm{d}Y \rangle|_p = \langle \mathrm{d}Y, \mathrm{d}Y \rangle|_{\gamma(t)} = \rho_t^2 \langle \mathrm{d}(T(t) \cdot Y(p)), \mathrm{d}(T(t) \cdot Y(p)) \rangle = \rho_t^2 \langle \mathrm{d}Y, \mathrm{d}Y \rangle|_p.$$

Thus we get $\rho_t = 1$ since $T(0) = \mathrm{id}$, i.e.,

$$T(t) \cdot Y(p) = Y(\gamma(t)).$$

Now we define a curve $c(s)$ passing through p such that $c(0) = p$ and $c'(0) = E_i$. So

$$Y(c(0)) = Y(p), \quad \frac{\partial}{\partial s} Y(c(s))\big|_{s=0} = Y_i(p) = E_i|_p(Y).$$

$T(t) \cdot Y(c(s))$ is obviously a curve passing through $T(t) \cdot Y(p)$ satisfying

$$\frac{\partial}{\partial s} T(t) \cdot Y(c(s))\big|_{s=0} = T(t) \cdot Y_i(p) = \widetilde{E}_i|_{\gamma_t}(Y), \tag{2.6}$$

in which $\widetilde{E}_i = \mathrm{d}x^{-1} \circ \mathrm{d}\sigma_t \circ \mathrm{d}x(E_i)$ is the tangent vector at $\gamma(t)$ determined by the tangent map of σ_t acting on E_i. Since conformal transformations will map principal directions to principal directions, we know that $\widetilde{E}_i$ is a principal vector at $\gamma(t)$. Due to

$$g_c(\widetilde{E}_i, \widetilde{E}_i)|_{\gamma_t} = \langle T(t) \cdot Y_i(p), T(t) \cdot Y_i(p) \rangle = \langle Y_i(p), Y_i(p) \rangle = \epsilon_i = g_c(E_i, E_i)|_{\gamma_t},$$

we obtain $\widetilde{E}_1 = \pm E_1$, $\{\widetilde{E}_2, \widetilde{E}_3\} = \{\pm E_2, \pm E_3\}$. By (2.6) and $T(0) = \mathrm{id}$ we have $\widetilde{E}_i = E_i$ and thus

$$T(t) \cdot Y_i(p) = Y_i(\gamma(t)).$$

Because conformal transformations map curvature spheres to curvature spheres, $\xi = \lambda_1 y + \xi_n$, which represents the curvature sphere corresponding to a time-like direction, must be mapped into a curvature sphere corresponding to a time-like direction, i.e.,

$$T(t) \cdot \xi(p) = \xi(\gamma(t)).$$

At last, $\widehat{Y}(\gamma(t))$ is uniquely determined by $\{Y, \xi, Y_1, Y_2, Y_3\}|_{\gamma(t)}$ and therefore

$$T(t) \cdot \widehat{Y}(p) = \widehat{Y}(\gamma(t)).$$

The proof of the complex case is similar.

As a result we have the following lemma which is crucial in our classification.

Lemma 2.3 *All the coefficients in structure equations are constant with respect to the conformal invariant frame.*

Proof The structure equations around point p are actually a differential system:

$$\mathrm{d}(Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_p = (Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_p \cdot \Lambda_p.$$

Analogously, around q there is

$$\mathrm{d}(Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_q = (Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_q \cdot \Lambda_q.$$

By Lemma 2.2,

$$\begin{aligned}\mathrm{d}(Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_q &= \mathrm{d}(T_{p,q} \cdot (Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_p) = T_{p,q} \cdot \mathrm{d}(Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_p \\ &= T_{p,q} \cdot (Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_p \cdot \Lambda_p = (Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_q \cdot \Lambda_p = (Y, \widehat{Y}, \xi, Y_1, Y_2, Y_3)|_q \cdot \Lambda_q.\end{aligned}$$

So $\Lambda_p = \Lambda_q$ for any two points.

3 Examples of Conformally Homogeouse Hypersurfaces

A basis $\{u_i\}$ of R^6_2 is called orthonormal or pseudo-orthonormal if the matrices of their inner products are respectively in form of

$$\begin{bmatrix} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} & -1 & & & & \\ -1 & & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & -1 \\ & & & & -1 & \end{bmatrix}.$$

In the following we take an orthonormal basis in the first two examples and pseudo-orthonormal basis in the others.

Example 3.1 For any $A \in (0, \frac{\pi}{2})$, we have a conformal homogeneous hypersurface as below:

$$\begin{aligned} G(\psi, \theta, \phi) &= \begin{bmatrix} \cosh\psi & \sinh\psi & & & & \\ \sinh\psi & \cosh\psi & & & & \\ & & \cos\theta & -\sin\theta & & \\ & & \sin\theta & \cos\theta & & \\ & & & & \cosh\phi & \sinh\phi \\ & & & & \sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \cos A \\ 0 \\ \sin A \\ 0 \end{bmatrix} \\ &= (\cosh\psi, \sinh\psi, \cos A\cos\theta, \cos A\sin\theta, \sin A\cosh\phi, \sin A\sinh\phi)^{\mathrm{T}}. \end{aligned}$$

If we define a hypersurface of R_1^4 by

$$x = \mathrm{e}^{-\psi}(\cos A\cos\theta, \cos A\sin\theta, \sin A\cosh\phi, \sin A\sinh\phi)^{\mathrm{T}},$$

then from

$$\begin{aligned} x_\psi &= -x, \\ x_\theta &= \mathrm{e}^{-\psi}\cos A(-\sin\theta, \cos\theta, 0, 0)^{\mathrm{T}}, \\ x_\phi &= \mathrm{e}^{-\psi}\sin A(0, 0, \sinh\phi, \cosh\phi)^{\mathrm{T}}, \end{aligned}$$

we know that the induced metric is $g = \mathrm{e}^{-2\psi}(\mathrm{d}\psi)^2 + \mathrm{e}^{-2\psi}\cos^2 A(\mathrm{d}\theta)^2 - \mathrm{e}^{-2\psi}\sin^2 A(\mathrm{d}\phi)^2$. The unit normal vector field can be chosen as $n = (\sin A\cos\theta, \sin A\sin\theta, -\cos A\cosh\phi, -\cos A\sinh\phi)^{\mathrm{T}}$. So the principal curvatures of x are $\lambda_1 = -\cot A\mathrm{e}^\psi$, $\lambda_2 = 0$, $\lambda_3 = \tan A\mathrm{e}^\psi$.

Thus, the conformal curvature is

$$W = \cos^2 A \in (0, 1)$$

and the conformal metric is $g_c = -\frac{1}{\cos^2 A}(\mathrm{d}\phi)^2 + \frac{1}{\sin^2 A\cos^2 A}(\mathrm{d}\psi)^2 + \frac{1}{\sin^2 A}(\mathrm{d}\theta)^2$. Obviously, the conformal orthonormal basis is $E_1 = \cos A\frac{\partial}{\partial\phi}$, $E_2 = \sin A\cos A\frac{\partial}{\partial\psi}$, $E_3 = \sin A\frac{\partial}{\partial\theta}$, satisfying

$$[E_i, E_j] = 0.$$

Example 3.2 For any $A \in (0, +\infty)$, we have a conformal homogeneous hypersurface

$$\begin{aligned} G(\psi,\theta,\phi) &= \begin{bmatrix} \cos\psi & & & & & -\sin\psi \\ & \cos\theta & & & -\sin\theta & \\ & & \cos\phi & -\sin\phi & & \\ & & \sin\phi & \cos\phi & & \\ & \sin\theta & & & \cos\theta & \\ \sin\psi & & & & & \cos\psi \end{bmatrix} \begin{bmatrix} \cosh A \\ \sinh A \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= (\cosh A\cos\psi, \sinh A\cos\theta, \cos\phi, \sin\phi, \sinh A\sin\theta, \cosh A\sin\psi)^{\mathrm{T}}. \end{aligned}$$

The conformal structure about this hypersurface is much easier to check in H_1^4 than in R_1^4. Extracting the component $\cos\phi$ we can define

$$x = (\tan\phi, \sec\phi(\sinh A\cos\theta, \sinh A\sin\theta, \cosh A\cos\psi, \cosh A\sin\psi))^T \in H_1^4 \subset R_2^5$$

provided $\cos\phi > 0$ (the case $\cos\phi < 0$ is actually $-x$ and $\cos\phi = 0$ can be similarly studied by extracting the component $\sin\phi$). Then from

$$\begin{aligned} x_\theta &= \sec\phi\sinh A(0, -\sin\theta, \cos\theta, 0, 0)^{\mathrm{T}}, \\ x_\psi &= \sec\phi\cosh A(0, 0, 0, -\sin\psi, \cos\psi)^{\mathrm{T}}, \\ x_\phi &= \sec\phi(\sec\phi, \tan\phi(\sinh A\cos\theta, \sinh A\sin\theta, \cosh A\cos\psi, \cosh A\sin\psi))^{\mathrm{T}}, \end{aligned}$$

we know the induced metric is $g = \sec^2\phi((\mathrm{d}\phi)^2 + \sinh^2 A(\mathrm{d}\theta)^2 - \cosh^2 A(\mathrm{d}\psi)^2)$. The unit normal vector is $n = -(0, \cosh A\cos\theta, \cosh A\sin\theta, \sinh A\cos\psi, \sinh A\sin\psi)^{\mathrm{T}}$. So the principal curvatures of x are $\lambda_1 = \cos\phi\tanh A$, $\lambda_2 = 0$, $\lambda_3 = \cos\phi\coth A$.

So it is easy to get the conformal curvature

$$W = -\sinh^2 A \in (-\infty, 0)$$

and the conformal metric $g_c = \frac{1}{\sinh^2 A\cosh^2 A}(\mathrm{d}\phi)^2 + \frac{1}{\cosh^2 A}(\mathrm{d}\theta)^2 - \frac{1}{\sinh^2 A}(\mathrm{d}\psi)^2$. Therefore, the conformal orthonormal basis is $E_1 = \sinh A\frac{\partial}{\partial\psi}$, $E_2 = \sinh A\cosh A\frac{\partial}{\partial\phi}$, $E_3 = \cosh A\frac{\partial}{\partial\theta}$, satisfying

$$[E_i, E_j] = 0.$$

Example 3.3 We define a homogeneous hypersurfaces as

$$G(\psi,\theta,\phi) = \begin{bmatrix} \mathrm{e}^{\psi} & \frac{\theta^2}{2}\mathrm{e}^{-\psi} & \theta & & -\psi\mathrm{e}^{\psi} & \\ 0 & \mathrm{e}^{-\psi} & 0 & & & \\ 0 & \theta\mathrm{e}^{-\psi} & 1 & & & \\ & & & 1 & \phi\mathrm{e}^{\psi} & 0 \\ & & & 0 & \mathrm{e}^{\psi} & 0 \\ & \psi\mathrm{e}^{-\psi} & & \phi & \frac{\phi^2}{2}\mathrm{e}^{\psi} & \mathrm{e}^{-\psi} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$
$$= \Big(\frac{\theta^2}{2}\mathrm{e}^{-\psi} - \psi\mathrm{e}^{\psi}, \mathrm{e}^{-\psi}, \theta\mathrm{e}^{-\psi}, \phi\mathrm{e}^{\psi}, \mathrm{e}^{\psi}, \frac{\phi^2}{2}\mathrm{e}^{\psi} + \psi\mathrm{e}^{-\psi}\Big)^{\mathrm{T}}.$$

One can verify that

$$x(\psi,\theta,\phi) = \Big(\theta, \phi\mathrm{e}^{2\psi}, \mathrm{e}^{2\psi}, \frac{\phi^2}{2}\mathrm{e}^{2\psi} + \psi\Big)^{\mathrm{T}}$$

is a hypersurface in R^4_1 and $G(\psi,\theta,\phi)$ is equivalent to $\mathrm{e}^{-\psi}\big(\frac{1-\langle x,x\rangle_1}{2}, x, \frac{1+\langle x,x\rangle_1}{2}\big)^{\mathrm{T}}$. The induced metric is $g = (\mathrm{d}\theta)^2 + \mathrm{e}^{4\psi}(\mathrm{d}\phi)^2 - 4\mathrm{e}^{2\psi}(\mathrm{d}\psi)^2$. The normal vector is $n = \mathrm{e}^{\psi}\big(0, \phi, 1, \frac{\phi^2}{2} - \frac{1}{2}\mathrm{e}^{-2\psi}\big)^{\mathrm{T}}$. It is easy to get that $n_\theta = 0$, $n_\phi = \mathrm{e}^{-\psi}x_\phi$, $n_\psi = \frac{1}{2}\mathrm{e}^{-\psi}x_\psi$. Therefore, $\lambda_1 = \frac{1}{2}\mathrm{e}^{-\psi}$, $\lambda_2 = 0$ and $\lambda_3 = \mathrm{e}^{-\psi}$. So the conformal curvature is

$$W = -1.$$

Hence, the conformal metric is $g_c = -(\mathrm{d}\psi)^2 + \frac{1}{4}\mathrm{e}^{-2\psi}(\mathrm{d}\theta)^2 + \frac{1}{4}\mathrm{e}^{2\psi}(\mathrm{d}\phi)^2$ and the conformal orthonormal basis can be chosen as $E_1 = -\frac{\partial}{\partial\psi}$, $E_2 = 2\mathrm{e}^{\psi}\frac{\partial}{\partial\theta}$, $E_3 = 2\mathrm{e}^{-\psi}\frac{\partial}{\partial\phi}$. The Lie brackets of basis are in form of

$$[E_1, E_2] = -E_2, \quad [E_1, E_3] = E_3, \quad [E_2, E_3] = 0.$$

Example 3.4 We define a homogeneous hypersurfaces for any constant $W \in (-1,0)\cup(0,1)$,

$$G(\psi,\theta,\phi) = \begin{bmatrix} \mathrm{e}^{-\psi} & \frac{\theta^2}{2}\mathrm{e}^{\psi} & \theta & & & \\ 0 & \mathrm{e}^{\psi} & 0 & & & \\ 0 & \theta\mathrm{e}^{\psi} & 1 & & & \\ & & & 1 & \phi\mathrm{e}^{W\psi} & 0 \\ & & & 0 & \mathrm{e}^{W\psi} & 0 \\ & & & \phi & \frac{\phi^2}{2}\mathrm{e}^{W\psi} & \mathrm{e}^{-W\psi} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$
$$= \Big(\frac{\theta^2}{2}\mathrm{e}^{\psi} - \mathrm{e}^{-\psi}, \mathrm{e}^{\psi}, \theta\mathrm{e}^{\psi}, \phi\mathrm{e}^{W\psi}, \mathrm{e}^{W\psi}, \frac{\phi^2}{2}\mathrm{e}^{W\psi} + \mathrm{e}^{-W\psi}\Big)^{\mathrm{T}}.$$

First we have

$$x(\psi,\theta,\phi)=\Big(\frac{\theta^2}{2}\mathrm{e}^{(1-W)\psi}-\mathrm{e}^{-(1+W)\psi},\mathrm{e}^{(1-W)\psi},\theta\mathrm{e}^{(1-W)\psi},\phi\Big)^{\mathrm{T}}$$

and $G(\psi,\theta,\phi)$ is equivalent to $\mathrm{e}^{W\psi}\big(\frac{1-\langle x,x\rangle_1}{2},x,\frac{1+\langle x,x\rangle_1}{2}\big)^{\mathrm{T}}$. One can see that

$$\begin{aligned}&x_\phi=(0,0,0,1)^{\mathrm{T}},\quad x_\theta=\mathrm{e}^{(1-W)\psi}(\theta,0,1,0)^{\mathrm{T}},\\&x_\psi=(1-W)\mathrm{e}^{(1-W)\psi}\Big(\frac{\theta^2}{2}+\frac{1+W}{1-W}\mathrm{e}^{-2\psi},1,\theta,0\Big)^{\mathrm{T}}\end{aligned}$$

and the induced metric is $g=(\mathrm{d}\phi)^2+\mathrm{e}^{2(1-W)\psi}(\mathrm{d}\theta)^2+2(W^2-1)\mathrm{e}^{-2W\psi}(\mathrm{d}\psi)^2$. The unit normal vector is $n=-\sqrt{\frac{1-W}{2(1+W)}}\mathrm{e}^{\psi}\big(\frac{\theta^2}{2}-\frac{1+W}{1-W}\mathrm{e}^{-2\psi},1,\theta,0\big)^{\mathrm{T}}$, by which we have $n_\phi=0$, $n_\theta=-\sqrt{\frac{1-W}{2(1+W)}}\mathrm{e}^{W\psi}x_\theta$, $n_\psi=\frac{1}{W-1}\sqrt{\frac{1-W}{2(1+W)}}\mathrm{e}^{W\psi}x_\psi$. So the principal curvatures are

$$\lambda_1=\frac{1}{W-1}\sqrt{\frac{1-W}{2(1+W)}}\mathrm{e}^{W\psi},\quad \lambda_2=-\sqrt{\frac{1-W}{2(1+W)}}\mathrm{e}^{W\psi},\quad \lambda_3=0.$$

Therefore, the conformal curvature is W and the conformal metric obviously is $g_c=-(\mathrm{d}\psi)^2+\frac{1}{2(1-W^2)}\mathrm{e}^{2W\psi}(\mathrm{d}\phi)^2+\frac{1}{2(1-W^2)}\mathrm{e}^{2\psi}(\mathrm{d}\theta)^2$. Finally we obtain a conformal orthonormal basis $E_1=\frac{\partial}{\partial\psi}$, $E_2=\sqrt{2(1-W^2)}\mathrm{e}^{-\psi}\frac{\partial}{\partial\theta}$, $E_3=\sqrt{2(1-W^2)}\mathrm{e}^{-W\psi}\frac{\partial}{\partial\phi}$, whose Lie brackets are in form of

$$[E_1,E_2]=-E_2,\quad [E_1,E_3]=-WE_3,\quad [E_2,E_3]=0.$$

Example 3.5 We define a family of homogeneous hypersurfaces for $A\in(0,\pi)$ as below:

$$\begin{aligned}G(\psi,\theta,\phi)&=\begin{bmatrix}\mathrm{e}^{-2\psi}&\mathrm{e}^{2\psi}\Big(\frac{\theta^2}{2}-\frac{\phi^4}{6}\Big)&\theta&\phi^2&\frac{\phi^3}{3}\mathrm{e}^{\psi}&2\phi\mathrm{e}^{-\psi}\\&\mathrm{e}^{2\psi}&&&&\\&\theta\mathrm{e}^{2\psi}&1&&&\\&-\phi^2\mathrm{e}^{2\psi}&&1&\phi e^{\psi}&\\&-2\phi\mathrm{e}^{2\psi}&&&e^{\psi}&\\&-\frac{\phi^3}{3}\mathrm{e}^{2\psi}&&\phi&\frac{\phi^2}{2}\mathrm{e}^{\psi}&\mathrm{e}^{-\psi}\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\1\\0\\1\\0\\\frac{2\cosh A}{3}\end{bmatrix}\\&=\Big(\mathrm{e}^{2\psi}\Big(\frac{\theta^2}{2}-\frac{\phi^4}{6}\Big)+\frac{1}{2}\mathrm{e}^{-2\psi}+\phi^2+\frac{4\phi\cosh A}{3}\mathrm{e}^{-\psi},\mathrm{e}^{2\psi},\theta\mathrm{e}^{2\psi},\\&\qquad 1-\phi^2\mathrm{e}^{2\psi},-2\phi\mathrm{e}^{2\psi},-\frac{\phi^3}{3}\mathrm{e}^{2\psi}+\phi+\frac{2\cosh A}{3}\mathrm{e}^{-\psi}\Big)^{\mathrm{T}}.\end{aligned}$$

A hypersurface

$$x(\psi,\theta,\phi)=\Big(\theta,(1-\phi^2)\mathrm{e}^{-2\psi},-2\phi\mathrm{e}^{-\psi},\Big(\frac{2\cosh A}{3}+\phi-\frac{\phi^3}{3}\Big)\mathrm{e}^{-3\psi}\Big)^{\mathrm{T}}$$

can be defined such that $G(\psi,\theta,\phi)$ is equivalent to $\mathrm{e}^{2\psi}\Big(\frac{1-\langle x,x\rangle_1}{2},x,\frac{1+\langle x,x\rangle_1}{2}\Big)^{\mathrm{T}}$.

One can check that

$$\begin{aligned}&x_\theta=(1,0,0,0)^{\mathrm{T}},\\&x_\psi=(0,-2\mathrm{e}^{-2\psi},0,-2\phi\mathrm{e}^{-2\psi}-2\cosh A\mathrm{e}^{-3\psi})^{\mathrm{T}},\\&x_\phi=(0,-2\phi,-2,-\phi^2+\mathrm{e}^{-2\psi})^{\mathrm{T}}\end{aligned}$$

and the induced metric is in form of

$$g = (\mathrm{d}\theta)^2 + 4\mathrm{e}^{-2\psi}(\mathrm{d}\phi)^2 - 8\cosh A\mathrm{e}^{-3\psi}\mathrm{d}\phi\mathrm{d}\psi + 4\mathrm{e}^{-4\psi}(\mathrm{d}\psi)^2.$$

Taking a unit normal $n = -\frac{\mathrm{e}^{\psi}}{\sinh A}\big(0, \phi + \cosh A\mathrm{e}^{-\psi}, 1, \frac{\phi^2+\mathrm{e}^{-2\psi}}{2} + \phi\cosh A\mathrm{e}^{-\psi}\big)^{\mathrm{T}}$, we have

$$n_\theta = 0, \quad n_\phi = \frac{\mathrm{e}^{3\psi}}{2\sinh A}x_\psi, \quad n_\psi = \frac{\mathrm{e}^{\psi}}{2\sinh A}x_\phi.$$

Therefore, $\lambda_1 = -\frac{\mathrm{e}^{2\psi}}{2\sinh A}$, $\lambda_2 = 0$, $\lambda_3 = \frac{\mathrm{e}^{2\psi}}{2\sinh A}$. The corresponding principal directions are respectively $\mathrm{e}^{-\psi}\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}$, $\frac{\partial}{\partial\theta}$, $\mathrm{e}^{-\psi}\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\psi}$. So the conformal curvature is

$$W = \frac{1}{2}.$$

The conformal metric is thus in form of

$$g_c = \frac{\mathrm{e}^{4\psi}}{\sinh^2 A}(\mathrm{d}\theta)^2 + \frac{4\mathrm{e}^{2\psi}}{\sinh^2 A}(\mathrm{d}\phi)^2 - \frac{8\cosh A\mathrm{e}^{\psi}}{\sinh^2 A}\mathrm{d}\psi\mathrm{d}\phi + \frac{4}{\sinh^2 A}(\mathrm{d}\psi)^2.$$

If we set the following conformal orthonormal basis

$$\begin{aligned} E_1 &= \frac{1}{2}\cosh\frac{A}{2}\Big(\mathrm{e}^{-\psi}\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}\Big),\\ E_2 &= \sinh A\mathrm{e}^{-2\psi}\frac{\partial}{\partial\theta},\\ E_3 &= \frac{1}{2}\sinh\frac{A}{2}\Big(\mathrm{e}^{-\psi}\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\psi}\Big),\end{aligned}$$

then we get

$$[E_1, E_2] = -\cosh\frac{A}{2}E_2, \quad [E_1, E_3] = -\frac{1}{2}\sinh\frac{A}{2}E_1 - \frac{1}{2}\cosh\frac{A}{2}E_3, \quad [E_2, E_3] = -\sinh\frac{A}{2}E_2.$$

Example 3.6 A homogeneous hypersurfaces for any $A \in (-\frac{\pi}{2}, \frac{\pi}{2})$ can be defined by

$$\begin{aligned} G &= \begin{bmatrix} \mathrm{e}^{\psi}\cos\phi & & & & & -\mathrm{e}^{\psi}\sin\phi\\ & \mathrm{e}^{-\psi}\cos\phi & & & -\mathrm{e}^{-\psi}\sin\phi & \\ & & \cos\theta & -\sin\theta & & \\ & & \sin\theta & \cos\theta & & \\ & \mathrm{e}^{-\psi}\sin\phi & & & \mathrm{e}^{-\psi}\cos\phi & \\ \mathrm{e}^{\psi}\sin\phi & & & & & \mathrm{e}^{\psi}\cos\phi \end{bmatrix} \begin{bmatrix} \cos A\\ \cos A\\ \sqrt{2}\cos A\\ 0\\ \sin A\\ 0 \end{bmatrix}\\ &= (\mathrm{e}^{\psi}\cos\phi\cos A, \mathrm{e}^{-\psi}\cos(\phi+A), \sqrt{2}\cos A\cos\theta,\\ &\qquad \sqrt{2}\cos A\sin\theta, \mathrm{e}^{-\psi}\sin(\phi+A), \mathrm{e}^{\psi}\sin\phi\cos A)^{\mathrm{T}}.\end{aligned}$$

We study this hypersurface in H_1^4 to obtain an immersion as

$$x(\psi,\theta,\phi) = \Big(\tan\theta, \frac{\sec\theta}{\sqrt{2}}(\mathrm{e}^{\psi}\cos\phi, \mathrm{e}^{-\psi}\frac{\cos(\phi+A)}{\cos A}, \mathrm{e}^{-\psi}\frac{\sin(\phi+A)}{\cos A}, \mathrm{e}^{\psi}\sin\phi)\Big)^{\mathrm{T}}.$$

One can check that

$$\begin{aligned} x_\theta &= \sec\theta\Big(\sec\theta, \frac{\tan\theta}{\sqrt{2}}(\mathrm{e}^{\psi}\cos\phi, e^{-\psi}\frac{\cos(\phi+A)}{\cos A}, \mathrm{e}^{-\psi}\frac{\sin(\phi+A)}{\cos A}, \mathrm{e}^{\psi}\sin\phi)\Big)^{\mathrm{T}},\\ x_\phi &= \Big(0, \frac{\sec\theta}{\sqrt{2}}(-\mathrm{e}^{\psi}\sin\phi, -\mathrm{e}^{-\psi}\frac{\sin(\phi+A)}{\cos A}, \mathrm{e}^{-\psi}\frac{\cos(\phi+A)}{\cos A}, \mathrm{e}^{\psi}\cos\phi)\Big)^{\mathrm{T}},\\ x_\psi &= \Big(0, \frac{\sec\theta}{\sqrt{2}}(\mathrm{e}^{\psi}\cos\phi, -\mathrm{e}^{-\psi}\frac{\cos(\phi+A)}{\cos A}, -\mathrm{e}^{-\psi}\frac{\sin(\phi+A)}{\cos A}, \mathrm{e}^{\psi}\sin\phi)\Big)^{\mathrm{T}}\end{aligned}$$

and the induced metric is $g = \sec^2\theta((\mathrm{d}\theta)^2 - (\mathrm{d}\phi)^2 + 2\tan A\mathrm{d}\phi\mathrm{d}\psi + (\mathrm{d}\psi)^2)$.

Set the unit normal vector by $n = \cos\theta(-\cos Ax_{\phi\psi} + \sin Ax_{\psi\psi})$. We get

$$n_\theta = 0, \quad n_\phi = \cos\theta(\sin Ax_\phi + \cos Ax_\psi), \quad n_\psi = \cos\theta(-\cos Ax_\phi + \sin Ax_\psi).$$

Therefore, $\lambda_1 = \cos\theta(-\sin A + \mathbf{i}\cos A)$, $\lambda_2 = \overline{\lambda_1}$ and $\lambda_3 = 0$, whose principal directions are respectively $\frac{\partial}{\partial\phi} + \mathbf{i}\frac{\partial}{\partial\psi}$, $\frac{\partial}{\partial\psi} + \mathbf{i}\frac{\partial}{\partial\phi}$, $\frac{\partial}{\partial\theta}$. Obviously, the conformal curvature is

$$W = \tan A \in (-\infty, +\infty).$$

The conformal metric is in form of $g_c = \cos^2 A((\mathrm{d}\theta)^2 - (\mathrm{d}\phi)^2 + 2\tan A\mathrm{d}\psi\mathrm{d}\phi + (\mathrm{d}\psi)^2)$. We choose the following conformal orthonormal basis

$$E_1 = C\Big(\frac{\partial}{\partial\phi} + \mathbf{i}\frac{\partial}{\partial\psi}\Big), \quad E_2 = \mathbf{i}\overline{E}_1, \quad E_3 = \frac{1}{\cos A}\frac{\partial}{\partial\theta}, \quad C = (2\cos A\mathrm{e}^{-\mathbf{i}A})^{-\frac{1}{2}}.$$

Then we have

$$[E_i, E_j] = 0.$$

Example 3.7 A family of homogeneous hypersurfaces can be defined by

$$\begin{aligned} G(\psi,\theta,\phi) &= \begin{bmatrix} \mathrm{e}^{-2\psi} & \mathrm{e}^{2\psi}\Big(\frac{\theta^2}{2} - \frac{\phi^4}{6}\Big) & \theta & \phi^2 & \frac{\phi^3}{3}\mathrm{e}^{\psi} & 2\phi\mathrm{e}^{-\psi} \\ & \mathrm{e}^{2\psi} & & & & \\ & \theta\mathrm{e}^{2\psi} & 1 & & & \\ & -\phi^2\mathrm{e}^{2\psi} & & 1 & \phi\mathrm{e}^{\psi} & \\ & -2\phi\mathrm{e}^{2\psi} & & & \mathrm{e}^{\psi} & \\ & -\frac{\phi^3}{3}\mathrm{e}^{2\psi} & & \phi & \frac{\phi^2}{2}\mathrm{e}^{\psi} & \mathrm{e}^{-\psi} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ -1 \\ 0 \\ \frac{2\sinh A}{3} \end{bmatrix} \\ &= \Big(\mathrm{e}^{2\psi}\Big(\frac{\theta^2}{2} - \frac{\phi^4}{6}\Big) + \frac{1}{2}\mathrm{e}^{-2\psi} - \phi^2 + \frac{4\phi\sinh A}{3}\mathrm{e}^{-\psi}, \mathrm{e}^{2\psi}, \theta\mathrm{e}^{2\psi}, \\ &\quad -1 - \phi^2\mathrm{e}^{2\psi}, -2\phi\mathrm{e}^{2\psi}, -\frac{\phi^3}{3}\mathrm{e}^{2\psi} - \phi + \frac{2\sinh A}{3}\mathrm{e}^{-\psi}\Big)^{\mathrm{T}}. \end{aligned}$$

It is easy to see that $G(\psi,\theta,\phi)$ is equivalent to $\mathrm{e}^{2\psi}\big(\frac{1-\langle x,x\rangle_1}{2}, x, \frac{1+\langle x,x\rangle_1}{2}\big)^{\mathrm{T}}$, in which

$$x(\psi,\theta,\phi) = \Big(\theta, -\mathrm{e}^{-2\psi} - \phi^2, -2\phi, -\frac{\phi^3}{3} - \phi\mathrm{e}^{-2\psi} + \frac{2\sinh A}{3}\mathrm{e}^{-3\psi}\Big)^{\mathrm{T}}.$$

From

$$\begin{aligned} x_\phi &= (0, -2\phi, -2, -\phi^2 - \mathrm{e}^{-2\psi})^{\mathrm{T}}, \\ x_\psi &= (0, 2\mathrm{e}^{-2\psi}, 0, 2\phi\mathrm{e}^{-2\psi} - 2\sinh A\mathrm{e}^{-3\psi})^{\mathrm{T}}, \\ x_\theta &= (1, 0, 0, 0)^{\mathrm{T}}, \end{aligned}$$

we know that the induced metric is $g = (\mathrm{d}\theta)^2 - 4\mathrm{e}^{-2\psi}(\mathrm{d}\phi)^2 - 8\sinh A\mathrm{e}^{-3\psi}\mathrm{d}\phi\mathrm{d}\psi + 4\mathrm{e}^{-4\psi}(\mathrm{d}\psi)^2$. The unit normal is $n = \frac{\mathrm{e}^{\psi}}{\cosh A}\big(0, \phi - \sinh A\mathrm{e}^{-\psi}, 1, \frac{\phi^2 - \mathrm{e}^{-2\psi}}{2} - \phi\sinh A\mathrm{e}^{-\psi}\big)^{\mathrm{T}}$, by which we get $n_\theta = 0$, $n_\phi = \frac{\mathrm{e}^{3\psi}}{2\cosh A}x_\psi$, $n_\psi = -\frac{\mathrm{e}^{\psi}}{2\cosh A}x_\phi$. So $\lambda_1 = \mathbf{i}\frac{\mathrm{e}^{2\psi}}{2\cosh A}$, $\lambda_2 = -\mathbf{i}\frac{\mathrm{e}^{2\psi}}{2\cosh A}$, $\lambda_3 = 0$, and the principal directions are respectively $\frac{\partial}{\partial\phi} + \mathbf{i}\mathrm{e}^{\psi}\frac{\partial}{\partial\psi}$, $\frac{\partial}{\partial\phi} - \mathbf{i}\mathrm{e}^{\psi}\frac{\partial}{\partial\psi}$, $\frac{\partial}{\partial\theta}$. The conformal curvature is

$$W = 0.$$

The conformal metric is in form of

$$g_c = \frac{1}{\cosh^2 A}\Big(\frac{e^{4\psi}}{4}(d\theta)^2 - e^{2\psi}(d\phi)^2 - 2\sinh A e^{\psi} d\psi d\phi + (d\psi)^2\Big).$$

The following conformal orthonormal basis

$$E_1 = C\Big(e^{-\psi}\frac{\partial}{\partial\phi} + \mathbf{i}\frac{\partial}{\partial\psi}\Big), \quad E_2 = \mathbf{i}\overline{E}_1, \quad E_3 = 2\cosh A e^{-2\psi}\frac{\partial}{\partial\theta}, \quad C = -\Big(\frac{1-\mathbf{i}\sinh A}{2}\Big)^{\frac{1}{2}}$$

yield

$$[E_1, E_2] = \overline{C}E_1 - \mathbf{i}CE_2, \quad [E_1, E_3] = -2\mathbf{i}CE_3, \quad [E_2, E_3] = -2\overline{C}E_3.$$

Remark 3.1 It is obvious that all the hypersurfaces come from the orbits of some 3-dimensional subgroup of $O(4,2)$. Thus their universal covering is R^3. in Example 3.1, the hypersurface is topologically $R^2\times S^1$, equivalent to a cone over a time-like homogenous cylinder in S^3_1; in Example 3.2 it is T^3, the only compact case here. Half part of it is equivalent to a cylinder in H^4_1 over a time-like homogenous torus in H^3_1; in Example 3.6 it is $T^2\times R$, half of which is equivalent to a cylinder over a time-like homogenous cylinder in H^3_1.

It should also be noted that Examples 3.5 and 3.7 are essentially generated by the same group.

4 Classification of Conformally Homogeneous Hypersurfaces

4.1 Real case

Since $W = \frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}$, one can see that $0 < W < 1$ when $\lambda_1 < \lambda_2 < \lambda_3$ and $W < 0$ when $\lambda_2 < \lambda_1 < \lambda_3$. For the latter case, we may choose $-n$ as normal vector so that $-1 \le W < 0$.

From structure equations it is easy to see that

$$c_i = \epsilon_i\langle E_i(\xi), E_i(Y)\rangle, \quad \Omega_i = \epsilon_j(c_j - c_i)\langle E_jE_j(Y), E_i(Y)\rangle,$$

which means that we may always assume $\Omega_i \ge 0$ by a change of $E_i \to -E_i$ if necessary.

For convenience we assume $\{i,j,k\} = \{1,2,3\}$. If we set $R = (c_2-c_1)\Omega^{12}_3$, then the integrable conditions (2.1)–(2.4) can be written as

$$\Omega^{ij}_k = \frac{R}{c_j - c_i}, \quad \Omega^{ij}_i = \frac{\epsilon_i\Omega_j}{c_j - c_i}, \tag{4.1}$$

$$\Theta^i_j = -\frac{2\Omega_i\Omega_j}{(c_i - c_j)^2} + \frac{\epsilon_k R\Omega_k}{(c_k - c_i)(c_k - c_j)}, \tag{4.2}$$

$$\epsilon_i\Theta^i_i + \epsilon_j\Theta^j_j = \epsilon_i(\Omega^{ij}_j)^2 + \epsilon_j(\Omega^{ji}_i)^2 - \Omega^{ki}_i\Omega^{kj}_j + 2\Omega^{ki}_j\Omega^{kj}_i + c_ic_j, \tag{4.3}$$

$$\epsilon_ic_i\Omega_j = (\epsilon_j\Theta^j_j - \epsilon_i\Theta^i_i)\Omega^{ij}_i + 2\epsilon_j\Theta^i_j\Omega^{ji}_j + \epsilon_k\Theta^i_k(2\Omega^{ki}_j - \Omega^{kj}_i) + \epsilon_k\Theta^k_j\Omega^{ik}_i, \tag{4.4}$$

$$(\epsilon_j\Theta^j_j - \epsilon_i\Theta^i_i)\Omega^{ji}_k - \epsilon_i(\Theta^j_i\Omega^{ik}_i + \Theta^k_i\Omega^{ij}_i) = (\epsilon_j\Theta^j_j - \epsilon_k\Theta^k_k)\Omega^{jk}_i - \epsilon_k(\Theta^j_k\Omega^{ki}_k + \Theta^i_k\Omega^{kj}_k). \tag{4.5}$$

Note that if we let F_{jik} denote the left-hand side of (4.5), then (4.5) becomes

$$F_{jik} = F_{jki}. \tag{4.6}$$

From the identity $[[E_i,E_j],E_k]+[[E_j,E_k],E_i]+[[E_k,E_i],E_j]=0$ we also have

$$\Big(\frac{1}{c_i-c_j}+\frac{1}{c_i-c_k}\Big)\frac{-\epsilon_i\Omega_i R}{(c_i-c_j)(c_i-c_k)}=\Big(\frac{1}{c_i-c_j}+\frac{1}{c_i-c_k}\Big)\frac{\Omega_j\Omega_k}{(c_k-c_j)^2}. \tag{4.7}$$

It is easy to check that $\frac{1}{c_i-c_j}+\frac{1}{c_i-c_k}=0$ if and only if

(1) $i=1$ and $W=-1$, in which case we have

$$\Omega_1\Omega_3=-\frac{\Omega_2 R}{2},\quad \Omega_1\Omega_2=-\frac{\Omega_3 R}{2}; \tag{4.8}$$

(2) $i=2$ and $W=\frac{1}{2}$, in which case we have

$$\Omega_2\Omega_3=\frac{\Omega_1 R}{2},\quad \Omega_1\Omega_2=-\frac{\Omega_3 R}{2}. \tag{4.9}$$

Otherwise we always have

$$\frac{-\epsilon_i\Omega_i R}{(c_i-c_j)(c_i-c_k)}=\frac{\Omega_j\Omega_k}{(c_k-c_j)^2}. \tag{4.10}$$

Lemma 4.1 *If all Ω_i vanish, then so does R and it is Example* 3.1 *or* 3.2.

Proof By (4.1)–(4.2), $\Omega_i=0$ immediately yields $\Omega^{ij}_j=0$ and $\Theta^i_j=0$. Then it is easy to see that F_{ijk} is totally symmetric to the indices. By (4.3) we get

$$F_{ikj}=(\epsilon_i\Theta^i_i-\epsilon_k\Theta^k_k)\Omega^{ik}_j=-R\Big(c_j+\frac{2R^2(2c_j-c_i-c_k)}{(c_i-c_k)^2(c_j-c_k)(c_j-c_i)}\Big).$$

Checking $F_{321}=F_{123}$ we have $R=0$. As a result, $\Omega^{ij}_k=0$ and by (2.5),

$$[E_i,E_j]=0.$$

Using Theorem 2.1 one can see that this is clearly the case of Example 3.1 or 3.2.

Moreover, we will show the following lemma.

Lemma 4.2 Ω^{ij}_k *always vanishes.*

Proof It is done by creating contradictions from the assumption $R\neq 0$.

When $W\neq\frac{1}{2}$ or -1, by (4.10) we know none of Ω_i vanishing and $-\frac{\Omega_i\Omega_j\Omega_k}{R}=\frac{\epsilon_i(\Omega_i)^2(c_j-c_k)^2}{(c_i-c_j)(c_i-c_k)}$. When $i=3$ one can see that it is positive, so we can define a function f such that

$$-\frac{\Omega_i\Omega_j\Omega_k}{R}=f^2(c_i-c_j)^2(c_i-c_k)^2(c_j-c_k)^2.$$

Then we get

$$(\Omega_i)^2=\epsilon_i f^2(c_i-c_j)^3(c_i-c_k)^3,\quad R^2=f^2(c_i-c_j)^2(c_i-c_k)^2(c_j-c_k)^2.$$

Applying these expressions in (4.3) we obtain

$$\epsilon_i\Theta^i_i-\epsilon_k\Theta^k_k=(c_i-c_k)(c_j+3f^2(c_i-c_j)(c_j-c_k)(c_i+c_k-2c_j)).$$

At the same time, (4.2) shows $\Theta^i_j=-3\frac{\Omega_i\Omega_j}{(c_i-c_j)^2}$. Finally, a straightforward calculation in (4.4) for $(i,j,k)=(3,2,1)$ yields a contradiction $1+9f^2(W^2-W+1)=0$ for $W\in[-1,0)$.

When $W=\frac{1}{2}$, from (4.9) we know $\Omega_2\neq 0$ and $\Omega_1=\Omega_3=0$. Then (4.2) leads to $\Theta^1_2=\Theta^3_2=0$, $\Theta^1_3=\frac{R\Omega_2}{2}$, and (4.3) leads to

$$\Theta^1_1=2(\Omega_2)^2+4R^2+\frac{1}{4},\quad \Theta^2_2=6(\Omega_2)^2+8R^2+\frac{1}{4},\quad \Theta^3_3=-2(\Omega_2)^2-4R^2+\frac{1}{4}.$$

Setting $(i,j,k)=(1,2,3)$ in (4.4) we get a contradiction $\Omega_2=0$.

When $W=-1$, from (4.8) we get $\Omega_1\neq 0$ and $\Omega_2=\Omega_3$. Then (4.2) yields $\Theta^2_1=\Theta^3_1=-3\Omega_1\Omega_2$, $\Theta^3_2=R\Omega_1-\frac{(\Omega_2)^2}{4}$. (4.3) becomes

$$\Theta^2_2-\Theta^1_1=\Theta^3_3-\Theta^1_1=R^2-(\Omega_1)^2+\frac{3}{2}(\Omega_2)^2, \tag{4.11}$$

$$\Theta^2_2+\Theta^3_3=-2R^2+(\Omega_1)^2+\frac{1}{2}(\Omega_2)^2-1. \tag{4.12}$$

Obviously we have $\Theta^2_2=\Theta^3_3$.

Checking $F_{123}=F_{132}$ we get

$$\Theta^1_1+\Theta^2_2=(\Omega_1)^2+\frac{7}{8}(\Omega_2)^2. \tag{4.13}$$

Setting $(i,j,k)=(3,1,2)$ in (4.4) we get

$$1=(\Omega_1)^2-\frac{29}{8}(\Omega_2)^2.$$

Then we use above two expressions in (4.4) for $(i,j,k)=(1,2,3)$ to obtain $\Omega_2=\Omega_3=0$, $(\Omega_1)^2=1$. From (4.11)–(4.13) we get a contradiction $R=0$.

Now we are going to prove a classification.

Proposition 4.1 *A time-like conformal homogeneous hypersurface in R^4_1, if admitting three distinct real principal curvatures, must be conformally equivalent to one of hypersurfaces from Examples* 3.1–3.5.

Proof Since $R=0$, by (4.8)–(4.10) we know at least one of Ω_i vanishing.

(i) If all Ω_i are zeroes then it corresponds to Examples 3.1 and 3.2.

(ii) If only one is not zero, say Ω_i, then every Θ^i_j vanishes and all $\Omega^{ij}_k=0$ except $\Omega^{ij}_j=\frac{\epsilon_j\Omega_i}{c_j-c_i}$ $(i\neq j)$. By (4.4) we get

$$\epsilon_i\Theta^i_i-\epsilon_k\Theta^k_k=(c_i-c_k)c_k.$$

Meanwhile, (4.3) yields

$$\epsilon_k\Theta^k_k-\epsilon_i\Theta^i_i=\frac{c_k-c_j}{(c_i-c_j)^2(c_i-c_k)}\epsilon_i(\Omega_i)^2-(c_i-c_k)c_j.$$

Comparing these two expressions we get

$$\Omega^2_i=-\epsilon_i(c_i-c_j)^2(c_i-c_k)^2.$$

Thus it has to be $\Omega_1=|W|$. So we have

$$[E_1,E_2]=-\frac{|W|}{W}E_2,\quad [E_1,E_3]=-|W|E_3,\quad [E_2,E_3]=0.$$

When $W < 0$, we can change the direction of E_1 so that there always have

$$[E_1, E_2] = -E_2, \quad [E_1, E_3] = -WE_3, \quad [E_2, E_3] = 0.$$

This leads to Examples 3.3 and 3.4.

(iii) If only one is zero, say Ω_k, then (4.10) cannot hold true. By (4.2) we have $\Theta_k^i = \Theta_k^j = 0$, $\Theta_j^i = -\frac{2\Omega_i\Omega_j}{(c_i-c_j)^2}$. Given the following equations from (4.4),

$$c_i = (\epsilon_j\Theta_j^j - \epsilon_i\Theta_i^i)\frac{\epsilon_i}{c_j - c_i} - \frac{4(\Omega_i)^2}{(c_i - c_j)^3}, \quad c_j = (\epsilon_i\Theta_i^i - \epsilon_j\Theta_j^j)\frac{\epsilon_j}{c_i - c_j} - \frac{4(\Omega_j)^2}{(c_j - c_i)^3},$$

we get

$$\epsilon_i(\Omega_i)^2 + \epsilon_j(\Omega_j)^2 = -\frac{(c_i - c_j)^4}{4}.$$

So either $i = 1$ or $j = 1$. Anyway, we have $\Omega_1 \neq 0$. By (4.8)–(4.9) we know $\Omega_2 = 0$ and therefore the above expression implies that there is constant A such that

$$\Omega_1 = \frac{1}{2}\cosh\frac{A}{2}, \quad \Omega_3 = \frac{1}{2}\sinh\frac{A}{2}.$$

Finally, we obtain Example 3.5 as we have the following Lie brackets:

$$[E_1, E_2] = -\cosh\frac{A}{2}E_2, \quad [E_1, E_3] = -\frac{1}{2}\sinh\frac{A}{2}E_1 - \frac{1}{2}\cosh\frac{A}{2}E_3, \quad [E_2, E_3] = -\sinh\frac{A}{2}E_2.$$

4.2 Complex case

First, one can see that, with respect to the conformal invariant frame, the structure equations and integrable conditions (4.1)–(4.5) are still valid. However, some coefficients may take complex values:

$$c_1 = \overline{c}_2 = W - \mathbf{i}, \quad c_3 = 0; \quad \overline{R} = -R; \quad \Omega_2 = \mathbf{i}\overline{\Omega}_1, \quad \overline{\Omega}_3 = \Omega_3.$$

Similarly we can prove the following lemma.

Lemma 4.3 *If* $\Omega_i = 0$*, then* $R = 0$ *and it is Example* 3.6*.*

When $W \neq 0$, we also have (4.10) valid for any i, j, k. So there is $(c_i - c_j)(c_j - c_k)(c_k - c_i)\Omega_i\Omega_j\Omega_k = \epsilon_j R(\Omega_i)^2(c_j - c_k)^3$, which yields

$$R(\Omega_1)^2c_2^3 = -R(\Omega_2)^2c_1^3 = -R(\Omega_3)^2(c_1 - c_2)^3.$$

From $R(\Omega_2)^2c_1^3 = \overline{R(\Omega_1)^2c_2^3}$, $R(\Omega_3)^2(c_1 - c_2)^3 = \overline{R(\Omega_3)^2(c_1 - c_2)^3}$, we know that the quantity in above expression is both real and imaginary. Thus, $R(\Omega_i)^2 = 0$ and then by Lemma 4.3 there must have $R = 0$. By (4.10) again we know that at least two of Ω_i are zero and obviously they have to be $\Omega_1 = \Omega_2 = 0$. It is easy to see that $\Theta_j^i = 0$ by (4.2), and $\Theta_1^1 + \Theta_2^2 = \frac{-4W\mathbf{i}}{(W^2+1)^2}(\Omega_3)^2$ is purely imaginary by (4.3). In the mean time, comparing $c_1\Omega_3$ and $c_2\Omega_3$ in (4.4) we get $(\Theta_1^1 + \Theta_2^2 + 2 - 2W^2)\Omega_3 = 0$. So $\Omega_3 = 0$ and it must be equivalent to Example 3.6.

Now we assume $W = 0$, in which case from (4.7) we have

$$R\Omega_1 = 2\Omega_2\Omega_3, \quad R\Omega_2 = -2\Omega_1\Omega_3. \tag{4.14}$$

By (4.2) we get

$$\Theta_3^1 = 3\Omega_1\Omega_3, \quad \Theta_3^2 = 3\Omega_2\Omega_3, \quad \Theta_2^1 = \frac{\Omega_1\Omega_2}{2} + R\Omega_3.$$

By (4.3) we get

$$\begin{aligned}\Theta_1^1 &= -\frac{3}{8}(\Omega_1)^2 - \frac{1}{8}(\Omega_2)^2 - \frac{1}{2}(\Omega_3)^2 - R^2 - \frac{1}{2},\\ \Theta_2^2 &= -\frac{1}{8}(\Omega_1)^2 - \frac{3}{8}(\Omega_2)^2 + \frac{1}{2}(\Omega_3)^2 + R^2 + \frac{1}{2},\\ \Theta_3^3 &= \frac{5}{8}(\Omega_1)^2 - \frac{5}{8}(\Omega_2)^2 - \frac{3}{2}(\Omega_3)^2 - 2R^2 - \frac{1}{2}.\end{aligned}$$

If $\Omega_1 = \mathbf{i}\overline{\Omega}_2 = 0$, then the calculation about $c_1\Omega_3$ in (4.4) shows $\Omega_3(3R^2 + 2(\Omega_3)^2 + 2) = 0$. While in (4.5) with $i = 3$, $j = 2$ we get $R(3R^2 + (\Omega_3)^2 + 1) = 0$. These two equations imply that Ω_3 has to be zero and hence by Lemma 4.3 we get Example 3.6 again.

If $\Omega_1 = \mathbf{i}\overline{\Omega}_2 \neq 0$, then the calculation about $c_1\Omega_2$ in (4.4) shows that

$$(\Omega_1)^2 - (\Omega_2)^2 = 4 - 92(\Omega_3)^2. \tag{4.15}$$

Since $|\Omega_1| = |\Omega_2| \neq 0$, by (4.14) we get $R = 2\mathbf{i}\Omega_3$. Taking $(i, j, k) = (1, 2, 3)$ and $(1, 3, 2)$ respectively in (4.5) we find that $R = \Omega_3 = 0$. Thus, from (4.15) there is a constant A such that $\Omega_1 = (2 - 2\mathbf{i}\sinh A)^{\frac{1}{2}}$, $\Omega_2 = \mathbf{i}(2 + 2\mathbf{i}\sinh A)^{\frac{1}{2}}$. This is exactly equivalent to Example 3.7. As a conclusion we have the following proposition.

Proposition 4.2 *If a regular conformal homogeneous hypersurface in R_1^4 has complex principal curvatures, then it must be conformally equivalent to Examples* 3.6 *or* 3.7*.*

Combining these two propositions we obtain the main theorem.

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