Chin. Ann. Math. Ser. B
41(5), 2020, 697–716
DOI: 10.1007/s11401-020-0228-3

Chinese Annals of Mathematics, Series B

Ghost Symmetries and Multi-fold Darboux Transformations of Extended Toda Hierarchy*

Chuanzhong LI[1]

Abstract In this paper, the author constructs ghost symmetries of the extended Toda hierarchy with their spectral representations. After this, two kinds of Darboux transformations in different directions and their mixed Darboux transformations of this hierarchy are constructed. These symmetries and Darboux transformations might be useful in Gromov-Witten theory of $\mathbb{C}P^1$.

Keywords Extended Toda hierarchy, Ghost symmetry, Spectral representations, Hirota quadratic equation, Darboux transformation
2010 MR Subject Classification 37K05, 37K10, 37K20

1 Introduction

The Toda lattice hierarchy (see [1]) as a completely integrable system has many important applications in mathematics and physics. Toda systems have many kinds of reductions or extensions, such as the extended Toda hierarchy (ETH for short) (see [2–4]), bigraded Toda hierarchy (BTH for short) (see [5–9]), extended multi-component Toda hierarchy (see [10]), extended Z_N-Toda hierarchy (see [11]) and so on.

With additional logarithm flows, the Toda lattice hierarchy becomes the extended Toda hierarchy (see [2]) which governs the Gromov-Witten invariant of $\mathbb{C}P^1$. That means the Gromov-Witten potential τ of $\mathbb{C}P^1$ satisfies the Hirota quadratic equations (see [4]) of the ETH. The extended bigraded Toda hierarchy (EBTH for short) (see [5]) is an extension of the ETH. The Hirota bilinear equation of the EBTH was equivalently constructed in [6, 12]. Meanwhile it was proved to govern Gromov-Witten invariants of the total descendent potential of $\mathbb{C}_{N,M}$ orbifolds (see [12]).

The systematical studies on symmetries on lattice equations can be seen in [13]. As one kind of symmetries depending on time variables explicitly, the ghost symmetry was discovered by Oevel [14]. After that, it attracts a lot of research (see [15–22]). Aratyn used the method of squared eigenfunction potentials to construct the ghost symmetry of the KP hierarchy and connect this kind of symmetry with constrained KP hierarchy (see [17–19]). One S function

Manuscript received June 28, 2018. Revised November 4, 2018.
[1]School of Mathematics and Statistics, Ningbo University, Ningbo 315211, Zhejiang, China; College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, China. E-mail: lichuanzhong@nbu.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11571192) and K. C. Wong Magna Fund in Ningbo University.

was used to represent $(2+1)$-dimensional hierarchies of the KP equation, the modified KP equation and the Dym equation (see [20–21]. Our group gave a good construction of the ghost symmetry of the discrete KP hierarchy (see [23]) and the BKP hierarchy (see [22]).

Among many analytical methods, the Darboux transformation is one of the efficient methods to generate the soliton solutions for integrable systems (see [16, 24]). The Darboux transformation for integrable coupled systems is studied in [25]. In [26], the two Darboux transforms on band matrices called LU and UL Darboux transformations are constructed, particularly for the $(2m+1)$-band matrix. The LU and UL Darboux transformations inspire us to consider two different Darboux transformations of the ETH. The determinant representation of multi-fold Darboux transformations gives a convenient tool to explicitly express new solutions (see [27–28]). This reminds us to consider the Darboux transformation and its determinant representation of the continuous interpolated ETH which will be used to generate new solutions from known solutions which include soliton solutions and solutions related to the Gromov-Witten theory of $\mathbb{C}P^1$.

This paper is arranged as follows. In the next section, we will give the Lax equations of the extended Toda hierarchy. In Section 3, the ghost symmetry of the ETH will be constructed in two directions. By Sato equations, Hirota bilinear equations of the ETH are recalled with the tau-function and the generalized vertex operators. Meanwhile we also compare our results from the reduction of the EBTH (see [6]) and the known results in [4]. In Section 5, multi-fold Darboux transformations of the ETH will be constructed using determinant techniques in [10–11, 28]. In another direction, another kind of multi-fold Darboux transformations of the ETH will be constructed in Section 6. Combining these two Darboux transformations together, we construct their mixed transformations in Section 7.

2 Extended Toda Hierarchy

Firstly we recall some basic notations of the extended Toda hierarchy. We introduce the following Lax operator L of the extended Toda hierarchy as in [2] by

$$L = \Lambda + u(x) + v(x)\Lambda^{-1}$$

with Λ being defined now as the shift operator

$$\Lambda = \mathrm{e}^{\varepsilon\partial_x}.$$

The dressing operators $\mathcal{P}_L$ and $\mathcal{P}_R$ as in [2],

$$\mathcal{P}_L = 1 + w_1\Lambda^{-1} + w_2\Lambda^{-2} + \cdots, \tag{2.1}$$

$$\mathcal{P}_R = \widetilde{w_0} + \widetilde{w_1}\Lambda^{-1} + \widetilde{w_2}\Lambda^{-2} + \cdots \tag{2.2}$$

can be formally defined by the following identities in the ring of Laurent series in Λ^{-1} and Λ respectively:

$$L = \mathcal{P}_L\Lambda\mathcal{P}_L^{-1} = \mathcal{P}_R\Lambda^{-1}\mathcal{P}_R^{-1}.$$

The pair is unique up to multiplying $\mathcal{P}_L$ from the right and $\mathcal{P}_R$ from the left by operators in the form respectively $1 + a_1\Lambda^{-1} + a_2\Lambda^{-2} + \cdots$ and $\widetilde{a}_0 + \widetilde{a}_1\Lambda + \widetilde{a}_2\Lambda^2 + \cdots$ with coefficients independent of x.

To construct an extension of the extended Toda hierarchy, one needs to introduce the following notion of the logarithm of the Lax operator L:

$$\log L := \frac{1}{2}(\mathcal{P}_L\varepsilon\partial_x\mathcal{P}_L^{-1} - \mathcal{P}_R\varepsilon\partial_x\mathcal{P}_R^{-1}).$$

Remarkably the above ambiguity in the choice of dressing operators is cancelled after the definition of the logarithmic operator $\log L$.

Definition 2.1 *The extended Toda hierarchy consists of the evolutionary equations which are represented in the following Lax pair*:

$$\frac{\partial L}{\partial t_{\beta,n}} = [B_{\beta,n}, L] := B_{\beta,n}L - LB_{\beta,n}, \quad \beta = 1, 2, \ n \geq 0. \tag{2.3}$$

Here the operators $B_{\beta,n}$ are defined by

$$B_{1,n} = \frac{2}{\varepsilon n!}[L^n(\log L - c_n)]_+, \quad B_{2,n} = \frac{1}{\varepsilon(n+1)!}(L^{n+1})_+, \tag{2.4}$$

and for any operator $B = \sum B_k\Lambda^k$, the operator B_+ is given by $\sum\limits_{k\geq 0} B_k\Lambda^k$ and $B_- = B - B_+$. Here the constants c_n are defined as follows:

$$c_0 = 0, \quad c_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \tag{2.5}$$

Also we define the operators $A_{\beta,q}$, $C_{\beta,q}$ by

$$A_{1,n} = \frac{2}{\varepsilon n!}[L^n(\log L - c_n)], \qquad A_{2,n} = \frac{1}{\varepsilon(n+1)!}(L^{n+1}), \tag{2.6}$$

$$C_{1,n} = -\frac{2}{\varepsilon n!}[L^n(\log L - c_n)]_-, \quad C_{2,n} = -\frac{1}{\varepsilon(n+1)!}(L^{n+1})_-. \tag{2.7}$$

Now we give the Sato equations of the extended Toda hierarchy (ETH for short).

Definition 2.2 *The extended Toda hierarchy is a hierarchy in which the dressing operators $\mathcal{P}_L, \mathcal{P}_R$ satisfy the following Sato equations*:

$$\partial_{t_{\gamma,j}}\mathcal{P}_L = C_{\gamma,j}\mathcal{P}_L, \quad \partial_{t_{\gamma,j}}\mathcal{P}_R = B_{\gamma,j}\mathcal{P}_R. \tag{2.8}$$

The dressing operators $\mathcal{P}_L, \mathcal{P}_R$ which satisfy the above Sato equations are called wave operators (see [4]). "$*$" is defined as an antiinvolution acting on the space of Laurent series in Λ by $x^* = x$ and $\Lambda^* = \Lambda^{-1}$. The Lax pair of the ETH can be written as the following linear equations.

Proposition 2.1 *The Lax equation of the ETH can have the following linear system on Baker-Akhiezer functions $\Phi_{BA}, \overline{\Phi}_{BA}$ and adjoint Baker-Akhiezer functions $\Psi_{BA}, \overline{\Psi}_{BA}$ as*

$$L\Phi_{BA} = \lambda\Phi_{BA}, \quad \frac{\partial\Phi_{BA}}{\partial t_{\beta,n}} = B_{\beta,n}\Phi_{BA};$$

$$L^*\Psi_{BA} = \lambda\Psi_{BA}, \quad \frac{\partial\Psi_{BA}}{\partial t_{\beta,n}} = -B^*_{\beta,n}\Psi_{BA};$$

$$L\overline{\Phi}_{BA} = \lambda\overline{\Phi}_{BA}, \quad \frac{\partial\overline{\Phi}_{BA}}{\partial t_{\beta,n}} = C_{\beta,n}\overline{\Phi}_{BA};$$

$$L^*\overline{\Psi}_{BA} = \lambda\overline{\Psi}_{BA}, \quad \frac{\partial\overline{\Psi}_{BA}}{\partial t_{\beta,n}} = -C^*_{\beta,n}\overline{\Psi}_{BA}.$$

Then we can define the following eigenfunctions $\phi, \overline{\phi}$ and adjoint eigenfunctions $\psi, \overline{\psi}$ as

$$\frac{\partial \phi}{\partial t_{\beta,n}} = B_{\beta,n}\phi, \quad \frac{\partial \psi}{\partial t_{\beta,n}} = -B^*_{\beta,n}\psi,$$
$$\frac{\partial \overline{\phi}}{\partial t_{\beta,n}} = C_{\beta,n}\overline{\phi}, \quad \frac{\partial \overline{\psi}}{\partial t_{\beta,n}} = -C^*_{\beta,n}\overline{\psi}.$$

With above preparations, we will construct the ghost symmetry of the extended Toda hierarchy in the next section.

3 The Ghost Symmetry of the Extended Toda Hierarchy

In this section, the ghost flows on the Lax operator of the extended Toda hierarchy will be introduced firstly. Then we will prove that they are symmetries of the extended Toda hierarchy. After this, we naturally further consider the action of ghost flows on Baker-Akhiezer functions and eigenfunctions. To define the ghost flows, we define the following two operators:

$$\frac{\Lambda^{-1}}{1-\Lambda^{-1}} := \sum_{i=1}^{+\infty}\Lambda^{-i}, \quad \frac{1}{1-\Lambda} := \sum_{i=0}^{+\infty}\Lambda^{i}.$$

Inspired by the definition of ghost flows of the KP hierarchy (see [19]), here we define the flows for the ghost symmetry as

$$\partial_Z L = \Big[\phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi, L\Big], \quad \partial_{\overline{Z}} L = \Big[\overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}, L\Big],$$

where functions $\phi, \overline{\phi}, \psi, \overline{\psi}$ are the eigenfunctions and adjoint eigenfunctions of the extended Toda hierarchy.

To prove that the above flows are symmetries of the ETH, we need to prove the following lemma.

Lemma 3.1 *For operators $B := \sum\limits_{n=0}^{\infty} b_n\Lambda^n$, $C := \sum\limits_{n=1}^{\infty} c_n\Lambda^{-n}$ and $f(x)$, $g(x)$ as two arbitrary functions, the following identities hold:*

$$\Big(Bf\frac{\Lambda^{-1}}{1-\Lambda^{-1}}g\Big)_- = B(f)\frac{\Lambda^{-1}}{1-\Lambda^{-1}}g, \quad \Big(f\frac{\Lambda^{-1}}{1-\Lambda^{-1}}gB\Big)_- = f\frac{\Lambda^{-1}}{1-\Lambda^{-1}}B^*(g), \tag{3.1}$$
$$\Big(Cf\frac{1}{1-\Lambda}g\Big)_+ = C(f)\frac{1}{1-\Lambda}g, \quad \Big(f\frac{1}{1-\Lambda}gC\Big)_+ = f\frac{1}{1-\Lambda}C^*(g). \tag{3.2}$$

Proof The proof is similar to that of [29, Lemma 1].

The following theorem will tell you why we call it the ghost symmetry.

Theorem 3.1 *The additional flows $\partial_Z, \partial_{\overline{Z}}$ commute with the extended Toda flows $\partial_{t_{\gamma,n}}$, i.e.,*

$$[\partial_Z, \partial_{t_{\gamma,n}}]L = 0, \quad [\partial_{\overline{Z}}, \partial_{t_{\gamma,n}}]L = 0. \tag{3.3}$$

Proof The commutativity between ghost flows and extended Toda flows is in fact equivalent to the following Zero-Curvature equation which includes the following detailed proof:

$$\partial_Z B_{\gamma,n} - \partial_{t_{\gamma,n}}\Big(\phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi\Big) + \Big[B_{\gamma,n}, \phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi\Big]$$

$$
\begin{aligned}
&= \Big[\phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi, A_{\gamma,n}\Big]_+ - \phi_{t_{\gamma,n}}\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi - \phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi_{t_n} + \Big[B_{\gamma,n}, \phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi\Big] \\
&= \Big[\phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi, B_{\gamma,n}\Big]_+ - P_0(B_{\gamma,n}\phi)\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi + \phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}P_0(B^*_{\gamma,n}\psi) + \Big[B_{\gamma,n}, \phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi\Big] \\
&= \Big(B_{\gamma,n}\phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi\Big)_- - \Big(\phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi B_{\gamma,n}\Big)_- - P_0(B_{\gamma,n}\phi)\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi + \phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}P_0(B^*_{\gamma,n}\psi) \\
&= 0, \\
&\quad \partial_{\overline{Z}}C_{\gamma,n} - \partial_{t_{\gamma,n}}\Big(\overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}\Big) + \Big[C_{\gamma,n}, \overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}\Big] \\
&= -\Big[\overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}, A_{\gamma,n}\Big]_- - \overline{\phi}_{t_{\gamma,n}}\frac{1}{1-\Lambda}\overline{\psi} - \overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}_{t_n} + \Big[C_{\gamma,n}, \overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}\Big] \\
&= \Big[\overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}, C_{\gamma,n}\Big]_- - P_0(C_{\gamma,n}\overline{\phi})\frac{1}{1-\Lambda}\overline{\psi} + \overline{\phi}\frac{1}{1-\Lambda}P_0(C^*_{\gamma,n}\overline{\psi}) + \Big[C_{\gamma,n}, \overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}\Big] \\
&= \Big(C_{\gamma,n}\overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}\Big)_- - \Big(\overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}C_{\gamma,n}\Big)_- - P_0(C_{\gamma,n}\overline{\phi})\frac{1}{1-\Lambda}\overline{\psi} + \overline{\phi}\frac{1}{1-\Lambda}P_0(C^*_{\gamma,n}\overline{\psi}) \\
&= 0.
\end{aligned}
$$

The above proposition tells us that the ghost flows are the symmetries of the extended Toda hierarchy.

The ghost symmetry on wave operators $\mathcal{P}_L, \mathcal{P}_R$ can be got as

$$\partial_Z \mathcal{P}_L = \phi\frac{\Lambda^{-1}}{1-\Lambda^{-1}}\psi\mathcal{P}_L, \quad \partial_{\overline{Z}}\mathcal{P}_R = \overline{\phi}\frac{1}{1-\Lambda}\overline{\psi}\mathcal{P}_R.$$

The ghost flows acting on the Baker-Akhiezer functions $\Phi_{BA}(t,z), \overline{\Phi}_{BA}(t,z)$ and adjoint Baker-Akhiezer functions $\Psi_{BA}(t,z), \overline{\Psi}_{BA}(t,z)$ will be defined as the following equations:

$$
\begin{aligned}
&\partial_Z\Phi_{BA}(t,z) = \phi H(\psi, \Phi_{BA}(t,z)), && H(\psi,\Phi_{BA}(t,z)) = \frac{\Lambda^{-1}}{1-\Lambda^{-1}}(\psi\Phi_{BA}(t,z)); \\
&\partial_Z\Psi_{BA}(t,z) = -\psi S(\phi, \Psi_{BA}(t,z)), && S(\phi,\Psi_{BA}(t,z)) = \frac{\Lambda}{1-\Lambda}(\phi\Psi_{BA}(t,z)); \\
&\partial_{\overline{Z}}\Phi_{BA}(t,z) = \overline{\phi}\,\overline{H}(\overline{\psi}, \Phi_{BA}(t,z)), && \overline{H}(\overline{\psi},\Phi_{BA}(t,z)) = \frac{1}{1-\Lambda}(\overline{\psi}\Phi_{BA}(t,z)); \\
&\partial_{\overline{Z}}\Psi_{BA}(t,z) = -\overline{\psi}\,\overline{S}(\overline{\phi}, \Psi_{BA}(t,z)), && \overline{S}(\overline{\phi},\Psi_{BA}(t,z)) = \frac{1}{1-\Lambda^{-1}}(\overline{\psi}\Psi_{BA}(t,z)); \\
&\partial_Z\overline{\Phi}_{BA}(t,z) = \phi H(\psi, \overline{\Phi}_{BA}(t,z)), && H(\psi,\overline{\Phi}_{BA}(t,z)) = \frac{\Lambda^{-1}}{1-\Lambda^{-1}}(\overline{\psi}\,\overline{\Phi}_{BA}(t,z)); \\
&\partial_Z\overline{\Psi}_{BA}(t,z) = -\psi S(\phi, \overline{\Psi}_{BA}(t,z)), && S(\phi,\overline{\Psi}_{BA}(t,z)) = \frac{\Lambda}{1-\Lambda}(\phi\overline{\Psi}_{BA}(t,z)); \\
&\partial_{\overline{Z}}\overline{\Phi}_{BA}(t,z) = \overline{\phi}\,\overline{H}(\overline{\psi}, \overline{\Phi}_{BA}(t,z)), && \overline{H}(\overline{\psi},\overline{\Phi}_{BA}(t,z)) = \frac{1}{1-\Lambda}(\overline{\psi}\,\overline{\Phi}_{BA}(t,z)); \\
&\partial_{\overline{Z}}\overline{\Psi}_{BA}(t,z) = -\overline{\psi}\,\overline{S}(\overline{\phi}, \overline{\Psi}_{BA}(t,z)), && \overline{S}(\overline{\phi},\overline{\Psi}_{BA}(t,z)) = \frac{1}{1-\Lambda^{-1}}(\overline{\phi}\,\overline{\Psi}_{BA}(t,z)).
\end{aligned}
$$

We consider the spectral representation of the eigenfunctions $\phi(t), \overline{\phi}(t)$ and adjoint eigenfunctions $\psi(t), \overline{\psi}(t)$ for the extended Toda hierarchy.

Proposition 3.1 *The eigenfunctions $\phi(t), \overline{\phi}(t)$ and adjoint eigenfunctions $\psi(t), \overline{\psi}(t)$ have the following spectral representation using the Baker-Akhiezer functions $\Phi_{BA}(t,z), \overline{\Phi}_{BA}(t,z)$*

and adjoint Baker-Akhiezer functions $\Psi_{BA}(t,z),\overline{\Psi}_{BA}(t,z)$:

$$\phi(t)=\int \mathrm{d}z\phi_s(z)(\Phi_{BA}(t,z)),\quad \overline{\phi}(t)=\int \mathrm{d}z\overline{\phi}_s(z)(\overline{\Phi}_{BA}(t,z)),$$
$$\psi(t)=\int \mathrm{d}z\psi_s(z)\Psi_{BA}(t,z),\qquad \overline{\psi}(t)=\int \mathrm{d}z\overline{\psi}_s(z)\overline{\Psi}_{BA}(t,z).$$

The spectral representation will help us to get the ghost flow of eigenfunctions $\phi(t),\overline{\phi}(t)$ and adjoint eigenfunctions $\psi(t),\overline{\psi}(t)$. Considering the ghost flows acting on the Baker-Akhiezer functions $\Phi_{BA}(t,z),\overline{\Phi}_{BA}(t,z)$, adjoint Baker-Akhiezer functions $\Psi_{BA}(t,z),\overline{\Psi}_{BA}(t,z)$ and the above spectral representation, this will lead to the following proposition which contains the ghost flows of the eigenfunctions $\phi(t),\overline{\phi}(t)$ and adjoint eigenfunctions $\psi(t),\overline{\psi}(t)$ as the following proposition.

Proposition 3.2 *The eigenfunctions* $\phi(t),\overline{\phi}(t)$ *and adjoint eigenfunctions* $\psi(t),\overline{\psi}(t)$ *satisfy the following equations*:

$$\begin{aligned}
&\partial_Z\phi=\phi H(\psi,\phi),\quad \partial_Z\psi=-\psi S(\phi,\psi),\\
&\partial_{\overline{Z}}\phi=\overline{\phi}\,\overline{H}(\overline{\psi},\phi),\quad \partial_{\overline{Z}}\psi=-\overline{\psi}\,\overline{S}(\overline{\phi},\psi),\\
&\partial_Z\overline{\phi}=\phi H(\psi,\overline{\phi}),\quad \partial_Z\overline{\psi}=-\psi S(\phi,\overline{\psi}),\\
&\partial_{\overline{Z}}\overline{\phi}=\overline{\phi}\,\overline{H}(\overline{\psi},\overline{\phi}),\quad \partial_{\overline{Z}}\overline{\psi}=-\overline{\psi}\,\overline{S}(\overline{\phi},\overline{\psi}).
\end{aligned}$$

4 Hirota Quadratic Equations and Tau Function

In this section, we will compare the results of the ETH derived in [4] and our paper [6]. Now we introduce the following free operators $W_0,\overline{W}_0$:

$$W_0:=\exp\Big(\sum_{j=0}^{\infty}t_{2,j}\frac{\Lambda^j}{\varepsilon j!}+t_{1,j}\frac{\Lambda^j}{\varepsilon j!}(\varepsilon\partial-c_j)\Big),\tag{4.1}$$

$$\overline{W}_0:=\exp\Big(\sum_{j=0}^{\infty}t_{2,j}\frac{\Lambda^{-j}}{\varepsilon j!}+t_{1,j}\frac{\Lambda^{-j}}{\varepsilon j!}(\varepsilon\partial-c_j)\Big),\tag{4.2}$$

$$\overline{W}_{0R}:=\exp\Big(-\sum_{j=0}^{\infty}t_{2,j}\frac{\Lambda^j}{\varepsilon j!}-t_{1,j}\frac{\Lambda^j}{\varepsilon j!}(\varepsilon\partial-c_j)\Big).\tag{4.3}$$

We define the dressing operators $W,\overline{W},W_L,W_R$ as follows:

$$W:=W_L:=\mathcal{P}_L\circ W_0,\quad \overline{W}:=\mathcal{P}_R\circ\overline{W}_0,\quad W_R:=\overline{W}_{0R}\circ\mathcal{P}_R^*.\tag{4.4}$$

For any operator $B=\sum B_k\Lambda^k$, the left symbol of the operator is defined by $\sum B_k\lambda^k$. For any operator $B=\sum\Lambda^kC_k$, the right symbol of the operator is defined by $\sum C_k\lambda^k$. Then we denote their corresponding left symbols $\mathcal{W}_L$ of W_L and right symbols $\mathcal{W}_R$ of W_R. Also we denote their corresponding left symbols $\mathcal{W}$, $\overline{\mathcal{W}}$ and right symbols $\mathcal{W}^{-1}$, $\overline{\mathcal{W}}^{-1}$.

To show the agreement between the results in [4] and [6], the following proposition can be proved.

Proposition 4.1 *The following several statements are equivalent*:

(1) $\mathcal{P}_L$ *and* $\mathcal{P}_R$ *are wave operators of the extended Toda hierarchy.*

(2) *The following Hirota bilinear equations hold for* $r \in \mathbb{N}$ (*see* [4]):

$$W_L \Lambda^r W_R = W_R^* \Lambda^{-r} W_L^*, \quad r \in \mathbb{N}. \tag{4.5}$$

(3) *The following Hirota bilinear equations hold* (*see* [4]):

$$\begin{aligned} &\mathrm{Res}_\lambda\{\lambda^{r+m-1}\ \mathcal{W}_L(x,t,\varepsilon\partial_x,\lambda)\mathcal{W}_R(x-m\varepsilon,t',\varepsilon\partial_x,\lambda)\} \\ &= \mathrm{Res}_\lambda\{\lambda^{r-m-1}\mathcal{W}_R^*(x,t,\varepsilon\partial_x,\lambda)\ \mathcal{W}_L^*(x-m\varepsilon,t',\varepsilon\partial_x,\lambda)\}. \end{aligned} \tag{4.6}$$

(4) *The following Hirota bilinear equations hold*:

$$W \Lambda^r W^{-1} = \overline{W} \Lambda^{-r} \overline{W}^{-1}, \quad r \in \mathbb{N}. \tag{4.7}$$

(5) *Let* $t_{1,0} = t'_{1,0}$. *For all* $m \in \mathbb{Z}$, $r \in \mathbb{N}$, *the following Hirota bilinear identity* (*HBI for short*) *holds*:

$$\begin{aligned} &\mathrm{Res}_\lambda\{\lambda^{r+m-1}\ \mathcal{W}(x,t,\varepsilon\partial_x,\lambda)\mathcal{W}^{-1}(x-m\varepsilon,t',\varepsilon\partial_x,\lambda)\} \\ &= \mathrm{Res}_\lambda\{\lambda^{-r+m-1}\overline{\mathcal{W}}(x,t,\varepsilon\partial_x,\lambda)\ \overline{\mathcal{W}}^{-1}(x-m\varepsilon,t',\varepsilon\partial_x,\lambda)\}. \end{aligned} \tag{4.8}$$

Proof The equivalences of statements (1), (2) and (3) were proved in [4]. The following proof is about the equivalences of (1), (4) and (5). Here we only give the proof of the equivalence between (1), (4) and (5).

(1) $\Rightarrow$ (4) Set

$$\gamma = (\gamma_0, \gamma_1, \gamma_2, \cdots;), \quad \beta = (\beta_1, \beta_2, \cdots) \tag{4.9}$$

to be a multi index and

$$\partial^\gamma := \partial_{t_{2,0}}^{\gamma_0} \partial_{t_{2,1}}^{\gamma_1} \partial_{t_{2,2}}^{\gamma_2} \cdots, \quad \partial^\beta := \partial_{t_{1,1}}^{\beta_1} \partial_{t_{1,2}}^{\beta_2} \cdots. \tag{4.10}$$

Suppose $\partial^\theta = \partial^\alpha \partial^\beta$. Firstly we shall prove that the left statement leads to

$$W(x,t,\Lambda)\Lambda^r W^{-1}(x,t',\Lambda) = \overline{W}(x,t,\Lambda)\Lambda^{-r}\overline{W}^{-1}(x,t',\Lambda) \tag{4.11}$$

for all integers $r \geq 0$. Using the same method as used in [4, 6], by induction on θ, we shall prove that

$$W(x,t,\Lambda)\Lambda^r(\partial^\theta W^{-1}(x,t,\Lambda)) = \overline{W}(x,t,\Lambda)\Lambda^{-r}(\partial^\theta \overline{W}^{-1}(x,t,\Lambda)). \tag{4.12}$$

When $\theta = 0$, it is obviously true according to the definition of wave operators.

Suppose that (4.12) is true in the case of $\theta \neq 0$. Note that

$$\partial_{p_j} W := \begin{cases} [(\partial_{t_{2,j}} \mathcal{P}_L)\mathcal{P}_L^{-1} + \mathcal{P}_L \Lambda^j \mathcal{P}_L^{-1}]W, & p_j = t_{2,j}, \\ [(\partial_{t_{1,j}} \mathcal{P}_L)\mathcal{P}_L^{-1} + \mathcal{P}_L \Lambda^j \varepsilon\partial_x \mathcal{P}_L^{-1}]W, & p_j = t_{1,j}, \end{cases}$$

and

$$\partial_{p_j} \overline{W} := \begin{cases} (\partial_{t_{2,j}} \mathcal{P}_R)\mathcal{P}_R^{-1}\overline{W}, & p_j = t_{2,j}, \\ [(\partial_{t_{1,j}} \mathcal{P}_R)\mathcal{P}_R^{-1} + \mathcal{P}_R \Lambda^{-j} \varepsilon\partial_x \mathcal{P}_R^{-1}]\overline{W}, & p_j = t_{1,j}, \end{cases}$$

which further lead to

$$\partial_{p_j} W := \begin{cases} (A_{2,j})_+ W, & p_j = t_{2,j}, \\ \Big[-(A_{1,j})_- + \dfrac{L^j}{\varepsilon j!}(\log_+ L - c_j) \Big] W, & p_j = t_{1,j}, \end{cases}$$

and

$$\partial_{p_j} \overline{W} := \begin{cases} (A_{2,j})_+ \overline{W}, & p_j = t_{2,j}, \\ \Big[(A_{1,j})_+ - \dfrac{L^j}{\varepsilon j!}(\log_- L - c_j) \Big] \overline{W}, & p_j = t_{1,j}. \end{cases}$$

This further implies

$$(\partial_{p_j} W)\Lambda^r(\partial^\theta W^{-1}) = (\partial_{p_j}\overline{W})\Lambda^{-r}(\partial^\theta \overline{W}^{-1})$$

by considering (4.12) and furthermore we get

$$W\Lambda^r(\partial_{p_j}\partial^\theta W^{-1}) = \overline{W}\Lambda^{-r}(\partial_{p_j}\partial^\theta \overline{W}^{-1}).$$

Thus if we increase the power of ∂_{p_j} by 1, (4.12) still holds. The induction is completed. By Taylor expanding both sides of (4.11) about $t = t'$, one can finish the proof of (4.11).

(4) $\Leftarrow$ (1) Vice versa, by separating the negative and the positive part of the equation, we can prove $\mathcal{P}_L,\ \mathcal{P}_R$ are a pair of wave operators.

To prove (4) $\Leftrightarrow$ (5), the following symbolics are needed.

If the series have forms

$$W(x,t,\Lambda) = \sum_{i\in\mathbb{Z}} a_i(x,t,\partial_x)\Lambda^i, \qquad \overline{W}(x,t,\Lambda) = \sum_{i\in\mathbb{Z}} b_i(x,t,\partial_x)\Lambda^i,$$
$$W^{-1}(x,t,\Lambda) = \sum_{i\in\mathbb{Z}} \Lambda^i a_i'(x,t,\partial_x), \quad \overline{W}^{-1}(x,t,\Lambda) = \sum_{j\in\mathbb{Z}} \Lambda^j b_j'(x,t,\partial_x),$$

then their corresponding left symbols $\mathcal{W}$, $\overline{\mathcal{W}}$ and right symbols $\mathcal{W}^{-1}$, $\overline{\mathcal{W}}^{-1}$ are as follows:

$$\mathcal{W}(x,t,\lambda) = \sum_{i\in\mathbb{Z}} a_i(x,t,\partial_x)\lambda^i, \quad \mathcal{W}^{-1}(x,t,\lambda) = \sum_{i\in\mathbb{Z}} a_i'(x,t,\partial_x)\lambda^i,$$
$$\overline{\mathcal{W}}(x,t,\lambda) = \sum_{i\in\mathbb{Z}} b_i(x,t,\partial_x)\lambda^i, \quad \overline{\mathcal{W}}^{-1}(x,t,\bar{t},\lambda) = \sum_{j\in\mathbb{Z}} b_j'(x,t,\partial_x)\lambda^j.$$

With the above preparation and defining residue as $\operatorname{Res}_\lambda \sum_{n\in\mathbb{Z}} \alpha_n\lambda^n = \alpha_{-1}$, the equivalence (4) $\Leftrightarrow$ (5) can be proved using the similar proof as [1, 4, 6].

We denote respectively $P_L,\ P_R$ as the left symbols $\mathcal{P}_L,\ \mathcal{P}_R$ and $P_L^{-1},\ P_R^{-1}$ as the right symbols $\mathcal{P}_L^{-1},\ \mathcal{P}_R^{-1}$. A function τ depending only on the dynamical variables t and ε is called the tau-function of the ETH if it provides symbols related to wave operators as

$$P_L := \frac{\tau\Big(t_{1,0} + x - \dfrac{\varepsilon}{2}, t_{2,j} - \dfrac{\varepsilon(j-1)!}{\lambda^j};\varepsilon\Big)}{\tau\Big(t_{1,0} + x - \dfrac{\varepsilon}{2}, t;\varepsilon\Big)}, \tag{4.13}$$

$$P_L^{-1} := \frac{\tau\Big(t_{1,0}+x+\frac{\varepsilon}{2}, t_{2,j}+\frac{\varepsilon(j-1)!}{\lambda^j};\varepsilon\Big)}{\tau\Big(t_{1,0}+x+\frac{\varepsilon}{2}, t;\varepsilon\Big)}, \tag{4.14}$$

$$P_R := \frac{\tau\Big(t_{1,0}+x+\frac{\varepsilon}{2}, t_{2,j}+\varepsilon(j-1)!\lambda^j;\varepsilon\Big)}{\tau\Big(t_{1,0}+x-\frac{\varepsilon}{2}, t;\varepsilon\Big)}, \tag{4.15}$$

$$P_R^{-1} := \frac{\tau\Big(t_{1,0}+x-\frac{\varepsilon}{2}, t_{2,j}-\varepsilon(j-1)!\lambda^j;\varepsilon\Big)}{\tau\Big(t_{1,0}+x+\frac{\varepsilon}{2}, t;\varepsilon\Big)}. \tag{4.16}$$

Then from the Hirota bilinear identity of the EBTH in [6], we can get the HBI of the ETH in the following proposition by taking $N = M = 1$ in [6].

Proposition 4.2 *Let $m \in \mathbb{Z}$, $r \in \mathbb{N}$. The HBI* (4.8) *leads to the following scalar-valued Hirota bilinear identities*:

$$\begin{aligned}
&\mathrm{Res}_\lambda \Big\{\lambda^{r+m-1}\Big[(\partial_{2,n}P_L(x,t,\lambda))P_L^{-1}(x-m\varepsilon,t,\lambda) + \frac{\lambda^{n+1}}{\varepsilon(n+1)!}P_L(x,t,\lambda)P_L^{-1}(x-m\varepsilon,t,\lambda)\Big]\Big\}\\
&= \mathrm{Res}_\lambda\{\lambda^{-r+m-1}(\partial_{2,n}P_R(x,t,\lambda))\ P_R^{-1}(x-m\varepsilon,t,\lambda)\}, \qquad (4.17)\\
&\mathrm{Res}_\lambda \Big\{\lambda^{r+m-1}\Big[(\partial_{1,n}P_L(x,t,\lambda))P_L^{-1}(x-m\varepsilon,t,\lambda) + \frac{\lambda^{n}}{n!}P_L(x,t,\lambda)P_{Lx}^{-1}(x-m\varepsilon,t,\lambda)\\
&- \frac{\lambda^{n}}{\varepsilon n!}c_n P_L(x,t,\lambda)P_L^{-1}(x-m\varepsilon,t,\lambda)\Big]\Big\}\\
&= \mathrm{Res}_\lambda \Big\{\lambda^{-r+m-1}\Big[(\partial_{1,n}P_R(x,t,\lambda))\ P_R^{-1}(x-m\varepsilon,t,\lambda) + \frac{\lambda^{-n}}{n!}P_R(x,t,\lambda)P_{Rx}^{-1}(x-m\varepsilon,t,\lambda)\\
&+ \frac{\lambda^{-n}}{\varepsilon n!}c_n P_R(x,t,\lambda)P_R^{-1}(x-m\varepsilon,t,\lambda)\Big]\Big\}, \qquad (4.18)\\
&\mathrm{Res}_\lambda\{\lambda^{r+m-1}P_L(x,t,\lambda)P_L^{-1}(x-m\varepsilon,t,\lambda)\}\\
&= \mathrm{Res}_\lambda\{\lambda^{-r+m-1}P_R(x,t,\lambda)\ P_R^{-1}(x-m\varepsilon,t,\lambda)\}. \qquad (4.19)
\end{aligned}$$

Proof The proof can be derived after taking $N = M = 1$ from the Propositions 3.2–3.3 of [6].

Moreover, the HBI (4.8) can imply other interesting identities of the ETH such as the following proposition.

Proposition 4.3 *Let $r \in \mathbb{N}$ and $x - x' = m\varepsilon$, $m \in \mathbb{Z}$. The HBI* (4.8) *leads to the following scalar-valued Hirota bilinear identities*:

$$\begin{aligned}
&\mathrm{Res}_\lambda \Big\{\lambda^{r-1}\Big[(\partial_{2,n}P_L(x,t,\lambda))P_L^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}} + \frac{\lambda^{n+1}}{\varepsilon(n+1)!}P_L(x,t,\lambda)P_L^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}}\Big]\Big\}\\
&= \mathrm{Res}_\lambda \Big\{\lambda^{-r-1}(\partial_{2,n}P_R(x,t,\lambda))\ P_R^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}}\Big\}, \qquad (4.20)\\
&\mathrm{Res}_\lambda \Big\{\lambda^{r-1}\Big[(\partial_{1,n}P_L(x,t,\lambda))P_L^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}}\\
&+ \frac{\lambda^{n}}{n!}P_L(x,t,\lambda)P_{Lx'}^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}} - \frac{\lambda^{n}}{\varepsilon n!}c_n P_L(x,t,\lambda)P_L^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}}\Big]\Big\}\\
&= \mathrm{Res}_{\lambda=\infty} \Big\{\lambda^{-r-1}\Big[(\partial_{1,n}P_R(x,t,\lambda))\ P_R^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}}
\end{aligned}$$

$$+\frac{\lambda^{-nM}}{n!}P_R(x,t,\lambda)P_{Rx'}^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}}+\frac{\lambda^{-n}}{\varepsilon n!}c_nP_R(x,t,\lambda)P_R^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}}\Big]\Big\}, \tag{4.21}$$

$$\begin{aligned}&\mathrm{Res}_\lambda\{\lambda^{r-1}P_L(x,t,\lambda)P_L^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}}\}\\ =\ &\mathrm{Res}_\lambda\{\lambda^{-r-1}P_R(x,t,\lambda)\ P_R^{-1}(x',t,\lambda)\lambda^{\frac{x-x'}{\varepsilon}}\}.\end{aligned} \tag{4.22}$$

In this section we continue to discuss on the fundamental properties of the tau-function of the ETH, i.e., the Hirota quadratic equations of the ETH as a reduction of the EBTH in [6]. We will show that the Hirota quadratic equations of the ETH in [4] agree with the HBEs from the reduction of the EBTH.

Basing on the vertex operators of the EBTH in [6], we introduce the following vertex operators:

$$\Gamma^{\pm a}:=\exp\Big(\pm\frac{1}{\varepsilon}\Big(\sum_{j=0}^{\infty}t_{2,j}\frac{\lambda^{j+1}}{(j+1)!}+t_{1,j}\frac{\lambda^j}{j!}(\log\lambda-c_j)\Big)\Big)\times\exp\Big(\mp\frac{\varepsilon}{2}\partial_{t_{1,0}}\mp[\lambda^{-1}]_\partial\Big),$$

$$\Gamma^{\pm b}:=\exp\Big(\pm\frac{1}{\varepsilon}\Big(\sum_{j=0}^{\infty}t_{2,j}\frac{\lambda^{-j-1}}{(j+1)!}-t_{1,j}\frac{\lambda^{-j}}{j!}(\log\lambda-c_j)\Big)\Big)\times\exp\Big(\mp\frac{\varepsilon}{2}\partial_{t_{1,0}}\mp[\lambda]_\partial\Big),$$

where

$$[\lambda]_\partial:=\varepsilon\sum_{j=0}^{\infty}j!\lambda^{j+1}\partial_{t_{2,j}}.$$

Because of the logarithm $\log\lambda$, the vertex operators $\Gamma^{\pm a}\otimes\Gamma^{\mp a}$ and $\Gamma^{\pm b}\otimes\Gamma^{\mp b}$ are multi-valued functions. There are monodromy factors M^a and M^b respectively as following among different branches around $\lambda=\infty$,

$$M^a=\exp\Big\{\pm\frac{2\pi\mathrm{i}}{\varepsilon}\sum_{j\geq0}\frac{\lambda^j}{j!}(t_{1,j}\otimes1-1\otimes t_{1,j})\Big\}, \tag{4.23}$$

$$M^b=\exp\Big(\pm\frac{2\pi\mathrm{i}}{\varepsilon}\sum_{j\geq0}\frac{\lambda^{-j}}{j!}(t_{1,j}\otimes1-1\otimes t_{1,j})\Big). \tag{4.24}$$

In order to offset the complication, we need to generalize the concept of vertex operators which leads it to be not scalar-valued any more but takes values in a differential operator algebra as shown in [4]. So we introduce the following vertex operators

$$\Gamma_a=\exp\Big(-\sum_{j>0}\frac{j!\lambda^{j+1}}{\varepsilon}(\varepsilon\partial_x)t_{1,j}\Big)\exp(x\partial_{t_{1,0}}), \tag{4.25}$$

$$\Gamma_b=\exp\Big(-\sum_{j>0}\frac{j!\lambda^{-(j+1)}}{\varepsilon}(\varepsilon\partial_x)t_{1,j}\Big)\exp(x\partial_{t_{1,0}}), \tag{4.26}$$

$$\Gamma_a^*=\exp(x\partial_{t_{1,0}})\exp\Big(\sum_{j>0}\frac{j!\lambda^{j+1}}{\varepsilon}(\varepsilon\partial_x)t_{1,j}\Big), \tag{4.27}$$

$$\Gamma_b^*=\exp(x\partial_{t_{1,0}})\exp\Big(\sum_{j>0}\frac{j!\lambda^{-(j+1)}}{\varepsilon}(\varepsilon\partial_x)t_{1,j}\Big). \tag{4.28}$$

Then

$$\Gamma_a^* \otimes \Gamma_a = \exp(x\partial_{t_{1,0}}) \exp\Big(\sum_{j>0} \frac{j!\lambda^{j+1}}{\varepsilon}(\varepsilon\partial_x)(t_{1,j} - t'_{1,j})\Big) \exp(x\partial_{t'_{1,0}}), \tag{4.29}$$

$$\Gamma_b^* \otimes \Gamma_b = \exp(x\partial_{t_{1,0}}) \exp\Big(\sum_{j>0} \frac{j!\lambda^{-(j+1)}}{\varepsilon}(\varepsilon\partial_x)(t_{1,j} - t'_{1,j})\Big) \exp(x\partial_{t'_{1,0}}). \tag{4.30}$$

After some computation we get

$$\begin{aligned}
(\Gamma_a^* \otimes \Gamma_a)M^a &= \exp\Big(\pm \frac{2\pi\mathrm{i}}{\varepsilon}\sum_{j>0}\frac{\lambda^j}{j!}(t_{1,j} - t'_{1,j})\Big) \\
&\qquad \exp\Big(\pm \frac{2\pi\mathrm{i}}{\varepsilon}\Big((t_{1,0}+x) - \Big(t'_{1,0} + x + \sum_{j>0}\frac{\lambda^j}{j!}(t_{1,j} - t'_{1,j})\Big)\Big)\Big)(\Gamma_a^* \otimes \Gamma_a)\Big) \\
&= \exp\Big(\pm\frac{2\pi\mathrm{i}}{\varepsilon}(t_{1,0} - t'_{1,0})\Big)(\Gamma_a^* \otimes \Gamma_a), \\
(\Gamma_b^* \otimes \Gamma_b)M^b &= \exp\Big(\pm \frac{2\pi\mathrm{i}}{\varepsilon}\sum_{j>0}\frac{\lambda^{-j}}{j!}(t_{1,j} - t'_{1,j})\Big) \\
&\qquad \exp\Big(\pm \frac{2\pi\mathrm{i}}{\varepsilon}\Big((t_{1,0}+x) - \Big(t'_{1,0} + x + \sum_{j>0}\frac{\lambda^{-j}}{j!}(t_{1,j} - t'_{1,j})\Big)\Big)\Big)(\Gamma_b^* \otimes \Gamma_b) \\
&= \exp\Big(\pm\frac{2\pi\mathrm{i}}{\varepsilon}(t_{1,0} - t'_{1,0})\Big)(\Gamma_b^* \otimes \Gamma_b).
\end{aligned}$$

Thus when $t_{1,0} - t'_{1,0} \in \mathbb{Z}\varepsilon$, $(\Gamma_a^* \otimes \Gamma_a)(\Gamma^a \otimes \Gamma^{-a})$ and $(\Gamma_b^* \otimes \Gamma_b)(\Gamma^{-b} \otimes \Gamma^b)$ are all single-valued near $\lambda = \infty$.

Now we should note that the above vertex operators take value in a differential operator algebra.

Theorem 4.1 *The invertible function $\tau(t, \varepsilon)$ is a tau-function of the ETH if and only if it satisfies the following Hirota quadratic equations of the ETH*:

$$\mathrm{Res}_\lambda\{\lambda^{r-1}(\Gamma_a^* \otimes \Gamma_a)(\Gamma^a \otimes \Gamma^{-a})(\tau\otimes\tau)\} = \mathrm{Res}_\lambda\{\lambda^{-r-1}(\Gamma_b^* \otimes \Gamma_b)(\Gamma^{-b} \otimes \Gamma^{b})(\tau\otimes\tau)\} \tag{4.31}$$

computed at $t_{1,0} - t'_{1,0} = m\varepsilon$ for each $m \in \mathbb{Z}$, $r \in \mathbb{N}$.

Proof The proof can be derived after taking $N = M = 1$ from HBEs of the EBTH in [6].

Taking a transformation on (4.31) by $\lambda \to \lambda^{-1}$, then (4.31) becomes

$$\mathrm{Res}_\lambda\{\lambda^{r-1}((\Gamma_a^* \otimes \Gamma_a)(\Gamma^a \otimes \Gamma^{-a} - \Gamma^{-a} \otimes \Gamma^a))(\tau\otimes\tau)\} = 0 \tag{4.32}$$

computed at $t_{1,0} - t'_{1,0} = m\varepsilon$ for each $m \in \mathbb{Z}$, $r \in \mathbb{N}$. That means

$$\frac{\mathrm{d}\lambda}{\lambda}((\Gamma_a^* \otimes \Gamma_a)(\Gamma^a \otimes \Gamma^{-a} - \Gamma^{-a} \otimes \Gamma^a))(\tau\otimes\tau) \tag{4.33}$$

is regular in λ computed at $t_{1,0} - t'_{1,0} = m\varepsilon$ for each $m \in \mathbb{Z}$. (4.33) is exactly the Hirota quadratic equation of the extended Toda hierarchy in [4]. In the next section, two different Darboux transformations and their mixed transformations of the ETH will be constructed using kernel determinant techniques as [27–28].

5 Multi-fold Darboux Transformation of the ETH

In this section, we consider the Darboux transformation of the ETH on the Lax operator

$$L^{[1]} = WLW^{-1},$$

where W is the Darboux transformation operator.

That means after the Darboux transformation, the spectral problem

$$L\phi = \Lambda\phi + u\phi + v\Lambda^{-1}\phi = \lambda\phi$$

will become

$$L^{[1]}\phi^{[1]} = \Lambda\phi^{[1]} + u^{[1]}\phi^{[1]} + v^{[1]}\Lambda^{-1}\phi^{[1]} = \lambda\phi^{[1]}.$$

To keep the Lax equation of the ETH invariant, i.e.,

$$\frac{\partial L}{\partial t_{\alpha,n}} = [(A_{\alpha,n})_+, L], \quad \frac{\partial L^{[1]}}{\partial t_{\alpha,n}} = [(A^{[1]}_{\alpha,n})_+, L^{[1]}], \quad A^{[1]}_{\alpha,n} := A_{\alpha,n}(L^{[1]}), \tag{5.1}$$

the dressing operator W should satisfy the following equation:

$$W_{t_{\gamma,n}} = -W(A_{\gamma,n})_+ + (WB_{\gamma,n}W^{-1})_+W, \quad \gamma = 0, 1, \ n \geq 0,$$

where $W_{t_{\gamma,n}}$ means the derivative of W by $t_{\gamma,n}$.

Now, we give the following important theorem which will be used to generate new solutions.

Theorem 5.1 *If ϕ is the first wave function of the ETH, the Darboux transformation operator of the ETH*

$$W(\lambda) = \Big(1 - \frac{\phi}{\Lambda^{-1}\phi}\Lambda^{-1}\Big) = \phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}$$

will generate new solutions

$$u^{[1]} = u + (\Lambda - 1)\frac{\phi}{\Lambda^{-1}\phi},$$
$$v^{[1]} = \frac{\phi}{\Lambda^{-1}\phi}(\Lambda^{-1}v)\frac{\Lambda^{-2}\phi}{\Lambda^{-1}\phi}.$$

Proof In the following proof, using Lemma 3.1, a direct computation will lead to the following:

$$\begin{aligned}
W_{t_{\gamma,n}}W^{-1} &= (\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1})_{t_{\gamma,n}}\phi \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1} \\
&= (((A_{\gamma,n})_+\phi) \circ (1 - \Lambda^{-1}) \circ \phi^{-1})\phi \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1} \\
&\quad - \phi \circ (1 - \Lambda^{-1}) \circ ((A_{\gamma,n})_+\phi)\phi^{-1} \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1} \\
&= ((A_{\gamma,n})_+\phi)\phi^{-1} - \phi \circ (1 - \Lambda^{-1}) \circ ((A_{\gamma,n})_+\phi)\phi^{-1} \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1} \\
&= -(\phi \circ [(1 - \Lambda^{-1}) \circ \phi^{-1}(x)((A_{\gamma,n}(x))_+ \circ \phi(x))] \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1})_- \\
&= -(\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}(x) \circ (A_{\gamma,n}(x))_+ \circ \phi(x) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1})_- \\
&= -\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}(x) \circ (A_{\gamma,n})_+(x) \circ \phi(x) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1} \\
&\quad + (\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}(x) \circ (A_{\gamma,n})_+(x) \circ \phi(x) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1})_+
\end{aligned}$$

$$= -W(A_{\gamma,n})_+W^{-1} + (WB_{\gamma,n}W^{-1})_+.$$

Therefore

$$W = \phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}$$

can be a Darboux transformation of the ETH.

Define $\phi_i = \phi_i^{[0]} := \phi|_{\lambda=\lambda_i}$. Then one can choose the specific one-fold Darboux transformation of the ETH as

$$W_1(\lambda_1) = 1 - \frac{\phi_1}{\Lambda^{-1}\phi_1}\Lambda^{-1} = \frac{\mathbb{T}_1}{\phi_1(x-\varepsilon)},$$

where

$$\mathbb{T}_1 = \begin{vmatrix} 1 & \Lambda^{-1} \\ \phi_1 & \phi_1(x-\varepsilon) \end{vmatrix}.$$

Meanwhile, we can also get the Darboux transformation on the wave function ϕ as

$$\phi^{[1]} = \Big(1 - \frac{\phi_1}{\Lambda^{-1}\phi_1}\Lambda^{-1}\Big)\phi.$$

Then using iterations on the Darboux transformation, the j-th Darboux transformation from the $(j-1)$-th solution is

$$\begin{aligned}
\phi^{[j]} &= \Big(1 - \frac{\phi_j^{[j-1]}}{\Lambda^{-1}\phi_j^{[j-1]}}\Lambda^{-1}\Big)\phi^{[j-1]}, \\
u^{[j]} &= u^{[j-1]} + (\Lambda - 1)\frac{\phi_j^{[j-1]}}{\Lambda^{-1}\phi_j^{[j-1]}}, \\
v^{[j]} &= \frac{\phi_j^{[j-1]}}{\Lambda^{-1}\phi_j^{[j-1]}}(\Lambda^{-1}v^{[j-1]})\frac{\Lambda^{-2}\phi_j^{[j-1]}}{\Lambda^{-1}\phi_j^{[j-1]}},
\end{aligned}$$

where $\phi_i^{[j-1]} := \phi^{[j-1]}|_{\lambda=\lambda_i}$ are wave functions corresponding to different spectrals with the $(j-1)$-th solutions $u^{[j-1]}, v^{[j-1]}$. It can be checked that $\phi_i^{[j-1]} = 0, \quad i = 1, 2, \cdots, j-1$.

After iteration on Darboux transformations, the following theorem about the two-fold Darboux transformation of the ETH can be derived by direct calculation.

Theorem 5.2 *The two-fold Darboux transformation of the ETH is as follows*:

$$W_2 = 1 + t_1^{[2]}\Lambda^{-1} + t_2^{[2]}\Lambda^{-2} = \frac{\mathbb{T}_2}{\Delta_2},$$

where

$$\Delta_2 = \begin{vmatrix} \phi_1(x-\varepsilon) & \phi_1(x-2\varepsilon) \\ \phi_2(x-\varepsilon) & \phi_2(x-2\varepsilon) \end{vmatrix}, \quad \mathbb{T}_2 = \begin{vmatrix} 1 & \Lambda^{-1} & \Lambda^{-2} \\ \phi_1 & \phi_1(x-\varepsilon) & \phi_1(x-2\varepsilon) \\ \phi_2 & \phi_2(x-\varepsilon) & \phi_2(x-2\varepsilon) \end{vmatrix}.$$

The Darboux transformation leads to new solutions from seed solutions

$$\begin{aligned}
u^{[2]} &= u + (\Lambda - 1)t_1^{[2]}, \\
v^{[2]} &= t_2^{[2]}(x)(\Lambda^{-2}v)t_2^{[2]-1}(x-\varepsilon).
\end{aligned}$$

Similarly, we can generalize the Darboux transformation to the n-fold case which is contained in the following theorem.

Theorem 5.3 *The n-fold Darboux transformation of ETH equation is as follows*:

$$W_n = 1 + t_1^{[n]}\Lambda^{-1} + t_2^{[n]}\Lambda^{-2} + \cdots + t_n^{[n]}\Lambda^{-n} = \frac{1}{\Delta_n}\mathbb{T}_n,$$

where

$$\Delta_n = \begin{vmatrix} \phi_1(x-\varepsilon) & \phi_1(x-2\varepsilon) & \phi_1(x-3\varepsilon) & \cdots & \phi_1(x-n\varepsilon) \\ \phi_2(x-\varepsilon) & \phi_2(x-2\varepsilon) & \phi_2(x-3\varepsilon) & \cdots & \phi_2(x-n\varepsilon) \\ \phi_3(x-\varepsilon) & \phi_3(x-2\varepsilon) & \phi_3(x-3\varepsilon) & \cdots & \phi_3(x-n\varepsilon) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_n(x-\varepsilon) & \phi_n(x-2\varepsilon) & \phi_n(x-3\varepsilon) & \cdots & \phi_n(x-n\varepsilon) \end{vmatrix}$$

$$\mathbb{T}_n = \begin{vmatrix} 1 & \Lambda^{-1} & \Lambda^{-2} & \Lambda^{-3} & \cdots & \Lambda^{-n} \\ \phi_1(x) & \phi_1(x-\varepsilon) & \phi_1(x-2\varepsilon) & \phi_1(x-3\varepsilon) & \cdots & \phi_1(x-n\varepsilon) \\ \phi_2(x) & \phi_2(x-\varepsilon) & \phi_2(x-2\varepsilon) & \phi_2(x-3\varepsilon) & \cdots & \phi_2(x-n\varepsilon) \\ \phi_3(x) & \phi_3(x-\varepsilon) & \phi_3(x-2\varepsilon) & \phi_3(x-3\varepsilon) & \cdots & \phi_3(x-n\varepsilon) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_n(x) & \phi_n(x-\varepsilon) & \phi_n(x-2\varepsilon) & \phi_n(x-3\varepsilon) & \cdots & \phi_n(x-n\varepsilon) \end{vmatrix}.$$

The Darboux transformation leads to new solutions from seed solutions

$$\begin{aligned} u^{[n]} &= u + (\Lambda - 1)t_1^{[n]}, \\ v^{[n]} &= t_n^{[n]}(x)(\Lambda^{-n}v)t_n^{[n]-1}(x-\varepsilon). \end{aligned}$$

It can be easily checked that $W_n\phi_i = 0,\ i = 1, 2, \cdots, n$.

Taking seed solution $u = 0, v = 1$, then using Theorem 5.3, one can get the n-th new solution of the ETH as

$$\begin{aligned} u^{[n]} &= (1 - \Lambda^{-1})\partial_{t_{2,0}} \log Wr(\phi_1, \phi_2, \cdots, \phi_n), \\ v^{[n]} &= \mathrm{e}^{(1-\Lambda^{-1})^2 \log Wr(\phi_1,\phi_2,\cdots,\phi_n)}, \end{aligned}$$

where $Wr(\phi_1, \phi_2, \cdots, \phi_n)$ is the discrete Wronskian, i.e., a Casorati determinant

$$Wr(\phi_1, \phi_2, \cdots, \phi_n) = \det(\Lambda^{-j+1}\phi_{n+1-i})_{1\le i,j\le n}.$$

Particularly for the ETH, choosing appropriate wave function ϕ, the n-th new solutions can be solitary wave solutions, i.e., n-soliton solutions.

In the next section, it is time to introduce another Darboux transformation of the ETH basing on another linear equation.

6 The Second Darboux Transformation of the ETH

In fact the ETH system can also be equivalently rewritten in form of the following linear differential system:

$$\begin{cases} L\psi = \lambda\psi, \\ \dfrac{\partial\psi}{\partial t_{\alpha,n}} = -(A_{\alpha,n})_-\psi, \quad \alpha = 1, 2,\ n \ge 0. \end{cases}$$

We call the function ψ in (6.1) the second wave function of the ETH.

In this section, we will consider the Darboux transformation of the ETH on the Lax operator

$$L^{[1]} = \overline{W} L \overline{W}^{-1},$$

where $\overline{W}$ is a Darboux transformation operator.

That means after Darboux transformation, the spectral problem

$$L\psi = \Lambda\psi + u\psi + v\Lambda^{-1}\psi = \lambda\psi$$

will become

$$L^{[1]}\psi^{[1]} = \Lambda\psi^{[1]} + u^{[1]}\psi^{[1]} + v^{[1]}\Lambda^{-1}\psi^{[1]} = \lambda\psi^{[1]}.$$

To keep the Lax pair of the ETH invariant, i.e.,

$$\frac{\partial L}{\partial t_{\alpha,n}} = [-(A_{\alpha,n})_-, L], \quad \frac{\partial L^{[1]}}{\partial t_{\alpha,n}} = [-(A^{[1]}_{\alpha,n})_-, L^{[1]}], \quad A^{[1]}_{\alpha,n} := A_{\alpha,n}(L^{[1]}), \tag{6.1}$$

the dressing operator $\overline{W}$ should satisfy the following dressing equation:

$$\overline{W}_{t_{\gamma,n}} = \overline{W}(A_{\gamma,n})_- - (\overline{W} C_{\gamma,n} \overline{W}^{-1})_- \overline{W}, \quad \gamma = 1, 2,\ n \geq 0.$$

Theorem 6.1 *If ψ is the second wave function of the ETH, the second Darboux transformation operator of the ETH*

$$\overline{W}(\lambda) = \Big(\frac{\Lambda\psi}{\psi} - \Lambda\Big) = \psi(x+\varepsilon) \circ (1-\Lambda) \circ \psi^{-1}(x)$$

will generate new solutions

$$\begin{aligned} u^{[1]} &= \Lambda u + (\Lambda - 1)\frac{\Lambda\psi}{\psi}, \\ v^{[1]} &= \frac{\Lambda\psi}{\psi} v \frac{\Lambda^{-1}\psi}{\psi}. \end{aligned}$$

Proof A direct computation yields

$$\begin{aligned} \overline{W}_{t_{\gamma,n}} \overline{W}^{-1} &= (\psi(x+\varepsilon) \circ (1-\Lambda) \circ \psi^{-1})_{t_{\gamma,n}} \psi \circ (1-\Lambda)^{-1} \circ \psi^{-1}(x+\varepsilon) \\ &= -(((A_{\gamma,n})_- \psi(x+\varepsilon)) \circ (1-\Lambda) \circ \psi^{-1})\psi \circ (1-\Lambda)^{-1} \circ \psi^{-1}(x+\varepsilon) \\ &\quad + \psi(x+\varepsilon) \circ (1-\Lambda) \circ ((A_{\gamma,n})_- \psi)\psi^{-1} \circ (1-\Lambda)^{-1} \circ \psi^{-1}(x+\varepsilon) \\ &= -((A_{\gamma,n})_- \psi(x+\varepsilon))\psi^{-1}(x+\varepsilon) \\ &\quad + \psi(x+\varepsilon) \circ (1-\Lambda) \circ ((A_{\gamma,n})_- \psi)\psi^{-1} \circ (1-\Lambda)^{-1} \circ \psi^{-1}(x+\varepsilon) \\ &= -\psi(x+\varepsilon) \circ [(\Lambda - 1) \circ (((A_{\gamma,n})_- \psi)\psi^{-1})]\Lambda \circ (1-\Lambda)^{-1} \circ \psi^{-1}(x+\varepsilon) \\ &= -(\psi(x+\varepsilon) \circ [(\Lambda - 1) \circ (\psi^{-1}(A_{\gamma,n})_- \circ \psi)]\Lambda \circ (1-\Lambda)^{-1} \circ \psi^{-1}(x+\varepsilon))_+ \\ &= \psi(x+\varepsilon) \circ (1-\Lambda) \circ \psi^{-1}(x) \circ (A_{\gamma,n})_-(x) \circ \psi(x) \circ (1-\Lambda)^{-1} \circ \psi^{-1}(x+\varepsilon) \\ &\quad - (\psi(x+\varepsilon) \circ (1-\Lambda) \circ \psi^{-1}(x) \circ A_{\gamma,n}(x) \circ \psi(x) \circ (1-\Lambda)^{-1} \circ \psi^{-1}(x+\varepsilon))_- \\ &= \overline{W}(A_{\gamma,n})_- \overline{W}^{-1} - (\overline{W} C_{\gamma,n} \overline{W}^{-1})_-. \end{aligned}$$

Therefore

$$\overline{W} = \psi(x+\varepsilon) \circ (1-\Lambda) \circ \psi^{-1}$$

can be as another Darboux transformation of the ETH. The new solutions can be got easily using the second dressing form (6.1).

Define $\psi_i = \psi_i^{[0]} := \psi|_{\lambda=\lambda_i}$. Then one can choose the specific one-fold Darboux transformation of the ETH equations as the following:

$$\overline{W}_1(\lambda_1) = \frac{\psi_1(x+\varepsilon)}{\psi_1(x)} - \Lambda = \frac{\mathbb{T}_1}{\psi_1(x)},$$

where

$$\mathbb{T}_1 = \begin{vmatrix} 1 & \Lambda \\ \psi_1 & \psi_1(x+\varepsilon) \end{vmatrix}.$$

Meanwhile, we can also get the second Darboux transformation on the wave function ψ as

$$\psi^{[1]} = \Big(\frac{\Lambda\psi_1}{\psi_1} - \Lambda\Big)\psi.$$

Then using iterations on the second Darboux transformation, the j-th Darboux transformation from the $(j-1)$-th solution is

$$\begin{aligned}
\psi^{[j]} &= \Big(\frac{\Lambda\psi^{[j-1]}}{\psi^{[j-1]}} - \Lambda\Big)\psi^{[j-1]}, \\
u^{[j]} &= \Lambda u^{[j-1]} + (\Lambda-1)\frac{\Lambda\psi^{[j-1]}}{\psi^{[j-1]}}, \\
v^{[j]} &= \frac{\Lambda\psi^{[j-1]}}{\psi^{[j-1]}} v^{[j-1]} \frac{\Lambda^{-1}\psi^{[j-1]}}{\psi^{[j-1]}},
\end{aligned}$$

where $\psi_i^{[j-1]} := \psi^{[j-1]}|_{\lambda=\lambda_i}$ are wave functions corresponding to different spectrals with the $(j-1)$-th solutions $u^{[j-1]}, v^{[j-1]}$. It can be checked that $\psi_i^{[j-1]} = 0, \quad i = 1, 2, \cdots, j-1$.

Theorem 6.2 *The second two-fold Darboux transformation of the ETH is as follows*:

$$\overline{W}_2 = t_0^{[2]} + t_1^{[2]}\Lambda + \Lambda^2 = \frac{\mathbb{T}_2}{\Delta_2},$$

where

$$\Delta_2 = \begin{vmatrix} \psi_1(x) & \psi_1(x+\varepsilon) \\ \psi_2(x) & \psi_2(x+\varepsilon) \end{vmatrix}, \quad \mathbb{T}_2 = \begin{vmatrix} 1 & \Lambda & \Lambda^2 \\ \psi_1 & \psi_1(x+\varepsilon) & \psi_1(x+2\varepsilon) \\ \psi_2 & \psi_2(x+\varepsilon) & \psi_2(x+2\varepsilon) \end{vmatrix}.$$

The Darboux transformation leads to new solutions from seed solutions

$$\begin{aligned}
u^{[2]} &= \Lambda^2 u - (\Lambda-1)t_1^{[2]}, \\
v^{[2]} &= t_0^{[2]}(x) v t_0^{[2]-1}(x-\varepsilon).
\end{aligned}$$

Similarly, we can generalize the Darboux transformation to n-fold case which is contained in the following theorem.

Theorem 6.3 *The second n-fold Darboux transformation of the ETH is as follows*:

$$\overline{W}_n = t_0^{[n]} + t_1^{[n]}\Lambda + t_2^{[n]}\Lambda^2 + \cdots + (-1)^n\Lambda^n = \frac{1}{\Delta_n}\mathbb{T}_n,$$

where

$$\Delta_n = \begin{vmatrix} \psi_1(x) & \psi_1(x+\varepsilon) & \psi_1(x+2\varepsilon) & \cdots & \psi_1(x+(n-1)\varepsilon) \\ \psi_2(x) & \psi_2(x+\varepsilon) & \psi_2(x+3\varepsilon) & \cdots & \psi_2(x+(n-1)\varepsilon) \\ \psi_3(x) & \psi_3(x+\varepsilon) & \psi_3(x+3\varepsilon) & \cdots & \psi_3(x+(n-1)\varepsilon) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_n(x) & \psi_n(x+\varepsilon) & \psi_n(x+3\varepsilon) & \cdots & \psi_n(x+(n-1)\varepsilon) \end{vmatrix},$$

$$\mathbb{T}_n = \begin{vmatrix} 1 & \Lambda & \Lambda^2 & \Lambda^3 & \cdots & \Lambda^n \\ \psi_1(x) & \psi_1(x+\varepsilon) & \psi_1(x+2\varepsilon) & \psi_1(x+3\varepsilon) & \cdots & \psi_1(x+n\varepsilon) \\ \psi_2(x) & \psi_2(x+\varepsilon) & \psi_2(x+2\varepsilon) & \psi_2(x+3\varepsilon) & \cdots & \psi_2(x+n\varepsilon) \\ \psi_3(x) & \psi_3(x+\varepsilon) & \psi_3(x+2\varepsilon) & \psi_3(x+3\varepsilon) & \cdots & \psi_3(x+n\varepsilon) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_n(x) & \psi_n(x+\varepsilon) & \psi_n(x+2\varepsilon) & \psi_n(x+3\varepsilon) & \cdots & \psi_n(x+n\varepsilon) \end{vmatrix}.$$

The second n-fold Darboux transformation leads to new solutions from seed solutions

$$\begin{aligned} u^{[n]} &= \Lambda^n u + (-1)^n(1-\Lambda)t_{n-1}^{[n]}, \\ v^{[n]} &= t_0^{[n]}(x)vt_0^{[n]-1}(x-\varepsilon). \end{aligned}$$

It can be easily checked that $\overline{W}_n\psi_i = 0,\ i = 1, 2, \cdots, n$.

Taking seed solution $u = 0, v = 1$, then using Theorem 6.3, one can get the n-th new solution of the ETH as

$$\begin{aligned} u^{[n]} &= (-1)^n(1-\Lambda)\partial_{t_{2,0}} \log Wr(\psi_1, \psi_2, \cdots, \psi_n), \\ v^{[n]} &= \mathrm{e}^{(\Lambda-1)(1-\Lambda^{-1})\log Wr(\psi_1,\psi_2,\cdots,\psi_n)}, \end{aligned}$$

where $Wr(\psi_1, \psi_2, \cdots, \psi_n)$ is the Casorati determinant

$$Wr(\psi_1, \psi_2, \cdots, \psi_n) = \det(\Lambda^{j-1}\psi_{n+1-i})_{1\le i,j\le n}.$$

7 Mixed Darboux Transformation of the ETH

In this section, we consider the mixed Darboux transformation of the ETH on the Lax operator

$$L^{[1]} = T_b L T_b^{-1},$$

where T_b is a mixed Darboux transformation operator as

$$T_b = \overline{W}(\psi^{[1]}) \circ W(\phi),$$

where $\psi^{[1]}$ satisfies

$$L^{[1]}\psi^{[1]} = \lambda\psi^{[1]},$$

$$\frac{\partial \psi^{[1]}}{\partial t_{\alpha,n}} = -(A^{[1]}_{\alpha,n})_- \psi^{[1]}, \quad A^{[1]}_{\alpha,n} := A_{\alpha,n}(L^{[1]}), \tag{7.1}$$
$$L^{[1]} = \Lambda + u + (\Lambda - 1)\frac{\phi}{\Lambda^{-1}\phi} + \frac{\phi}{\Lambda^{-1}\phi}(\Lambda^{-1}v)\frac{\Lambda^{-2}\phi}{\Lambda^{-1}\phi}\Lambda^{-1}.$$

To keep the Lax pair invariant, the following equations hold:

$$\frac{\partial L}{\partial t_{\alpha,n}} = [-(A_{\alpha,n})_-, \mathcal{L}], \quad \frac{\partial L^{[1]}}{\partial t_{\alpha,n}} = [-(A^{[1]}_{\alpha,n})_-, L^{[1]}], \quad A^{[1]}_{\alpha,n} := A_{\alpha,n}(L^{[1]}). \tag{7.2}$$

Theorem 7.1 *The ETH has the following mixed Darboux transformation*:

$$T_b(\lambda) = \Big(\frac{\Lambda\psi^{[1]}}{\psi^{[1]}} - \Lambda\Big)\Big(1 - \frac{\phi}{\Lambda^{-1}\phi}\Lambda^{-1}\Big) = \psi^{[1]}(x+\varepsilon) \circ (1-\Lambda) \circ \psi^{[1]-1}(x)\phi \circ (1-\Lambda^{-1}) \circ \phi^{-1},$$

which generaters new solutions from seed solutions u, v.

We can also generalize the above one-fold mixed Darboux transformation to the following n-fold mixed Darboux transformation

$$T_b^{[2n]} = [\overline{W}(\psi^{[2n]}) \circ W(\phi^{[2n-1]})] \circ \cdots \circ [\overline{W}(\psi^{[3]}) \circ W(\phi^{[2]})] \circ [\overline{W}(\psi^{[1]}) \circ W(\phi^{[0]})],$$

where $\psi^{[2i-1]}$ satisfies

$$L^{[2i-1]}\psi^{[2i-1]} = \lambda\psi^{[2i-1]},$$
$$\frac{\partial \psi^{[2i-1]}}{\partial t_{\alpha,n}} = -\big(A^{[2i-1]}_{\alpha,n}\big)_- \psi^{[2i-1]}, \quad A^{[2i-1]}_{\alpha,n} := A_{\alpha,n}(L^{[2i-1]}), \tag{7.3}$$
$$L^{[2i-1]} = \Lambda + u^{[2i-2]} + (\Lambda-1)\frac{\phi^{[2i-2]}}{\Lambda^{-1}\phi^{[2i-2]}} + \frac{\phi^{[2i-2]}}{\Lambda^{-1}\phi^{[2i-2]}}(\Lambda^{-1}v^{[2i-2]})\frac{\Lambda^{-2}\phi^{[2i-2]}}{\Lambda^{-1}\phi^{[2i-2]}}\Lambda^{-1},$$

and $\phi^{[2i]}$ satisfies

$$L^{[2i]}\phi^{[2i]} = \lambda\phi^{[2i]},$$
$$\frac{\partial \phi^{[2i]}}{\partial t_{\alpha,n}} = (A^{[2i]}_{\alpha,n})_+ \phi^{[2i]}, \quad A^{[2i]}_{\alpha,n} := A_{\alpha,n}(L^{[2i]}), \tag{7.4}$$
$$L^{[2i]} = \Lambda + \Lambda u^{[2i-1]} + (\Lambda-1)\frac{\Lambda\psi^{[2i-1]}}{\psi^{[2i-1]}} + \frac{\Lambda\psi^{[2i-1]}}{\psi^{[2i-1]}}v^{[2i-1]}\frac{\Lambda^{-1}\psi^{[2i-1]}}{\psi^{[2i-1]}}\Lambda^{-1}.$$

This n-fold mixed Darboux transformation will generate new solutions under the following iterate Darboux transformation:

$$u^{[2i-1]} = u^{[2i-2]} + (\Lambda-1)\frac{\phi^{[2i-2]}}{\Lambda^{-1}\phi^{[2i-2]}},$$
$$v^{[2i-1]} = \frac{\phi^{[2i-2]}}{\Lambda^{-1}\phi^{[2i-2]}}(\Lambda^{-1}v^{[2i-2]})\frac{\Lambda^{-2}\phi^{[2i-2]}}{\Lambda^{-1}\phi^{[2i-2]}},$$
$$u^{[2i]} = \Lambda u^{[2i-1]} + (\Lambda-1)\frac{\Lambda\psi^{[2i-1]}}{\psi^{[2i-1]}},$$
$$v^{[2i]} = \frac{\Lambda\psi^{[2i-1]}}{\psi^{[2i-1]}}v^{[2i-1]}\frac{\Lambda^{-1}\psi^{[2i-1]}}{\psi^{[2i-1]}}.$$

$\psi^{[2i-1]}$ and $\phi^{[2i]}$ satisfy

$$\psi^{[2i-1]} = [W(\phi^{[2i-1]})] \circ \cdots \circ [\overline{W}(\psi^{[3]}) \circ W(\phi^{[2]})] \circ [\overline{W}(\psi^{[1]}) \circ W(\phi^{[0]})]\phi^{[2i-2]},$$
$$\phi^{[2i]} = [\overline{W}(\psi^{[2i]}) \circ W(\phi^{[2i-1]})] \circ \cdots \circ [\overline{W}(\psi^{[3]}) \circ W(\phi^{[2]})] \circ [\overline{W}(\psi^{[1]}) \circ W(\phi^{[0]})]\psi^{[2i-1]}.$$

In the specific computation, the n-fold mixed Darboux transformation will be chosen as

$$T_b^{[2n]} = [\overline{W}(\psi_{2n+1}^{[2n]}) \circ W(\phi_{2n}^{[2n-1]})] \circ \cdots \circ [\overline{W}(\psi_4^{[3]}) \circ W(\phi_3^{[2]})] \circ [\overline{W}(\psi_2^{[1]}) \circ W(\phi_1^{[0]})].$$

Of course, we can also construct the mixed Darboux transformations in all kinds of orders of the first and the second Darboux transformations. These Darboux transformations can help us to get different solutions from different seed solutions.

Very recently, lump solutions [30–32] and interaction solutions [33–34] are presented for many integrable continuous equations. It would be interesting to see if there exist similar solution situations for integrable Toda type lattice equations.

Acknowledgement Thanks to the referees for their valuable suggestions.

References

[1] Ueno, K. and Takasaki, K., Toda lattice hierarchy, Group Representations and Systems of Differential Equations, Tokyo, 1982, 1–95; Adv. Stud. Pure Math., **4**, North-Holland, Amsterdam, 1984.

[2] Carlet, G., Dubrovin, B. and Zhang, Y., The extended Toda hierarchy, *Mosc. Math. J.*, **4**, 2004, 313–332.

[3] Dubrovin, B. and Zhang, Y., Virasoro symmetries of the extended Toda hierarchy, *Comm. Math. Phys.*, **250**, 2004, 161–193.

[4] Milanov, T. E., Hirota quadratic equations for the extended Toda hierarchy, *Duke Math. J.*, **138**, 2007, 161–178.

[5] Carlet, G., The extended bigraded Toda hierarchy, *J. Phys. A*, **39**, 2006, 9411–9435.

[6] Li, C. Z., He, J. S., Wu, K. and Cheng, Y., Tau function and Hirota bilinear equations for the extended bigraded Toda hierarchy, *J. Math. Phys.*, **51**, 2010, 043514.

[7] Li, C. Z., Solutions of bigraded Toda hierarchy, *Journal of Physics A*, **44**, 2011, 255201.

[8] Li, C. Z. and He, J. S., Dispersionless bigraded Toda hierarchy and its additional symmetry, *Reviews in Mathematical Physics*, **24**, 2012, 1230003.

[9] Li, C. Z., He, J. S. and Su, Y. C., Block type symmetry of bigraded Toda hierarchy, *J. Math. Phys.*, **53**, 2012, 013517.

[10] Li, C. Z. and He, J. S., The extended multi-component Toda hierarchy, *Math. Phys., Analyis and Geometry*, **17**, 2014, 377–407.

[11] Li, C. Z. and He, J. S., On the extended Z_N-Toda hierarchy, *Theoretical and Mathematical Physics*, **185**, 2015, 1–22, arXiv: 1403.0684

[12] Carlet, G. and van de Leur, J., Hirota equations for the extended bigraded Toda hierarchy and the total descendent potential of $\mathbb{P}^1$ orbifolds, *J. Phys. A: Math. Theor.*, **46**, 2013, 405205.

[13] Ma, W. X. and Fuchssteiner, B., Algebraic structure of discrete zero curvature equations and master symmetries of discrete evolution equations, *Journal of Mathematical Physics*, **40**, 1999, 2400.

[14] Oevel, W., Darboux theorems and Wronskian formulas for integrable systems I: Constrained KP flows, *Physica A*, **195**, 1993, 533–576.

[15] Oevel, W. and Schief, W., Squared eigenfunctions of the (modified) KP hierarchy and scattering problems of Loewner type, *Rev. Math. Phys.*, **6**, 1994, 1301–1338.

[16] Oevel, W., Darboux transformations for integrable lattice systems, Nonlinear Physics: Theory and Experiment, Alfinito, E., Martina, L. and Pempinelli, F. (eds.), World Scientific, Singapore, 1996, 233–240.

[17] Aratyn. H., Nissimov, E. and Pacheva, S., Virasoro symmetry of constrained KP hierarchies, *Phys. Lett. A*, 1997, 164–175.

[18] Aratyn, H., Nissimov, E. and Pacheva, S., A new "dual" symmetry structure of the KP hierarchy, *Phys. Lett. A*, **244**, 1998, 245–255.

[19] Aratyn, H., Nissimov, E. and Pacheva, S., Method of squared eigenfunction potentials in integrable hierarchies of KP type, *Comm. Math. Phys.*, **193**, 1998, 493–525.

[20] Oevel, W. and Carillo, S., Squared eigenfunction symmetries for soliton equations, Part I, *J. Math. Anal. Appl.*, **217**, 1998, 161–178.

[21] Oevel, W. and Carillo, S., Squared eigenfunction symmetries for soliton equations, Part II, *J. Math. Anal. Appl.*, **217**, 1998, 179–199.

[22] Cheng, J. P., He, J. S. and Hu, S., The "ghost" symmetry of the BKP hierarchy, *J. Math. Phys.*, **51**, 2010, 053514.

[23] Li, C. Z., Cheng, J. P., Tian, K. L., et al., Ghost symmetry of the discrete KP hierarchy, *Monatsh. Math.*, **180**, 2016, 815–832.

[24] Mateev, V. B. and Salle, M. A., Darboux Transfromations and Solitons, Springer-Verlag, Berlin, 1991.

[25] Ma, W. X. and Zhang, Y. J., Darboux transformations of integrable couplings and applications, *Reviews in Mathematical Physics*, **30**, 2018, 1850003.

[26] Adler, M. and Moerbeke, P. V., Darboux transforms on band matrices, weights and associated polynomials, *International Mathematics Research Notices*, **18**, 2001, 935–984.

[27] He, J. S., Zhang, L., Cheng, Y. and Li, Y. S., Determinant representation of Darboux transformation for the AKNS system, *Sci. China A*, **12**, 2006, 1867–1878.

[28] Li, C. Z., He, J. S. and Porsezian, K., Rogue waves of the Hirota and the Maxwell-Bloch equation, *Physical Review E*, **87**, 2013, 012913.

[29] Li, C. Z., Sato theory on the q-Toda hierarchy and its extension, *Chaos, Solitons and Fractals*, **76**, 2015, 10–23.

[30] Ma, W. X., Abundant lumps and their interaction solutions of $(3+1)$-dimensional linear PDEs, *Journal of Geometry and Physics*, **133**, 2018, 10–16.

[31] Yang, J. Y., Ma, W. X. and Qin, Z. Y., Lump and lump-soliton solutions to the $(2+1)$-dimensional Ito equation, *Analysis and Mathematical Physics*, **8**, 2018, 427–436.

[32] Zhao, H. Q. and Ma, W. X., Mixed lump-kink solutions to the KP equation, *Computers and Mathematics with Applications*, **74**, 2017, 1399–1405.

[33] Zhang, J. B. and Ma, W. X., Mixed lump-kink solutions to the BKP equation, *Computers Mathematics with Applications*, **74**, 2017, 591–596.

[34] Ma, W. X., Yong, X. L. and Zhang, H. Q., Diversity of interaction solutions to the $(2+1)$-dimensional Ito equation, *Computers Mathematics with Applications*, **75**, 2018, 289–295.