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Commutators and Semi-commutators of Monomial Toeplitz Operators on the Pluriharmonic Hardy Space*

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Abstract In this paper, the authors completely characterize the finite rank commutator and semi-commutator of two monomial Toeplitz operators on the pluriharmonic Hardy space of the torus or the unit sphere. As a consequence, many non-trivial examples of (semi-)commuting Toeplitz operators on the pluriharmonic Hardy spaces are given.

Keywords Toeplitz operator, Pluriharmonic Hardy space, Commutator, Semi-commutator, Monomial symbol
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1 Introduction

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$ and the torus $\mathbb{T}^n$ be the Cartesian product of n copies of $\mathbb{T}$. Let $d\mu$ be the normalized Haar measure on $\mathbb{T}^n$. The Hardy space $H^2(\mathbb{T}^n)$ is the closure of the analytic polynomials in $L^2(\mathbb{T}^n, d\mu)$ (or $L^2(\mathbb{T}^n)$). It is well known that $H^2(\mathbb{T}) + \overline{H^2(\mathbb{T})} \cong L^2(\mathbb{T})$. However, for $n \geq 2$, $H^2(\mathbb{T}^n) + \overline{H^2(\mathbb{T}^n)} \subsetneqq L^2(\mathbb{T}^n)$. So we shall suppose $n \geq 2$ to avoid trivialities throughout the paper and define the pluriharmonic Hardy space $h^2(\mathbb{T}^n)$ by

$$h^2(\mathbb{T}^n) = H^2(\mathbb{T}^n) + \overline{H^2(\mathbb{T}^n)}.$$

See [3] for more information about the pluriharmonic Hardy space $h^2(\mathbb{T}^n)$. Similarly, let $d\sigma$ be the surface area measure on the unit sphere $\mathbb{S}_n$, the pluriharmonic Hardy space $h^2(\mathbb{S}_n)$ denotes the closed subspaces of all pluriharmonic functions in $L^2(\mathbb{S}_n, d\sigma)$ (or $L^2(\mathbb{S}_n)$).

Let Q be the orthogonal projection from $L^2(\Omega_n)$ onto $h^2(\Omega_n)$, where Ω_n denotes $\mathbb{T}^n$ or $\mathbb{S}_n$. The Toeplitz operator with symbol f in $L^\infty(\Omega_n)$ is defined by $T_f(h) = Q(fh)$ for functions $h \in h^2(\Omega_n)$. It is safe to use the same notation T_f to denote the Toeplitz operators on both $h^2(\mathbb{T}^n)$ and $h^2(\mathbb{S}_n)$, as we will always specify the space on which the operator T_f acts. For two Toeplitz operators T_{f_1} and T_{f_2} on $h^2(\Omega_n)$, we define their commutator and the semi-commutator by $[T_{f_1}, T_{f_2}] = T_{f_1}T_{f_2} - T_{f_2}T_{f_1}$ and $(T_{f_1}, T_{f_2}] = T_{f_1}T_{f_2} - T_{f_1f_2}$, respectively.

On the Hardy space $H^2(\mathbb{T})$, Brown and Halmos [2] first obtained a complete description of bounded symbols of (semi-)commuting Toeplitz operators. Later, some related problems

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were studied by many authors (see [7] and references there). However, the function theory on Ω_n is quite different from and much less understood than that on $\mathbb{T}$. For example, the complete characterization of (semi-)commuting Toeplitz operators on the Hardy space $H^2(\mathbb{T}^n)$ were obtained only when $n = 2$ (see [4, 8]). Zheng [11] characterized commuting Toeplitz operators with bounded pluriharmonic symbols on $H^2(\mathbb{S}_n)$.

In the setting of pluriharmonic Hardy spaces, Liu and Ding [9] obtained a characterization of (semi-)commuting Toeplitz operators with holomorphic symbols on $h^2(\mathbb{T}^2)$. Recently, Ding and Sang [10] first gave a necessary and sufficient condition for an analytic Toeplitz operator that commutes with another co-analytic Toeplitz operator on $h^2(\mathbb{T}^2)$, and then characterized (semi-)commuting Toeplitz operators on $h^2(\mathbb{T}^2)$ with bounded pluriharmonic symbols in [3].

In this paper, we are concerned with the finite rank problem of the commutator and semi-commutator of two monomial Toeplitz operators (namely, Toeplitz operators with symbol functions of the form $z^p\overline{z}^q$) on both $h^2(\mathbb{T}^n)$ and $h^2(\mathbb{S}_n)$. Recall that an operator A on a Hilbert space $\mathcal{H}$ is said to have finite rank if the closure of $\mathrm{Ran}(A)$ which is the range of the operator has finite dimension. For a bounded finite rank operator A on $\mathcal{H}$, we define $\mathrm{rank}(A) = \dim \mathrm{Ran}(A)$. In particular, the problem of determining when the commutator or semi-commutator of two Toeplitz operators has finite rank on the Hardy space $H^2(\mathbb{T})$ was completely solved in [1, 5].

In order to describe our main results, we first recall some standard multi-index notations. For $z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n$ and $p = (p_1, p_2, \cdots, p_n) \in \mathbb{N}^n$, where $\mathbb{N}$ is the set of all non-negative integers, we write $z^p = z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}$. For $l = (l_1, l_2, \cdots, l_n)$ and $m = (m_1, m_2, \cdots, m_n)$ in $\mathbb{N}^n \cup (-\mathbb{N})^n$, we write $l \succeq m$ if $l_i \geq m_i$ for all $i \in \{1, 2, \cdots, n\}$. If $l \succeq m$ and $|l| > |m|$, where $|l| = |l_1| + |l_2| \cdots + |l_n|$, then we write $l \succneqq m$.

The following two theorems completely solve the finite rank problem of the commutator and semi-commutator of two monomial Toeplitz operators on $h^2(\mathbb{T}^n)$, respectively.

Theorem 1.1 *Let $p, q, s, t \in \mathbb{N}^n$. Then the following statements are equivalent*:

(a) *The commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{T}^n)$.*

(b) *$(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$.*

Furthermore, $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ commute on $h^2(\mathbb{T}^n)$ if and only if one of the following conditions holds:

(1) *$p = q$, $s = t$, or $p - q = s - t$.*

(2) *There exists an $i_0 \in \{1, 2, \cdots, n\}$ such that $(p_{i_0} - q_{i_0})(s_{i_0} - t_{i_0}) > 0$, and $p_j - q_j = s_j - t_j = 0$ for any $j \in \{1, 2, \cdots, n\}$ with $j \neq i_0$.*

(3) *Neither $p \succeq q, s \succeq t$ nor $p \preceq q, s \preceq t$, and $(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$.*

Theorem 1.2 *Let $p, q, s, t \in \mathbb{N}^n$. Then the following statements are equivalent*:

(a) *The semi-commutator $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{T}^n)$.*

(b) *The semi-commutator $(T_{z^s\overline{z}^t}, T_{z^p\overline{z}^q}]$ has finite rank on $h^2(\mathbb{T}^n)$.*

(c) *$(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$.*

Furthermore, the semi-commutator $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ on $h^2(\mathbb{T}^n)$ is zero if and only if one of the following conditions holds:

(1) *Either $p = q$ or $s = t$.*

(2) *There exists an $i_0 \in \{1, 2, \cdots, n\}$ such that $(p_{i_0} - q_{i_0})(s_{i_0} - t_{i_0}) > 0$, and $p_j - q_j = s_j - t_j = 0$ for any $j \in \{1, 2, \cdots, n\}$ with $j \neq i_0$.*

(3) *Neither $p \succeq q, s \succeq t$ nor $p \preceq q, s \preceq t$, and $(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$.*

Some interesting higher-dimensional phenomena appear on the torus. For example, as a direct consequence of Theorems 1.1–1.2, on $h^2(\mathbb{T}^n)$ we have

(1) $T_{z^p}T_{\overline{z}^t} = T_{\overline{z}^t}T_{z^p}$ if and only if $T_{z^p}T_{\overline{z}^t} = T_{z^p\overline{z}^t}$ if and only if $p \perp t$ (i.e., $p_1t_1+\cdots+p_nt_n = 0$).

(2) $T_{z_i^{p_i}}T_{z_i^{s_i}} = T_{z_i^{s_i}}T_{z_i^{p_i}} = T_{z_i^{p_i+s_i}}$ for any $p_i, s_i \in \mathbb{N}$, but $T_{z_j^{p_j}}T_{z_k^{s_k}} \neq T_{z_k^{s_k}}T_{z_j^{p_j}}$ for any $j \neq k$ and positive natural numbers p_j, s_k.

(3) $T_{z_i^{p_i}}T_{z_j^{s_j}\overline{z}_k^{t_k}} = T_{z_j^{s_j}\overline{z}_k^{t_k}}T_{z_i^{p_i}} = T_{z_i^{p_i}z_j^{s_j}\overline{z}_k^{t_k}}$ for any different indexes i, j and k, and any positive natural numbers p_i, s_j and t_k.

Recently, the second author and Zhu [6] completely characterized finite rank commutator and semi-commutator of two monomial Toeplitz operators on the pluriharmonic Bergman s-paces of the unit ball. By the same argument as that in [6], we completely characterize when the commutator and semi-commutator of two monomial Toeplitz operators have finite rank on $h^2(\mathbb{S}_n)$. To simplify the presentation, we say that a tuple $(n_1, n_2, m_1, m_2) \in \mathbb{N}^4$ satisfies Condition (I) (see [6]) if at least one of the following conditions holds:

(i) $n_1 = n_2 = 0$,

(ii) $m_1 = m_2 = 0$,

(iii) $n_1 = m_1 = 0$,

(iv) $n_2 = m_2 = 0$,

(v) $n_1 = n_2$ and $m_1 = m_2$,

(vi) $n_1 = m_1$ and $n_2 = m_2$.

Theorem 1.3 *Let $p, q, s, t \in \mathbb{N}^n$. Then the following statements are equivalent*:

(a) *The commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{S}_n)$.*

(b) *$(|p|, |q|, |s|, |t|)$ and (p_i, q_i, s_i, t_i) satisfy Condition* (I) *for all $i \in \{1, 2, \cdots, n\}$.*

Furthermore, $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ commute on $h^2(\mathbb{S}_n)$ if and only if one of the following conditions holds:

(1) *Either $p = q = 0$ or $s = t = 0$.*

(2) *Either $p \preceq q$, $s \preceq t$ or $q \preceq p$, $t \preceq s$, $(|p|, |q|, |s|, |t|)$ and (p_i, q_i, s_i, t_i) satisfy either* (v) *or* (vi) *of Condition* (I) *for all $i \in \{1, 2, \cdots, n\}$.*

(3) *Neither $p \preceq q$, $s \preceq t$ nor $q \preceq p$, $t \preceq s$, $(|p|, |q|, |s|, |t|)$ and (p_i, q_i, s_i, t_i) satisfy Condition* (I) *for all $i \in \{1, 2, \cdots, n\}$.*

Theorem 1.4 *Let $p, q, s, t \in \mathbb{N}^n$. Then the following statements are equivalent*:

(a) *The semi-commutator $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{S}_n)$.*

(b) *The semi-commutator $(T_{z^s\overline{z}^t}, T_{z^p\overline{z}^q}]$ has finite rank on $h^2(\mathbb{S}_n)$.*

(c) *$p = q = 0$, $s = t = 0$, $p = s = 0$, or $q = t = 0$.*

Furthermore, the semi-commutator $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ on $h^2(\mathbb{S}_n)$ is zero if and only if either $p = q = 0$ or $s = t = 0$.

As a natural extension of the classical bilateral shift operator, monomial Toeplitz operators on pluriharmonic Hardy spaces enjoy some very interesting properties. For example, the symmetry property for the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ (see Corollary 2.1), and the close relationship between the semi-commutators $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ and $(T_{z^s\overline{z}^t}, T_{z^p\overline{z}^q}]$ (see Corollary 2.2). Also, our main theorems produce many non-trivial examples of commuting Toeplitz operators on the pluriharmonic Hardy spaces of the torus or the unit sphere (see Example 6.1). As applications, we make some interesting comparison of our main results. For example, if $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ commute on $h^2(\mathbb{S}_n)$, then $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ also commute on $h^2(\mathbb{T}^n)$. But the converse is not true. Indeed, if $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ commute on $h^2(\mathbb{T}^n)$, then the rank of the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ on $h^2(\mathbb{S}_n)$ may be nonzero or even infinite (see Example 6.2). These results further reveal the obvious differences in operator theory on the torus and on the unit sphere.

We end this introduction by mentioning that the proofs of our main theorems on the torus and on the unit sphere are quite different. In fact, the method of characterizing (semi-) commuting monomial Toeplitz operators on $h^2(\mathbb{S}_n)$ relies on explicit formulas for the action of the operators on the monomial orthonormal basis. This action leads to a holomorphic identity on a domain in the complex n-space (see [6, Equation (6)] for example). However, without such identity on $h^2(\mathbb{T}^n)$, substantial amount of different analysis is required.

2 Basic Results of Monomial Toeplitz Operators on $h^2(\mathbb{T}^n)$

In this section we study some basic properties of monomial Toeplitz operators on the pluriharmonic Hardy space $h^2(\mathbb{T}^n)$. We first begin with the following lemma.

Lemma 2.1 *Let $p, q \in \mathbb{N}^n$. Then on $h^2(\mathbb{T}^n)$, for each $\gamma \in \mathbb{N}^n$, we have*

$$T_{z^p\overline{z}^q}(z^\gamma) = \begin{cases} z^{\gamma+p-q}, & \gamma + p \succeq q, \\ \overline{z}^{q-\gamma-p}, & \gamma + p \preceq q, \\ 0, & \textit{otherwise} \end{cases}$$

and

$$T_{z^p\overline{z}^q}(\overline{z}^\gamma) = \begin{cases} \overline{z}^{\gamma+q-p}, & \gamma + q \succeq p, \\ z^{p-\gamma-q}, & \gamma + q \preceq p, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof For each $\lambda \in \mathbb{N}^n$, we have

$$\langle T_{z^p\overline{z}^q}(z^\gamma), z^\lambda \rangle = \langle z^{\gamma+p}\overline{z}^q, z^\lambda \rangle = \langle z^{\gamma+p}, z^{\lambda+q} \rangle$$

and

$$\langle T_{z^p\overline{z}^q}(z^\gamma), \overline{z}^\lambda \rangle = \langle z^{\gamma+p}\overline{z}^q, \overline{z}^\lambda \rangle = \langle \overline{z}^q, \overline{z}^{\lambda+\gamma+p} \rangle,$$

where the notation $\langle\, , \rangle$ denotes the inner product in $L^2(\mathbb{T}^n)$ with respect to the measure $d\mu$.

First we assume $\gamma + p \succeq q$. Recall from [3] that

$$\{z^\alpha\}_{\alpha\in\mathbb{N}^n} \cup \{\overline{z}^\beta\}_{\beta\in\mathbb{N}^n}$$

is an orthogonal basis of $h^2(\mathbb{T}^n)$. It follows that $T_{z^p\overline{z}^q}(z^\gamma)$ is orthogonal to every element of the basis except the holomorphic monomial $z^{\gamma+p-q}$, and hence $T_{z^p\overline{z}^q}(z^\gamma) = z^{\gamma+p-q}$.

Next, we assume $\gamma+p \preceq q$. Just like the previous case, we can show that $T_{z^p\overline{z}^q}(z^\gamma) = \overline{z}^{q-\gamma-p}$.

Finally, we assume that $\gamma + p \not\succeq q$ and $\gamma + p \not\preceq q$. Then

$$\langle T_{z^p\overline{z}^q}(z^\gamma), z^\lambda \rangle = \langle T_{z^p\overline{z}^q}(z^\gamma), \overline{z}^\lambda \rangle = 0, \quad \lambda \in \mathbb{N}^n,$$

which shows that $T_{z^p\overline{z}^q}(z^\gamma) = 0$.

The computation for $T_{z^p\overline{z}^q}(\overline{z}^\gamma)$ is similar and we leave the details to the interested reader.

The following remark provides a great convenience for our next content.

Remark 2.1 To simplify notation, let us write $z^l = \overline{z}^{-l}$ for any $l \in (-\mathbb{N})^n$. Then on $h^2(\mathbb{T}^n)$, for each $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$, it follows from Lemma 2.1 that

$$T_{z^p\overline{z}^q}(z^l) = \begin{cases} z^{l+p-q}, & l+p-q \in \mathbb{N}^n \cup (-\mathbb{N})^n, \\ 0, & l+p-q \notin \mathbb{N}^n \cup (-\mathbb{N})^n. \end{cases} \tag{2.1}$$

The next two propositions will be essential for our arguments in Sections 3 and 4, respectively.

Proposition 2.1 *Let $p, q, s, t \in \mathbb{N}^n$. Then on $h^2(\mathbb{T}^n)$, for any $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$, the following statements are equivalent*:

(a) $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0$.

(b) *One of the following conditions holds*:

(b1) $l+p-q+s-t \in \mathbb{N}^n \cup (-\mathbb{N})^n$, $l+p-q \in \mathbb{N}^n \cup (-\mathbb{N})^n$, *and* $l+s-t \notin \mathbb{N}^n \cup (-\mathbb{N})^n$.

(b2) $l+p-q+s-t \in \mathbb{N}^n \cup (-\mathbb{N})^n$, $l+s-t \in \mathbb{N}^n \cup (-\mathbb{N})^n$, *and* $l+p-q \notin \mathbb{N}^n \cup (-\mathbb{N})^n$.

Proof We will prove the equivalence of (a) and (b) by a direct calculation of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l)$ for any fixed $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$.

First we assume $l+p-q+s-t \notin \mathbb{N}^n \cup (-\mathbb{N})^n$. Then it follows from (2.1) that

$$T_{z^p\overline{z}^q}T_{z^s\overline{z}^t}(z^l) = 0 = T_{z^s\overline{z}^t}T_{z^p\overline{z}^q}(z^l),$$

which implies that $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0$.

Next we assume $l+p-q+s-t \in \mathbb{N}^n \cup (-\mathbb{N})^n$. Then there are three possibilities for $l+p-q$ and $l+s-t$.

Case 1 Both $l+p-q$ and $l+s-t$ belong to $\mathbb{N}^n \cup (-\mathbb{N})^n$. Then it follows from (2.1) that

$$[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = z^{l+s-t+p-q} - z^{l+p-q+s-t} = 0.$$

Case 2 Only one of $l+p-q$ and $l+s-t$ belongs to $\mathbb{N}^n \cup (-\mathbb{N})^n$. Without loss of generality, we may assume that $l+s-t \in \mathbb{N}^n \cup (-\mathbb{N})^n$ and $l+p-q \notin \mathbb{N}^n \cup (-\mathbb{N})^n$. By (2.1), we obtain

$$[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = z^{l+p-q+s-t} \neq 0.$$

Case 3 Neither $l+p-q$ nor $l+s-t$ is in $\mathbb{N}^n \cup (-\mathbb{N})^n$. Then

$$T_{z^p\overline{z}^q}T_{z^s\overline{z}^t}(z^l) = 0 = T_{z^s\overline{z}^t}T_{z^p\overline{z}^q}(z^l),$$

and hence $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0$. The proposition is now evident from what we have proved.

Corollary 2.1 *Let $p, q, s, t \in \mathbb{N}^n$. Then on $h^2(\mathbb{T}^n)$,*

$$l \in \mathbb{N}^n \cup (-\mathbb{N})^n \quad \textit{and} \quad [T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0,$$

if and only if $-l-p+q-s+t \in \mathbb{N}^n \cup (-\mathbb{N})^n$ and $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^{-l-(p-q)-(s-t)}) \neq 0$.

Proof Applying Proposition 2.1 with l replaced by $-l-p+q-s+t$, we obtain that if $-l-p+q-s+t \in \mathbb{N}^n \cup (-\mathbb{N})^n$ and $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^{-l-(p-q)-(s-t)}) \neq 0$, then $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$ and only one of $l+p-q$ and $l+s-t$ belongs to $\mathbb{N}^n \cup (-\mathbb{N})^n$. The corollary is now a direct consequence of Proposition 2.1.

Moreover, if $l = \frac{-(p-q)-(s-t)}{2} \in \mathbb{N}^n \cup (-\mathbb{N})^n$, then l does not satisfy condition (b) of Proposition 2.1, and hence $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]\big(z^{\frac{-(p-q)-(s-t)}{2}}\big) = 0$. Similar to the case on the pluriharmonic Bergman space of the unit ball (see [6]), we will call $\frac{-(p-q)-(s-t)}{2}$ the symmetry multi-index of the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ on $h^2(\mathbb{T}^n)$, and the finite rank commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ on $h^2(\mathbb{T}^n)$ can not have an odd rank.

Proposition 2.2 *Let $p, q, s, t \in \mathbb{N}^n$. Then on $h^2(\mathbb{T}^n)$, for any $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$, the following statements are equivalent*:

(a) $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0$.

(b) $l + p - q + s - t \in \mathbb{N}^n \cup (-\mathbb{N})^n$ *and* $l + s - t \notin \mathbb{N}^n \cup (-\mathbb{N})^n$.

Proof By a simple calculation, we obtain from (2.1) that

$$\begin{aligned}&(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = T_{z^p\overline{z}^q}T_{z^s\overline{z}^t}(z^l) - T_{z^{p+s}\overline{z}^{q+t}}(z^l)\\ &= \begin{cases} 0, & l+p-q+s-t \notin \mathbb{N}^n \cup (-\mathbb{N})^n,\\ 0, & l+p-q+s-t \in \mathbb{N}^n \cup (-\mathbb{N})^n \text{ and } l+s-t \in \mathbb{N}^n \cup (-\mathbb{N})^n,\\ -z^{l+p-q+s-t}, & l+p-q+s-t \in \mathbb{N}^n \cup (-\mathbb{N})^n \text{ and } l+s-t \notin \mathbb{N}^n \cup (-\mathbb{N})^n.\end{cases}\end{aligned}$$

This easily implies the desired result.

Corollary 2.2 *Let $p, q, s, t \in \mathbb{N}^n$. Then on $h^2(\mathbb{T}^n)$,*

$$l \in \mathbb{N}^n \cup (-\mathbb{N})^n \quad \text{and} \quad (T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0,$$

if and only if $-l - p + q - s + t \in \mathbb{N}^n \cup (-\mathbb{N})^n$ *and* $(T_{z^s\overline{z}^t}, T_{z^p\overline{z}^q}](z^{-l-(p-q)-(s-t)}) \neq 0$.

Proof Applying Proposition 2.2 to the semi-commutator $(T_{z^s\overline{z}^t}, T_{z^p\overline{z}^q}]$, we obtain that if $-l - p + q - s + t \in \mathbb{N}^n \cup (-\mathbb{N})^n$ and $(T_{z^s\overline{z}^t}, T_{z^p\overline{z}^q}](z^{-l-(p-q)-(s-t)}) \neq 0$, then $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$ and $l + s - t \notin \mathbb{N}^n \cup (-\mathbb{N})^n$. The corollary is now a direct consequence of Proposition 2.2.

Therefore, it follows that the semi-commutator $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{T}^n)$ if and only if the semi-commutator $(T_{z^s\overline{z}^t}, T_{z^p\overline{z}^q}]$ has finite rank on $h^2(\mathbb{T}^n)$. In this case,

$$\operatorname{rank}((T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]) = \operatorname{rank}((T_{z^s\overline{z}^t}, T_{z^p\overline{z}^q}]), \tag{2.2}$$

and the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ also has finite rank on $h^2(\mathbb{T}^n)$ with

$$\operatorname{rank}([T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]) \leq 2\operatorname{rank}((T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]). \tag{2.3}$$

We would like to point out that the above results hold for more general Toeplitz operates on the pluriharmonic Hardy space $h^2(\Omega_n)$. More specifically, we consider the trivial complex conjugation operator C on $L^2(\Omega_n)$ defined by $Cf = \overline{f}$. Then $CQ(g) = \overline{Q(g)} = Q(\overline{g})$ for every $g \in L^2(\Omega_n)$. Let f_1, f_2 and f be bounded functions on Ω_n, then it follows that

$$\begin{aligned}C(T_{f_2}T_{f_1} - T_f)^*C(h) &= C(T_{\overline{f}_1}T_{\overline{f}_2} - T_{\overline{f}})(\overline{h}) = C[Q(\overline{f}_1Q(\overline{f}_2\overline{h})) - Q(\overline{fh})]\\ &= Q(f_1Q(f_2h)) - Q(fh) = (T_{f_1}T_{f_2} - T_f)(h)\end{aligned}$$

for any $h \in h^2(\Omega_n)$. Then $T_{f_1}T_{f_2} - T_f$ is called a transpose of $T_{f_2}T_{f_1} - T_f$ on $h^2(\Omega_n)$, and by [6, Theorem 12], we have that $T_{f_1}T_{f_2} - T_f$ has finite rank on $h^2(\Omega_n)$ if and only if $T_{f_2}T_{f_1} - T_f$ has finite rank on $h^2(\Omega_n)$. More details can be found in [6].

3 The Commutator of Monomial Toeplitz Operators on $h^2(\mathbb{T}^n)$

In this section we study the finite rank problem of the commutator of two monomial Toeplitz operators on the pluriharmonic Hardy space $h^2(\mathbb{T}^n)$.

According to Proposition 2.1, the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{T}^n)$ if and only if there are finite multi-indexes $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$ satisfying condition (b) of Proposition 2.1. With the help of this result, we first give a specific necessary condition for the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ to be finite rank.

Lemma 3.1 *Let $p, q, s, t \in \mathbb{N}^n$. If the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{T}^n)$, then $(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$.*

Proof If there exists an $i_1 \in \{1, 2, \cdots, n\}$ such that $(p_{i_1} - q_{i_1})(s_{i_1} - t_{i_1}) < 0$, then we must show that the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is infinite.

If $p_{i_1} - q_{i_1} > 0$, then $s_{i_1} - t_{i_1} < 0$. So we consider infinitely many multi-indexes $l \in \mathbb{N}^n$ such that

$$\begin{cases} \max\{0, -(p_{i_1} - q_{i_1} + s_{i_1} - t_{i_1})\} \leq l_{i_1} < -(s_{i_1} - t_{i_1}), \\ l_j > \max\{0, -(p_j - q_j), -(s_j - t_j), -(p_j - q_j + s_j - t_j)\}, \end{cases}$$

where $j \in \{1, 2, \cdots, n\}$ with $j \neq i_1$. It is easy to check that

$$l + p - q + s - t \in \mathbb{N}^n, \quad l + p - q \in \mathbb{N}^n, \quad l + s - t \notin \mathbb{N}^n \cup (-\mathbb{N})^n.$$

So all such l satisfy condition (b1) of Proposition 2.1, and hence the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is infinite.

Similarly, if $p_{i_1} - q_{i_1} < 0$, then consider infinitely many multi-indexes $l \in (-\mathbb{N})^n$ such that

$$\begin{cases} -(s_{i_1} - t_{i_1}) < l_{i_1} \leq \min\{0, -(p_{i_1} - q_{i_1} + s_{i_1} - t_{i_1})\}, \\ l_j < \min\{0, -(p_j - q_j), -(s_j - t_j), -(p_j - q_j + s_j - t_j)\}, \end{cases}$$

where $j \in \{1, 2, \cdots, n\}$ and $j \neq i_1$. Then all such l satisfy condition (b1) of Proposition 2.1, and hence the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is infinite. This completes the proof.

Next, we give some sufficient conditions for the commutativity of two monomial Toeplitz operators on $h^2(\mathbb{T}^n)$, which simplifies the proof of Theorem 1.1.

Lemma 3.2 *Let $p, q, s, t \in \mathbb{N}^n$ with $(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$. Then on $h^2(\mathbb{T}^n)$, the following statements hold*:

(a) *If $s - t \notin \mathbb{N}^n \cup (-\mathbb{N})^n$, then the rank of commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is zero.*

(b) *If $p - q \notin \mathbb{N}^n \cup (-\mathbb{N})^n$, then the rank of commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is zero.*

Proof We first assume that $s - t \notin \mathbb{N}^n \cup (-\mathbb{N})^n$. Denote

$$J = \{j \in \{1, 2, \cdots, n\} : s_j - t_j > 0\}$$

and

$$K = \{k \in \{1, 2, \cdots, n\} : s_k - t_k < 0\},$$

both of which are nonempty. Since $(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$, it follows that $p_j - q_j \geq 0$ for all $j \in J$, $p_k - q_k \leq 0$ for all $k \in K$, and $s_d - t_d = 0$ for all possible indexes $d \in \{1, 2, \cdots, n\}$ with $d \notin J \cup K$.

Suppose that there exists an $l \in \mathbb{N}^n$ such that $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0$. Since

$$l_j + p_j - q_j + s_j - t_j > 0,$$

it follows from Proposition 2.1 that $l + p - q + s - t \in \mathbb{N}^n$. Note that

$$\begin{cases} l_j + p_j - q_j \geq 0, \\ l_k + p_k - q_k > l_k + p_k - q_k + s_k - t_k \geq 0, \\ l_d + p_d - q_d = l_d + p_d - q_d + s_d - t_d \geq 0 \end{cases}$$

and

$$\begin{cases} l_j + s_j - t_j > 0, \\ l_k + s_k - t_k \geq l_k + p_k - q_k + s_k - t_k \geq 0, \\ l_d + s_d - t_d = l_d \geq 0. \end{cases}$$

Thus, both $l + p - q$ and $l + s - t$ belong to $\mathbb{N}^n$, which contradicts Proposition 2.1.

Similarly, if there exists an $l \in (-\mathbb{N})^n$ such that $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0$, then by

$$l_k + p_k - q_k + s_k - t_k < 0,$$

we deduce from Proposition 2.1 that $l + p - q + s - t \in (-\mathbb{N})^n$. However,

$$\begin{cases} l_j + p_j - q_j < l_j + p_j - q_j + s_j - t_j \leq 0, \\ l_k + p_k - q_k \leq 0, \\ l_d + p_d - q_d = l_d + p_d - q_d + s_d - t_d \leq 0 \end{cases}$$

and

$$\begin{cases} l_j + s_j - t_j \leq l_j + p_j - q_j + s_j - t_j \leq 0, \\ l_k + s_k - t_k < 0, \\ l_d + s_d - t_d = l_d \leq 0. \end{cases}$$

Thus both $l+p-q$ and $l+s-t$ belong to $(-\mathbb{N})^n$, which also contradicts Proposition 2.1. Thus we have derived that $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0$ for any $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$, and hence condition (a) holds.

Observe that

$$[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}] = -[T_{z^s\overline{z}^t}, T_{z^p\overline{z}^q}]. \tag{3.1}$$

Combining this with condition (a), we conclude that condition (b) holds. This completes the proof.

We are now ready to prove Theorem 1.1 stated in the introduction.

Proof of Theorem 1.1 It is clear from Lemma 3.1 that (a) implies (b). To show that (b) implies (a), we assume that $(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$. If at least one of $p - q$ and $s - t$ does not belong to $\mathbb{N}^n \cup (-\mathbb{N})^n$, then by Lemma 3.2 we have that the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is 0. To consider the remaining case when both $p - q$ and $s - t$ belong to $\mathbb{N}^n \cup (-\mathbb{N})^n$, we break the discussion into three cases.

Case a $p = q$, $s = t$, or $p - q = s - t$. Obviously, there is no multi-index $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$ satisfying condition (b) of Proposition 2.1. Therefore, the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is 0.

Case b Either $p \gneqq q$, $s \lneqq t$ or $p \lneqq q$, $s \gneqq t$. By (3.1), we need only consider the case $p \gneqq q$, $s \lneqq t$. Recalling that $(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$, it follows that

$$(p_i - q_i)(s_i - t_i) = 0$$

for all $i \in \{1, 2, \cdots, n\}$. Denote

$$J = \{j \in \{1, 2, \cdots, n\} : p_j - q_j > 0\}$$

and

$$K = \{k \in \{1, 2, \cdots, n\} : s_k - t_k < 0\}.$$

Since $p \gneqq q$ and $s \lneqq t$, both J and K are nonempty. Then $s_j - t_j = 0$ for all $j \in J$, $p_k - q_k = 0$ for all $k \in K$ and $s_d - t_d = p_d - q_d = 0$ for all possible indexes $d \in \{1, 2, \cdots, n\}$ with $d \notin J \cup K$. We now prove that $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0$ for any $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$. Using the same argument as in the proof of Lemma 3.2, we assume the contrary and let $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0$ for some $l \in \mathbb{N}^n$. Since

$$l_j + p_j - q_j + s_j - t_j > 0,$$

it follows from Proposition 2.1 that $l + p - q + s - t \in \mathbb{N}^n$. Then

$$\begin{cases} l_j + s_j - t_j = l_j \geq 0, \\ l_k + s_k - t_k = l_k + p_k - q_k + s_k - t_k \geq 0, \\ l_d + s_d - t_d = l_d \geq 0, \end{cases}$$

which implies that $l + s - t \in \mathbb{N}^n$. Since $p \gneqq q$ and $l \in \mathbb{N}^n$, it follows that $l + p - q \in \mathbb{N}^n$, which leads to a contradiction. Similarly, it can easily be verified that $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0$ for any $l \in (-\mathbb{N})^n$. Therefore, the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is 0.

Case c Either $p \gneqq q$, $s \gneqq t$ or $p \lneqq q$, $s \lneqq t$, and $p - q \neq s - t$. Since

$$[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]^* = -[T_{z^q\overline{z}^p}, T_{z^t\overline{z}^s}],$$

we may assume, without loss of generality, that $p \gneqq q$ and $s \gneqq t$. Then for any $l \in \mathbb{N}^n$, it is obvious that

$$\begin{cases} l + p - q + s - t \succeq 0, \\ l + p - q \succeq 0, \\ l + s - t \succeq 0. \end{cases}$$

Consequently, by condition (b) of Proposition 2.1, we have

$$[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0, \quad \forall l \in \mathbb{N}^n.$$

Since $\frac{-(p-q)-(s-t)}{2}$ is the symmetry multi-index of the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ on $h^2(\mathbb{T}^n)$, we obtain

$$[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0, \quad \forall l \in (-\mathbb{N})^n,\ l \preceq -(p - q + s - t).$$

Moreover, Proposition 2.1 implies that

$$[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0, \quad \forall l \in (-\mathbb{N})^n,\ -(p - q + s - t) \npreceq l \npreceq -(p - q + s - t).$$

Thus we arrive at the conclusion that if $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0$, then

$$-(p-q+s-t) \precneqq l \precneqq 0. \tag{3.2}$$

Obviously, the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is finite.

In each of the cases above, we have shown that $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{T}^n)$, and hence condition (a) holds. In order to characterize when the rank of the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is actually 0, we need consider two possibilities for Case c.

Case c1 Either $p-q \succneqq s-t \succneqq 0$ or $0 \precneqq p-q \precneqq s-t$. By (3.1), without loss of generality, we only consider $p-q \succneqq s-t \succneqq 0$.

First, we assume that there is only one $i_0 \in \{1,2,\cdots,n\}$ such that $p_{i_0}-q_{i_0}>0$. Then for any l satisfying (3.2), we have

$$-(p_{i_0}-q_{i_0}+s_{i_0}-t_{i_0}) \leq l_{i_0} \leq 0, \quad l_j = 0,$$

where $j \in \{1,2,\cdots,n\}$ with $j \neq i_0$. Consequently, both $l+p-q$ and $l+s-t$ belong to $(-\mathbb{N})^n$. Then by Proposition 2.1, we have that the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is 0.

Next, we assume that at least two of $i \in \{1,2,\cdots,n\}$ satisfy $p_i-q_i>0$. We consider two cases. Suppose that $p_{i_1}-q_{i_1}>0$ for some $i_1 \in \{1,2,\cdots,n\}$ with $s_{i_1}-t_{i_1}=0$. Since $s-t \succneqq 0$, there exists another $i_2 \in \{1,2,\cdots,n\}$ such that $p_{i_2}-q_{i_2} \geq s_{i_2}-t_{i_2}>0$. Choose l such that

$$\begin{cases} -(p_{i_1}-q_{i_1}) < l_{i_1} \leq 0, \\ -(p_{i_2}-q_{i_2}+s_{i_2}-t_{i_2}) \leq l_{i_2} < -(p_{i_2}-q_{i_2}), \\ l_k = -(p_k-q_k+s_k-t_k), \end{cases}$$

where $k \in \{1,2,\cdots,n\}$ with $k \neq i_1, i_2$. Then $l \in (-\mathbb{N})^n$ and satisfies condition (b2) of Proposition 2.1. So the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is nonzero. We now suppose that $p_i-q_i>0$ implies $s_i-t_i>0$ for all $i \in \{1,2,\cdots,n\}$. Since $p-q \succneqq s-t$, there exists an $i_1 \in \{1,2,\cdots,n\}$ such that $p_{i_1}-q_{i_1}>s_{i_1}-t_{i_1}$, and hence $s_{i_1}-t_{i_1}>0$. According to assumptions, there exists another $i_2 \in \{1,2,\cdots,n\}$ such that $p_{i_2}-q_{i_2}>0$, and hence $0<s_{i_2}-t_{i_2} \leq p_{i_2}-q_{i_2}$. Similarly, we choose l such that

$$\begin{cases} -(p_{i_1}-q_{i_1}) < l_{i_1} \leq -(s_{i_1}-t_{i_1}), \\ -(p_{i_2}-q_{i_2}+s_{i_2}-t_{i_2}) \leq l_{i_2} < -(p_{i_2}-q_{i_2}), \\ l_k = -(p_k-q_k+s_k-t_k), \end{cases}$$

where $k \in \{1,2,\cdots,n\}$ with $k \neq i_1, i_2$. Then $l \in (-\mathbb{N})^n$ and satisfies condition (b2) of Proposition 2.1, and hence the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is nonzero.

Case c2 $p \succneqq q$, $s \succneqq t$ and $(p-q)-(s-t) \notin \mathbb{N}^n \cup (-\mathbb{N})^n$. Then there exist some $i_1, i_2 \in \{1,2,\cdots,n\}$ such that $p_{i_1}-q_{i_1}>s_{i_1}-t_{i_1}$ and $p_{i_2}-q_{i_2}<s_{i_2}-t_{i_2}$, respectively. Choose l such that

$$\begin{cases} -(p_{i_1}-q_{i_1}+s_{i_1}-t_{i_1}) \leq l_{i_1} \leq -(p_{i_1}-q_{i_1}), \\ -(s_{i_2}-t_{i_2}) < l_{i_2} \leq -(p_{i_2}-q_{i_2}), \\ l_k = -(p_k-q_k+s_k-t_k), \end{cases}$$

where $k \in \{1,2,\cdots,n\}$ with $k \neq i_1, i_2$. Then $l \in (-\mathbb{N})^n$ and satisfies condition (b1) of Proposition 2.1, and hence the rank of $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is nonzero finite.

In view of the above discussion, we can summarize what we have proved as follows. If both $p-q$ and $s-t$ belong to $\mathbb{N}^n \cup (-\mathbb{N})^n$, then $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ commute on $h^2(\mathbb{T}^n)$ if and only if at least one of the following conditions holds:

(1) $p = q$, $s = t$, or $p - q = s - t$.

(2) There exists an $i_0 \in \{1, 2, \cdots, n\}$ such that $(p_{i_0} - q_{i_0})(s_{i_0} - t_{i_0}) > 0$, and $p_j - q_j = s_j - t_j = 0$ for any $j \in \{1, 2, \cdots, n\}$ with $j \neq i_0$.

(3) Either $p \gneqq q, s \ngeqq t$ or $p \ngeqq q, s \gneqq t$, and $(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$.

This combined with Lemma 3.1 completes the proof.

4 The Semi-commutator of Monomial Toeplitz Operators on $h^2(\mathbb{T}^n)$

In this section we study the finite rank problem of the semi-commutator of two monomial Toeplitz operators on the pluriharmonic Hardy space $h^2(\mathbb{T}^n)$.

It follows from Proposition 2.2 that the semi-commutator $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{T}^n)$ if and only if there are finite multi-indexes $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$ satisfying condition (b) of Proposition 2.2. Then we begin with the following two lemmas, which will simplify the proof of Theorem 1.2.

Lemma 4.1 *Let $p, q, s, t \in \mathbb{N}^n$. Then the following statements hold*:

(a) *If the semi-commutator $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{T}^n)$, then $(p_i-q_i)(s_i-t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$.*

(b) *If the semi-commutator $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is zero on $h^2(\mathbb{T}^n)$, then the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is also zero on $h^2(\mathbb{T}^n)$.*

Proof This is a direct consequence of (2.3) and Lemma 3.1.

Lemma 4.2 *Let $p, q, s, t \in \mathbb{N}^n$ with $(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$. Then on $h^2(\mathbb{D}^n)$, the following statements hold*:

(a) *If there exists an $i_1 \in \{1, 2, \cdots, n\}$ such that $s_{i_1} - t_{i_1} < 0$ or $p_{i_1} - q_{i_1} < 0$, then $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0$ for any $l \in (-\mathbb{N})^n$.*

(b) *If there exists an $i_1 \in \{1, 2, \cdots, n\}$ such that $s_{i_1} - t_{i_1} > 0$ or $p_{i_1} - q_{i_1} > 0$, then $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0$ for any $l \in \mathbb{N}^n$.*

Proof First assume that there exists an $i_1 \in \{1, 2, \cdots, n\}$ such that $s_{i_1} - t_{i_1} < 0$ or $p_{i_1} - q_{i_1} < 0$. Let us assume the contrary, namely, there exists an $l \in (-\mathbb{N})^n$ such that $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0$. Since $(p_{i_1} - q_{i_1})(s_{i_1} - t_{i_1}) \geq 0$, it is easy to check that

$$l_{i_1} + p_{i_1} - q_{i_1} + s_{i_1} - t_{i_1} < 0.$$

Combining this with Proposition 2.2, we see that $l + p - q + s - t \in (-\mathbb{N})^n$ and $l + s - t \notin \mathbb{N}^n \cup (-\mathbb{N})^n$. Thus there exists an $i_2 \in \{1, 2, \cdots, n\}$ such that

$$l_{i_2} + s_{i_2} - t_{i_2} > 0,$$

which implies that $s_{i_2} - t_{i_2} > 0$, and hence $p_{i_2} - q_{i_2} \geq 0$. Consequently,

$$l_{i_2} + p_{i_2} - q_{i_2} + s_{i_2} - t_{i_2} \geq l_{i_2} + s_{i_2} - t_{i_2} > 0,$$

which leads to a contradiction. This shows that $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0$ for any $l \in (-\mathbb{N})^n$.

Condition (b) can be proved in a similar way as shown before.

We are now ready to prove Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2 In view of (2.2) and Lemma 4.1, we just need to show that condition (c) implies (a). So we assume $(p_i - q_i)(s_i - t_i) \geq 0$ for all $i \in \{1, 2, \cdots, n\}$. If at least one of $p - q$ and $s - t$ does not belong to $\mathbb{N}^n \cup (-\mathbb{N})^n$, then by Lemma 4.2 we have that the rank of $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is 0. Therefore, we only need to consider the remaining case when both $p - q$ and $s - t$ belong to $\mathbb{N}^n \cup (-\mathbb{N})^n$. To this end, we break the discussion into three cases.

Case a Either $p = q$ or $s = t$. Obviously, it follows from Proposition 2.2 that

$$(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) = 0$$

for any $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$. Therefore, the rank of $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is 0.

Case b Either $p \succneqq q, s \precneqq t$ or $p \precneqq q, s \succneqq t$. By (2.2), we need only consider the case that $p \succneqq q, s \precneqq t$. Clearly, there exist $i_1, i_2 \in \{1, 2, \cdots, n\}$ such that

$$p_{i_1} - q_{i_1} > 0, \quad s_{i_2} - t_{i_2} < 0.$$

Then it follows from Lemma 4.2 that the rank of $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is zero.

Case c Either $p \succneqq q, s \succneqq t$ or $p \precneqq q, s \precneqq t$. Since

$$(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]^* = (T_{z^t\overline{z}^s}, T_{z^q\overline{z}^p}],$$

we may assume, without loss of generality, that $p \succneqq q$ and $s \succneqq t$. Obviously,

$$l + s - t \succeq 0$$

for any $l \in \mathbb{N}^n$,

$$l + s - t \preceq l + s - t + p - q \preceq 0$$

for any $l \in (-\mathbb{N})^n$ with $l \preceq -(p - q + s - t)$, and

$$l + p - q + s - t \notin \mathbb{N}^n \cup (-\mathbb{N})^n$$

for any $l \in (-\mathbb{N})^n$ with $-(p-q+s-t) \not\preceq l \not\preceq -(p-q+s-t)$. Thus it follows from Proposition 2.2 that if there exists an $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$ such that $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0$, then

$$-(p - q + s - t) \precneqq l \precneqq 0. \tag{4.1}$$

Therefore, the rank of $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is finite.

In each of the cases above, we have shown that $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{T}^n)$, and hence condition (a) holds.

To finish the characterization of when the rank of the commutator $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is actually 0, it suffices to consider the remaining case $p \succneqq q$ and $s \succneqq t$. Recall from condition (b) of Lemma 4.1 that

$$(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}] \neq 0,$$

provided that $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}] \neq 0$. Combining this with the proof of Theorem 1.1, we need only focus on the following cases:

Case I There exists only one $i_0 \in \{1, 2, \cdots, n\}$ such that $(p_{i_0} - q_{i_0})(s_{i_0} - t_{i_0}) > 0$, and $p_j - q_j = s_j - t_j = 0$ for any $j \in \{1, 2, \cdots, n\}$ with $j \neq i_0$. If $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0$ for some $l \in \mathbb{N}^n \cup (-\mathbb{N})^n$, then it follows from (4.1) that

$$\begin{cases} -(p_{i_0} - q_{i_0} + s_{i_0} - t_{i_0}) \leq l_{i_0} \leq 0, \\ l_j = 0, \end{cases}$$

which implies that $l + s - t \in \mathbb{N}^n \cup (-\mathbb{N})^n$. This contradicts Proposition 2.2. So the rank of $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is zero.

Case II $p - q = s - t$ and there exist at least two of $i \in \{1, 2, \cdots, n\}$ such that $p_i - q_i > 0$. So $p_{i_1} - q_{i_1} = s_{i_1} - t_{i_1} > 0$ and $p_{i_2} - q_{i_2} = s_{i_2} - t_{i_2} > 0$ for some $i_1 \neq i_2 \in \{1, 2, \cdots, n\}$. Choose l such that

$$\begin{cases} -2(s_{i_1} - t_{i_1}) \leq l_{i_1} < -(s_{i_1} - t_{i_1}), \\ -(s_{i_2} - t_{i_2}) < l_{i_2} \leq 0, \\ l_k = -2(s_k - t_k), \end{cases}$$

where $k \in \{1, 2, \cdots, n\}$ with $k \neq i_1, i_2$. Then it follows from Proposition 2.2 that

$$(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}](z^l) \neq 0.$$

Therefore, the rank of $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ is nonzero finite. It is now obvious that the theorem holds.

5 Results about Monomial Toeplitz Operators on $h^2(\mathbb{S}_n)$

In this section we study the problem of when the commutator or the semi-commutator of two monomial Toeplitz operators has finite rank on the pluriharmonic Hardy space $h^2(\mathbb{S}_n)$. We first start with the following lemma.

Lemma 5.1 *Let $p, q \in \mathbb{N}^n$. Then on $h^2(\mathbb{S}_n)$, for each $\gamma \in \mathbb{N}^n$, we have*

$$T_{z^p\overline{z}^q}(z^\gamma) = \begin{cases} \dfrac{(n + |\gamma| + |p| - |q| - 1)!(\gamma + p)!}{(n + |\gamma| + |p| - 1)!(\gamma + p - q)!} z^{\gamma+p-q}, & \gamma + p \succeq q, \\[2ex] \dfrac{(n - |\gamma| - |p| + |q| - 1)!q!}{(n + |q| - 1)!(q - \gamma - p)!} \overline{z}^{q-\gamma-p}, & \gamma + p \preceq q, \\[2ex] 0, & \textit{otherwise} \end{cases}$$

and

$$T_{z^p\overline{z}^q}(\overline{z}^\gamma) = \begin{cases} \dfrac{(n + |\gamma| + |q| - |p| - 1)!(\gamma + q)!}{(n + |\gamma| + |q| - 1)!(\gamma - p + q)!} \overline{z}^{\gamma+q-p}, & \gamma + q \succeq p, \\[2ex] \dfrac{(n - |\gamma| - |q| + |p| - 1)!p!}{(n + |p| - 1)!(-\gamma + p - q)!} z^{p-\gamma-q}, & \gamma + q \preceq p, \\[2ex] 0, & \textit{otherwise.} \end{cases}$$

Proof First we assume $\gamma + p \succeq q$. Then for each $\lambda \in \mathbb{N}^n$, we use formula (1.22) of [12] twice to obtain

$$\langle T_{z^p\overline{z}^q}(z^\gamma), z^\lambda \rangle = \frac{(n + |\gamma| + |p| - |q| - 1)!(\gamma + p)!}{(n + |\gamma| + |p| - 1)!(\gamma + p - q)!} \langle z^{\gamma+p-q}, z^\lambda \rangle.$$

Moreover, for any nonzero $\lambda \in \mathbb{N}^n$, we have

$$\langle T_{z^p\overline{z}^q}(z^\gamma), \overline{z}^\lambda\rangle = \langle z^{\gamma+p-q}, \overline{z}^\lambda\rangle = 0$$

as $\gamma + p \succeq q$. Therefore,

$$T_{z^p\overline{z}^q}(z^\gamma) = \frac{(n+|\gamma|+|p|-|q|-1)!(\gamma+p)!}{(n+|\gamma|+|p|-1)!(\gamma+p-q)!}\, z^{\gamma+p-q}.$$

Next we assume $\gamma + p \preceq q$. Just like the previous case, we can show that

$$\langle T_{z^p\overline{z}^q}(z^\gamma), \overline{z}^\lambda\rangle = \frac{(n-|\gamma|-|p|+|q|-1)!q!}{(n+|q|-1)!(q-\gamma-p)!}\, \langle \overline{z}^{q-\gamma-p}, \overline{z}^\lambda\rangle$$

and

$$\langle T_{z^p\overline{z}^q}(z^\gamma), z^\lambda\rangle = \langle \overline{z}^{q-\gamma-p}, z^\lambda\rangle = 0$$

for any nonzero $\lambda \in \mathbb{N}^n$. Thus

$$T_{z^p\overline{z}^q}(z^\gamma) = \frac{(n-|\gamma|-|p|+|q|-1)!q!}{(n+|q|-1)!(q-\gamma-p)!}\, \overline{z}^{q-\gamma-p}.$$

Finally, we assume that $\gamma + p \not\succeq q$ and $\gamma + p \not\preceq q$. Then $\gamma_j + p_j < q_j$ and $\gamma_i + p_i > q_i$ for some $i, j \in \{1, 2, \cdots, n\}$, which implies $\gamma + p - \lambda \neq q$ and $\gamma + p + \lambda \neq q$ for any $\lambda \in \mathbb{N}^n$. Consequently,

$$\langle T_{z^p\overline{z}^q}(z^\gamma), z^\lambda\rangle = \langle T_{z^p\overline{z}^q\varphi}(z^\gamma), \overline{z}^\lambda\rangle = 0, \quad \lambda \in \mathbb{N}^n,$$

which shows that $T_{z^p\overline{z}^q}(z^\gamma) = 0$.

The computation for $T_{z^p\overline{z}^q}(\overline{z}^\gamma)$ is similar and we leave the details to the interested reader.

Comparing Lemma 5.1 with [6, Lemma 8], we obtain that the formulas for $T_{z^p\overline{z}^q}(z^\gamma)$ and $T_{z^p\overline{z}^q}(\overline{z}^\gamma)$ on $h^2(\mathbb{S}_n)$ are exactly the same as those on the pluriharmonic Bergman space of the unit ball with the parameter $\alpha = -1$. So we omit the details of the proof of Theorem 1.3 and Theorem 1.4, which can be easily obtained by the same argument of [6, Theorem 15] and [6, Theorem 22].

Moreover, we can obtain the following corollaries, which provides convenience for the next section.

Corollary 5.1 *Let $p, q, s, t \in \mathbb{N}^n$. Then the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has nonzero finite rank on $h^2(\mathbb{S}_n)$ if and only if one of the following conditions holds*:

(1) *$p = s = 0$, $q \neq 0$, $t \neq 0$, and $q \neq t$.*

(2) *$q = t = 0$, $p \neq 0$, $s \neq 0$, and $p \neq s$.*

(3) *Either $p \preceq q$, $s \preceq t$ or $q \preceq p$, $t \preceq s$, $|p| = |s|$, $|q| = |t|$, (p_i, q_i, s_i, t_i) satisfies Condition* (I) *for all $i \in \{1, 2, \cdots, n\}$ but not both of* (v) *and* (vi) *in Condition* (I).

Corollary 5.2 *Let $p, q, s, t \in \mathbb{N}^n$. Then the semi-commutator $(T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has nonzero finite rank on $h^2(\mathbb{S}_n)$ if and only if one of the following conditions holds*:

(i) *$p = s = 0$, $q \neq 0$, and $t \neq 0$.*

(ii) *$q = t = 0$, $p \neq 0$, and $s \neq 0$.*

6 Examples and Comparison

In this section we will give some interesting examples and make comparison of the operator theory on the torus and on the unit sphere. First, we have the following proposition.

Proposition 6.1 *For $p, q, s, t \in \mathbb{N}^n$, the following statements hold:*

(1) *If the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ has finite rank on $h^2(\mathbb{S}_n)$, then it also has finite rank on $h^2(\mathbb{T}^n)$.*

(2) *If the operators $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ commute on $h^2(\mathbb{S}_n)$, then they also commute on $h^2(\mathbb{T}^n)$.*

Proof Note that if (p_i, q_i, s_i, t_i) satisfies Condition (I), then

$$(p_i - q_i)(s_i - t_i) \geq 0,$$

and if (p_i, q_i, s_i, t_i) satisfies either (v) or (vi) of Condition (I) for any $i \in \{1, 2, \cdots, n\}$, then

$$p - q = s - t.$$

So the proposition is now a direct consequence of Theorem 1.1 and Theorem 1.3.

As a consequence of Theorem 1.3 and Proposition 6.1, we present two classes of examples, which correspond to cases (v) and (vi) in Condition (I) for the tuple $(|p|, |q|, |s|, |t|)$, of non-trivial monomial Toeplitz operators that commute on both $h^2(\mathbb{S}_n)$ and $h^2(\mathbb{T}^n)$.

Example 6.1 Fix $a, b, c, d, e, f \in \mathbb{N}$ and let

$$\begin{aligned} p &= (0, d, 0, e, a, c), \\ q &= (0, c, e, 0, a, d), \\ s &= (d, 0, 0, f, b, c), \\ t &= (f, 0, c, 0, b, d), \end{aligned}$$

or

$$\begin{aligned} p &= (0, e, 0, b, a, c), \\ q &= (0, b, f, 0, a, d), \\ s &= (e, 0, 0, a, b, c), \\ t &= (f, 0, a, 0, b, d). \end{aligned}$$

Then in each case $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ commute on both $h^2(\mathbb{S}_6)$ and $h^2(\mathbb{T}^6)$.

We would like to mention that the converse of Proposition 6.1 is false. For any positive natural numbers a and b with $a \neq b$, it is easy to check that $T_{z_1^a}$ and $T_{z_1^b}$ commute on $h^2(\mathbb{T}^n)$, but Corollary 5.1 shows that the commutator $[T_{z_1^a}, T_{z_1^b}]$ has nonzero finite rank on $h^2(\mathbb{S}_n)$. Moreover, the following example shows that the commutator $[T_{z^p\overline{z}^q}, T_{z^s\overline{z}^t}]$ does not have finite rank on $h^2(\mathbb{S}_n)$, even if $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ commute on $h^2(\mathbb{T}^n)$.

Example 6.2 Suppose $p = (a, 0, \cdots, 0)$, $q = (0, b, 0, \cdots, 0)$, $s = (0, \cdots, 0)$ and $t = (0, c, 0, \cdots, 0)$ for some positive natural numbers a, b, and c. Then $p - q \notin \mathbb{N}^n \cup (-\mathbb{N})^n$ and

$$(p_i - q_i)(s_i - t_i) \geq 0$$

for all $i \in \{1, 2, \cdots, n\}$. So it follows from Theorem 1.1 that $T_{z_1^a \overline{z}_2^b}$ and $T_{\overline{z}_2^c}$ commute on $h^2(\mathbb{T}^n)$. However, $(|p|, |q|, |s|, |t|)$ does not satisfy Condition (I). Thus from Theorem 1.3 we see that the rank of $[T_{z_1^a \overline{z}_2^b}, T_{\overline{z}_2^c}]$ on $h^2(\mathbb{S}_n)$ is infinite.

Similarly, we have the following result for the semi-commutator of two monomial Toeplitz operators.

Example 6.3 Let $p, q, s, t \in \mathbb{N}^n$. If the semi-commutator $(T_{z^p \overline{z}^q}, T_{z^s \overline{z}^t}]$ has finite rank on $h^2(\mathbb{S}_n)$, then it also has finite rank on $h^2(\mathbb{T}^n)$. Moreover, if the rank of the semi-commutator $(T_{z^p \overline{z}^q}, T_{z^s \overline{z}^t}]$ is zero on $h^2(\mathbb{S}_n)$, then on $h^2(\mathbb{T}^n)$ it is also zero. On $h^2(\mathbb{T}^n)$, it follows that

$$\operatorname{rank}((T_{\overline{z}_1^a}, T_{\overline{z}_1^b}]) = \operatorname{rank}((T_{z_1 \overline{z}_2}, T_{z_1}]) = 0$$

for any positive natural numbers a and b, but on $h^2(\mathbb{S}_n)$ the semi-commutator $(T_{\overline{z}_1^a}, T_{\overline{z}_1^b}]$ has nonzero finite rank, and the semi-commutator $(T_{z_1 \overline{z}_2}, T_{z_1}]$ does not have finite rank.

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