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Chen-Ruan Cohomology and Stringy Orbifold K-Theory for Stable Almost Complex Orbifolds*

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Abstract Comparing to the construction of stringy cohomology ring of equivariant stable almost complex manifolds and its relation with the Chen-Ruan cohomology ring of the quotient almost complex orbifolds, the authors construct in this note a Chen-Ruan cohomology ring for a stable almost complex orbifold. The authors show that for a finite group G and a G-equivariant stable almost complex manifold X, the G-invariant part of the stringy cohomology ring of (X, G) is isomorphic to the Chen-Ruan cohomology ring of the global quotient stable almost complex orbifold $[X/G]$. Similar result holds when G is a torus and the action is locally free. Moreover, for a compact presentable stable almost complex orbifold, they study the stringy orbifold K-theory and its relation with Chen-Ruan cohomology ring.

Keywords Stable almost complex orbifolds, Chen-Ruan cohomology, Orbifold K-theory, Stringy product
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1 Introduction

Since the introduction of Chen-Ruan cohomology ring (see [9]) of almost complex orbifolds and orbifold Gromov-Witten theory (see [8]) of compact symplectic orbifolds, there are lots of works on related area. The most simple orbifolds are global quotient orbifolds. Let G be a finite group and X be a G-equivariant almost complex manifold, the global quotient orbifold $[X/G]$ is an almost complex orbifold. In 2003, Fantechi-Göttsche [14] constructed a stringy cohomology ring $\mathscr{H}^*(X, G)$, which they called orbifold cohomology, for the pair (X, G) by following the construction of Chen-Ruan cohomology ring in [9], and showed that $\mathscr{H}^*(X, G)^G$, the G-invariant part of $\mathscr{H}^*(X, G)$, is isomorphic to the Chen-Ruan cohomology ring $H^*_{CR}([X/G])$ as Frobenius algebras. Their construction of stringy cohomology ring works for general G-equivariant stable almost complex manifolds. In 2007, Jarvis-Kaufmann-Kimura [17] constructed the stringy Chow ring and stringy K-theory for G-varieties when G is finite.

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They also constructed a stringy Chern character which gives rise to an isomorphism between the stringy Chow ring and stringy K-theory. When the group G is not finite but a torus T, Goldin-Holm-Knutson [15] constructed a stringy cohomology ring $NH_T^{*,\diamond}(Y)$ over the T-equivariant cohomology for a T-equivariant stable almost complex manifold Y. When the T-action is locally free and the quotient orbifold $[Y/T]$ is almost complex, they proved that $NH_T^{*,\diamond}(Y)$ is isomorphic to $H^*_{CR}([Y/T])$. Recently, when G is a non-abelian Lie group and Y is a G-equivariant almost complex manifold, Chen and the authors [6] constructed an equivariant commutative stringy cohomology ring for (Y, G).

The construction of stringy cohomology ring in [14–15] works for general G-equivariant stable almost complex manifolds. For both cases the quotient orbifolds are stable almost complex orbifolds (see Definition 2.1). In [10], Ding-Jiang-Pan constructed Chen-Ruan cohomology ring for an almost contact orbifold X, by applying Chen-Ruan's construction in [9] to the almost complex orbifold $\mathsf{X}\times\mathbb{R}$. However, there is still a lack of a Chen-Ruan cohomology ring for stable almost complex orbifolds. In this note we will construct a Chen-Ruan cohomology ring for a stable almost complex orbifold, and extend the isomorphism between (the invariant part of) stringy cohomology ring and Chen-Ruan cohomology ring to stable almost complex orbifolds for both cases of G being finite or abelian. As the almost complex case, for a stable almost complex orbifold X there is also an associated inertia orbifold IX and the underlying group of its Chen-Ruan cohomology ring is defined to be the de Rham cohomology group of IX with a degree shifting. As the almost complex case, we need an obstruction bundle $\mathsf{E}^{[2]}$ over the 2-sector $\mathsf{X}^{[2]}$ to define the ring structure. In this note we adapt the K-theoretical definition of obstruction bundles in [16–17] to define $\mathsf{E}^{[2]}$ (see Definition 3.1). We could prove that $\mathsf{E}^{[2]}$ is in fact an honest bundle over $\mathsf{X}^{[2]}$, not just an element in the K-group of orbifold bundles over $\mathsf{X}^{[2]}$.

On the other hand, there are also a lot of works on orbifold K-theory. For example, Adem-Ruan [2] studied orbifold K-theory and orbifold K-theory twisted by discrete torsion, Lupercio-Uribe [19] studied orbifold K-theory twisted by general $U(1)$-gerbes. Moreover, Adem-Ruan-Zhang [3] defined a stringy product over the twisted orbifold K-theory ${}^{\tau}K_{\mathrm{orb}}(\mathsf{IX})$, where $\tau = \theta(\varphi)$ is in the image of the inverse transgression map and φ is a 2-gerbe over X. Moreover, as noted by Hu-Wang [16], when φ is trivial, the stringy product of Adem-Ruan-Zhang induced a stringy product on the orbifold K-theory $K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C})$ of X. In [17], Jarvis-Kaufmann-Kimura also defined a full orbifold K-theory $\mathsf{K}_{\mathrm{orb}}(\mathsf{X})$, for an orbifold X, and showed that for a global quotient orbifold $\mathsf{X} = [X/G]$ of a G-variety X, the G-invariant part of the stringy K-theory $\mathscr{K}(X, G)$ is a sub-algebra of $\mathsf{K}_{\mathrm{orb}}(\mathsf{X})$, see also [11–13]. Becerra-Uribe [4] studied the stringy products of twisted orbifold K-theory for abelian quotient orbifolds. The most impressive result is that Hu-Wang [16] showed for every compact presentable almost complex orbifold there is a modified delocalized Chern character which gives rise to a ring isomorphism between the Chen-Ruan cohomology ring and the orbifold K-theory equipped with the stringy product of Adem-Ruan-Zhang (twisted by trivial 2-gerbe). This result was extended to twisted Chen-Ruan cohomology ring and twisted stringy orbifold K-theory by Lin [18] for global quotient orbifolds.

Following the construction in [1], we define an associative stringy product over $K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C})$ for a compact presentable stable almost complex orbifold X. We then show that there is a modified delocalized Chern character which is a ring isomorphism from $K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C})$ to $H^*_{CR}(\mathsf{X}, \mathbb{C})$.

Furthermore we could extend the stringy product over twisted orbifold K-theory of almost complex orbifolds of Adem-Ruan-Zhang [3] to stable almost complex orbifolds.

This note is organized as follows. In Section 2 we give the definition of stable almost complex orbifolds, and define the Chen-Ruan cohomology group of them. In Section 3 we first construct the obstruction bundle and define the product, prove the associativity, then we study the relation between the Chen-Ruan cohomology rings of (global) quotient stable almost complex orbifolds and the stringy cohomology rings of equivariant stable almost complex manifolds. In Section 4 we study the stringy product over $K^*_{\mathrm{orb}}(\mathsf{X},\mathbb{C})$ for a compact stable almost complex orbifold X. For the compactness of this note we put the construction of the stringy product over twisted orbifold K-theory of stable almost complex orbifolds in the appendix.

2 Stable Almost Complex Orbifolds

In this note we study orbifolds via orbifold groupoids, i.e., proper étale Lie groupoids. We assume that the readers are familiar with orbifold groupoids. One can see [1, 3] for example and references therein for basic concepts on orbifolds, orbifold groupoids, coarse space, de Rham cohomology of orbifolds, morphisms between orbifold groupoids, quasi-suborbifolds, intersection of quasi-suborbifolds and etc.. We next give the definition of stable almost complex orbifold groupoids.

We first give the definition of stable complex orbifold bundles. Let $\mathsf{X} = (X^1 \rightrightarrows X^0)$ be an orbifold groupoids, with $s, t : X^1 \to X^0$ be its source and target maps from the arrow space to the object space. An orbifold vector bundle E over X consists of a vector bundle $\pi^0 : E^0 \to X^0$ and a section of $\sigma \in \mathrm{Hom}(s^*E^0, t^*E^0)$ such that for every arrow $g \in X^1$,

$$\sigma(g) : s^*E^0_g = E^0_{s(g)} \to t^*E^0_g = E^0_{t(g)}$$

is an isomorphism of vector space, and $\sigma(h) \circ \sigma(g) = \sigma(gh)$ for any two composable[1] arrows in X^1. The σ gives us a left X-action[2] on E^0 with anchor map given by $\pi^0 : E^0 \to X^0$ and action map given by

$$X^1 \ {}_s\!\times_{\pi^0} E^0 \to E^0, \quad (g, v) \mapsto \sigma(g)(v).$$

Then we get an action groupoid $\mathsf{E} = \mathsf{X} \ltimes E^0 = (E^1 \rightrightarrows E^0)$ with

$$E^1 := \{(v, g, w) \in s^*E^0 \times X^1 \times t^*E^0 \mid \sigma(g)(v) = w\}.$$

It is also an orbifold groupoid. The source and target maps are given by $(v, g, w) \mapsto v$ and w respectively. The projection $\pi^1 : E^1 \to X^1$, $\pi^1(v, g, w) = g$ gives us a vector bundle, which together with the bundle map $\pi^0 : E^0 \to X^0$ gives us an orbifold groupoid morphism $\pi : \mathsf{E} \to \mathsf{X}$. However, sometimes it is more convenience to view an orbifold vector bundle E over X as a vector bundle $\pi^0 : E^0 \to X^0$ with a left X-action, and then when we talk about the fiber of E we mean the fiber of $\pi^0 : E^0 \to X^0$. The definitions of direct sum and tensor product of orbifold vector bundles are similar to the manifold case.

We denote by $\underline{\mathbb{R}}^m \to \mathsf{X}$ or simply by $\underline{\mathbb{R}}^m$ the trivial bundle of rank m over X. Here by trivial we mean that both $E^i = X^i \times \mathbb{R}^m$ are trivial bundles and the section σ is identity, i.e., $\sigma(g) = \mathrm{id}_{\mathbb{R}^m}$ for every $g \in X^1$. Hence the action of X on $X^0 \times \mathbb{R}^m$ is trivial on fibers.

[1] This means $t(g) = s(h)$, the source of the arrow h is the target of the arrow g.

[2] See [7, Section 2.2] for example for the definition of groupoid action on manifolds.

A complex structure J over $\mathsf{E} \to \mathsf{X}$ consists of a pair of complex structures J^i over $E^i \to X^i$, $i = 0, 1$, such that $s^*J^0 = t^*J^0 = J^1$. Similarly, a hermitian metric over (E, J) is a pair of hermitian metrics over (E^i, J^i) that are compatible with s^* and t^*.

The tangent bundle of X is

$$\mathsf{TX} := (TX^1 \rightrightarrows TX^0)$$

with source and target maps being the differentials of s and t of X.

Definition 2.1 *Let $\mathsf{E} \to \mathsf{X}$ be an orbifold vector bundle. A stable complex structure over E consists of a trivial bundle $\underline{\mathbb{R}}^m \to \mathsf{X}$ and a complex structure J over $\mathsf{E} \oplus \underline{\mathbb{R}}^m$.*

A stable almost complex structure over X is a stable complex structure over TX. When X is equipped with a stable almost complex structure we call it a stable almost complex orbifold groupoid.

Stable almost complex orbifold groupoids are direct generalizations of stable almost complex manifolds in [5].

We abbreviate "stable almost complex orbifold groupoid" as "SACOG".

2.1 *k*-Sectors

For $k \in \mathbb{Z}_{\geq 1}$, the k-sector $\mathsf{X}^{[k]}$ of X is an action groupoid $\mathsf{X}^{[k]} := \mathsf{X} \ltimes \mathcal{S}^k$ obtained from a left X-action on the space

$$\mathcal{S}^k := \{(g_1, \cdots, g_k) \in (X^1)^k \mid s(g_1) = t(g_1) = \cdots = s(g_k) = t(g_k)\},$$

where the left X-action on $\mathcal{S}^k$ has

(i) an anchor map: $\pi^k : \mathcal{S}^k \to X^0,\ (g_1, \cdots, g_k) \mapsto s(g_1)$, and

(ii) an action map: $\rho_k : X^1 \ {}_s\!\times_{\pi^k} \mathcal{S}^k \to \mathcal{S}^k,\ (h, (g_1, \cdots, g_k)) = (h^{-1}g_1h, \cdots, h^{-1}g_kh)$.

Then $\mathsf{X}^{[k]} = \mathsf{X} \ltimes \mathcal{S}^k = (X^1 \ {}_s\!\times_{\pi^k} \mathcal{S}^k \rightrightarrows \mathcal{S}^k)$; the source map is the projection to the second factor and the target map is ρ_k. When $k = 1$, $\mathsf{X}^{[1]}$ is called the inertia groupoid of X and is denoted by IX.

There are several evaluation morphisms between various k-sectors and X. We list them by only writing down the maps on object spaces. The maps on arrows are obvious.

(i) For $l \leq k$, $e_{i_1, \cdots, i_l} : \mathsf{X}^{[k]} \to \mathsf{X}^{[l]}$ is given by $e_{i_1, \cdots, i_l}(g_1, \cdots, g_k) = (g_{i_1}, \cdots, g_{i_l})$.

(ii) For $k \geq 1$, $e_{1 \cdots k} : \mathsf{X}^{[k]} \to \mathsf{X}^{[1]} = \mathsf{IX}$ is given by $e_{1 \cdots k}(g_1, \cdots, g_k) = (g_1 \cdot \cdots \cdot g_k)$.

(iii) For $k \geq 2$, $\mu_i : \mathsf{X}^{[k]} \to \mathsf{X}^{[k-1]}$, $1 \leq i \leq k-1$ is given by $\mu_i(g_1, \cdots, g_k) = (g_1, \cdots, g_i g_{i+1}, \cdots, g_k)$.

(iv) For $k \geq 1$, $e : \mathsf{X}^{[k]} \to \mathsf{X}$ is given by $e(g_1, \cdots, g_k) = s(g_1)$.

All these evaluation morphisms are quasi-embeddings (see [3, Definition 2.7]).

There is also an involution morphism $I : \mathsf{IX} \to \mathsf{IX}, (g) \mapsto (g^{-1})$.

For each $k \geq 1$, according to the decomposition of connected components of the coarse space $|\mathsf{X}^{[k]}|$, we have a disjoint union decomposition

$$\mathsf{X}^{[k]} := \bigsqcup_{[\vec{g} = (g_1, \cdots, g_k)] \in \mathcal{T}^k} \mathsf{X}_{[\vec{g}]},$$

where $\mathcal{T}^k$ is the index set of components and the set of equivalence classes of $\vec{g}$ w.r.t conjugations. Then the above evaluation maps and the involution map also decompose into components.

Now suppose that X has a stable almost complex structure, hence is a SACOG. Then there is a complex bundle $\mathsf{V} := \mathsf{TX} \oplus \underline{\mathbb{R}}^m$ for some $m \in \mathbb{Z}_{\geq 0}$. For each $k \in \mathbb{Z}_{\geq 1}$, we pull back V via the morphism $e : \mathsf{X}^{[k]} \to \mathsf{X}$ to $\mathsf{X}^{[k]}$ to get a pull-back bundle

$$\mathsf{V}^{[k]} := e^*\mathsf{V} = e^*\mathsf{TX} \oplus \underline{\mathbb{R}}^m.$$

The complex structure over V pulls back to a complex structure over each $\mathsf{V}^{[k]}$. Denote by $\mathsf{V}^{[k]}_{[\vec{g}]}$ the restriction of the bundle $\mathsf{V}^{[k]}$ over a component $\mathsf{X}_{[\vec{g}]}$. Choose a hermitian metric on V. Then all $\mathsf{V}^{[k]}$ have induced hermitian metrics.

Take a point $\vec{g} = (g_1, \cdots, g_k) \in \mathcal{S}^k$ with $s(g_1) = x \in X^0$. Then the fiber of $\mathsf{V}^{[k]}$ over $\vec{g}$ is

$$\mathsf{V}^{[k]}_{\vec{g}} = \mathsf{V}_x = T_x X^0 \oplus \mathbb{R}^m.$$

It has a natural $Z\langle\vec{g}\rangle$-action, where $\langle\vec{g}\rangle$ is the subgroup of G_x generated by $\vec{g}$, $Z\langle\vec{g}\rangle$ is its center and G_x is the local (or isotropy) group of x in X. Since we have chosen a hermitian metric, $\mathsf{V}^{[k]}_{\vec{g}}$ is a unitary representation of $Z\langle\vec{g}\rangle$, we could decompose it into irreducible $Z\langle\vec{g}\rangle$-representations

$$\mathsf{V}^{[k]}_{\vec{g}} = \bigoplus_{\lambda \in \widehat{Z\langle\vec{g}\rangle}} \mathsf{V}^{[k]}_{\vec{g},\lambda}.$$

Then one can see that the decomposition of fibers forms a decomposition of the bundle $\mathsf{V}^{[k]}_{[\vec{g}]}$,

$$\mathsf{V}^{[k]}_{[\vec{g}]} = \bigoplus_{\lambda \in \widehat{Z\langle\vec{g}\rangle}} \mathsf{V}^{[k]}_{[\vec{g}],\lambda}. \tag{2.1}$$

On the other hand, note that each g_i in $\vec{g}$ acts on $\mathsf{V}^{[k]}_{\vec{g}}$, and also on each irreducible representation $\mathsf{V}^{[k]}_{\vec{g},\lambda}$ of $Z\langle\vec{g}\rangle$. Since $\mathrm{ord}(g_i)$ is finite, the g_i-action on $\mathsf{V}^{[k]}_{\vec{g},\lambda}$ is by multiplying $\exp^{2\pi\sqrt{-1}m_{\lambda,i}}$ for some

$$m_{i,\lambda} \in \mathbb{Q} \cap [0,1).$$

These numbers are constant over each component $\mathsf{X}_{[\vec{g}]}$. When $k = 1$, we omit the i, and write them as m_λ.

Note that the tangent space of $\mathcal{S}^k$ at $\vec{g}$ is just the fixed part of $T_x X^0$ under the action of $\langle\vec{g}\rangle$. Therefore the irreducible representation $\mathsf{V}^{[k]}_{\vec{g},\lambda}$ with zero weight $m_{i,\lambda} = 0$ for all $i = 1, \cdots, n$ corresponds to the tangent space of $\mathcal{S}^k$ and the fiber of the trivial bundle $\underline{\mathbb{R}}^m$. So we see that over each component $\mathsf{X}_{[\vec{g}]}$,

$$\bigoplus_{m_{\lambda,i}=0, \forall i=1,\cdots,k} \mathsf{V}^{[k]}_{[\vec{g}],\lambda} = \mathsf{TX}^{[k]} \oplus \underline{\mathbb{R}}^m. \tag{2.2}$$

This is a complex bundle over $\mathsf{X}^{[k]}$, since each $\mathsf{V}^{[k]}_{[\vec{g}],\lambda}$ is complex. Therefore we have proved the following result.

Proposition 2.1 *When X is a SACOG, for every $k \geq 1$, every component $\mathsf{X}^{[k]}_{[\vec{g}]}$ of $\mathsf{X}^{[k]}$ is a SACOG with stable almost complex structures inherited from X via* (2.2).

On the other hand,

$$\bigoplus_{m_{\lambda,1}+\cdots+m_{\lambda,k}>0} \mathsf{V}^{[k]}_{[\vec{g}],\lambda} \tag{2.3}$$

is the normal bundle $\mathsf{N}^{[k]}_{[\vec{g}]}$ of the evaluation morphism $e : \mathsf{X}^{[k]} \to \mathsf{X}$ over the component $\mathsf{X}_{[\vec{g}]}$. It is a complex orbifold vector bundle. We denote the disjoint union of these $\mathsf{N}^{[k]}_{[\vec{g}]}$ by $\mathsf{N}^{[k]}$.

Remark 2.1 From the analysis above, we see that if the stable almost complex structure is given by the complex orbifold bundle $\mathsf{V} = \mathsf{TX} \oplus \underline{\mathbb{R}}^m$, then for each k-sector $\mathsf{X}^{[k]}$, we have the decomposition of the complex bundle $\mathsf{V}^{[k]} = e^*\mathsf{V}$:

$$\mathsf{V}^{[k]} = (\mathsf{TX}^{[k]} \oplus \underline{\mathbb{R}}^m) \oplus \mathsf{N}^{[k]}$$

with $\mathsf{N}^{[k]}$ and $\mathsf{TX}^{[k]} \oplus \underline{\mathbb{R}}^m$ being two complex sub-bundles. And the stable almost complex structure associated to $\mathsf{X}^{[k]}$ is given by the complex bundle $\mathsf{TX}^{[k]} \oplus \underline{\mathbb{R}}^m$.

Remark 2.2 We also need the normal bundle $\mathsf{N}^{e_{12}}$ for the evaluation morphism $e_{12} : \mathsf{X}^{[2]} \to \mathsf{X}^{[1]}$ in the following. It is a complex sub-bundle of $\mathsf{V}^{[2]}$, moreover, a sub-bundle of $\mathsf{N}^{[2]}$. Consider a component $\mathsf{X}_{[g_1,g_2]}$ of $\mathsf{X}^{[2]}$. Denote the component of $\mathsf{N}^{e_{12}}$ over $\mathsf{X}_{[g_1,g_2]}$ by $\mathsf{N}^{e_{12}}_{[g_1,g_2]}$. Then the fiber of $\mathsf{N}^{e_{12}}_{[g_1,g_2]}$ over a point (g_1, g_2) with $s(g_1) = x$ is the subspace of V_x over which the action is trivial for g_1g_2 and nontrivial for g_1 or g_2.

Consider the diagram

$$\begin{tikzcd} \mathsf{X}^{[2]} \arrow[r, "e_{12}"] \arrow[dr, "e"'] & \mathsf{X}^{[1]} \arrow[d, "e"] \\ & \mathsf{X}. \end{tikzcd}$$

Then we have

$$\mathsf{N}^{[2]} = \mathsf{N}^{e_{12}} \oplus e_{12}^*\mathsf{N}^{[1]}. \tag{2.4}$$

2.2 Chen-Ruan cohomology of SACOGs

As for almost complex orbifolds, the Chen-Ruan cohomology group of a SACOG X is also defined as the de Rham cohomology group of its inertia groupoid IX with a degree shifting. We first define the degree shifting. Note that, every complex orbifold bundle over $\mathsf{X}^{[1]} = \mathsf{IX}$ has a canonical finite order automorphism, given by the action of $g \in \mathcal{S}^1$ on the fiber over g.

Definition 2.2 (see [16]) *Let E be any complex orbifold vector bundle with an automorphism Φ of finite order over an orbifold groupoid X. Choose a hermitian metric on E preserved by Φ. Then E has an eigen-bundle decomposition*

$$\mathsf{E} = \bigoplus_{m_j \in \mathbb{Q}\cap[0,1)} \mathsf{E}(m_j),$$

where Φ acts on $\mathsf{E}(m_j)$ as multiplication by $\exp^{2\pi\sqrt{-1}m_j}$ for $m_j \in \mathbb{Q}\cap[0,1)$. We define

$$\mathsf{E}_\Phi = \bigoplus_{m_j \in \mathbb{Q}\cap(0,1)} m_j\mathsf{E}(m_j), \quad \mathsf{E}_{\Phi^{-1}} = \bigoplus_{m_j \in \mathbb{Q}\cap(0,1)} (1-m_j)\mathsf{E}(m_j)$$

as a linear combination of vector bundles with rational coefficients or as an element in $K^0_{\mathrm{orb}}(\mathsf{X})\otimes\mathbb{Q}$.

One immediately sees that

$$\mathsf{E}_{\Phi} \oplus \mathsf{E}_{\Phi^{-1}} = \bigoplus_{m_j \in \mathbb{Q} \cap (0,1)} \mathsf{E}(m_j) \tag{2.5}$$

is the sub-bundle over which the Φ-action is nontrivial.

Suppose that the stable almost complex structure over X is given by $\mathsf{V} = \mathsf{TX} \oplus \underline{\mathbb{R}}^m$. Now consider the complex bundle $\mathsf{V}^{[1]} = e^*\mathsf{TX} \oplus \underline{\mathbb{R}}^m \to \mathsf{IX}$. It has a canonical automorphism Φ of finite order given by the action of $g \in \mathcal{S}^1$ on the fiber over g. Then by taking a Φ-invariant hermitian metric, we could decompose it into eigen-bundles

$$\mathsf{V}^{[1]} = \bigoplus_{m_j \in \mathbb{Q} \cap [0,1)} \mathsf{V}^{[1]}(m_j).$$

This decomposition varies over different connected component $\mathsf{X}_{[g]}$ of IX. So we write

$$\mathsf{V}^{[1]}_{[g]} := \mathsf{V}^{[1]}|_{\mathsf{X}_{[g]}} = \bigoplus_{m_{j,[g]} \in \mathbb{Q} \cap [0,1)} \mathsf{V}^{[1]}_{[g]}(m_{j,[g]}). \tag{2.6}$$

Definition 2.3 *For each $[g] \in \mathcal{T}^1$, we define the degree shifting number of $\mathsf{X}_{[g]}$ to be*

$$\iota(\mathsf{X}_{[g]}) = \iota([g]) = \sum_{m_{j,[g]}} m_{j,[g]} \cdot \mathrm{rank}_{\mathbb{C}} \mathsf{V}^1_{[g]}(m_{j,[g]}).$$

Note that every hermitian metric over $\mathsf{V}^{[1]}$ is Φ-invariant. So we could use the same metric for the decomposition (2.6) of $\mathsf{V}^{[1]}$ and the decomposition (2.1) with $k = 1$. Then the summands in the irreducible decomposition (2.1) of $\mathsf{V}^{[k]}_{[\vec{g}]}$ with $k = 1$ combine into the summands of the decomposition (2.6). So we also have

$$\iota([g]) = \sum_{\lambda \in \widehat{Z\langle \vec{g} \rangle}} m_\lambda \cdot \mathrm{rank}_{\mathbb{C}} \mathsf{V}^1_{[g],\lambda}.$$

Moreover, the analysis above shows that (comparing (2.6) with (2.1)–(2.3) for $k = 1$)

$$\mathsf{N}^{[1]}_{\Phi} = \mathsf{V}^{[1]}_{\Phi}, \quad \mathsf{N}^{[1]}_{\Phi^{-1}} = \mathsf{V}^{[1]}_{\Phi^{-1}}. \tag{2.7}$$

Definition 2.4 *For a SACOG X, we define the Chen-Ruan cohomology group of X as*

$$H^*_{CR}(\mathsf{X}, \mathbb{C}) := H^{*-2\iota(\mathsf{IX})}_{dR}(\mathsf{IX}, \mathbb{C}) = \bigoplus_{[g] \in \mathcal{T}^1} H^{*-2\iota([g])}_{dR}(\mathsf{X}_{[g]}, \mathbb{C}),$$

and the Chen-Ruan cohomology group with compact support of X as

$$H^*_{CR,c}(\mathsf{X}, \mathbb{C}) := H^{*-2\iota(\mathsf{IX})}_{c,dR}(\mathsf{IX}, \mathbb{C}) = \bigoplus_{[g] \in \mathcal{T}^1} H^{*-2\iota([g])}_{c,dR}(\mathsf{X}_{[g]}, \mathbb{C}).$$

Since the de Rham cohomology (with compact support) of an orbifold is canonically isomorphic to the singular cohomology (with compact support) of its coarse space (see [1, Section 2.1]), in the following we will omit the subscript "dR".

For an oriented orbifold X, the orbifold Poincaré pairing

$$\int^{\mathrm{orb}}_{\mathsf{X}} : H^k(|\mathsf{X}|, \mathbb{C}) \times H^{\dim \mathsf{X} - k}_c(|\mathsf{X}|, \mathbb{C}) \to \mathbb{C}$$

is nondegenerate. A SACOG X is natural orientated. So is its inertia orbifold IX. The involution map $I : \mathsf{IX} \to \mathsf{IX}$ and the paring above induce a nondegenerate pairing

$$\langle \cdot, \cdot \rangle_{\mathsf{X}} : H^*_{CR}(\mathsf{X}, \mathbb{C}) \times H^*_{CR,c}(\mathsf{X}, \mathbb{C}) \to \mathbb{C}, \quad \langle \alpha, \beta \rangle_{\mathsf{X}} := \int^{\mathrm{orb}}_{\mathsf{IX}} \alpha \cup I^* \beta.$$

3 Chen-Ruan Product for SACOGs

3.1 Obstruction bundle and Chen-Ruan product

Let X be a SACOG. We next define an obstruction bundle over $\mathsf{X}^{[2]}$. Recall that we have $e_i : \mathsf{X}^{[2]} \to \mathsf{IX}$, $i = 1, 2$, and $e_{12} : \mathsf{X}^{[2]} \to \mathsf{IX}$. Recall also that $\mathsf{V}^{[1]}$ has a canonical finite order automorphism Φ.

Definition 3.1 *We define the obstruction bundle over* $\mathsf{X}^{[2]}$ *to be*

$$\mathsf{E}^{[2]} : = e_1^* \mathsf{V}^{[1]}_{\Phi} \oplus e_2^* \mathsf{V}^{[1]}_{\Phi} \oplus e_{12}^* \mathsf{V}^{[1]}_{\Phi^{-1}} \ominus \mathsf{N}^{[2]} \tag{3.1}$$

as an element in $K^0_{\mathrm{orb}}(\mathsf{X}^{[2]}) \otimes \mathbb{Q}$. *The* $\mathsf{N}^{[2]}$ *is given by* (2.3).

Remark 3.1 By (2.7) we have

$$\mathsf{E}^{[2]} : = e_1^* \mathsf{N}^{[1]}_{\Phi} \oplus e_2^* \mathsf{N}^{[1]}_{\Phi} \oplus e_{12}^* \mathsf{N}^{[1]}_{\Phi^{-1}} \ominus \mathsf{N}^{[2]}. \tag{3.2}$$

Remark 3.2 We could also define this obstruction bundle via the original construction of Chen-Ruan in [9] by taking the invariant part $H^{0,1}(\Sigma, T_x X^0 \oplus \mathbb{R}^m)^{\langle g_1, g_2 \rangle}$ as the fiber of $\mathsf{E}^{[2]}$ over the point $(g_1, g_2) \in \mathcal{S}^2$, where $x = s(g_1)$ and we have replaced the tangent space $T_x X^0$ by the complex linear space $T_x X^0 \oplus \mathbb{R}^m$. By the proof of [16, Theorem 3.2], we could show that this construction will also give rise to the definition formula (3.1) of $\mathsf{E}^{[2]}$. On the other hand, by similar computation as (3.5) in the proof of Lemma 3.1 below we can show that over each component of $\mathsf{X}^{[2]}$, $\mathsf{E}^{[2]}$ is a direct sum of certain bundles. Hence $\mathsf{E}^{[2]}$ is not just an element in $K^0_{\mathrm{orb}}(\mathsf{X}^{[2]}) \otimes \mathbb{Q}$, but an honest bundle over $\mathsf{X}^{[2]}$. So we can take the Euler class of $\mathsf{E}^{[2]}$.

Definition 3.2 *We define a 3-point function for* $\alpha, \beta \in H^*_{CR}(\mathsf{X}, \mathbb{C})$ *and* $\gamma \in H^*_{CR,c}(\mathsf{X}, \mathbb{C})$ *by*

$$\langle \alpha, \beta, \gamma \rangle := \int^{\mathrm{orb}}_{\mathsf{X}^{[2]}} e_1^* \alpha \cup e_2^* \beta \cup (I \circ e_{12})^* \gamma \cup e(\mathsf{E}^{[2]}).$$

The product is defined as follows.

Definition 3.3 *Given* $\alpha, \beta \in H^*_{CR}(\mathsf{X}, \mathbb{C})$, *the product* $\alpha \cup_{CR} \beta$ *is defined by requiring that for every* $\gamma \in H^*_c(\mathsf{X}, \mathbb{C})$, *the following equality*

$$\langle \alpha \cup_{CR} \beta, \gamma \rangle_{\mathsf{X}} = \langle \alpha, \beta, \gamma \rangle$$

holds. Equivalently, this product is also given by

$$\alpha \cup_{CR} \beta = e_{12,*}(e_1^* \alpha \wedge e_2^* \beta \wedge e(\mathsf{E}^{[2]})).$$

Theorem 3.1 *The product "* $\cup_{CR}$ *" over* $H^*_{CR}(\mathsf{X}, \mathbb{C})$ *is associative.*

Proof To prove the associativity we need to show that for any $\alpha, \beta, \gamma \in H^*_{CR}(\mathsf{X}, \mathbb{C})$, the following equality

$$\begin{aligned} & e_{12,*}(e_1^*(e_{12,*}(e_1^*\alpha \wedge e_2^*\beta \wedge e(\mathsf{E}^{[2]}))) \wedge e_2^*\gamma \wedge e(\mathsf{E}^{[2]})) \\ = {} & e_{12,*}(e_1^*\alpha \wedge e_2^*(e_{12,*}(e_1^*\beta \wedge e_2^*\gamma \wedge e(\mathsf{E}^{[2]}))) \wedge e(\mathsf{E}^{[2]})) \end{aligned} \tag{3.3}$$

holds. For simplicity, we could assume $\alpha \in H^*(\mathsf{X}_{[g_1]}, \mathbb{C})$, $\beta \in H^*(\mathsf{X}_{[g_2]}, \mathbb{C})$, $\gamma \in H^*(\mathsf{X}_{[g_3]}, \mathbb{C})$. Then we see that both left and right sides support in a neighborhood of the intersection of quasi-suborbifolds[3] of IX,

$$\mathsf{X}_{[g_1,g_2]} \cap \mathsf{X}_{[g_1g_2,g_3]} = \bigsqcup_{[h_1,h_2,h_3],[h_i]=[g_i]} \mathsf{X}_{[h_1,h_2,h_3]} = \mathsf{X}_{[g_1,g_2g_3]} \cap \mathsf{X}_{[g_2,g_3]}$$

in IX. Set $\vec{h} = (h_1, h_2, h_3)$. We could assume that the neighborhoods of all different $\mathsf{X}_{[\vec{h}]}$ above in IX do not intersect with each other, since their images in IX are closed and do not intersect with each other.

For a fixed $\mathsf{X}_{[\vec{h}]}$ we have the following commutative diagram:

$$\begin{tikzcd}
 & & \mathsf{X}_{[h_1]} & & \\
 & \mathsf{X}_{[h_2]} & \mathsf{X}_{[h_1,h_2]} \arrow[l, "e_2"'] \arrow[u, "e_1"] \arrow[r, "e_{12}"] & \mathsf{X}_{[h_{12}]} & \\
\mathsf{X}_{[h_3]} & \mathsf{X}_{[h_2,h_3]} \arrow[l, "e_2"'] \arrow[u, "e_1"] \arrow[d, "e_{12}"'] & \mathsf{X}_{[\vec{h}]} \arrow[l, "e_{2,3}"'] \arrow[u, "e_{1,2}"'] \arrow[r, "\mu_1"] \arrow[d, "\mu_2"'] \arrow[dr, "e_{123}"] & \mathsf{X}_{[h_{12},h_3]} \arrow[u, "e_1"'] \arrow[r, "e_2"] \arrow[d, "e_{12}"] & \mathsf{X}_{[h_3]} \\
 & \mathsf{X}_{[h_{23}]} & \mathsf{X}_{[h_1,h_{23}]} \arrow[l, "e_2"'] \arrow[r, "e_{12}"] \arrow[d, "e_1"'] & \mathsf{X}_{[h_{123}]} & \\
 & & \mathsf{X}_{[h_1]} & &
\end{tikzcd} ,$$

where $h_{12} = h_1h_2$, $h_{23} = h_2h_3$ and $h_{123} = h_1h_2h_3$.

Fix a small neighborhood $\mathsf{U}_{[\vec{h}]}$ of $\mathsf{X}_{[\vec{h}]}$ in IX. By the clean intersection formula in [3, Lemma 7.2], the restriction of the LHS of (3.3) in the neighborhood $\mathsf{U}_{[\vec{h}]}$ of $\mathsf{X}_{[\vec{h}]}$ is

$$\begin{aligned} & e_{12,*}(e_1^*(e_{12,*}(e_1^*\alpha \wedge e_2^*\beta \wedge e(\mathsf{E}^{[2]}))) \wedge e_2^*\gamma \wedge e(\mathsf{E}^{[2]}))|_{\mathsf{U}_{[\vec{h}]}} \\ = {} & e_{123,*}(e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge \mathsf{E}^{[2]}_{[h_1,h_2]}|_{\mathsf{X}_{[\vec{h}]}} \wedge \mathsf{E}^{[2]}_{[h_1h_2,h_3]}|_{\mathsf{X}_{[\vec{h}]}} \wedge e(E(\mathsf{X}_{[h_1h_2]}, \mathsf{X}_{[h_1,h_2]}, \mathsf{X}_{[h_1h_2,h_3]}))), \end{aligned}$$

and the restriction of the RHS of (3.3) in the neighborhood $\mathsf{U}_{[\vec{h}]}$ of $\mathsf{X}_{[\vec{h}]}$ is

$$\begin{aligned} & e_{12,*}(e_1^*\alpha \wedge e_2^*(e_{12,*}(e_1^*\beta \wedge e_2^*\gamma \wedge e(\mathsf{E}^{[2]}))) \wedge e(\mathsf{E}^{[2]}))|_{\mathsf{U}_{[\vec{h}]}} \\ = {} & e_{123,*}(e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge \mathsf{E}^{[2]}_{[h_1,h_2h_3]}|_{\mathsf{X}_{[\vec{h}]}} \wedge \mathsf{E}^{[2]}_{[h_2,h_3]}|_{\mathsf{X}_{[\vec{h}]}} \wedge e(E(\mathsf{X}_{[h_2h_3]}, \mathsf{X}_{[h_1,h_2h_3]}, \mathsf{X}_{[h_2,h_3]}))), \end{aligned}$$

where $E(\mathsf{X}_{[h_1h_2]}, \mathsf{X}_{[h_1,h_2]}, \mathsf{X}_{[h_1h_2,h_3]})$ and $E(\mathsf{X}_{[h_2h_3]}, \mathsf{X}_{[h_1,h_2h_3]}, \mathsf{X}_{[h_2,h_3]})$ are the excess bundles for the non-transversal clean intersections of quasi-suborbifolds. Then the theorem follows from the following Lemma 3.1.

[3] See [3, Definition 2.12, Example 2.14].

Remark 3.3 One can see [3, 17, 20] for the concept of excess bundle. For example, consider the intersection of quasi-suborbifolds $\mathsf{X}_{[h_1,h_2]}, \mathsf{X}_{[h_1h_2,h_3]}$ in $\mathsf{X}_{[h_1h_2]}$. Suppose that this intersection has a component $\mathsf{X}_{[\vec{h}]}$ with $\vec{h}=(h_1,h_2,h_3)$. We draw them as

$$\begin{tikzcd} \mathsf{X}_{[\vec{h}]} \arrow[r, hook, "e_{1,2}"] \arrow[d, hook, "\mu_1"'] & \mathsf{X}_{[h_1,h_2]} \arrow[d, hook, "e_{12}"] \\ \mathsf{X}_{[h_1h_2,h_3]} \arrow[r, hook, "e_1"] & \mathsf{X}_{[h_1h_2]}. \end{tikzcd}$$

Then the excess bundle over the component $\mathsf{X}_{[\vec{h}]}$ for this intersection is

$$E(\mathsf{X}_{[h_1h_2]}, \mathsf{X}_{[h_1,h_2]}, \mathsf{X}_{[h_1h_2,h_3]}) = e_{1,2}^*\mathsf{N}_{\mathsf{X}_{[h_1,h_2]}|\mathsf{X}_{[h_1h_2]}} \ominus \mathsf{N}_{\mathsf{X}_{[\vec{h}]}|\mathsf{X}_{[h_1h_2,h_3]}},$$

where $\mathsf{N}_{\mathsf{X}_{[h_1,h_2]}|\mathsf{X}_{[h_1h_2]}}$ is the normal bundle of the quasi-embedding $e_{12}:\mathsf{X}_{[h_1,h_2]}\to\mathsf{X}_{[h_1h_2]}$ and $\mathsf{N}_{\mathsf{X}_{[\vec{h}]}|\mathsf{X}_{[h_1h_2,h_3]}}$ is the normal bundle of the quasi-embedding $\mu_1:\mathsf{X}_{[\vec{h}]}\to\mathsf{X}_{[h_1h_2,h_3]}$.

Lemma 3.1 *Over* $\mathsf{X}_{[\vec{h}]}$ *we have*

$$\begin{aligned} &\mathsf{E}^{[2]}_{[h_1,h_2]}|_{\mathsf{X}_{[\vec{h}]}} \oplus \mathsf{E}^{[2]}_{[h_1h_2,h_3]}|_{\mathsf{X}_{[\vec{h}]}} \oplus E(\mathsf{X}_{[h_1h_2]}, \mathsf{X}_{[h_1,h_2]}, \mathsf{X}_{[h_1h_2,h_3]}) \\ &= \mathsf{E}^{[2]}_{[h_1,h_2h_3]}|_{\mathsf{X}_{[\vec{h}]}} \oplus \mathsf{E}^{[2]}_{[h_2,h_3]}|_{\mathsf{X}_{[\vec{h}]}} \oplus E(\mathsf{X}_{[h_2h_3]}, \mathsf{X}_{[h_1,h_2h_3]}, \mathsf{X}_{[h_2,h_3]}). \end{aligned} \tag{3.4}$$

Proof This follows from the definition of $\mathsf{E}^{[2]}$. We first compute the LHS. Recall that from (2.1), over each component $\mathsf{X}_{[\vec{h}]}$ of $\mathsf{X}^{[3]}$, $\mathsf{V}^{[3]}$ has a decomposition

$$\mathsf{V}^{[3]}_{[\vec{h}]} = \bigoplus_{\lambda\in\widehat{Z\langle\vec{h}\rangle}} \mathsf{V}^{[3]}_{[\vec{h}],\lambda},$$

and the action weight of each $h_i\in\vec{h}$ on $\mathsf{V}^{[3]}_{[\vec{h}],\lambda}$ does not change over the component $\mathsf{X}_{[\vec{h}]}$. We denote them by

$$m_{i,\lambda}\in\mathbb{Q}\cap[0,1),\quad i=1,2,3,$$

i.e., h_i acts on the fiber of $\mathsf{V}^{[3]}_{[\vec{h}],\lambda}$ over $\vec{h}$ by multiplying $\exp^{2\pi\sqrt{-1}m_{i,\lambda}}$. On the other hand h_{12}, h_{23} and h_{123} also act on the fiber of each $\mathsf{V}^{[3]}_{[\vec{h}],\lambda}$. We denote the corresponding action weights by

$$m_{12,\lambda},\ m_{23,\lambda},\ m_{123,\lambda}\in\mathbb{Q}\cap[0,1).$$

Then since $h_1h_2h_{12}^{-1}=h_{12}h_3h_{123}^{-1}=1$ and $h_1h_2h_3h_{123}^{-1}=1$, we have

$$\left.\begin{aligned} m_{1,\lambda}+m_{2,\lambda}+\{1-m_{12,\lambda}\} \\ m_{12,\lambda}+m_{3,\lambda}+\{1-m_{123,\lambda}\} \end{aligned}\right\} = 0,\ \text{or } 1,\ \text{or } 2,$$

and

$$m_{1,\lambda}+m_{2,\lambda}+m_{3,\lambda}+\{1-m_{123,\lambda}\}=0,\ \text{or } 1,\ \text{or } 2,\ \text{or } 3,$$

where $\{\cdot\}$ means the fractional part of a real number.

Then by the definition of $\mathsf{E}^{[2]}$ we have

$$\mathsf{E}^{[2]}_{[h_1,h_2]}|_{\mathsf{X}_{[\vec{h}]}} = (e_1^*\mathsf{V}^{[1]}_{\Phi} \oplus e_2^*\mathsf{V}^{[1]}_{\Phi} \oplus e_{12}^*\mathsf{V}^{[1]}_{\Phi^{-1}} \ominus \mathsf{N}^{[2]})|_{\mathsf{X}_{[\vec{h}]}}$$

$$
\begin{aligned}
&= \bigoplus_{\lambda}(m_{1,\lambda} + m_{2,\lambda} + \{1 - m_{12,\lambda}\})\mathsf{V}^{[3]}_{[\vec{h}],\lambda} \ominus \mathsf{V}^{[3]}_{[\vec{h}]} \oplus \bigoplus_{m_{1,\lambda}=m_{2,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda} \\
&= \bigoplus_{\lambda}(m_{1,\lambda} + m_{2,\lambda} + \{1 - m_{12,\lambda}\})\mathsf{V}^{[3]}_{[\vec{h}],\lambda} \ominus \bigoplus_{m_{1,\lambda}+m_{2,\lambda}+\{1-m_{12,\lambda}\}=1} \mathsf{V}^{[3]}_{[\vec{h}],\lambda} \\
&= \bigoplus_{m_{1,\lambda}+m_{2,\lambda}+\{1-m_{12,\lambda}\}=2} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}.
\end{aligned} \tag{3.5}
$$

Similarly,

$$
\begin{aligned}
\mathsf{E}^{[2]}_{[h_1h_2,h_3]}\big|_{\mathsf{X}_{[\vec{h}]}} &= (e_1^*\mathsf{V}^{[1]}_{\Phi} \oplus e_2^*\mathsf{V}^{[1]}_{\Phi} \oplus e_{12}^*\mathsf{V}^{[1]}_{\Phi^{-1}} \ominus \mathsf{N}^{[2]})\big|_{\mathsf{X}_{[\vec{h}]}} \\
&= \bigoplus_{\lambda}(m_{12,\lambda} + m_{3,\lambda} + \{1 - m_{123,\lambda}\})\mathsf{V}^{[3]}_{[\vec{h}],\lambda} \ominus \mathsf{V}^{[3]}_{[\vec{h}]} \oplus \bigoplus_{m_{12,\lambda}=m_{3,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda} \\
&= \bigoplus_{m_{12,\lambda}+m_{3,\lambda}+\{1-m_{123,\lambda}\}=2} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}.
\end{aligned}
$$

On the other hand, the excess bundle for the intersection of $\mathsf{X}_{[h_1,h_2]}$ and $\mathsf{X}_{[h_1h_2,h_3]}$ in $\mathsf{X}_{[h_1h_2]}$ is

$$
\begin{aligned}
&E(\mathsf{X}_{[h_1h_2]}, \mathsf{X}_{[h_1,h_2]}, \mathsf{X}_{[h_1h_2,h_3]}) \\
&= \Big[\Big(\bigoplus_{m_{12,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \ominus \underline{\mathbb{R}}^m\Big] \ominus \Big[\Big(\bigoplus_{m_{1,\lambda}=m_{2,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \ominus \underline{\mathbb{R}}^m\Big] \\
&\quad \ominus \Big[\Big(\bigoplus_{m_{12,\lambda}=m_{3,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \ominus \underline{\mathbb{R}}^m\Big] \oplus \Big[\Big(\bigoplus_{m_{1,\lambda}=m_{2,\lambda}=m_{3,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \ominus \underline{\mathbb{R}}^m\Big] \\
&= \Big(\bigoplus_{m_{12,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \oplus \Big(\bigoplus_{\substack{m_{1,\lambda}=m_{2,\lambda}\\=m_{3,\lambda}=0}} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \ominus \Big(\bigoplus_{m_{1,\lambda}=m_{2,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \ominus \Big(\bigoplus_{m_{12,\lambda}=m_{3,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big).
\end{aligned}
$$

Therefore the LHS of (3.4) is

$$
\begin{aligned}
&\bigoplus_{\lambda}(m_{1,\lambda} + m_{2,\lambda} + \{1 - m_{12,\lambda}\})\mathsf{V}^{[3]}_{[\vec{h}],\lambda} \ominus \mathsf{V}^{[3]}_{[\vec{h}]} \oplus \bigoplus_{m_{1,\lambda}=m_{2,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda} \\
&\oplus \bigoplus_{\lambda}(m_{12,\lambda} + m_{3,\lambda} + \{1 - m_{123,\lambda}\})\mathsf{V}^{[3]}_{[\vec{h}],\lambda} \ominus \mathsf{V}^{[3]}_{[\vec{h}]} \oplus \bigoplus_{m_{12,\lambda}=m_{3,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda} \\
&\oplus \Big(\bigoplus_{m_{12,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \oplus \Big(\bigoplus_{\substack{m_{1,\lambda}=m_{2,\lambda}\\=m_{3,\lambda}=0}} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \ominus \Big(\bigoplus_{m_{1,\lambda}=m_{2,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \ominus \Big(\bigoplus_{m_{12,\lambda}=m_{3,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda}\Big) \\
&= \bigoplus_{\lambda}(m_{1,\lambda} + m_{2,\lambda} + m_{3,\lambda} + \{1 - m_{123,\lambda}\})\mathsf{V}^{[3]}_{[\vec{h}],\lambda} \ominus \mathsf{V}^{[3]}_{[\vec{h}]} \oplus \bigoplus_{m_{1,\lambda}=m_{2,\lambda}=m_{3,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda} \\
&\quad \oplus \bigoplus_{\lambda}(m_{12,\lambda} + \{1 - m_{12,\lambda}\})\mathsf{V}^{[3]}_{[\vec{h}],\lambda} \oplus \bigoplus_{m_{12,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda} \ominus \mathsf{V}^{3}_{[\vec{h}]} \\
&= \bigoplus_{\lambda}(m_{1,\lambda} + m_{2,\lambda} + m_{3,\lambda} + \{1 - m_{123,\lambda}\})\mathsf{V}^{[3]}_{[\vec{h}],\lambda} \ominus \mathsf{V}^{[3]}_{[\vec{h}]} \oplus \bigoplus_{m_{1,\lambda}=m_{2,\lambda}=m_{3,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda} \\
&= \bigoplus_{m_{1,\lambda}+m_{2,\lambda}+m_{3,\lambda}+\{1-m_{123,\lambda}\}\geq 2} (m_{1,\lambda} + m_{2,\lambda} + m_{3,\lambda} + \{1 - m_{123,\lambda}\} - 1)\mathsf{V}^{[3]}_{[\vec{h}],\lambda} \\
&= (e_1^*\mathsf{V}^{[1]}_{\Phi} \oplus e_2^*\mathsf{V}^{[1]}_{\Phi} \oplus e_3^*\mathsf{V}^{[1]} \oplus e_{123}^*\mathsf{V}^{[1]}_{\Phi^{-1}} \ominus \mathsf{N}^{[3]})|_{\mathsf{X}_{[\vec{h}]}},
\end{aligned}
$$

where for the second equality we have used the fact that

$$
\bigoplus_{\lambda}(m_{12,\lambda} + \{1 - m_{12,\lambda}\})\mathsf{V}^{[3]}_{[\vec{h}],\lambda} \oplus \bigoplus_{m_{12,\lambda}=0} \mathsf{V}^{[3]}_{[\vec{h}],\lambda} = \mathsf{V}^{3}_{[\vec{h}]}.
$$

Similarly, the RHS of (3.4) is also

$$\begin{aligned}&(e_1^*\mathsf{V}_\Phi^{[1]} \oplus e_2^*\mathsf{V}_\Phi^{[1]} \oplus e_3^*\mathsf{V}^{[1]} \oplus e_{123}^*\mathsf{V}_{\Phi^{-1}}^{[1]} \ominus \mathsf{N}^{[3]})|_{\mathsf{X}_{[\vec{h}]}}\\ =&\bigoplus_{m_{1,\lambda}+m_{2,\lambda}+m_{3,\lambda}+\{1-m_{123,\lambda}\}\geq 2}(m_{1,\lambda}+m_{2,\lambda}+m_{3,\lambda}+\{1-m_{123,\lambda}\}-1)\mathsf{V}_{[\vec{h}],\lambda}^{[3]}.\end{aligned}$$

The lemma follows.

3.2 Relation with stringy cohomology ring

We next study the relation between Chen-Ruan cohomology ring of stable almost complex (global) quotient orbifolds and stringy cohomology of the ambient equivariant stable almost complex manifolds.

3.2.1 Global quotient orbifolds

Suppose that G is a finite group and X is a compact G-equivariant stable almost complex manifolds, i.e., there is a trivial bundle $\underline{\mathbb{R}}^m = X \times \mathbb{R}^m$ over X such that $V := TX \oplus \underline{\mathbb{R}}^m$ is a complex bundle and the G-action on TX together with trivial action on $\mathbb{R}^m$ gives rise to a complex linear action of G on V. We refer the reader to [14, 17] for the explicit construction of stringy cohomology ring of (X, G).

The global quotient orbifold $\mathsf{X} := [X/G]$ has a natural orbifold groupoid representation $\mathsf{X} = (X \times G \rightrightarrows X)$. It is a SACOG. In fact the stable almost complex structure is obtained via the equality

$$[V/G] = [(TX \oplus \underline{\mathbb{R}}^m)/G] = \mathsf{TX} \oplus \underline{\mathbb{R}}^m,$$

and the complex structure over $V = TX \oplus \underline{\mathbb{R}}^m$ induces a complex structure over $\mathsf{V} := \mathsf{TX} \oplus \underline{\mathbb{R}}^m$.

The stringy cohomology group $\mathscr{H}^*(X, G)$ of (X, G) is the cohomology of its inertia manifold $I_G X = \bigsqcup_{g\in G} X^g$, with a degree shifting

$$\mathscr{H}^*(X, G) = \bigoplus_{g\in G} H^{*-2\iota(g)}(X^g, \mathbb{C}),$$

where X^g is the fixed locus of g-action on X. The degree shifting $\iota(g)$ is defined to be

$$\iota(g) = \sum_{j=1}^{\mathrm{ord}(g)} \frac{j}{\mathrm{ord}(g)} \mathrm{rank}_{\mathbb{C}}\, V|_{X^g}(j), \tag{3.6}$$

where $V|_{X^g}(j)$ is the eigen-bundle with eigen value $\exp^{2\pi\sqrt{-1}\frac{j}{\mathrm{ord}(g)}}$ of the g-action on $V|_{X^g}$. In fact there is a formal bundle

$$\mathscr{S}_g := \sum_{j=1}^{\mathrm{ord}(g)} \frac{j}{\mathrm{ord}(g)} V|_{X^g}(j) \in K(X^g) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The pairing over $\mathscr{H}^*(X, G)$ is the direct sum of

$$\langle\cdot,\cdot\rangle_{X,G} : H^*(X^g, \mathbb{C}) \otimes H^*(X^{g^{-1}}, \mathbb{C}) \to \mathbb{C}, \quad \langle\alpha_g, \beta_{g^{-1}}\rangle_{X,G} = \int_{X^g} \alpha_g \wedge I^*\beta_{g^{-1}},$$

where $I : X^g \to X^{g^{-1}},\ x \mapsto x$.

G acts on $I_G X$ via $h : X^g \to X^{hgh^{-1}},\ x \mapsto h \cdot x$. This action gives rise to an averaging map

$$\mathscr{H}^*(X,G) \to \mathscr{H}^*(X,G), \quad \alpha_g \mapsto \frac{1}{|G|}\sum_{h\in G} h^*(\alpha_g).$$

The image of this averaging map $\overline{\mathscr{H}^*(X,G)}$ is the G-invariant part of $\mathscr{H}^*(X,G)$ w.r.t. the G-action on IX. The pairing $\langle\cdot,\cdot\rangle_{X,G}$ induces a pairing $\frac{1}{|G|}\langle\cdot,\cdot\rangle_{X,G}$ over $\overline{\mathscr{H}^*(X,G)}$.

The ring structure over $\mathscr{H}^*(X,G)$ is defined as follows. The 2-sector of (X,G) is

$$\mathbb{I}^2_G(X) = \bigsqcup_{(g_1,g_2)\in G^2} X^{g_1,g_2} \tag{3.7}$$

with $X^{g_1,g_2} = X^{g_1} \cap X^{g_2}$. We also have obvious maps $e_i : \mathbb{I}^2_G(X) \to I_G X,\ i = 1,2$ and $e_{12} : \mathbb{I}^2_G(X) \to I_G X$ with $e_i : X^{g_1,g_2} \hookrightarrow X^{g_i},\ e_{12} : X^{g_1,g_2} \hookrightarrow X^{g_1g_2}$. Over a component $X^{\vec g}$ with $\vec g = (g_1,g_2)$, by setting $g_3 = (g_1g_2)^{-1}$, the component $\mathscr{R}(\vec g)$ over $X^{\vec g}$ of the obstruction bundle $\mathscr{R}$ is

$$\mathscr{R}(\vec g) = TX^{\vec g} \ominus TX|_{X^{\vec g}} \oplus \bigoplus_{i=1}^{3} \mathscr{S}_{g_i}|_{X^{\vec g}}.$$

Then the product over $\mathscr{H}^*(X,G)$ is defined by requiring that the equality

$$\langle \alpha \star \beta, \gamma\rangle_{X,G} = \int_{\mathbb{I}^2_G(X)} e_1^*\alpha \wedge e_2^*\beta \wedge (I \circ e_{12}^*)\gamma \wedge e(\mathscr{R})$$

holds for all $\gamma \in H^*(I_G X)$. This is equivalent to the formula

$$\alpha \star \beta = e_{12,*}(e_1^*\alpha \wedge e_2^*\beta \wedge e(\mathscr{R})). \tag{3.8}$$

Since

$$\mathsf{IX} = \bigsqcup_{[g]\in[G]} [X^g/Z_G(g)] = [I_G X/G],$$

and G is a finite group, we see that there is a group isomorphism

$$\overline{\mathscr{H}^*(X,G)} = \mathscr{H}^*(X,G)^G = H^*(I_G X,\mathbb{C})^G \cong H^*(I_G X/G,\mathbb{C}) = H^*_{CR}(\mathsf{X},\mathbb{C}).$$

This isomorphism identifies the degree shifting, and the pairing by the definition of orbifold integration. Therefore, to show the isomorphism between ring structure, we only need to show that the isomorphism identifies the Euler class of the obstruction bundles. Note that

$$\mathsf{X}^{[2]} = \bigsqcup_{[\vec g]\in[G^2]} [X^{\vec g}/Z_G(\vec g)] = [\mathbb{I}^2_G(X)/G].$$

Then from the definition of $\mathsf{E}^{[2]}$ we immediately get $[\mathscr{R}/G] = \mathsf{E}^{[2]}$. In fact, $\mathscr{S}_{g_i}$ corresponds to $e_i^*\mathsf{V}^{[1]}_\Phi$, and $TX|_{X^{\vec g}} \ominus TX^{\vec g}$ corresponds to $\mathsf{N}^{[2]}$. Therefore we have the following theorem.

Theorem 3.2 *We have a ring isomorphism*[4] $\overline{\mathscr{H}^*(X,G)} \cong H^*_{CR}(\mathsf{X},\mathbb{C})$.

[4] In fact, this is a Frobenius algebra isomorphism, since this isomorphism also identifies the pairing.

3.2.2 Torus quotient orbifolds

Now suppose that Y is a compact T-equivariant stable almost complex manifold, T is a torus and the T-action on Y is locally free. Then $\mathsf{X} = [Y/T]$ is a SACOG since the adjoint action of T on its Lie algebra is trivial. The construction of stringy cohomology group of (Y, T) is identical to the previous case except that singular cohomology is replaced by T-equivariant cohomology (see [15]). The stringy cohomology group of (Y, T) is the T-equivariant cohomology group of $I_T Y := \bigsqcup_{t\in T} Y^t$ with a degree shifting

$$NH_T^{*,\diamond}(Y) = \bigoplus_{t\in T} H_T^{*-2\iota(t)}(Y^t, \mathbb{C}),$$

where the sum indicates the $\diamond$-grading, i.e., $NH_T^{*,t}(Y) = H_T^*(Y^t, \mathbb{C})$. The degree shifting $\iota(t)$ is defined in the same form as (3.6). We also have $\mathsf{IX} = [I_T Y/T]$. It is also direct to see that the degree shifting of $NH_T^{*,\diamond}(Y)$ is the same as the degree shifting of $H_{CR}^*(\mathsf{X})$ since T is abelian.

Since for a locally free action $H_T^*(Y^t, \mathbb{C}) \cong H^*(Y^t/T, \mathbb{C})$, we get a group isomorphism

$$NH_T^{*,\diamond}(Y) \cong H_{CR}^*(\mathsf{X}, \mathbb{C}).$$

The 2-sector of (Y, T) is also of the form $\mathbb{I}_T^2(Y) := \bigsqcup_{(t_1,t_2)\in T^2} Y^{t_1,t_2}$ as (3.7) and $\mathsf{X}^{[2]} = [\mathbb{I}_T^2(Y)/T]$. The stringy product over $NH_T^{*,\diamond}(Y)$ is defined by the same formula as (3.8) (see [15, Definition 3.2]) with the obstruction bundle E (see [15, Definition 3.1]) constructed as follows. Over a component Z of Y^{g_1,g_2}, the normal bundle νZ of Z in Y splits into irreducible representation of $\langle g_1, g_2\rangle < T$,

$$\nu Z = \bigoplus_{\lambda} I_\lambda.$$

Denote the action weights of $g_1, g_2, (g_1g_2)^{-1}$ over each I_λ by $m_{1,\lambda}, m_{2,\lambda}, \overline{m}_{12,\lambda}$, respectively. Then the obstruction bundle is given by

$$E|_Z := \bigoplus_{m_{1,\lambda}+m_{2,\lambda}+\overline{m}_{12,\lambda}=2} I_\lambda.$$

Comparing with (3.5) $[E/T] = \mathsf{E}^{[2]}$. Therefore, we have the following theorem.

Theorem 3.3 *We have a ring isomorphism* $NH_T^{*,\diamond}(Y) \cong H_{CR}^*(\mathsf{X}, \mathbb{C})$.

4 Stringy Product over Orbifold K-Theory of SACOGs

In this section we generalize the stringy product on orbifold K-theory and the modified delocalized Chern character in [16] for compact presentable[5] almost complex orbifolds to compact presentable SACOGs. Suppose that X is compact, i.e., its coarse space is compact. Then there is a delocalized Chern character.

Theorem 4.1 (see [16, Proposition 2.5]) *For any compact presentable orbifold groupoid* X*, the delocalized Chern character gives a ring isomorphism*

$$ch_{\text{deloc}} : K_{\text{orb}}^*(\mathsf{X}) \otimes_{\mathbb{Z}} \mathbb{C} \to H^*(\mathsf{IX}, \mathbb{C})$$

[5] An orbifold groupoid is presentable if it is Morita equivalent to a quotient groupoid $G \ltimes X$ with G being a Lie group and X being a smooth manifold equipped with a smooth, almost free G-action.

over $\mathbb{C}$. Here the ring structure over $K^*_{\mathrm{orb}}(\mathsf{X}) \otimes_{\mathbb{Z}} \mathbb{C}$ is tensor product of orbifold vector bundles, and the ring structure on $H^*(\mathsf{IX}, \mathbb{C})$ is the wedge product of differential forms.

We recall the definition of ch_{deloc} (see [16, Section 2.2]). Given a complex orbifold vector bundle E over X, pull it back to IX by the evaluation map $e : \mathsf{IX} \to \mathsf{X}$ to get $e^*\mathsf{E}$. Then $e^*\mathsf{E}$ has the canonical automorphism Φ. Decompose it into eigen-bundles

$$e^*\mathsf{E} = \bigoplus_{\theta \in \mathbb{Q} \cap [0,1)} e^*\mathsf{E}(\theta).$$

Then $ch_{\mathrm{deloc}} : K^0_{\mathrm{orb}}(\mathsf{X}) \to H^*_{CR}(\mathsf{X}, \mathbb{C})$ is defined to be

$$ch_{\mathrm{deloc}}(\mathsf{E}) = \sum_{\theta} \mathrm{e}^{2\pi\sqrt{-1}\theta} ch(e^*\mathsf{E}(\theta)). \tag{4.1}$$

ch_{deloc} over $K^1_{\mathrm{orb}}(\mathsf{X})$ is defined in the usual way.

For a compact presentable orbifold groupoid X we also have the following result.

Proposition 4.1 (see [16, Proposition 4.2]) *There exists a canonical ring homomorphism*

$$ch_\Phi : K^*_{\mathrm{orb}}(\mathsf{IX}) \to H^*(\mathsf{IX}, \mathbb{C})$$

such that the diagram

$$\begin{tikzcd} K^*_{\mathrm{orb}}(\mathsf{IX}) \arrow[r, "ch_\Phi"] & H^*(\mathsf{IX}, \mathbb{C}) \\ K^*_{\mathrm{orb}}(\mathsf{X}) \arrow[u, "e^*"] \arrow[ur, "ch_{\mathrm{deloc}}"'] & \end{tikzcd}$$

commutes.

Proof The proof is similar to the proof of [16, Proposition 4.2]. In fact, since any complex orbifold bundle E over IX has a canonical automorphism Φ, we can always decompose it into eigen-bundles, then ch_Φ is defined in a similar way as ch_{deloc} in (4.1).

Let $K^*_{\mathrm{orb}}(\mathsf{IX}, \mathbb{C}) := K^*_{\mathrm{orb}}(\mathsf{IX}) \otimes_{\mathbb{Z}} \mathbb{C}$. Then the previous proposition implies the commutative diagram of linear maps between vector spaces over $\mathbb{C}$,

$$\begin{tikzcd} K^*_{\mathrm{orb}}(\mathsf{IX}, \mathbb{C}) \arrow[r, "ch_\Phi"] & H^*(\mathsf{IX}, \mathbb{C}) \\ K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C}) \arrow[u, "e^*"] \arrow[ur, "ch_{\mathrm{deloc}}"'] & \end{tikzcd}$$

with ch_{deloc} being isomorphism. Following the same argument as in [16, p.6330] we get a left inverse of e^*. First of all e^* is injective. Hence $K^*_{\mathrm{orb}}(\mathsf{IX}, \mathbb{C}) = \mathrm{Im}(e^*) + \ker ch_\Phi$. Now suppose $\widetilde{\alpha} \in \mathrm{Im}(e^*) \cap \ker ch_\Phi$. Then there is an $\alpha \in K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C})$ such that $e^*\alpha = \widetilde{\alpha}$ and $ch_\Phi(\widetilde{\alpha}) = 0$. Then $ch_{\mathrm{deloc}}(\alpha) = ch_\Phi \circ e^*(\alpha) = ch_\Phi(\widetilde{\alpha}) = 0$. Since ch_{deloc} is an isomorphism, $\alpha = 0$ and hence $\widetilde{\alpha} = 0$. Therefore

$$K^*_{\mathrm{orb}}(\mathsf{IX}, \mathbb{C}) \cong \mathrm{Im}(e^*) \oplus \ker ch_\Phi.$$

Hence, each element $\widetilde{\alpha} \in K^*_{\mathrm{orb}}(\mathsf{IX}, \mathbb{C})$ can be uniquely written as $\widetilde{\alpha} = e^*\alpha + \beta$ for a unique element $\alpha \in K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C})$ and $\beta \in \ker ch_\Phi$. We then can take a left inverse

$$e_\# : K^*_{\mathrm{orb}}(\mathsf{IX}, \mathbb{C}) \to K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C})$$

of e^*, i.e., $e_\# \circ e^* = \mathrm{id}$ over $K^*_{\mathrm{orb}}(\mathsf{X})$, which maps $\widetilde{\alpha} = e^*\alpha + \beta$ to α. So we have $ch_{\mathrm{deloc}} \circ e_\# = ch_\Phi$.

Definition 4.1 *Let X be a compact presentable SACOG and IX be its inertia orbifold. The stringy product on $K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C})$ is defined by*

$$\alpha_1 \circ \alpha_2 := e_\#[e_{12,*}(e_1^*(e^*\alpha_1) \cdot e_2^*(e^*\alpha_2) \cdot \lambda_{-1}(\mathsf{E}^{[2]}))]$$

for $\alpha_1, \alpha_2 \in K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C})$. Here

$$e_{12,*}(e_1^*(e^*\alpha_1) \cdot e_2^*(e^*\alpha_2) \cdot \lambda_{-1}(\mathsf{E}^{[2]}))$$

is similar to the Adem-Ruan-Zhang stringy product on $K^*_{\mathrm{orb}}(\mathsf{IX}, \mathbb{C})$ for almost complex orbifold (see [3, 16] and Appendix), "$\cdot$" is the tensor product of bundles, and λ_{-1} is the K-theory Euler class.

Remark 4.1 We could directly prove the associativity of this product by using (3.4) in Lemma 3.1 (see Theorem 4.3). However, this also follows from the isomorphism between $K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C})$ and $H^*(\mathsf{IX}, \mathbb{C}) = H^*_{CR}(\mathsf{X}, \mathbb{C})$ in Theorem 4.2, which identifies the product $\circ$ with $\cup_{CR}$.

Next, we follow [16] to define a modified version of the delocalized Chern character,

$$\widetilde{ch_{\mathrm{deloc}}} : K^*_{\mathrm{orb}}(\mathsf{X}, \mathbb{C}) \to H^*(\mathsf{IX}, \mathbb{C}) = H^*_{CR}(\mathsf{X}, \mathbb{C}).$$

For a complex orbifold vector bundle E over an orbifold groupoid Y, we can assign it a characteristic class $\mathscr{T}(\mathsf{E}) \in H^*(\mathsf{Y}, \mathbb{C})$ associated to the formal power series $\mathscr{T}(x) = \frac{1-\mathrm{e}^x}{x}$. It assigns E,

$$\mathscr{T}(\mathsf{E}) = \frac{ch(\lambda_{-1}(\mathsf{E}))}{e(\mathsf{E})} \in H^*(\mathsf{Y}, \mathbb{C}).$$

On the other hand, an orbifold complex vector bundle E over IY has a canonical automorphism Φ, and an eigen-bundle decomposition w.r.t this Φ,

$$\mathsf{E} = \bigoplus_{m_i \in \mathbb{Q} \cap [0,1)} \mathsf{E}(m_i),$$

where Φ acts on $\mathsf{E}(m_i)$ as multiplication by $\exp^{2\pi\sqrt{-1}m_i}$. Define a cohomology class

$$\mathscr{T}(\mathsf{E}, \Phi) := \prod_{m_i} \mathscr{T}(\mathsf{E}_{m_i})^{m_i} \in H^*(\mathsf{IY}, \mathbb{C}),$$

where $\mathscr{T}(\mathsf{E}_{m_i})^{m_i}$ is the characteristic class associated to the formal power series $\mathscr{T}(x)^m := \big(\frac{1-\mathrm{e}^x}{x}\big)^m$. Then $\mathscr{T}(\mathsf{E}, \Phi)$ is an invertible element in $H^*(\mathsf{IY}, \mathbb{C})$, as the degree zero component is 1.

Now for the normal bundle

$$\mathsf{N}^{[1]} = \coprod_{[g] \in \mathcal{T}^1} \mathsf{N}^{[1]}_{[g]}$$

over IX of the evaluation map $e : \mathsf{IX} \to \mathsf{X}$, we have the cohomology class $\mathscr{T}(\mathsf{N}^{[1]}, \Phi)$ in $H^*(\mathsf{IX})$ whose component in $H^*(\mathsf{X}_{[g]}, \mathbb{C})$ is given by $\mathscr{T}(\mathsf{N}^{[1]}_{[g]}, \Phi)$. For the bundle $\mathsf{E}^{[2]} \oplus \mathsf{N}^{e_{12}}$ over $\mathsf{X}^{[2]}$ we also have the cohomology class $\mathscr{T}(\mathsf{E}^{[2]} \oplus \mathsf{N}^{e_{12}})$ in $H^*(\mathsf{X}^{[2]}, \mathbb{C})$ whose component in $H^*(\mathsf{X}_{[g_1,g_2]}, \mathbb{C})$ is given by $\mathscr{T}(\mathsf{E}^{[2]}_{[g_1,g_2]} \oplus \mathsf{N}^{e_{12}}_{[g_1,g_2]})$.

By Definition 3.1 of the obstruction bundle $\mathsf{E}^{[2]}$ over $\mathsf{X}^{[2]}$ and (3.2) we have

$$\begin{aligned}\mathsf{E}^{[2]} \oplus \mathsf{N}^{[2]} \oplus e_{12}^*\mathsf{N}^{[1]}_{\Phi} &= e_1^*\mathsf{N}^{[1]}_{\Phi} \oplus e_2\mathsf{N}^{[1]}_{\Phi} \oplus e_{12}^*\mathsf{N}^{[1]}_{\Phi^{-1}} \oplus e_{12}^*\mathsf{N}^{[1]}_{\Phi}\\ &= e_1^*\mathsf{N}^{[1]}_{\Phi} \oplus e_2\mathsf{N}^{[1]}_{\Phi} \oplus e_{12}^*\mathsf{N}^{[1]},\end{aligned}$$

which together with (2.4), i.e., $\mathsf{N}^{[2]} = \mathsf{N}^{e_{12}} \oplus e_{12}^*\mathsf{N}^{[1]}$, implies that

$$\mathsf{E}^{[2]} \oplus \mathsf{N}^{e_{12}} \oplus e_{12}^*\mathsf{N}^{[1]}_{\Phi} = e_1^*\mathsf{N}^{[1]}_{\Phi} \oplus e_2\mathsf{N}^{[1]}_{\Phi}.$$

Therefore we have the following identity:

$$\mathscr{T}(\mathsf{E}^{[2]}_{[g_1,g_2]} \oplus \mathsf{N}^{e_{12}}_{[g_1,g_2]}) \wedge e_{12}^*\mathscr{T}(\mathsf{N}^{[1]}_{[g_1g_2]},\Phi) = e_1^*\mathscr{T}(\mathsf{N}^{[1]}_{[g_1]},\Phi) \wedge e_2^*(\mathscr{T}(\mathsf{N}^{[1]}_{[g_2]},\Phi)) \tag{4.2}$$

in $H^*(\mathsf{X}_{[g_1,g_2]},\mathbb{C})$ for any connected component $\mathsf{X}_{[g_1,g_2]}$ of $\mathsf{X}^{[2]}$.

Definition 4.2 *The modified delocalized Chern character on the orbifold K-theory $K^*_{\mathrm{orb}}(\mathsf{X})$ is defined to be*

$$\widetilde{ch_{\mathrm{deloc}}} := \mathscr{T}(\mathsf{N}^{[1]},\Phi) \wedge ch_{\mathrm{deloc}} : K^*_{\mathrm{orb}}(\mathsf{X}) \to H^*_{CR}(\mathsf{X},\mathbb{C}).$$

Theorem 4.2 *Let X be a compact presentable SACOG. The modified delocalized Chern character*

$$\widetilde{ch_{\mathrm{deloc}}} : (K^*_{\mathrm{orb}}(\mathsf{X},\mathbb{C}),\circ) \to (H^*_{CR}(\mathsf{X},\mathbb{C}),\cup_{CR})$$

is a vector space isomorphism that identifies the two products $\circ$ and $\cup_{CR}$.

Proof The isomorphism between vector space follows from the fact that the degree 0 component of $\mathscr{T}(\mathsf{N}^{[1]},\Phi)$ is 1 over each component of IX and ch_{deloc} is an isomorphism of linear spaces (see Theorem 4.1). The equality (4.2) and the same computation as the proof of [16, Theorem 4.5] would prove this theorem.

First of all by the definition of stringy product we have

$$\begin{aligned}ch_{\mathrm{deloc}}(\alpha_1 \circ \alpha_2) &= (ch_{\mathrm{deloc}} \circ e_{\#})(e_{12,*}(e_1^*\alpha_1 \cdot e_2^*\alpha_2 \cdot \lambda_{-1}(\mathsf{E}^{[2]})))\\ &= ch_{\Phi}[e_{12,*}(e_1^*\alpha_1 \cdot e_2^*\alpha_2 \cdot \lambda_{-1}(\mathsf{E}^{[2]}))]\end{aligned}$$

for $\alpha_1,\alpha_2 \in K^*_{\mathrm{orb}}(\mathsf{X},\mathbb{C})$.

The inertia orbifold $\mathsf{IX} = \bigsqcup_{[g]\in\mathcal{T}^1} \mathsf{X}_{[g]}$, the evaluation map $e:\mathsf{IX}\to\mathsf{X}$ decompose into $e^* = \oplus_{[g]\in\mathcal{T}^1} e^*_{[g]}$. We next compute the $\mathsf{X}_{[g]}$-component of $ch_{\mathrm{deloc}}(\alpha_1\circ\alpha_2)$. It is given by

$$\sum_{[g_1,g_2]\in\mathcal{T}^2,[g_1g_2]=[g]} ch_{\Phi}[e_{12,*}(e_1^*e^*_{[g_1]}\alpha_1 \cdot e_2^*e^*_{[g_2]}\alpha_2 \cdot \lambda_{-1}(\mathsf{E}^{[2]}_{[g_1,g_2]}))].$$

Here the pushforward map $e_{12,*} : K^*_{\mathrm{orb}}(\mathsf{X}_{[g_1,g_2]},\mathbb{C}) \to K^*_{\mathrm{orb}}(\mathsf{X}_{[g_1g_2]},\mathbb{C})$ is obtained by composing the Thom isomorphism for the normal bundle $\mathsf{N}^{e_{12}}_{[g_1,g_2]}$ of $e_{12}:\mathsf{X}_{[g_1,g_2]}\to\mathsf{X}_{[g_1g_2]}$ with the natural extension for open embeddings.

Using the fact that the Thom class of $\mathsf{N}^{e_{12}}_{[g_1,g_2]}$ has trivial automorphism (see [16, p.6332]), we obtain

$$\widetilde{ch_{\mathrm{deloc}}}(\alpha_1\circ\alpha_2)$$

$$
\begin{aligned}
&= ch_{\mathrm{deloc}}(\alpha_1 \circ \alpha_2) \wedge \mathscr{T}(\mathsf{N}^{[1]}_{[g_1g_2]}, \Phi) \\
&= \sum_{[g_1,g_2]\in\mathcal{T}^2} ch_\Phi[e_{12,*}(e_1^*e_{[g_1]}^*\alpha_1 \cdot e_2^*e_{[g_2]}^*\alpha_2 \cdot \lambda_{-1}(\mathsf{E}^{[2]}_{[g_1,g_2]}))] \wedge \mathscr{T}(\mathsf{N}^{[1]}_{[g_1g_2]}, \Phi) \\
&= \sum_{[g_1,g_2]\in\mathcal{T}^2} e_{12,*}[e_1^*ch_\Phi e_{[g_1]}^*\alpha_1 \wedge e_2^*ch_\Phi e_{[g_2]}^*\alpha_2 \wedge ch(\lambda_{-1}(\mathsf{E}^{[2]}_{[g_1,g_2]})) \wedge \mathscr{T}(\mathsf{N}^{e_{12}}_{[g_1,g_2]})] \wedge \mathscr{T}(\mathsf{N}^{[1]}_{[g_1g_2]}, \Phi) \\
&= \sum_{[g_1,g_2]\in\mathcal{T}^2} e_{12,*}[e_1^*ch_\Phi e_{[g_1]}^*\alpha_1 \wedge e_2^*ch_\Phi e_{[g_2]}^*\alpha_2 \wedge e(\mathsf{E}^{[2]}_{[g_1,g_2]}) \wedge \mathscr{T}(\mathsf{E}^{[2]}_{[g_1,g_2]} \oplus \mathsf{N}^{e_{12}}_{[g_1,g_2]})] \wedge \mathscr{T}(\mathsf{N}^{[1]}_{[g_1g_2]}, \Phi) \\
&= \sum_{[g_1,g_2]\in\mathcal{T}^2} e_{12,*}[e_1^*ch_\Phi e_{[g_1]}^*\alpha_1 \wedge e_2^*ch_\Phi e_{[g_2]}^*\alpha_2 \wedge e(\mathsf{E}^{[2]}_{[g_1,g_2]}) \wedge \mathscr{T}(\mathsf{E}^{[2]}_{[g_1,g_2]} \oplus \mathsf{N}^{e_{12}}_{[g_1,g_2]}) \wedge e_{12}^*\mathscr{T}(\mathsf{N}^{[1]}_{[g_1g_2]}, \Phi)] \\
&= \sum_{[g_1,g_2]\in\mathcal{T}^2} e_{12,*}[e_1^*ch_\Phi e_{[g_1]}^*\alpha_1 \wedge e_2^*ch_\Phi e_{[g_2]}^*\alpha_2 \wedge e(\mathsf{E}^{[2]}_{[g_1,g_2]}) \wedge e_1^*\mathscr{T}(\mathsf{N}^{[1]}_{[g_1]}, \Phi) \wedge e_2^*\mathscr{T}(\mathsf{N}^{[1]}_{[g_2]}, \Phi)] \\
&= \sum_{[g_1,g_2]\in\mathcal{T}^2} e_{12,*}[e_1^*\widetilde{ch_{\mathrm{deloc}}}(\alpha_1) \wedge e_2^*\widetilde{ch_{\mathrm{deloc}}}(\alpha_2) \wedge e(\mathsf{E}^{[2]}_{[g_1,g_2]})] \\
&= e_{12,*}[e_1^*\widetilde{ch_{\mathrm{deloc}}}(\alpha_1) \wedge e_2^*\widetilde{ch_{\mathrm{deloc}}}(\alpha_2) \wedge e(\mathsf{E}^{[2]})] \\
&= \widetilde{ch_{\mathrm{deloc}}}(\alpha_1) \cup_{CR} e_2^*\widetilde{ch_{\mathrm{deloc}}}(\alpha_2),
\end{aligned}
$$

where we have used (4.2). This shows that $\widetilde{ch_{\mathrm{deloc}}}$ preserves the product.

Appendix Stringy Product over Twisted Orbifold K-theory of SACOGs

In the appendix we extend the stringy product over twisted orbifold K-theory of almost complex orbifolds of Adem-Ruan-Zhang [3] to twisted orbifold K-theory of SACOGs. We refer the reader to [3] for the definition of twisted orbifold K-theory for general orbifolds.

Let φ be a $U(1)$-valued 3-cocycle for X, i.e., a 2-gerbe over X, and $\theta(\varphi)$ be its inverse transgression, hence a $U(1)$-valued 2-cocycle over IX, i.e., a 1-gerbe over IX. Then there is a twisted orbifold K-theory of IX,

$$
{}^{\theta(\varphi)}K^*_{\mathrm{orb}}(\mathsf{IX}).
$$

Definition 4.3 *Suppose that X is a SACOG. Let $\alpha, \beta \in {}^{\theta(\varphi)}K^*_{\mathrm{orb}}(\mathsf{IX})$. We define*

$$
\alpha \star \beta = e_{12,*}(e_1^*\alpha \cdot e_2^*\beta \cdot \lambda_{-1}(\mathsf{E}^{[2]})),
$$

where "$\cdot$" is the product in twisted orbifold K-theory, and $\lambda_{-1}(\mathsf{E}^{[2]})$ is the K-theory Euler class. When φ is trivial, we get a stringy product over $K^*_{\mathrm{orb}}(\mathsf{IX})$.

Remark 4.2 Note that here α, β are elements in the twisted orbifold K-theory, however $\lambda_{-1}(\mathsf{E}^{[2]})$ is an element in $K^0_{\mathrm{orb}}(\mathsf{X}^{[2]})$. The product between $e_1^*\alpha$ and $e_2^*\beta$ is the product in twisted orbifold K-theory, the product between $(e_1^*\alpha \cdot e_2^*\beta)$ and $\lambda_{-1}(\mathsf{E}^{[2]})$ is obtained from the natural module structure of twisted orbifold K-theory over K^0_{orb} (see [3, Section 3]).

Theorem 4.3 *The product $\star$ over ${}^{\theta(\varphi)}K^*_{\mathrm{orb}}(\mathsf{IX})$ is associative.*

Proof The proof is similar to that of Theorem 3.1. Take $\alpha, \beta, \gamma \in {}^{\theta(\varphi)}K^*_{\mathrm{orb}}(\mathsf{IX})$. Then we have

$$
(\alpha \star \beta) \star \gamma = e_{12,*}[e_1^*(e_{12,*}(e_1^*\alpha \cdot e_2^*\beta \cdot \lambda_{-1}(\mathsf{E}^{[2]}))) \cdot e_2^*\gamma \cdot \lambda_{-1}(\mathsf{E}^{[2]})]
$$

and

$$
\alpha \star (\beta \star \gamma) = e_{12,*}[e_1^*\alpha \cdot e_2^*(e_{12,*}(e_1^*\beta \cdot e_2^*\gamma \cdot \lambda_{-1}(\mathsf{E}^{[2]}))) \cdot \lambda_{-1}(\mathsf{E}^{[2]})].
$$

For simplicity we assume that $\alpha \in {}^{\theta(\varphi)}K^*_{\rm orb}(\mathsf{X}_{[g_1]})$, $\beta \in {}^{\theta(\varphi)}K^*_{\rm orb}(\mathsf{X}_{[g_2]})$, $\gamma \in {}^{\theta(\varphi)}K^*_{\rm orb}(\mathsf{X}_{[g_3]})$. Then $(\alpha\star\beta)\star\gamma$ supports over a neighborhood of

$$\bigsqcup_{[\vec{h}]=[h_1,h_2,h_3],[h_i]=[g_i]} \mathsf{X}_{[\vec{h}]}.$$

As the proof of Theorem 3.1, by the clean intersection formula (see [3, Lemma 7.2]), for a fixed $\mathsf{X}_{[\vec{h}]}$ the restriction of $(\alpha\star\beta)\star\gamma$ in a small neighborhood $\mathsf{U}_{[\vec{h}]}$ of $\mathsf{X}_{[\vec{h}]}$ is

$$\begin{aligned}
&e_{12,*}[e_1^*(e_{12,*}(e_1^*\alpha\cdot e_2^*\beta\cdot\lambda_{-1}(\mathsf{E}^{[2]})))\cdot e_2^*\gamma\cdot\lambda_{-1}(\mathsf{E}^{[2]})]|_{\mathsf{U}_{[\vec{h}]}}\\
&= e_{123,*}[e_1^*\alpha\cdot e_2^*\beta\cdot e_3^*\gamma\cdot\lambda_{-1}(\mathsf{E}^{[2]}_{[h_1,h_2]})|_{\mathsf{X}_{[\vec{h}]}}\cdot\lambda_{-1}(\mathsf{E}^{[2]}_{[h_{12},h_3]})|_{\mathsf{X}_{[\vec{h}]}}\cdot\lambda_{-1}(E(X_{[h_{12}]},\mathsf{X}_{[h_1,h_2]},\mathsf{X}_{[h_{12},h_3]}))].
\end{aligned}$$

Similarly, the restriction of $\alpha\star(\beta\star\gamma)$ in a small neighborhood $\mathsf{U}_{[\vec{h}]}$ of $\mathsf{X}_{[\vec{h}]}$ is

$$\begin{aligned}
&e_{12,*}[e_1^*\alpha\cdot e_2^*(e_{12,*}(e_1^*\beta\cdot e_2^*\gamma\cdot\lambda_{-1}(\mathsf{E}^{[2]})))\cdot\lambda_{-1}(\mathsf{E}^{[2]})]|_{\mathsf{U}_{[\vec{h}]}}\\
&= e_{123,*}[e_1^*\alpha\cdot e_2^*\beta\cdot e_3^*\gamma\cdot\lambda_{-1}(\mathsf{E}^{[2]}_{[h_2,h_3]})|_{\mathsf{X}_{[\vec{h}]}}\cdot\lambda_{-1}(\mathsf{E}^{[2]}_{[h_1,h_{23}]})|_{\mathsf{X}_{[\vec{h}]}}\cdot\lambda_{-1}(E(X_{[h_{23}]},\mathsf{X}_{[h_2,h_3]},\mathsf{X}_{[h_1,h_{23}]}))].
\end{aligned}$$

Then by Lemma 3.1 we get

$$\begin{aligned}
&e_{12,*}[e_1^*(e_{12,*}(e_1^*\alpha\cdot e_2^*\beta\cdot\lambda_{-1}(\mathsf{E}^{[2]})))\cdot e_2^*\gamma\cdot\lambda_{-1}(\mathsf{E}^{[2]})]\\
&= e_{12,*}[e_1^*\alpha\cdot e_2^*(e_{12,*}(e_1^*\beta\cdot e_2^*\gamma\cdot\lambda_{-1}(\mathsf{E}^{[2]})))\cdot\lambda_{-1}(\mathsf{E}^{[2]})].
\end{aligned}$$

This finishes the proof.

By using this stringy product, we could rewrite the stringy product over $K^*_{\rm orb}(\mathsf{X},\mathbb{C})$ as

$$\alpha\circ\beta = e_\#(e_1^*\alpha\star e_2^*\beta).$$

Here the stringy product "$\star$" is defined over non-twisted orbifold K-theory of IX by using the trivial gerbe $\varphi = 1$.

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