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On Gorenstein Projective Dimensions of Unbounded Complexes*

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Abstract Let $R \to S$ be a ring homomorphism and X be a complex of R-modules. Then the complex of S-modules $S \otimes_R^{\mathbf{L}} X$ in the derived category $\mathbf{D}(S)$ is constructed in the natural way. This paper is devoted to dealing with the relationships of the Gorenstein projective dimension of an R-complex X (possibly unbounded) with those of the S-complex $S \otimes_R^{\mathbf{L}} X$. It is shown that if R is a Noetherian ring of finite Krull dimension and $\phi : R \to S$ is a faithfully flat ring homomorphism, then for any homologically degree-wise finite complex X, there is an equality $\mathrm{Gpd}_R X = \mathrm{Gpd}_S(S \otimes_R^{\mathbf{L}} X)$. Similar result is obtained for Ding projective dimension of the S-complex $S \otimes_R^{\mathbf{L}} X$.

Keywords Gorenstein projective dimension, Ding projective dimension, Faithfully flat ring homomorphism
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1 Introduction

Let $R \to S$ be a ring homomorphism. This paper is devoted to continuing the study for relationships of the Gorenstein dimension of an R-complex X with those of the S-complex $S \otimes_R^{\mathbf{L}} X$.

We use abbreviations $\mathrm{Gpd}_R X$, $\mathrm{Gid}_R X$ and $\mathrm{Gfd}_R X$ for Gorenstein projective, Gorenstein injective and Gorenstein flat dimension of an R-complex X, respectively. By G-dim$X$ we denote the G-dimension of an R-complex X. We recall some results on transfer of Gorenstein dimensions of complexes along ring homomorphism $\phi : R \to S$. If ϕ is local and flat, then it was shown in [15] that for every homologically bounded complex X with finitely generated homology modules, there is an equality $\text{G-dim}_S(X \otimes_R S) = \text{G-dim}_R X$. This result was generalized in [4] as follows. If $\phi : R \to S$ is local and flat and C is a semi-dualizing complex for R, then for every homologically bounded complex X with finitely generated homology modules, there is an equality $\text{G-dim}_{C\otimes_R S}(X \otimes_R S) = \text{G-dim}_C X$. Let $\phi : R \to S$ be a homomorphism of commutative Noetherian rings with fdϕ finite. It was shown in [5, Theorem 5.3] that if R has a dualizing

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complex D and $D \otimes_R^{\mathbf{L}} S$ is dualizing for S, then for each homologically bounded complex X,

$$\mathrm{Gfd}_R X < \infty \Longrightarrow \mathrm{Gfd}_S(S \otimes_R^{\mathbf{L}} X) < \infty,$$
$$\mathrm{Gid}_R X < \infty \Longrightarrow \mathrm{Gid}_S(S \otimes_R^{\mathbf{L}} X) < \infty,$$

and the implication may be reversed if ϕ is faithfully flat. Christensen and Sather-Wagstaff [8] showed that if $\phi : R \to S$ is a ring homomorphism of finite flat dimension then for every homologically bounded R-complex X there is an inequality $\mathrm{Gfd}_S(S \otimes_R^{\mathbf{L}} X) \leq \mathrm{Gfd}_R X$. Also some sufficient conditions are given in [8] under which the equalities hold. It was proved in [3] that if X is a bounded below complex then $\mathrm{Gfd}_R(U \otimes_R^{\mathbf{L}} X) \leq \mathrm{Gfd}_R X + \mathrm{fd}_R U$ for all R-complexes U with finite flat dimension. Note that the complexes under consideration are homologically bounded above or homologically bounded below and this explains the interest in establishing corresponding results for homologically unbounded complexes. It was shown in [16] that $\mathrm{Gfd}_S(S \otimes_R^{\mathbf{L}} X) \leq \mathrm{Gfd}_R X$ for every R-complex X (possibly unbounded) when $\phi : R \to S$ is a ring homomorphism of finite flat dimension. If $\phi : R \to S$ is faithfully flat then $\mathrm{Gfd}_S(S \otimes_R^{\mathbf{L}} X) = \mathrm{Gfd}_R X$ (see [16, Theorem 15]). This result was generalized by Christensen, Koksal and Liang [7]. Let R be commutative coherent, let S be a faithfully flat R-algebra which is left GF-closed, and let X be an R-complex. Then it was shown in [7] that $\mathrm{Gfd}_S(S \otimes_R^{\mathbf{L}} X) = \mathrm{Gfd}_R X$.

Note that most of the above results pay only close attention to Gorenstein injective and Gorenstein flat dimensions of complexes. For Gorenstein projective dimensions, it was shown in [21] that if $\phi : R \to S$ is faithfully flat and module-finite, and $\dim R$ is finite, then for every homologically bounded R-complex M there is an equality $\mathrm{Gpd}_S(S \otimes_R^{\mathbf{L}} M) = \mathrm{Gpd}_R M$. In this paper we will consider Gorenstein projective dimensions of complexes in general. We give some sufficient conditions under which the equality $\mathrm{Gpd}_S(S \otimes_R^{\mathbf{L}} X) = \mathrm{Gpd}_R X$ holds. For Ding projective dimensions we prove that $\mathrm{Dpd}_S(S \otimes_R^{\mathbf{L}} X) = \mathrm{Dpd}_R X$ if $\phi : R \to S$ is faithfully flat and X is a homologically degree-wise finite complex.

2 Preliminaries

Let R be a ring with unity. A complex

$$\cdots \longrightarrow X_1 \xrightarrow{\delta_1^X} X_0 \xrightarrow{\delta_0^X} X_{-1} \xrightarrow{\delta_{-1}^X} \cdots$$

of R-modules will be denoted by (X, δ) or X. We set $\sup X = \sup\{i \in \mathbb{Z} \mid X_i \neq 0\}$ and $\inf X = \inf\{i \in \mathbb{Z} \mid X_i \neq 0\}$. The complex X is called bounded above (resp. bounded below) if $\sup X < \infty$ (resp. $\inf X > -\infty$). It is bounded when it is bounded below and bounded above. For an R-complex X and $i \in \mathbb{Z}$, we set $Z_i^X = \mathrm{Ker}\,\delta_i^X$, $B_i^X = \mathrm{Im}\,\delta_{i+1}^X$, $C_i^X = \mathrm{Coker}\,\delta_{i+1}^X$. The module $\mathrm{H}_i(X) = Z_i^X / B_i^X$ is called the homology module of X in degree i. The homology complex $\mathrm{H}(X)$ is defined by setting $\mathrm{H}(X)_i = \mathrm{H}_i(X)$ and $\delta_i^{\mathrm{H}(X)} = 0$ for all $i \in \mathbb{Z}$. A complex X is said to be homologically trivial if $\mathrm{H}(X) = 0$ and homologically degree-wise finite if $\mathrm{H}_i(X)$ is finitely generated for each $i \in \mathbb{Z}$. The homological infimum and supremum of a complex X are $\inf \mathrm{H}(X)$ and $\sup \mathrm{H}(X)$, respectively. A complex X is called homologically bounded above (resp. bounded below, bounded) if the complex $\mathrm{H}(X)$ is so. In the following $\mathcal{C}(R)$ will be the category of R-complexes. The symbol "$\simeq$" is used to designate quasi-isomorphism in the category $\mathcal{C}(R)$ and isomorphisms in the derived category $\mathbf{D}(R)$.

A DG-projective resolution of X is a quasi-isomorphism of complexes $i : P \to X$ with P DG-projective. By Avramov and Foxby [1, 1.6], every complex has a DG-projective resolution. The notion of the projective dimension of an unbounded complex X, denoted by $\mathrm{pd}_R X$, is defined in Avramov and Foxby [1] by the formula

$$\mathrm{pd}_R X = \inf\{\sup\{n \mid P_n \neq 0\} \mid P \text{ is a DG-projective resolution of } X\}.$$

The flat dimension and injective dimension of an unbounded complex are defined similarly. See [1, 3, 5] for details on homological dimensions of complexes.

All rings in this paper are assumed to be commutative and unital; throughout, R and S denote such rings. All modules are unitary. We will use the notations $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$, to denote the full subcategory of R-modules of finite projective, injective and flat dimension, respectively. Let $\phi : R \to S$ be a ring homomorphism. ϕ is said to be of finite flat dimension (or with $\mathrm{fd}\phi$ finite) if flat dimension of S is finite as an R-module. We say ϕ is faithfully flat if S is a faithfully flat R-module (that is, S_R satisfies the condition that $0 \to A \to B$ is an exact sequence of R-modules if and only if $0 \to S \otimes_R A \to S \otimes_R B$ is exact). The examples of faithfully flat ring homomorphisms include the natural ring homomorphisms $R \to R[[x_1, \cdots, x_n]]$ and $R \to \widehat{R}$ when R is Noetherian, where $\widehat{R}$ is the I-adic completion for an ideal I of R with $I \subseteq \mathrm{rad}(R)$.

3 Gorenstein Projective Dimension

The following lemma appeared in [11].

Lemma 3.1 *Let R be a Noetherian ring and M be an R-module.*

(1) *Assume that M is embedded in an R-module with finite flat dimension and that $\mathrm{Tor}_i^R(M, I) = 0$ for every injective R-module I and all $i \geq 1$. Then M possesses a monic $\mathcal{F}(R)$-preenvelope $M \to F$ with F flat.*

(2) *Assume that M is embedded in an R-module with finite projective dimension and that $\mathrm{Ext}_R^i(M, P) = 0$ for all projective R-modules P and all $i > 0$. If M has a monic $\mathcal{P}(R)$-preenvelope, then there is a monic $\mathcal{P}(R)$-preenvelope $M \to P$ of M such that P is projective.*

A complex T of projective R-modules is called totally acyclic if T is homologically trivial, and $\mathrm{Hom}_R(T, P)$ is homologically trivial for every projective module P. The syzygies of a totally acyclic complex of projective modules are called Gorenstein projective R-modules.

The definition of Gorenstein injective modules is dual to the one of Gorenstein projective modules. An R-module M is Gorenstein flat if there exists an exact complex F of flat modules, such that M is isomorphic to a cokernel of F, and $H(J \otimes_R F) = 0$ for all injective right R-modules J. The definitions of Gorenstein projective dimension, Gorenstein injective dimension and Gorenstein flat dimension of modules and their general background material can be found in [3, 5, 13].

Let $\phi : R \to S$ be a faithfully flat ring homomorphism. In [8], it is shown that an R-module M is Gorenstein flat if and only if $S \otimes_R M$ is a Gorenstein flat S-module and $\mathrm{Tor}_i^R(E, M) = 0$ for every injective R-module E and all $i \geq 1$, and that if $\dim(S)$ is finite then an R-module M is Gorenstein injective if and only if $\mathrm{Hom}_R(S, M)$ is a Gorenstein injective S-module and

$\mathrm{Ext}_R^i(F, M) = 0$ for every flat R-module F and all $i \geq 1$. Let R be coherent and let S be a faithfully flat R-algebra that is left GF-closed (possibly noncommutative). Then it was proved in [7, Lemma 5.3] that an R-module M is Gorenstein flat if and only if the S-module $S \otimes_R M$ is Gorenstein flat. If R is Noetherian and S is a faithfully flat R-algebra (possibly noncommutative), then an R-module M is Gorenstein injective if and only if it is cotorsion and the S-module $\mathrm{Hom}_R(S, M)$ is Gorenstein injective (see [7, Theorem 3.5]). For Gorenstein projectivity we have the following result.

Lemma 3.2 *Let R be a Noetherian ring, let $\phi : R \to S$ be a faithfully flat ring homomorphism and M be an R-module. If R has finite Krull dimension, then the following are equivalent*:

(1) *M is Gorenstein projective.*

(2) *$S \otimes_R M$ is a Gorenstein projective S-module and $\mathrm{Ext}_R^i(M, P) = 0$ for all projective R-modules P and all $i > 0$.*

(3) *$S \otimes_R M$ is an S-module with finite Gorenstein projective dimension and $\mathrm{Ext}_R^i(M, P) = 0$ for all projective R-modules P and all $i > 0$.*

Proof (1)⇒(2) follows from [6, Ascent table II (b)] and (2)⇒(3) is clear. So it is enough to prove (3)⇒(1).

Suppose that $S \otimes_R M$ is an S-module with finite Gorenstein projective dimension. Then, by [5, Lemma 2.17], there is an exact sequence of S-modules $0 \to S \otimes_R M \to H \to B \to 0$ where B is Gorenstein projective and $\mathrm{pd}_S H = \mathrm{Gpd}_S(S \otimes_R M)$. Since $\phi : R \to S$ is a faithfully flat ring homomorphism there is an exact sequence of R-modules $0 \to M \to S \otimes_R M$. Thus we have an exact sequence of R-modules

$$0 \to M \to H$$

with $\mathrm{pd}_S H < \infty$. Note that S is a flat R-module and each flat R-module has finite projective dimension. Thus it is easy to see that every projective S-module has finite projective dimension as an R-module. Hence $\mathrm{pd}_R H < \infty$.

Standard discussion yields $\mathrm{Ext}_R^i(M, P) = 0$ for every R-module P with finite projective dimension and all $i > 0$. Let I and E be injective R-modules. Then $\mathrm{Hom}_R(I, E)$ is flat and so $\mathrm{pd}_R \mathrm{Hom}_R(I, E) < \infty$. If we take E as the injective cogenerator then from $\mathrm{Hom}_R(\mathrm{Tor}_i^R(M, I), E) \cong \mathrm{Ext}_R^i(M, \mathrm{Hom}_R(I, E))$ it follows that $\mathrm{Tor}_i^R(M, I) = 0$ for every injective R-module I and all $i \geq 1$. Thus, by Lemma 3.1(1), M possesses a monic $\mathcal{F}(R)$-preenvelope $M \to F$ with F flat. Since $\mathcal{F}(R) = \mathcal{P}(R)$, M possesses a monic $\mathcal{P}(R)$-preenvelope. It follows from Lemma 3.1(2) that there is a monic $\mathcal{P}(R)$-preenvelope $M \to P$ of M such that P is projective. Now consider exact sequence

$$0 \to M \to P \to M_1 \to 0.$$

Let Q be a projective R-module. Then the sequence

$$0 \to \mathrm{Hom}_R(M_1, Q) \to \mathrm{Hom}_R(P, Q) \to \mathrm{Hom}_R(M, Q) \to 0$$

is exact, which implies $\mathrm{Ext}_R^1(M_1, Q) = 0$. Now it is easy to see that $\mathrm{Ext}_R^i(M_1, Q) = 0$ for all projective R-modules Q and all $i > 0$.

From

$$0 \to S \otimes_R M \to S \otimes_R P \to S \otimes_R M_1 \to 0$$

it follows that $S \otimes_R M_1$ has finite Gorenstein projective dimension since $S \otimes_R P$ is a projective S-module and $S \otimes_R M$ has finite Gorenstein projective dimension.

Now proceeding in this manner, we get the desired co-proper right projective resolution of M.

Lemma 3.3 (see [16, Lemma 10]) *Let $\phi : R \to S$ be a faithfully flat ring homomorphism. If E is an injective cogenerator in S-*Mod *then it is an injective cogenerator in R-*Mod.

For unbounded complexes, Veliche [18] introduced and studied Gorenstein projective dimension over an arbitrary associative ring A. Let X be a complex of A-modules. A complete resolution of X is a diagram

$$T \xrightarrow{\tau} P \to X$$

with $P \to X$ a DG-projective resolution, T a totally acyclic complex of projective modules, and τ a map of complexes such that τ_i is bijective for $i \gg 0$. The Gorenstein projective dimension of an A-complex X is defined in [18, Definition 3.1] by

$$\mathrm{Gpd}_A X = \inf \left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{l} T \xrightarrow{\tau} P \to X \text{ is a complete resolution such} \\ \text{that } \tau_i \text{ is bijective for each } i \geq n \end{array} \right\}.$$

It was shown in [18] that the Gorenstein projective dimension of complexes in the sense of Christensen [3], when they can be applied, agrees with Veliche's notion.

For Gorenstein projective dimensions of complexes we have the following result.

Theorem 3.1 *Let R be a Noetherian ring and $\phi : R \to S$ be a faithfully flat ring homomorphism. If R has finite Krull dimension, then for any homologically degree-wise finite complex X, there is an equality*

$$\mathrm{Gpd}_R X = \mathrm{Gpd}_S(S \otimes^{\mathbf{L}}_R X).$$

Proof Firstly we show $\mathrm{Gpd}_S(S \otimes^{\mathbf{L}}_R X) \leq \mathrm{Gpd}_R X$. If $X \simeq 0$ or $\mathrm{Gpd}_R X = \infty$ then there is nothing to do. So we assume that $X \not\simeq 0$ and $\mathrm{Gpd}_R X < \infty$. Denote $g = \mathrm{Gpd}_R X$. Then $g \in \mathbb{Z}$. Thus, by [18, Theorem 3.4], there is a DG-projective resolution $P \to X$ such that $\sup \mathrm{H}(P) \leq g$ and C^P_j is Gorenstein projective for any $j \geq g$. It is clear that $S \otimes^{\mathbf{L}}_R X = S \otimes_R P$. For each $l \in \mathbb{Z}$, $(S \otimes_R P)_l = S \otimes_R P_l$ is projective as an S-module by [6, 0.5]. For any homologically trivial S-complex E,

$$\mathrm{Hom}_S(S \otimes_R P, E) \cong \mathrm{Hom}_R(P, E)$$

is homologically trivial since P is DG-projective and E is homologically trivial as an R-complex. This means that $S \otimes_R P$ is a DG-projective resolution of $S \otimes^{\mathbf{L}}_R X$ over S. Since S is flat as an R-module, we have the following exact sequence

$$\cdots \to S \otimes_R P_{g+1} \to S \otimes_R P_g \to S \otimes_R C^P_g \to 0.$$

This means that $\sup \mathrm{H}(S \otimes_R P) \leq g$. Also $C^{S \otimes_R P}_j \cong S \otimes_R C^P_j$ is a Gorenstein projective S-module for each $j \geq g$ by [6, Lemma 5.6]. Thus $\mathrm{Gpd}_S(S \otimes^{\mathbf{L}}_R X) \leq g = \mathrm{Gpd}_R X$ by [18, Theorem 3.4] again.

Now it is enough to show that $\mathrm{Gpd}_R X \leq \mathrm{Gpd}_S(S \otimes_R^{\mathbf{L}} X)$. If $\mathrm{Gpd}_S(S \otimes_R^{\mathbf{L}} X) = \infty$ then there is nothing to do. Thus we consider two possibilities as follows.

(i) Suppose that $\mathrm{Gpd}_S(S \otimes_R^{\mathbf{L}} X) = g \in \mathbb{Z}$. Since X is a homologically degree-wise finite complex, there is a DG-projective resolution $P \to X$ of X with each P_l finitely generated by [18, 1.3.4]. Then $S \otimes_R P$ is a DG-projective resolution of $S \otimes_R^{\mathbf{L}} X$, which implies by [18, Theorem 3.4] that $\sup \mathrm{H}(S \otimes_R P) \leq g$ and $C_j^{S \otimes_R P}$ is Gorenstein projective for any $j \geq g$. Thus the sequence

$$\cdots \to S \otimes_R P_{g+2} \to S \otimes_R P_{g+1} \to S \otimes_R P_g \tag{$*$}$$

is exact.

Let $j \geq g$. From the isomorphism $S \otimes_R C_j^P \cong C_j^{S \otimes_R P}$ it follows that $S \otimes_R C_j^P$ is a Gorenstein projective S-module. For every projective R-module Q, there is an isomorphism by [10, Theorem 3.2.5]:

$$S \otimes_R \mathrm{Ext}_R^i(C_j^P, Q) \cong \mathrm{Ext}_S^i(S \otimes_R C_j^P, S \otimes_R Q)$$

by noting that C_j^P is finitely generated. Since $S \otimes_R C_j^P$ is a Gorenstein projective S-module and $S \otimes_R Q$ is a projective S-module, we have $\mathrm{Ext}_S^i(S \otimes_R C_j^P, S \otimes_R Q) = 0$. Thus $\mathrm{Ext}_R^i(C_j^P, Q) = 0$ by the faithful flatness of the ring homomorphism $\phi : R \to S$. Now, by Lemma 3.2, C_j^P is Gorenstein projective.

Let E be an injective cogenerator in S-Mod. Then the first row of the following commutative diagram is exact:

$$\begin{array}{ccccccc}
\mathrm{Hom}_S(S \otimes_R P_g, E) & \longrightarrow & \mathrm{Hom}_S(S \otimes_R P_{g+1}, E) & \longrightarrow & \mathrm{Hom}_S(S \otimes_R P_{g+2}, E) & \longrightarrow & \cdots \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_R(P_g, E) & \longrightarrow & \mathrm{Hom}_R(P_{g+1}, E) & \longrightarrow & \mathrm{Hom}_R(P_{g+2}, E) & \longrightarrow & \cdots
\end{array}$$

Thus the second row is exact. By Lemma 3.3, E is an injective cogenerator in R-Mod. Hence, from the second exact row, we have the following exact sequence:

$$\cdots \to P_{g+2} \to P_{g+1} \to P_g.$$

This shows that $\sup \mathrm{H}(P) \leq g$. Thus $\mathrm{Gpd}_R X \leq g = \mathrm{Gpd}_S(S \otimes_R^{\mathbf{L}} X)$.

(ii) Assume $\mathrm{Gpd}_S(S \otimes_R^{\mathbf{L}} X) = -\infty$. Then $S \otimes_R^{\mathbf{L}} X$ is homologically trivial. Consider a DG-projective resolution $P \to X$. Then $S \otimes_R^{\mathbf{L}} X = S \otimes_R P$ is homologically trivial. That is, the sequence

$$\cdots \to S \otimes_R P_{l+1} \to S \otimes_R P_l \to S \otimes_R P_{l-1} \to \cdots$$

is exact. By analogy with the proof of (i), it follows that

$$\cdots \to P_{l+1} \to P_l \to P_{l-1} \to \cdots$$

is exact, which implies that X is homologically trivial. Thus $\mathrm{Gpd}_R X = -\infty = \mathrm{Gpd}_S(S \otimes_R^{\mathbf{L}} X)$.

A ring R is called a GF-closed ring (see [2, 14]) if the class of Gorenstein flat modules is closed under extensions. The class of GF-closed rings includes (strictly) the one of right coherent rings and the one of rings of finite weak dimension (for examples of GF-closed rings that are neither right coherent nor of finite weak dimension see [2]). Let R be a GF-closed ring. Let X be a complex of R-modules. According to [14], the Gorenstein flat dimension of X is defined by: $\mathrm{Gfd}_R X \leq g$ if there is a DG-flat resolution $F \to X$ such that $\sup \mathrm{H}(F) \leq g$ and C_j^F is Gorenstein flat for any $j \geq g$. If $\mathrm{Gfd}_R X \leq g$ but $\mathrm{Gfd}_R X \leq g-1$ does not hold then $\mathrm{Gfd}_R X = g$. If $\mathrm{Gfd}_R X \leq g$ for any g then $\mathrm{Gfd}_R X = -\infty$. If $\mathrm{Gfd}_R X \leq g$ does not hold for any g then $\mathrm{Gfd}_R X = \infty$.

Remark 3.1 We compare Theorem 3.1 with the result of [16]. Let $\phi : R \to S$ be a faithfully flat ring homomorphism. It was proved in [16, Theorem 15] that if R and S are Noetherian rings then for each R-complex X, there is an equality $\mathrm{Gfd}_R X = \mathrm{Gfd}_S(S \otimes_R^{\mathbf{L}} X)$. Thus it is natural to asker whether or not [16, Theorem 15] implies Theorem 3.1 when S is Noetherian. This obviously connects the relationship of Gorenstein projective modules with Gorenstein flat modules. Holm [13] gave a clear and useful condition under which all Gorenstein projective modules are Gorenstein flat. This is the case if the ring R is right coherent and has finite left finitistic dimension. If R is a Noetherian ring of finite Krull dimension and M is an R-module, then the results we know are only that

$$\mathrm{Gpd}_R M < \infty \Longleftrightarrow \mathrm{Gfd}_R M < \infty$$

(see, for example [11, Theorem 3.4]). In the following example, we do not know whether or not all Gorenstein projective T-modules are Gorenstein flat. Thus Theorem 3.1 is not implied by the result of [16, Theorem 15].

Example 3.1 Let T be Nagata's Noetherian regular ring of infinite Krull dimension and consider the natural inclusion $k \to T$ where k is the field over which T is built. Then k is a Noetherian ring with finite Krull dimension and $k \to T$ is faithfully flat (see [6, Remark of Theorem 3.2]). Recall that over a Noetherian ring, the finitistic dimension is equal to the Krull dimension.

The following is another result concerning Gorenstein projective dimensions of unbounded complexes. It was proved in [3, Theorem 6.4.7] that

$$\mathrm{Gpd}_R(\mathbf{R}\mathrm{Hom}_R(U, X)) \leq \mathrm{Gpd}_R(X) - \inf(U)$$

for any homologically bounded below complex X and any complex U with finite homology and finite projective dimension. Here we will show the same result for general complexes X (not necessary homologically bounded below). We will need the following lemma. It is well known that the same isomorphism holds when two of the complexes P, Q and E are bounded and, Q consists of finitely generated projective modules, or R is Noetherian and Q consists of finitely generated R-modules and E consists of injective R-modules (see, for example [6, 0.3]).

Lemma 3.4 *Let P and E be complexes and Q be a bounded complex consisting of finitely generated projective modules. Then there exists an isomorphism*

$$\mathrm{Hom}_R(\mathrm{Hom}_R(Q, P), E) \cong Q \otimes \mathrm{Hom}_R(P, E).$$

Proof The proof is modeled. Suppose that $u = \sup(Q)$ and $v = \inf(Q)$. Then, for any

$l \in \mathbb{Z}$,

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Hom}_R(Q,P),E)_l &= \prod_{s\in\mathbb{Z}} \operatorname{Hom}_R\Big(\bigoplus_{t=v}^{u} \operatorname{Hom}_R(Q_t, P_{t+s}), E_{s+l}\Big)\\ &\cong \prod_{s\in\mathbb{Z}} \bigoplus_{t=v}^{u} \operatorname{Hom}_R(\operatorname{Hom}_R(Q_t, P_{t+s}), E_{s+l}).\end{aligned}$$

On the other hand,

$$\begin{aligned}(Q\otimes \operatorname{Hom}_R(P,E))_l &= \bigoplus_{t=v}^{u}\Big(Q_t \otimes \prod_{s\in\mathbb{Z}} \operatorname{Hom}_R(P_s, E_{l-t+s})\Big)\\ &\cong \bigoplus_{t=v}^{u}\prod_{s\in\mathbb{Z}}(Q_t\otimes \operatorname{Hom}_R(P_s, E_{l-t+s}))\\ &\cong \bigoplus_{t=v}^{u}\prod_{s\in\mathbb{Z}}\operatorname{Hom}_R(\operatorname{Hom}_R(Q_t,P_s), E_{l-t+s})\\ &\cong \prod_{s\in\mathbb{Z}}\bigoplus_{t=v}^{u}\operatorname{Hom}_R(\operatorname{Hom}_R(Q_t,P_s), E_{l-t+s}).\end{aligned}$$

The first isomorphism holds since Q_t is finitely presented and the second holds since Q_t is also projective. Note that all isomorphisms are natural. Thus the result follows.

Proposition 3.1 *Let X be a complex of R-modules and U be a complex with finite homology and finite projective dimension. Then*

$$\operatorname{Gpd}_R(\mathbf{R}\operatorname{Hom}_R(U,X)) \le \operatorname{Gpd}_R(X) - \inf(U).$$

Proof If $\inf(U) = -\infty$ then there is nothing to do. Now suppose $\inf(U) > -\infty$. Then $\inf \mathrm{H}(U) > -\infty$. Since U is a complex with finite homology and finite projective dimension, there exists a complex Q consisting of finitely generated projective R-modules such that $Q \simeq U$ and $Q_l = 0$ when $l < v = \inf(U)$ or $l > u = \operatorname{pd}(U)$ by [3, A.5.4]. If $X \simeq 0$ then the result is clear. If $\operatorname{Gpd}_R(X) = \infty$ then the result is also clear. So we assume $X \not\simeq 0$ and $\operatorname{Gpd}_R(X) < \infty$. Denote $g = \operatorname{Gpd}_R(X)$. Then $g \in \mathbb{Z}$. Thus there exists a complete resolution $T \xrightarrow{\tau} P \to X$ with $P \xrightarrow{\simeq} X$ a DG-projective resolution, T a totally acyclic complex of projective R-modules, and τ a map of complexes such that τ_i is bijective for $i \ge g$. Now consider

$$\operatorname{Hom}_R(Q,T) \xrightarrow{\operatorname{Hom}_R(Q,\tau)} \operatorname{Hom}_R(Q,P) \longrightarrow \operatorname{Hom}_R(Q,X). \qquad (**)$$

We will show that $(**)$ is a complete resolution of $\operatorname{Hom}_R(Q,X)$.

Clearly $\operatorname{Hom}_R(Q,T)$ is an exact complex of projective modules since Q is a DG-projective complex. For any projective module W, from [6, 0.3], it follows that

$$\operatorname{Hom}_R(\operatorname{Hom}_R(Q,T),W) \cong Q\otimes \operatorname{Hom}_R(T,W).$$

Now $Q\otimes \operatorname{Hom}_R(T,W)$ and thus, $\operatorname{Hom}_R(\operatorname{Hom}_R(Q,T),W)$, is exact since Q is DG-flat. Hence $\operatorname{Hom}_R(Q,T)$ is a totally acyclic complex of projective R-modules.

From Lemma 3.4 we have an isomorphism of complexes $\operatorname{Hom}_R(\operatorname{Hom}_R(Q,P),E) \cong Q\otimes \operatorname{Hom}_R(P,E)$. Thus it follows that $\operatorname{Hom}_R(\operatorname{Hom}_R(Q,P),E)$ is exact for any exact complex E.

Note that $\mathrm{Hom}_R(Q,P)_l$ is projective for all $l \in \mathbb{Z}$. Thus $\mathrm{Hom}_R(Q,P)$ is a DG-projective complex. From $P \xrightarrow{\simeq} X$ it follows that $\mathrm{Hom}_R(Q,P) \xrightarrow{\simeq} \mathrm{Hom}_R(Q,X)$ since Q is a DG-projective complex. This shows that $\mathrm{Hom}_R(Q,P)$ is a DG-projective resolution of $\mathrm{Hom}_R(Q,X)$.

The following discussion shows that the map of complexes $\mathrm{Hom}_R(Q,\tau)$ is bijective for $i \gg 0$ and, thus $(**)$ is a complete resolution of $\mathrm{Hom}_R(Q,X)$. If $l > g-v$, then, for any $k \in \mathbb{Z}$ with $k \geq v$, $k+l > k+g-v \geq g$. Thus we have

$$\begin{aligned}\mathrm{Hom}_R(Q,T)_l &= \prod_{v\leq k\in\mathbb{Z}} \mathrm{Hom}_R(Q_k, T_{l+k})\\ &\cong \prod_{v\leq k\in\mathbb{Z}} \mathrm{Hom}_R(Q_k, P_{l+k})\\ &= \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(Q_k, P_{l+k})\\ &= \mathrm{Hom}_R(Q,P)_l.\end{aligned}$$

This shows that $\mathrm{Gpd}_R(\mathbf{R}\mathrm{Hom}_R(U,X)) = \mathrm{Gpd}_R(\mathrm{Hom}_R(Q,X)) \leq g-v = \mathrm{Gpd}_R(X) - \inf(U)$.

4 Ding Projective Dimension

The following definition appeared in [20, Definition 2.1].

Definition 4.1 *A complex of R-modules P is said to be totally $\mathcal{F}$-acyclic if the following conditions are satisfied*:

(1) P_n *is projective for every* $n \in \mathbb{Z}$;

(2) P *is exact*;

(3) $\mathrm{Hom}_R(P,F)$ *is exact for every flat R-module F.*

An R-module M is called strongly Gorenstein flat (see [9]) if there exists a totally $\mathcal{F}$-acyclic complex P such that $C_0^P = M$. These modules were studied by Gillespie [12] with different name of Ding projective modules. In the following we prefer to use the name Ding projective modules.

According to [20, Definition 2.11], the strongly Gorenstein flat dimension, or Ding projective dimension, of an R-module M is defined by

$$\mathrm{Dpd}_R(M) = \inf\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{l}\text{there is an exact sequence } 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0\\ \text{with each } P_i \text{ Ding projective}\end{array} \right\}.$$

See [9, 17, 19, 22–23] for details of Ding projective modules and Ding projective dimensions.

Lemma 4.1 *Let R be a Noetherian ring, let $\phi : R \to S$ be a faithfully flat ring homomorphism and M be an R-module. Then the following are equivalent*:

(1) *M is Ding projective*;

(2) *$S \otimes_R M$ is a Ding projective S-module and* $\mathrm{Ext}^i_R(M,F) = 0$ *for all flat R-modules F and all $i > 0$*;

(3) *$S \otimes_R M$ is an S-module with finite Ding projective dimension and* $\mathrm{Ext}^i_R(M,F) = 0$ *for all flat R-modules F and all $i > 0$.*

Proof (1)⇒(2). Suppose that there exists a totally $\mathcal{F}$-acyclic complex

$$P: \quad \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$$

such that $C_0^P = M$. Then, for each flat S-module F, from

$$\mathrm{Hom}_S(S \otimes_R P_i, F) \cong \mathrm{Hom}_R(P_i, \mathrm{Hom}_S(S, F)) \cong \mathrm{Hom}_R(P_i, F)$$

it follows that the complex $S \otimes P$ is $\mathcal{F}$-acyclic. Thus $S \otimes_R M$ is a Ding projective S-module. Another conclusion follows from [17, Proposition 2.1].

(3)⇒(1). Suppose that $S \otimes_R M$ is an S-module with finite Ding projective dimension. Then, by [17, Corollary 2.2], there is an exact sequence of S-modules $0 \to S \otimes_R M \to H \to B \to 0$ where B is Ding projective and $\mathrm{pd}_S H = \mathrm{Dpd}_S(S \otimes_R M)$. Thus $\mathrm{fd}_S H < \infty$. Since $\phi : R \to S$ is a faithfully flat ring homomorphism there is an exact sequence $0 \to M \to S \otimes_R M$ of R-modules. Thus M embeds into H as R-modules with $\mathrm{fd}_S H < \infty$. Note that S is a flat R-module. Thus it is easy to see that $\mathrm{fd}_R H < \infty$.

By analogy with discussion of Lemma 3.2, we get a monic $\mathcal{F}(R)$-preenvelope $M \to F$ of M with F flat. Now, let $0 \to K \to P \to F \to 0$ be an exact sequence such that P is a projective R-module. Then K is flat. Thus $\mathrm{Ext}_R^i(M, K) = 0$, which implies that $f : M \to F$ has a lifting $M \to P$ which is monic and still an $\mathcal{F}(R)$-preenvelope.

Now consider exact sequence

$$0 \to M \to P \to M_1 \to 0.$$

Let G be a flat R-module. Then the sequence

$$0 \to \mathrm{Hom}_R(M_1, G) \to \mathrm{Hom}_R(P, G) \to \mathrm{Hom}_R(M, G) \to 0$$

is exact, which implies $\mathrm{Ext}_R^1(M_1, G) = 0$. Now it is easy to see that $\mathrm{Ext}_R^i(M_1, G) = 0$ for all flat R-modules G and all $i > 0$. From

$$0 \to S \otimes_R M \to S \otimes_R P \to S \otimes_R M_1 \to 0$$

and [17, Proposition 2.3], it follows that $S \otimes_R M_1$ has finite Ding projective dimension since $S \otimes_R P$ is a projective S-module and $S \otimes_R M$ has finite Ding projective dimension. Now proceeding in this manner, we get that M is Ding projective by [17, Proposition 2.1].

Let X be a complex of R-modules. An $\mathcal{F}$-complete resolution of X is a diagram of morphisms of complexes

$$T \xrightarrow{\tau} P \xrightarrow{\pi} X,$$

where $P \xrightarrow{\pi} X$ is a DG-projective resolution, T is a totally $\mathcal{F}$-acyclic complex, and τ_i is bijective for all $i \gg 0$. The Ding projective dimension, or strongly Gorenstein flat dimension, of X is defined by

$$\mathrm{Dpd}_R X = \inf \left\{ n \in \mathbb{Z} \,\middle|\, \begin{array}{l} T \xrightarrow{\tau} P \to X \text{ is an } \mathcal{F}\text{-complete resolution} \\ \text{such that } \tau_i \text{ is bijective for each } i \geq n \end{array} \right\}.$$

Lemma 4.2 (see [20, Theorem 2.13]) *Let X be a complex and n be an integer. Then the following assertions are equivalent*:

(1) $\mathrm{Dpd}_R X \leq n$;

(2) $\sup \mathrm{H}(X) \leq n$ *and there exists a DG-projective resolution* $P \to X$ *such that the module* C_n^P *is Ding projective*;

(3) $\sup \mathrm{H}(X) \leq n$ *and for every DG-projective resolution* $P \to X$, *the module* C_n^P *is Ding projective.*

Theorem 4.1 *Let* R *be a Noetherian ring and let* $\phi : R \to S$ *be a faithfully flat ring homomorphism. Then for any homologically degree-wise finite complex* X, *there is an equality*

$$\mathrm{Dpd}_R X = \mathrm{Dpd}_S(S \otimes_R^{\mathbf{L}} X).$$

Proof It follows by analogy with the proof of Theorem 3.1; only one has to invoke Lemma 4.1 instead of Lemma 3.2 and Lemma 4.2 instead of [18, Theorem 3.4].

Corollary 4.1 *Let* R *be a Noetherian ring and let* $S = R[[x_1, \cdots, x_n]]$. *Then for any homologically degree-wise finite complex* X, *there is an equality* $\mathrm{Dpd}_R X = \mathrm{Dpd}_S(S \otimes_R^{\mathbf{L}} X)$.

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