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Results on Uniqueness Problem for Meromorphic Mappings Sharing Moving Hyperplanes in General Position Under More General and Weak Conditions*

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Abstract The aim of the paper is to deal with the algebraic dependence and uniqueness problem for meromorphic mappings by using the new second main theorem with different weights involved the truncated counting functions, and some interesting uniqueness results are obtained under more general and weak conditions where the moving hyperplanes in general position are partly shared by mappings from $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, which can be seen as the improvements of previous well-known results.

Keywords Algebraic dependence, Uniqueness problem, Meromorphic mapping, Moving hyperplane
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1 Introduction and Main Results

In 1926, Nevanlinna [9] proved that for two non-constant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images (ignoring multiplicities) for five distinct values in $\mathbb{P}^1(\mathbb{C})$, then $f = g$. If they have the same inverse images, counted with multiplicities, for four distinct values, then g is a special type of a linear fractional transformation of f. We know that the number five of distinct values in Nevanlinna's five-value theorem cannot be reduced to four. These results are usually called the five-value theorem and four-value theorem, respectively.

Nevanlinna theory for meromorphic mappings of $\mathbb{C}^n$ into the complex projective space $\mathbb{P}^N(\mathbb{C})$ intersecting a finite set of fixed hyperplanes or moving hyperplanes was studied deeply as many years previously due to their important values in complex analysis in several variables, and many interesting results were established, please see [3, 10, 26] for example. Over the last few decades, there have been several generalizations of Nevanlinna's five-value theorem to the case of meromorphic mappings from $\mathbb{C}^n$ into the complex projective space $\mathbb{P}^N(\mathbb{C})$. Fujimoto [7] generalized the Nevanlinna's well-known five-value theorem to the case of meromorphic

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mappings from $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ and obtained that for two linearly non-degenerate meromorphic mappings f, g of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, if they have the same inverse images of q $(\geq 3N+2)$ hyperplanes counted with multiplicities located in general position, then $f = g$. After that, many significant contributions along this line were made to find the smaller "q" (see [5, 20, 23]). In recent years, Chen and Yan [4] considered the case of ignoring the multiplicities and verified that q can be relaxed to $2N+3$ which can be seen an accurate result and improve the previous results under the weak condition.

Stoll [21] and Ji [8] studied the theory of algebraic dependence of meromorphic mappings from $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ by using the original idea of Cartan in 1988. Later, Ru [18] considered the case of holomorphic curves for moving targets, which can be seen as the generalization of Stoll's result. Many authors including Pham et al. [11] and Thoan et al. [25], have a great interest in the theory of algebraic dependence of meromorphic mappings and obtained a lot of meaningful results. By using the results of Quang [14], Cao [1] obtained the interesting result which was the improvement of Thoan [25] and Quang [13].

Theorem 1.1 (see [1, Theorem 1]) *Let $f_1, f_2, \cdots, f_\lambda$ be λ non-constant meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{H_j\}_{j=1}^q$ be slowly moving hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position with coefficients $\{a_j\}_{j=1}^q$ such that $T(r, a_j) = o\big(\max_{1\leq i\leq q} T(r, f_i)\big)$, $1 \leq j \leq q$. Assume that $(f_i, a_j) \not\equiv 0$ $(1 \leq i \leq \lambda, 1 \leq j \leq q)$, and the following conditions are satisfied*:

(1) $\nu^1_{(f_1,a_j)} = \nu^1_{(f_2,a_j)} = \cdots = \nu^1_{(f_\lambda,a_j)}$, $1 \leq j \leq q$,

(2) $\dim\{z \mid (f_1, a_{j_1})(z) = (f_1, a_{j_2})(z) = 0\} \leq n-2$ *for any* $1 \leq j_1 < j_2 \leq q$,

(3) *there exists an integer number l satisfying $2 \leq l \leq \lambda$ such that for any increasing sequence $1 \leq i_1 < i_2 < \cdots < i_l \leq \lambda$, $f_{i_1}(z) \wedge f_{i_2}(z) \wedge \cdots \wedge f_{i_l}(z) = 0$ for all $z \in \bigcup_{j=1}^q A_j$, where $A_j = \{z \mid (f, a_j)(z) = 0\}$.*

If
$$q > \frac{3N(N+1)\lambda - 2(N-1)(\lambda-1)}{2(\lambda - l + 1)},$$
then $f_1, f_2, \cdots, f_\lambda$ are algebraically dependent over $\mathcal{R}$, i.e., $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \equiv 0$.

In 2011, Cao and Yi [2] gave some uniqueness theorems for meromorphic mappings sharing fixed hyperplanes where all intersecting points more than a certain number are omitted. Actually, there are many authors who consider the multiple values for meromorphic mappings sharing hyperplanes, i.e., consider only the intersecting points of the mappings f_i and the hyperplanes a_j with the multiplicity not exceeding a certain number $m_j \leq \infty$. In 2017, Quynh also considered the case of meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ and obtained the result as showed in Theorem 1.2. Before we introduce the result, we need to do some brief explanation of symbol "d".

Assume that every analytic set A_j has the irreducible decomposition as follows: $A_j = \bigcup_{k=1}^{t_j} A_{jk}$ for $1 \leq j \leq q$. Set $A_0 = \bigcup_{A_{il}\not\equiv A_{jk}} A_{il} \cap A_{jk}$ with $1 \leq i, j \leq q$, $1 \leq l \leq t_i$ and $1 \leq$

$k \le t_j$. And set $T_a = \bigcup_{\tau \in T[N+1,q]} \{z \mid a_{\tau(1)}(z) \wedge \cdots \wedge a_{\tau(N+1)}(z) = 0\}$ and $I(f) := \{z \in \mathbb{C} \mid f_1 = f_2 = \cdots = f_{N+1} = 0\}$, where $T[N+1,q]$ denotes the set of all injective maps from $\{1,2,\cdots,N+1\}$ to $\{1,2,\cdots,q\}$. For each $z \in \mathbb{C}^n \setminus \{T_a \cup A_0 \cup \bigcup_{i=1}^{\lambda} I(f_i)\}$, we define $\chi(z) = \sharp\{j \mid z \in A_j\}$. If $\{H_j\}_{j=1}^q$ are located in general position with coefficient vectors $\{a_j\}_{j=1}^q$, then $\chi(z) \le N$. Indeed, suppose that $z_0 \in A_j$ for all $1 \le j \le N+1$. Then $(f_{i_0}, a_j)(z_0) = 0$, i.e., $f_{i_0 1}(z_0) a_{j1}(z_0) + \cdots + f_{i_0 N+1}(z_0) a_{jN+1}(z_0) = 0$. By the assumption that we know $\{a_j\}_{j=1}^q$ are linearly independent, i.e., $a_1(z_0) \wedge a_2(z_0) \wedge \cdots \wedge a_{N+1}(z_0) \ne 0$, it implies that $f_{i_0 1}(z_0) = f_{i_0 2}(z_0) = \cdots = f_{i_0 N+1}(z_0) = 0$. Hence, $z_0 \in I(f_{i_0})$, which is a contradiction. For any positive number $r \ge 0$, define $d(r) = \sup\{\chi(z) \mid \|z\| \le r\}$, where the supremum is taken over all $z \in \mathbb{C}^n \setminus \{T_a \cup A_0 \cup \bigcup_{i=1}^{\lambda} I(f_i)\}$. Then $d(r)$ is a increasing function. Let $d := \lim_{r \to \infty} d(r)$, then $d \le N$. Note that if for each $j_1 \ne j_2$, $\dim A_{j_1} \cap A_{j_2} \le n-2$, then $d = 1$.

Theorem 1.2 (see [16, Theorem 1.1]) *Let $f_1, f_2, \cdots, f_\lambda$ be λ non-constant meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{H_j\}_{j=1}^q$ be slowly moving hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position with coefficients $\{a_j\}_{j=1}^q$ such that $T(r, a_j) = o\big(\max_{1 \le i \le q} T(r, f_i)\big)$, $1 \le j \le q$. Let m_j $(1 \le j \le q)$ be q positive integers or $+\infty$, $A_j := \operatorname{Supp} \nu_{(f_1,a_j),\le m_j} = \operatorname{Supp} \nu_{(f_2,a_j),\le m_j} = \cdots = \operatorname{Supp} \nu_{(f_\lambda,a_j),\le m_j}$ $(1 \le j \le q)$. Assume that $(f_i, a_j) \not\equiv 0$ $(1 \le i \le \lambda, 1 \le j \le q)$. There exists an integer number l satisfying $2 \le l \le \lambda$ such that for any increasing sequence $1 \le i_1 < i_2 < \cdots < i_l \le \lambda$, $f_{i_1}(z) \wedge f_{i_2}(z) \wedge \cdots \wedge f_{i_l}(z) = 0$ for all $z \in \bigcup_{j=1}^q A_j$. If*

$$\sum_{j=1}^q \frac{1}{m_j} < \frac{q}{N(N+2)} - \frac{d\lambda}{\lambda - l + 1},$$

then $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \equiv 0$.

In Theorem 1.2, there is no restriction on the dimension of the images of the mappings f_i of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. In other words, if we consider the case that the images of all mappings f_i have the same dimension, then whether there is a better result? Quynh [16] answered the question and obtained the meaningful result as follows.

Theorem 1.3 (see [16, Theorem 1.3]) *Let $f_1, f_2, \cdots, f_\lambda$ be λ non-constant meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{H_j\}_{j=1}^q$ be slowly moving hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position with coefficients $\{a_j\}_{j=1}^q$ such that $T(r, a_j) = o\big(\max_{1 \le i \le q} T(r, f_i)\big)$, $1 \le j \le q$. Let m_j $(1 \le j \le q)$ be q positive integers or $+\infty$, $A_j := \operatorname{Supp} \nu_{(f_1,a_j),\le m_j} = \operatorname{Supp} \nu_{(f_2,a_j),\le m_j} = \cdots = \operatorname{Supp} \nu_{(f_\lambda,a_j),\le m_j}$ $(1 \le j \le q)$. Assume that $(f_i, a_j) \not\equiv 0$ $(1 \le i \le \lambda, 1 \le j \le q)$, and the following conditions are satisfied*:

(1) $\nu^1_{(f_1,a_j),\le m_j} = \nu^1_{(f_2,a_j),\le m_j} = \cdots = \nu^1_{(f_\lambda,a_j),\le m_j}$, $1 \le j \le q$,

(2) $\dim\{z \mid (f_1, a_{j_1})(z) = (f_1, a_{j_2})(z) = 0\} \le n-2$ *for any* $1 \le j_1 < j_2 \le q$,

(3) *there exists an integer numbers l satisfying $2 \le l \le \lambda$ such that for any increasing sequence $1 \le i_1 < i_2 < \cdots < i_l \le \lambda$, $f_{i_1}(z) \wedge f_{i_2}(z) \wedge \cdots \wedge f_{i_l}(z) = 0$ for all $z \in \bigcup_{j=1}^{q} A_j$.*

We assume further that $\operatorname{rank}_{\mathcal{R}} f_1 = \cdots = \operatorname{rank}_{\mathcal{R}} f_\lambda = s+1$, *where s is a positive integer. If*

$$\sum_{j=1}^{q} \frac{1}{m_j + 1 - s} < \frac{q}{s(2N - s + 2)} - \frac{\lambda q}{q(\lambda - l + 1) + \lambda(s-1)},$$

then $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \equiv 0$.

Recently, Thai-Quang [24] proved the new second main theorem, and many results were obtained by Thoan, Duc and Quang (see [6, 13, 25]), which extended the results of Stoll [21] and Ru [17]. For the case of non-degenerate meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ intersecting moving hyperplanes, Ru [17] and Thai-Quang [22] established the second main theorem with truncated counting functions for the case of $n = 1$ and more general case, respectively. Corresponding to the case of non-degenerate meromorphic mappings, Ru, Thai, Quang et al. considered the case of degenerate meromorphic mappings and obtained many important results including the second main theorem (see [12, 14, 19, 24]). In particular, in 2016, Quang [15] studied the case where the truncated counting functions involve the second main theorem with different weights, which improved and extended the previous results as follows.

Theorem 1.4 (see [15, Theorem 1.1]) *Let f be a meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{H_j\}_{j=1}^{q}$ be $q(\ge 2N-s+2)$ moving hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position with coefficients $\{a_j\}_{j=1}^{q}$ such that $(f, a_j) \not\equiv 0$ $(1 \le j \le q)$. Assume that $s + 1 = \operatorname{rank}_{\mathcal{R}\{a_j\}}(f)$. Let $\lambda_1, \cdots, \lambda q$ be q positive numbers with $(2N - s + 2) \max\limits_{1\le j\le q} \lambda_j \le \sum\limits_{j=1}^{q} \lambda_j$. Then the following assertion holds*:

$$\Big\| \frac{\sum_{j=1}^{q} \lambda_j}{2N - s + 2} T(r, f) \le \sum_{j=1}^{q} \lambda_j N^{s}_{(f,a_j)}(r) + o(T(r,f)) + O\Big(\max_{1\le j\le q} T(r, a_j)\Big).$$

As usual, by the notation "$\| P$", we mean the assertion P holds for all $r \in [0, +\infty)$ excluding a Borel subset E of the interval $[0, +\infty)$ with $\int_E \mathrm{d}r < +\infty$.

The aim of the paper is to explore the algebraic dependence and uniqueness problem for meromorphic mappings by using the second main theorem with different weights involved the truncated counting functions and obtain some interesting uniqueness results under the more general and weak conditions as showed in Theorems 1.5–1.6 where the moving hyperplanes are partly shared by every mappings from $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, i.e., only the intersecting points of the mappings f_i and hyperplanes H_j with the multiplicity not exceeding m_j are considered, which can be seen as the meaningful generalizations and accurate improvements of Theorems 1.1–1.3.

Theorem 1.5 *Let $f_1, f_2, \cdots, f_\lambda$ be λ non-constant meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{H_j\}_{j=1}^{q}$ be slowly moving hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position with coefficients $\{a_j\}_{j=1}^{q}$ such that $T(r, a_j) = o\big(\max\limits_{1\le i\le q} T(r, f_i)\big)$, $1 \le j \le q$. Let M, m_j $(1 \le j \le q)$ be $q+1$ positive integers or $+\infty$, $A_j := \operatorname{Supp} \nu_{(f_1,a_j),\le m_j} = \operatorname{Supp} \nu_{(f_2,a_j),\le m_j} = \cdots = \operatorname{Supp} \nu_{(f_\lambda,a_j),\le m_j}$ $(1 \le$*

$j \le q$). Assume that $(f_i, a_j) \not\equiv 0$ $(1 \le i \le \lambda, 1 \le j \le q)$, and the following conditions are satisfied:

(1) $\nu^M_{(f_1,a_j),\le m_j} = \nu^M_{(f_2,a_j),\le m_j} = \cdots = \nu^M_{(f_\lambda,a_j),\le m_j}$, $1 \le j \le q$,

(2) *there exist q integer numbers $l_1, l_2, \cdots, l_q$ satisfying $2 \le l_j \le \lambda$ $(1 \le j \le q)$ such that for any increasing sequence $1 \le i_1 < i_2 < \cdots < i_{l_j} \le \lambda$, $f_{i_1}(z) \wedge f_{i_2}(z) \wedge \cdots \wedge f_{i_{l_j}}(z) = 0$ for all $z \in A_j$ $(1 \le j \le q)$.*

If

$$d\lambda M < \sum_{j=1}^{q} \frac{\lambda - l_j + 1}{m_j + 1 - M}\Big(\frac{m_j + 1}{s(2N - s + 2)} - M\Big),$$

where $s = \max\limits_{1\le i\le\lambda} \operatorname{rank}_{\mathcal{R}} f_i$, then $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \equiv 0$.

From Theorem 1.5, for the case of $m_1 = \cdots = m_q = \infty$, $l_1 = \cdots = l_q = l$ and $M = 1$, we can obtain the unicity result as showed in the following corollary, which may be regarded as an accurate improvement of Theorem 1.1. Note that the condition $\dim\{z \mid (f, a_{j_1})(z) = (f, a_{j_2})(z) = 0\} \le n - 2$ for any $1 \le j_1 < j_2 \le q$ implies $d = 1$.

Corollary 1.1 *Let $f_1, f_2, \cdots, f_\lambda$ be λ non-constant meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{H_j\}_{j=1}^q$ be slowly moving hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position with coefficients $\{a_j\}_{j=1}^q$ such that $T(r, a_j) = o\big(\max\limits_{1\le i\le q} T(r, f_i)\big)$, $1 \le j \le q$. Assume that $(f_i, a_j) \not\equiv 0$ $(1 \le i \le \lambda, 1 \le j \le q)$, and the following conditions are satisfied:*

(1) $\nu^1_{(f_1,a_j)} = \nu^1_{(f_2,a_j)} = \cdots = \nu^1_{(f_\lambda,a_j)}$, $1 \le j \le q$,

(2) $\dim\{z \mid (f_1, a_{j_1})(z) = (f_1, a_{j_2})(z) = 0\} \le n - 2$ *for any* $1 \le j_1 < j_2 \le q$,

(3) *there exists an integer number l satisfying $2 \le l \le \lambda$ such that for any increasing sequence $1 \le i_1 < i_2 < \cdots < i_l \le \lambda$, $f_{i_1}(z) \wedge f_{i_2}(z) \wedge \cdots \wedge f_{i_l}(z) = 0$ for all $z \in \bigcup\limits_{j=1}^{q} A_j$, where $A_j = \{z \mid (f_i, a_j)(z) = 0\}$.*

If

$$q > \frac{\lambda s(2N - s + 2)}{\lambda - l + 1}\Big(< \frac{3N(N + 1)\lambda - 2(N - 1)(\lambda - 1)}{2(\lambda - l + 1)}\Big),$$

where $s + 1 = \max\limits_{1\le i\le\lambda} \operatorname{rank}_{\mathcal{R}} f_i$, then $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \equiv 0$.

From Theorem 1.5, for the case of $l_1 = \cdots = l_j = l$ $(1 \le j \le q)$ and $M = 1$, we can obtain the unicity result as showed in the following corollary, which may be regarded as a meaningful improvement of Theorem 1.2.

Corollary 1.2 *Let $f_1, f_2, \cdots, f_\lambda$ be λ non-constant meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{H_j\}_{j=1}^q$ be slowly moving hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position with coefficients $\{a_j\}_{j=1}^q$ such that $T(r, a_j) = o\big(\max\limits_{1\le i\le q} T(r, f_i)\big)$, $1 \le j \le q$. Let m_j $(1 \le j \le q)$ be q positive integers or $+\infty$, $A_j := \operatorname{Supp}\nu_{(f_1,a_j),\le m_j} = \operatorname{Supp}\nu_{(f_2,a_j),\le m_j} = \cdots = \operatorname{Supp}\nu_{(f_\lambda,a_j),\le m_j}$ $(1 \le j \le q)$. Assume that $(f_i, a_j) \not\equiv 0$ $(1 \le i \le \lambda, 1 \le j \le q)$, and there exists an integer number l satisfying $2 \le l \le \lambda$ such that for any increasing sequence $1 \le i_1 < i_2 < \cdots < i_l \le \lambda$,*

$f_{i_1}(z) \wedge f_{i_2}(z) \wedge \cdots \wedge f_{i_l}(z) = 0$ for all $z \in \bigcup_{j=1}^{q} A_j$. If

$$\Big(1 + \frac{1}{s(2N-s+2)}\Big) \sum_{j=1}^{q} \frac{1}{m_j} < \frac{q}{s(2N-s+2)} - \frac{d\lambda}{\lambda - l + 1},$$

where $s + 1 = \max_{1 \le i \le \lambda} \operatorname{rank}_{\mathcal{R}} f_i$, then $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \equiv 0$.

For the special case of $s = N$, we have

$$\Big(1 + \frac{1}{N(N+2)}\Big) \sum_{j=1}^{q} \frac{1}{m_j} < \frac{q}{N(N+2)} - \frac{d\lambda}{\lambda - l + 1},$$

and the "q" of Corollary 1.2 can be smaller than that of Theorem 1.2.

From Theorem 1.5, for the case of $m_1 = \cdots = m_q = \infty$, $M = 1$ we have the following corollary, which extends the result of [15] obtained by Quang.

Corollary 1.3 *Let $f_1, f_2, \cdots, f_\lambda$ be λ non-constant meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{H_j\}_{j=1}^q$ be slowly moving hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position with coefficients $\{a_j\}_{j=1}^q$ such that $T(r, a_j) = o\big(\max_{1 \le i \le q} T(r, f_i)\big)$, $1 \le j \le q$. Let $A_j := \operatorname{Supp} \nu_{(f_1, a_j)} = \operatorname{Supp} \nu_{(f_2, a_j)} = \cdots = \operatorname{Supp} \nu_{(f_\lambda, a_j)}$ $(1 \le j \le q)$. Assume that $(f_i, a_j) \not\equiv 0$ $(1 \le i \le \lambda, 1 \le j \le q)$, and there exist q integer numbers $l_1, l_2, \cdots, l_q$ satisfying $2 \le l_j \le \lambda$ $(1 \le j \le q)$ such that for any increasing sequence $1 \le i_1 < i_2 < \cdots < i_{l_j} \le \lambda$, $f_{i_1}(z) \wedge f_{i_2}(z) \wedge \cdots \wedge f_{i_{l_j}}(z) = 0$ for all $z \in A_j$ $(1 \le j \le q)$. If*

$$q > \frac{d\lambda s(2N-s+2) + \sum_{j=1}^{q} l_j}{\lambda + 1},$$

where $s + 1 = \max_{1 \le i \le \lambda} \operatorname{rank}_{\mathcal{R}} f_i$, then $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \equiv 0$.

In Theorem 1.5, for the dimension of the linearly closures of the images of the mappings f_i $(1 \le i \le \lambda)$, there may be different from each other. In other words, we can obtain the same uniqueness results for different cases if they have the same maximum number of their dimensions. Next, corresponding to Theorem 1.3, we consider the special case that $s + 1 = \operatorname{rank}_{\mathcal{R}} f_1 = \cdots = \operatorname{rank}_{\mathcal{R}} f_\lambda$ in which more accurate result can be obtained as showed in Theorem 1.6.

Theorem 1.6 *Let $f_1, f_2, \cdots, f_\lambda$ be λ non-constant meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{H_j\}_{j=1}^q$ be slowly moving hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position with coefficients $\{a_j\}_{j=1}^q$ such that $T(r, a_j) = o\big(\max_{1 \le i \le q} T(r, f_i)\big)$, $1 \le j \le q$. Let M, m_j $(1 \le j \le q)$ be $q+1$ positive integers or $+\infty$, $A_j := \operatorname{Supp} \nu_{(f_1, a_j), \le m_j} = \operatorname{Supp} \nu_{(f_2, a_j), \le m_j} = \cdots = \operatorname{Supp} \nu_{(f_\lambda, a_j), \le m_j}$ $(1 \le j \le q)$. Assume that $(f_i, a_j) \not\equiv 0$ $(1 \le i \le \lambda, 1 \le j \le q)$, and the following conditions are satisfied:*

(1) $\nu^M_{(f_1, a_j), \le m_j} = \nu^M_{(f_2, a_j), \le m_j} = \cdots = \nu^M_{(f_\lambda, a_j), \le m_j}$, $1 \le j \le q$,

(2) $\dim\{z \mid (f_1, a_{j_1})(z) = (f_1, a_{j_2})(z) = 0\} \le n - 2$ *for any* $1 \le j_1 < j_2 \le q$,

(3) *there exist q integer numbers $l_1, l_2, \cdots, l_q$ satisfying $2 \le l_j \le \lambda$ $(1 \le j \le q)$ such that for any increasing sequence $1 \le i_1 < i_2 < \cdots < i_{l_j} \le \lambda$, $f_{i_1}(z) \wedge f_{i_2}(z) \wedge \cdots \wedge f_{i_{l_j}}(z) = 0$ for all $z \in A_j$ $(1 \le j \le q)$.*

If $\operatorname{rank}_{\mathcal{R}} f_1 = \cdots = \operatorname{rank}_{\mathcal{R}} f_\lambda = s+1$ *and*

$$\lambda((\lambda-1)s+1) \le q(\lambda - l_j + 1), \quad 1 \le j \le q, \tag{$*$}$$

$$1 < \sum_{j=1}^{q} \frac{q(\lambda - l_j + 1) + \lambda(s-1)}{\lambda s q(m_j + 1 - s)} \Big(\frac{m_j + 1}{2N - s + 2} - s \Big), \tag{$**$}$$

then $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \equiv 0$.

Remark 1.1 (a) The condition $(*)$ of Theorem 1.6 can be omited when $l_1 = l_2 = \cdots = l_q$ holds. In fact, the condition $(**)$ of Theorem 1.6 implies that the condition $(*)$ holds for the case of $l_1 = l_2 = \cdots = l_q = l$. However, let $m_1 = \cdots = m_q = +\infty, s = N = 2, \lambda = 3$, $l_1 = 2, l_2 = \cdots = l_q = 3$. If $q = 12$, then q satisfies the condition $(**)$, but it fails to the condition $(*)$ for $2 \le j \le q$. Hence, there exist some differences between $(*)$ and $(**)$, and the condition $(*)$ is necessary for the proof of Theorem 1.6.

(b) Theorem 1.6 also holds for any positive integer M of the condition (1), i.e., we can take $M = 1$.

(c) The "q" in Theorem 1.6 can be rewritten as follows:

$$\lambda < \sum_{j=1}^{q} \Big(\frac{\lambda - l_j + 1}{m_j + 1 - s} + \frac{\lambda(s-1)}{q(m_j + 1 - s)} \Big) \Big(\frac{m_j + 1}{s(2N - s + 2)} - 1 \Big).$$

And letting $M = 1, d = 1$, the "q" in Theorem 1.5 can be rewritten as follows:

$$\lambda < \sum_{j=1}^{q} \frac{\lambda - l_j + 1}{m_j} \Big(\frac{m_j + 1}{s(2N - s + 2)} - 1 \Big).$$

As can be seen from the above comparison, the "q" of Theorem 1.6 may be smaller than that of Theorem 1.5. Thus, Theorem 1.6 can be regarded as an accurate improvement of Theorem 1.5 for the case of $M = 1, d = 1$.

For the case of $l_1 = \cdots = l_q = l$, we can obtain the unicity result as showed in the following corollary. Note that the condition $\dim\{z \mid (f, a_{j_1})(z) = (f, a_{j_2})(z) = 0\} \le n - 2$ for any $1 \le j_1 < j_2 \le q$ implies $d = 1$.

Corollary 1.4 *Let $f_1, f_2, \cdots, f_\lambda$ be λ non-constant meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{H_j\}_{j=1}^q$ be slowly moving hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position with coefficients $\{a_j\}_{j=1}^q$ such that $T(r, a_j) = o\big(\max_{1 \le i \le q} T(r, f_i)\big)$, $1 \le j \le q$. Let M, m_j $(1 \le j \le q)$ be $q+1$ positive integers or $+\infty$, $A_j := \operatorname{Supp} \nu_{(f_1, a_j), \le m_j} = \operatorname{Supp} \nu_{(f_2, a_j), \le m_j} = \cdots = \operatorname{Supp} \nu_{(f_\lambda, a_j), \le m_j}$ $(1 \le j \le q)$. Assume that $(f_i, a_j) \not\equiv 0$ $(1 \le i \le \lambda, 1 \le j \le q)$, and the following conditions are satisfied*:

(1) $\nu^1_{(f_1, a_j), \le m_j} = \nu^1_{(f_2, a_j), \le m_j} = \cdots = \nu^1_{(f_\lambda, a_j), \le m_j}$, $1 \le j \le q$,

(2) $\dim\{z \mid (f_1, a_{j_1})(z) = (f_1, a_{j_2})(z) = 0\} \le n-2$ *for any* $1 \le j_1 < j_2 \le q$,

(3) *there exists an integer numbers* l *satisfying* $2 \le l \le \lambda$ *such that for any increasing sequence* $1 \le i_1 < i_2 < \cdots < i_l \le \lambda$, $f_{i_1}(z) \wedge f_{i_2}(z) \wedge \cdots \wedge f_{i_l}(z) = 0$ *for all* $z \in \bigcup_{j=1}^{q} A_j$.

If $\operatorname{rank}_{\mathcal{R}} f_1 = \cdots = \operatorname{rank}_{\mathcal{R}} f_\lambda = s+1$ *and*

$$1 < \sum_{j=1}^{q} \frac{q(\lambda-l+1)+\lambda(s-1)}{\lambda s q(m_j+1-s)}\Big(\frac{m_j+1}{2N-s+2} - s\Big),$$

then $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \equiv 0$.

Noting that the above inequality of "q" in Corollary 1.4, we can rewrite it as follows:

$$\sum_{j=1}^{q} \frac{1}{m_j+1-s} < \frac{1}{s(2N-s+2)} \sum_{j=1}^{q} \frac{m_j+1}{m_j+1-s} - \frac{\lambda q}{q(\lambda-l+1)+\lambda(s-1)},$$

and we know the fact that the scope of "q" in Corollary 1.4 is broader than that of Theorem 1.3. For example, we let $m_1 = m_2 = \cdots = m_q = 5, s = N = 1$ and $\lambda = l = 2$, Corollary 1.4 implies that $q > 10$ and Theorem 1.3 implies that $q > 15$. Thus, Theorem 1.6 may be regarded as the slight improvement of Theorem 1.3.

The remainder of this paper is organized as follows. In Section 2, we show basic notions and some necessary auxiliary results including some further instructions which play important roles and are used frequently in the later proofs. In Section 3, we give the proof of Theorem 1.5. And Theorem 1.6 is proved in the last section.

2 Basic Notions and Auxiliary Results from Nevanlinna Theory

Set $\|z\|^2 = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)$ for $z = (z_1, z_2, \cdots, z_n)$. For $r > 0$, we define

$$B_n(r) := \{z \in \mathbb{C}^n \mid \|z\| < r\}, \quad S_n(r) := \{z \in \mathbb{C}^n \mid \|z\| = r\}.$$

Let $d = \partial + \overline{\partial}$, $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \overline{\partial})$. Write

$$\upsilon_n(z) := (dd^c\|z\|^2)^{n-1}, \quad \sigma_n(z) := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{n-1}$$

for $z \in \mathbb{C}^n \backslash \{\mathbf{0}\}$.

Recall that the N-dimensional complex projective space is $\mathbb{P}^N(\mathbb{C}) = \mathbb{C}^{N+1} - \{\mathbf{0}\}/\sim$, where $(\alpha_1, \cdots, \alpha_{N+1}) \sim (\beta_1, \cdots, \beta_{N+1})$ if and only if $(\alpha_1, \cdots, \alpha_{N+1}) = \gamma(\beta_1, \cdots, \beta_{N+1})$ for some $\gamma \in \mathbb{C}$. We denote by $[\alpha_1 : \alpha_2 : \cdots : \alpha_{N+1}]$ the equivalent class of $(\alpha_1, \alpha_2, \cdots, \alpha_{N+1})$. Let f be a non-constant meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. We can choose holomorphic functions $f_1, f_2, \cdots, f_{N+1}$ on $\mathbb{C}^n$ such that $I(f) := \{z \in \mathbb{C} \mid f_1 = f_2 = \cdots = f_{N+1} = 0\}$ is of dimension at most $n-2$ and $f = [f_1 : f_2 : \cdots : f_{N+1}]$. Usually, $(f_1, f_2, \cdots, f_{N+1})$ is called a reduced representation of f. The characteristic function of f is defined by

$$T(r, f) = \int_{S_n(r)} \log\|f\|\sigma_n - \int_{S_n(1)} \log\|f\|\sigma_n, \quad r > 1,$$

where $\|f\| = \big(\sum_{i=1}^{N+1} |f_i|^2\big)^{\frac{1}{2}}$. Note that $T(r,f)$ is independent of the choice of the reduced representation of f.

Let H_j, $1 \le j \le q$ be the moving hyperplanes in $\mathbb{P}^N(\mathbb{C})$, which are given by

$$H_j = \Big\{[x_1 : x_2 : \cdots : x_{N+1}] \in \mathbb{P}^N(\mathbb{C}) \Big| \sum_{i=1}^{N+1} a_{ji}(z)x_i = 0\Big\},$$

where $[x_1 : x_2 : \cdots : x_{N+1}]$ is a homogeneous coordinate system of $\mathbb{P}^N(\mathbb{C})$, $a_{j1}(z), a_{j2}(z), \cdots, a_{jN+1}(z)$ are $N+1$ entire functions of $\mathbb{C}^n$ without common zeros. Denote by $\mathbf{a}_j = (a_{j1}, a_{j2}, \cdots, a_{jN+1}) : \mathbb{C}^n \to \mathbb{C}^{N+1} \setminus \{\mathbf{0}\}$ the non-zero moving vector associated with H_j. Let $a_j = \mathbb{P}(\mathbf{a}_j)$ and $a_j = [a_{j1} : a_{j2} : \cdots : a_{jN+1}]$, where a_j can be seen as meromorphic mappings of $\mathbb{C}^n$ into the dual space $\mathbb{P}^N(\mathbb{C})^*$. In this paper, we also call a_j the coefficient associated with H_j for the convenience of description. In particular, we can say H_j is a fixed hyperplane if $\mathbf{a}_j$ is a constant vector. Given a meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, we say that f and H_j are free if $(f, a_j) \not\equiv 0$, where

$$(f, a_j)(z) = f_1(z)a_{j1}(z) + f_2(z)a_{j2}(z) + \cdots + f_{N+1}(z)a_{jN+1}(z).$$

If f and H_j are free, i.e., (f, a_j) is a non-zero holomorphic function, the proximity function of f and a_j is defined by

$$m_{(f,a_j)}(r) = \int_{S_n(r)} \log \frac{\|f\|\|a_j\|}{|(f,a_j)|}\sigma_n - \int_{S_n(1)} \log \frac{\|f\|\|a_j\|}{|(f,a_j)|}\sigma_n, \quad r > 1,$$

where $\|a_j\| = (|a_{j1}|^2 + |a_{j2}|^2 + \cdots + |a_{jN+1}|^2)^{\frac{1}{2}}$.

The moving hyperplanes $H_1, H_2, \cdots, H_q$ are said to be located in general position if for any $l(\le N+1)$ non-zero moving vectors $\{\mathbf{a}_{j_t}\}_{t=1}^{l}$, we have $\mathbf{a}_{j_1}(z) \wedge \cdots \wedge \mathbf{a}_{j_2}(z) \wedge \mathbf{a}_{j_l}(z) \not\equiv 0$, where $1 \le j_1 < j_2 < \cdots < j_l \le q$ and the non-zero moving vectors $\{\mathbf{a}_j\}_{j=1}^{q}$ are associated with $\{H_j\}_{j=1}^{q}$. Here, we also say that the non-zero moving vectors $\{\mathbf{a}_j\}_{j=1}^{q}$ are located in general position. The non-zero vectors $\{\mathbf{a}_j\}_{j=1}^{q}$ are said to be located in special position if they are not located in general position. Take $1 \le l \le q$. Then $\{\mathbf{a}_j\}_{j=1}^{q}$ are said to be in l-special position if for each selection of integers $1 \le j_1 \le j_2 \le \cdots \le j_l \le q$ the vectors $\{\mathbf{a}_{j_t}\}_{t=1}^{l}$ are located in special position.

Denote by $\mathcal{M}$ the field of meromorphic functions on $\mathbb{C}^n$ and denote by $\mathcal{R}$ the smallest subfield of $\mathcal{M}$ which contains $\mathbb{C}$ and all $\frac{a_{jt_1}}{a_{jt_2}}$ with $a_{jt_2} \not\equiv 0$. We set $\operatorname{rank}_{\mathcal{R}}(f_i) := \operatorname{rank}\{f_{i1}, f_{i2}, \cdots, f_{iN+1}\}$ over $\mathcal{R}$ for all $1 \le i \le \lambda$. It is easy to see that the definition of $\operatorname{rank}_{\mathcal{R}}(f_i)$ does not depend on the choice of the reduced representation of f_i for all $1 \le i \le \lambda$.

Let $f(z)$ be a non-zero entire function on $\mathbb{C}^n$. For a point $z_0 \in \mathbb{C}^n$, we write $f(z) = \sum_{i=0}^{\infty} P_i(z - z_0)$, where the term $P_i(z)$ is homogeneous polynomial of degree i. We denote the zero-multiplicity of f at z_0 by $\nu_f(z_0) = \min\{i \mid P_i \ne 0\}$. Set $|\nu_f| := \operatorname{Supp} \nu_f$, which is a purely $(n-1)$-dimensional analytic subset or empty set.

Let $f(z)$ be a non-zero meromorphic function on $\mathbb{C}^n$. For each $z_0 \in \mathbb{C}^n$, we choose non-zero holomorphic functions f_1, f_2 on a neighborhood U of z_0 such that $f = \frac{f_1}{f_2}$ on U and

$\dim\{z \in \mathbb{C}^n \mid f_1(z) = f_2(z) = 0\} \le n-2$. We define $\nu_f = \nu_{f_1}, \nu_f^{\infty} = \nu_{f_2}$, which are independent of the choice of f_1, f_2.

For a divisor ν on $\mathbb{C}^n$ and letting m, M be positive integers or ∞, we define the following counting functions of ν by

$$\nu^M(z) = \min\{\nu(z), M\}, \quad \nu_m^M(z) = \begin{cases} 0, & \text{if } \nu(z) > m, \\ \nu^M(z), & \text{if } \nu(z) \le m, \end{cases}$$

$$n(t) = \begin{cases} \displaystyle\int_{|\nu| \cap B(t)} \nu(z) \upsilon_n(z), & \text{if } n \ge 2, \\ \displaystyle\sum_{|z| \le t} \nu(z), & \text{if } n = 1. \end{cases}$$

Similarly, we define $n^M(t)$, $n^M_{>m}(t)$ and $n^M_{\le k}(t)$. Define

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2m-1}} \mathrm{d}t, \quad 1 < r < \infty.$$

Similarly, we define $N(r, \nu^M)$, $N(r, \nu^M_{>m})$ and $N(r, \nu^M_{\le m})$ and denote them by $N^M(r, \nu)$, $N^M_{>m}(r, \nu)$ and $N^M_{\le m}(r, \nu)$, respectively. For a meromorphic function f on $\mathbb{C}^n$, we denote by

$$N_f(r) = N(r, \nu_f), \quad N_f^M(r) = N^M(r, \nu_f),$$
$$N^M_{f, \le m}(r) = N^M_{\le m}(r, \nu_f), \quad N^M_{f, >m}(r) = N^M_{>m}(r, \nu_f).$$

In addition, if $M = \infty$, we will omit the superscript M for brevity. On the other hand, we have the following Jensen's formula:

$$N_f(r) - N_{\frac{1}{f}}(r) = \int_{S_n(r)} \log |f| \sigma_n.$$

Theorem 2.1 (see [21, Theorem 2.1]) *Let M be a connected complex manifold of dimension n. Let A be a pure $(n-1)$-dimensional analytic subset of M. Let V be a complex vector space of dimension $N+1 (> 1)$. Let l and λ be integers with $1 \le l \le \lambda \le N+1$. Let $f_1, f_2, \cdots, f_\lambda$ be the λ meromorphic mappings of M into $\mathbb{P}(V)$. Assume that $f_1, f_2, \cdots, f_\lambda$ are in general position. Also assume $f_1, f_2, \cdots, f_\lambda$ are in l-special position on A. Then we have*

$$\mu_{f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda} \ge (\lambda - l + 1) \nu_A.$$

Theorem 2.2 (see [21]) *Assume that $1 \le \lambda \le N+1$, $f_i : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ $(1 \le i \le \lambda)$ are λ meromorphic mappings located in general position. Then*

$$N_{f_1 \wedge \cdots \wedge f_\lambda}(r) + m_{f_1 \wedge \cdots \wedge f_\lambda}(r) \le \sum_{1 \le i \le \lambda} T_{f_i}(r) + O(1).$$

3 Proof of Theorem 1.5

It suffices to prove Theorem 1.5 in the case of $\lambda \le N+1$. Assume that $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \not\equiv 0$. We denote by $\mu_{f_1 \wedge \cdots \wedge f_\lambda}$ the divisor associated with $f_1 \wedge \cdots \wedge f_\lambda$ and denote by $N_{f_1 \wedge \cdots \wedge f_\lambda}(r)$ the counting function associated with the divisor $\mu_{f_1 \wedge \cdots \wedge f_\lambda}$. Let $A = \bigcup\limits_{j=1}^{q} A_j$.

Take $z_0 \in A \setminus \Big(A_0 \cup \bigcup_{i=1}^{\lambda} I(f_i) \cup T_a\Big)$. Then there exists at least j_0 such that $z_0 \in A_{j_0}$, $1 \le j_0 \le q$. By the given condition, we know that for any increasing sequence $1 \le i_1 < i_2 < \cdots < i_{l_{j_0}} \le \lambda$, $f_{i_1}(z_0) \wedge f_{i_2}(z_0) \wedge \cdots \wedge f_{i_{l_{j_0}}}(z_0) = 0$. It follows from Theorem 2.1 that we have $\mu_{f_1\wedge\cdots\wedge f_\lambda}(z_0) \ge \lambda - l_{j_0} + 1$, which implies that

$$\sum_{j=1}^{q}(\lambda - l_j + 1)\nu^M_{(f_i,a_j),\le m_j}(z_0) \le dM\mu_{f_1\wedge\cdots\wedge f_\lambda}(z_0)$$

for all $z_0 \in A \setminus \Big(A_0 \cup \bigcup_{i=1}^{\lambda} I(f_i) \cup T_a\Big)$. For $z_0 \in T_a$, we have

$$\begin{aligned}
&\sum_{j=1}^{q}(\lambda - l_j + 1)\nu^M_{(f_i,a_j),\le m_j}(z_0)\\
&\le M\sum_{j=1}^{q}(\lambda - l_j + 1)\nu^1_{(f_i,a_j)}(z_0)\\
&\le M\sum_{j=1}^{q}(\lambda - l_j + 1)\sum_{\tau\in T[N+1,q]}\mu_{a_{\tau(1)}\wedge\cdots\wedge a_{\tau(N+1)}}(z_0).
\end{aligned}$$

Note that $\nu^M_{(f_i,a_j),\le m_j}(z) \equiv 0$ for all $\mathbb{C}^n \setminus A$. Therefore, for $z \notin A_0 \cup \bigcup_{i=1}^{\lambda} I(f_i)$, we can conclude that

$$\begin{aligned}
&\Big\| \sum_{j=1}^{q}(\lambda - l_j + 1)\nu^M_{(f_i,a_j),\le m_j}(z)\\
&\le dM\mu_{f_1\wedge\cdots\wedge f_\lambda}(z) + M\sum_{j=1}^{q}(\lambda - l_j + 1)\sum_{\tau\in T[N+1,q]}\mu_{a_{\tau(1)}\wedge\cdots\wedge a_{\tau(N+1)}}(z).
\end{aligned}$$

Combining with Theorem 2.2, we know

$$\begin{aligned}
&\Big\| \sum_{j=1}^{q}(\lambda - l_j + 1)N^M_{(f_i,a_j),\le m_j}(r)\\
&\le dMN_{f_1\wedge\cdots\wedge f_\lambda}(r) + M\sum_{j=1}^{q}(\lambda - l_j + 1)\sum_{\tau\in T[N+1,q]}N_{a_{\tau(1)}\wedge\cdots\wedge a_{\tau(N+1)}}(z)\\
&\le dM\sum_{i=1}^{\lambda}T(r,f_i) + o\Big(\max_{1\le i\le\lambda}T(r,f_i)\Big).
\end{aligned}$$

Thus, by summing up both sides of the above inequalities, we have

$$\Big\| \sum_{i=1}^{\lambda}\sum_{j=1}^{q}(\lambda - l_j + 1)N^M_{(f_i,a_j),\le m_j}(r) \le d\lambda M\sum_{i=1}^{\lambda}T(r,f_i) + o\Big(\max_{1\le i\le\lambda}T(r,f_i)\Big). \tag{3.1}$$

On the other hand, for any $1 \le i \le \lambda, 1 \le j \le q$,

$$N^M_{(f_i,a_j),\le m_j}(r) = N^M_{(f_i,a_j)}(r) - N^M_{(f_i,a_j),>m_j}(r)$$

$$
\begin{aligned}
&\geq N^M_{(f_i,a_j)}(r) - \frac{M}{m_j+1} N_{(f_i,a_j),>m_j}(r) \\
&= N^M_{(f_i,a_j)}(r) - \frac{M}{m_j+1} N_{(f_i,a_j)}(r) + \frac{M}{m_j+1} N^M_{(f_i,a_j),\leq m_j}(r) \\
&\geq N^M_{(f_i,a_j)}(r) - \frac{M}{m_j+1} T(r,f_i) + \frac{M}{m_j+1} N^M_{(f_i,a_j),\leq m_j}(r).
\end{aligned}
$$

Let $s_i + 1 = \operatorname{rank}_{\mathcal{R}} f_i$, from the above inequalities, we know

$$
\begin{aligned}
N^M_{(f_i,a_j),\leq m_j}(r) &\geq \frac{m_j+1}{m_j+1-M} N^M_{(f_i,a_j)}(r) - \frac{M}{m_j+1-M} T(r,f_i) \\
&\geq \frac{m_j+1}{m_j+1-M} N^1_{(f_i,a_j)}(r) - \frac{M}{m_j+1-M} T(r,f_i) \\
&\geq \frac{m_j+1}{s_i(m_j+1-M)} N^{s_i}_{(f_i,a_j)}(r) - \frac{M}{m_j+1-M} T(r,f_i).
\end{aligned}
$$

Furthermore, combining with (3.1), we obtain

$$
\begin{aligned}
\Big\| d\lambda M \sum_{i=1}^{\lambda} T(r,f_i) &\geq \sum_{i=1}^{\lambda} \sum_{j=1}^{q} (\lambda - l_j + 1) N^M_{(f_i,a_j),\leq m_j}(r) + o\Big(\max_{1\leq i\leq \lambda} T(r,f_i) \Big) \\
&\geq \sum_{i=1}^{\lambda} \sum_{j=1}^{q} \frac{(m_j+1)(\lambda - l_j + 1)}{s_i(m_j+1-M)} N^{s_i}_{(f_i,a_j)}(r) \\
&\quad - \sum_{i=1}^{\lambda} \sum_{j=1}^{q} \frac{M(\lambda - l_j + 1)}{m_j+1-M} T(r,f_i) + o\Big(\max_{1\leq i\leq \lambda} T(r,f_i) \Big).
\end{aligned}
$$

Next, we verify the fact that for $1 \leq i \leq \lambda$,

$$
(2N - s_i + 2) \max_{1\leq j\leq q} \frac{(m_j+1)(\lambda - l_j + 1)}{s_i(m_j+1-M)} < \sum_{j=1}^{q} \frac{(m_j+1)(\lambda - l_j + 1)}{s_i(m_j+1-M)}.
$$

By the given assumption that

$$
d\lambda M < \sum_{j=1}^{q} \frac{\lambda - l_j + 1}{m_j+1-M} \Big(\frac{m_j+1}{s(2N-s+2)} - M \Big),
$$

we know that

$$
M < \frac{m_j+1}{s(2N-s+2)}, \tag{3.2}
$$

$$
\lambda(2N - s_i + 2) < \sum_{j=1}^{q} \frac{(m_j+1)(\lambda - l_j + 1)}{s_i(m_j+1-M)}. \tag{3.3}
$$

On the other hand, we know $\frac{m_j+1}{m_j+1-M}$ is a decreasing function of m_j. Thus, by (3.2), we obtain

$$
\begin{aligned}
\max_{1\leq j\leq q} \frac{(m_j+1)(\lambda - l_j + 1)}{s_i(m_j+1-M)} &\leq \frac{Ms_i(2N-s_i+2)}{Ms_i(2N-s_i+2) - M} \cdot \frac{\lambda - l_j + 1}{s_i} \\
&\leq \frac{(2N-s_i+2)}{s_i(2N-s_i+2) - 1} \cdot (\lambda - 1)
\end{aligned}
$$

$$\begin{aligned}&\leq \frac{(2N-s_i+2)(\lambda-1)}{(2N-s_i+2)-1}\\&=\frac{\lambda(2N-s_i+2)}{2N-s_i+2}\cdot\frac{(\lambda-1)(2N-s_i+2)}{\lambda((2N-s_i+2)-1)}.\end{aligned}$$

It follows from $2N-s_i+2>\lambda$ that $\frac{2N-s_i+2}{(2N-s_i+2)-1}<\frac{\lambda}{\lambda-1}$ and

$$(2N-s_i+2)\max_{1\leq j\leq q}\frac{(m_j+1)(\lambda-l_j+1)}{s_i(m_j+1-M)}<\lambda(2N-s_i+2).$$

By (3.3), we have

$$(2N-s_i+2)\max_{1\leq j\leq q}\frac{(m_j+1)(\lambda-l_j+1)}{s_i(m_j+1-M)}<\sum_{j=1}^{q}\frac{(m_j+1)(\lambda-l_j+1)}{s_i(m_j+1-M)}.$$

Thus, we can apply Theorem 1.4 and estimate the above inequality as follows:

$$\begin{aligned}\Big\| d\lambda M\sum_{i=1}^{\lambda}T(r,f_i)\geq{}&\sum_{i=1}^{\lambda}\frac{\sum\limits_{j=1}^{q}\frac{(m_j+1)(\lambda-l_j+1)}{(m_j+1-M)}}{s_i(2N-s_i+2)}T(r,f_i)\\&-\sum_{i=1}^{\lambda}\sum_{j=1}^{q}\frac{M(\lambda-l_j+1)}{m_j+1-M}T(r,f_i)+o\Big(\max_{1\leq i\leq\lambda}T(r,f_i)\Big).\end{aligned}\tag{3.4}$$

Let $s=\max\limits_{1\leq i\leq\lambda}s_i$, and note that $s_i(2N-s_i+2)\leq s(2N-s+2)(s\leq N)$. Then (3.4) can be rewritten as

$$\| d\lambda MT(r)\geq\Big(\sum_{j=1}^{q}\frac{\lambda-l_j+1}{m_j+1-M}\Big(\frac{m_j+1}{s(2N-s+2)}-M+o(1)\Big)\Big)T(r),$$

which implies that $d\lambda M\geq\sum\limits_{j=1}^{q}\frac{\lambda-l_j+1}{m_j+1-M}\big(\frac{m_j+1}{s(2N-s+2)}-M\big)$, where $T(r)=\sum\limits_{i=1}^{\lambda}T(r,f_i)$. By the given assumption, it is a contradiction. Hence, $f_1\wedge f_2\wedge\cdots\wedge f_\lambda\equiv 0$. Thus, we complete the proof of Theorem 1.5.

4 Proof of Theorem 1.6

It suffices to prove Theorem 1.5 in the case of $\lambda\leq N+1$. Assume that $f_1\wedge f_2\wedge\cdots\wedge f_\lambda\not\equiv 0$. We can choose $N+1$ appropriate vectors from the set of $\{a_j\}_{j=1}^{q}$ that can be determined later. Here, we may assume that $\{a_j\}_{j=1}^{N+1}$ are suitable vectors that we want. For any $1\leq i\leq\lambda$, we define

$$\widetilde{f}_i(z)=((f_i,a_1)(z),(f_i,a_2)(z),\cdots,(f_i,a_{N+1})(z)).\tag{4.1}$$

Then, we have

$$\begin{pmatrix}\widetilde{f}_1(z)\\\widetilde{f}_2(z)\\\vdots\\\widetilde{f}_\lambda(z)\end{pmatrix}=\begin{pmatrix}(f_1,a_1)(z)&(f_1,a_2)(z)&\cdots&(f_1,a_{N+1})(z)\\(f_2,a_1)(z)&(f_2,a_2)(z)&\cdots&(f_2,a_{N+1})(z)\\\vdots&\vdots&\vdots&\vdots\\(f_\lambda,a_1)(z)&(f_\lambda,a_2)(z)&\cdots&(f_\lambda,a_{N+1})(z)\end{pmatrix}.\tag{4.2}$$

On the other hand, we have another expression for (4.2) as follows:

$$\begin{pmatrix} (f_1,a_1) & \cdots & (f_1,a_{N+1}) \\ (f_2,a_1) & \cdots & (f_2,a_{N+1}) \\ \vdots & & \vdots \\ (f_\lambda,a_1) & \cdots & (f_\lambda,a_{N+1}) \end{pmatrix}^{\mathrm{T}} = \Lambda_0 \begin{pmatrix} f_{11} & \cdots & f_{\lambda 1} \\ f_{12} & \cdots & f_{\lambda 2} \\ \vdots & & \vdots \\ f_{1N+1} & \cdots & f_{\lambda N+1} \end{pmatrix}, \tag{4.3}$$

where

$$\Lambda_0 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N+1} \\ a_{21} & a_{22} & \cdots & a_{2N+1} \\ \vdots & \vdots & & \vdots \\ a_{N+11} & a_{N+12} & \cdots & a_{N+1N+1} \end{pmatrix}.$$

By the given condition that $a_1, a_2, \cdots, a_q$ are located in general position, then we have

$$\operatorname{rank}\{\widetilde{f}_1, \widetilde{f}_2, \cdots, \widetilde{f}_\lambda\} = \operatorname{rank}\{f_1, f_2, \cdots, f_\lambda\}.$$

From $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \not\equiv 0$, we know that the rank of the matrix of the left of (4.2) is λ, i.e., $\widetilde{f}_1 \wedge \widetilde{f}_2 \wedge \cdots \wedge \widetilde{f}_\lambda \not\equiv 0$. Without loss of generality, we may assume that the matrix

$$\Lambda = \begin{pmatrix} (f_1,a_1) & \cdots & (f_\lambda,a_1) \\ (f_1,a_2) & \cdots & (f_\lambda,a_2) \\ \vdots & & \vdots \\ (f_1,a_\lambda) & \cdots & (f_\lambda,a_\lambda) \end{pmatrix} \tag{4.4}$$

is non-degenerate. Set $B = \bigcup\limits_{j=1}^{\lambda} A_j$ and $C = \bigcup\limits_{j=\lambda+1}^{q} A_j$. It is obvious that $A = B \cup C$. Take $z_0 \in A \setminus \big(A_0 \cup \bigcup\limits_{i=1}^{\lambda} I(f_i) \cup \{z \mid a_1(z) \wedge a_2(z) \wedge \cdots \wedge a_\lambda(z) = 0\}\big)$. Then $z_0 \in B$ or $z_0 \in C$.

First, we consider the case of $z_0 \in B$. Without loss of generality, we may assume $z_0 \in A_1$. Let S be an irreducible component of B containing z_0. Suppose that U is an open neighborhood of z_0 in $\mathbb{C}^n$ such that $U \cap B \subseteq S$. Select a holomorphic function $u(z)$ on an open neighborhood $U_0 \subseteq U$ of z_0 such that

$$\nu_u(z) = \begin{cases} \min\limits_{1 \le i \le \lambda} \{\nu_{(f_i,a_1),\le m_1}(z)\}, & \text{if } z \in S, \\ 0, & \text{if } z \notin S, \end{cases}$$

which yields that $(f_i,a_1)(z) = u(z)g_i(z)$ for $1 \le i \le \lambda$, where $\{g_i(z)\}_{i=1}^{\lambda}$ are holomorphic functions. On the other hand, we obtain a new matrix Λ_1 after removing the first row of (4.4) as follows:

$$\Lambda_1 = \begin{pmatrix} (f_1,a_2) & \cdots & (f_\lambda,a_2) \\ (f_1,a_3) & \cdots & (f_\lambda,a_3) \\ \vdots & & \vdots \\ (f_1,a_\lambda) & \cdots & (f_\lambda,a_\lambda) \end{pmatrix},$$

and we know that the rank of Λ_1 is less than λ strictly. Hence, we can find λ not all zero homomorphic functions $c_1(z), c_2(z), \cdots, c_\lambda(z)$ such that for all $2 \le j \le \lambda$,

$$c_1(z)(f_1,a_j)(z) + c_2(z)(f_2,a_j)(z) + \cdots + c_\lambda(z)(f_\lambda,a_j)(z) = 0.$$

Here, we may assume that the set of common zeros of $\{c_i(z)\}$ is an analytic subset of codimension no less than 2. Thus, there exists an index $1 \le i_0 \le \lambda$ such that $S \not\subseteq c_{i_0}^{-1}(0)$. We may assume that $i_0 = \lambda$. For any $1 \le i \le \lambda$, we define

$$\widehat{f}_i(z) = ((f_i, a_1)(z), (f_i, a_2)(z), \cdots, (f_i, a_\lambda)(z)). \tag{4.5}$$

Note that the definitions of $\widehat{f}_i$ in (4.5) are different from $\widetilde{f}_i$ in (4.1) for $1 \le i \le \lambda$. Then for all $z \in U_0 \cap S \setminus c_\lambda^{-1}(0)$, we obtain

$$\begin{aligned}\widehat{f}_1 \wedge \widehat{f}_2 \wedge \cdots \wedge \widehat{f}_\lambda &= \widehat{f}_1 \wedge \widehat{f}_2 \wedge \cdots \wedge \widehat{f}_{\lambda-1} \wedge \Big(\widehat{f}_\lambda + \sum_{i=1}^{\lambda-1} \frac{c_i}{c_\lambda} \widehat{f}_i\Big) \\ &= \widehat{f}_1 \wedge \widehat{f}_2 \wedge \cdots \wedge \widehat{f}_{\lambda-1} \wedge (W(z)u(z)) \\ &= u(z) \cdot \widehat{f}_1 \wedge \widehat{f}_2 \wedge \cdots \wedge \widehat{f}_{\lambda-1} \wedge W(z),\end{aligned} \tag{4.6}$$

where $W(z) = \big(\sum\limits_{i=1}^{\lambda} \frac{c_i}{c_\lambda} g_i, 0, \cdots, 0\big)$. From (4.1) and (4.5), it is easy to see that $\widehat{f}_i(z)$ has been regarded the "part" of $\widetilde{f}_i(z)$ for all $1 \le i \le \lambda$. That is to say, $\widehat{f}_1 \wedge \widehat{f}_2 \wedge \cdots \wedge \widehat{f}_\lambda \equiv 0$ if $\widetilde{f}_1 \wedge \widetilde{f}_2 \wedge \cdots \wedge \widetilde{f}_\lambda \equiv 0$ holds.

Next, we will show the fact that $\widetilde{f}_1 \wedge \widetilde{f}_2 \wedge \cdots \wedge \widetilde{f}_\lambda \equiv 0$ if $f_1 \wedge f_2 \wedge \cdots \wedge f_\lambda \equiv 0$ holds. In fact, if $\widetilde{f}_1 \wedge \widetilde{f}_2 \wedge \cdots \wedge \widetilde{f}_\lambda \not\equiv 0$, then there exists a point z_* such that $\widetilde{f}_1(z_*) \wedge \widetilde{f}_2(z_*) \wedge \cdots \wedge \widetilde{f}_\lambda(z_*) \ne 0$. On the other hand, by the given assumption that $a_1, a_2, \cdots, a_q$ are located in general position, we have

$$\begin{pmatrix} (f_1, a_1) & \cdots & (f_\lambda, a_1) \\ (f_1, a_2) & \cdots & (f_\lambda, a_2) \\ \vdots & & \vdots \\ (f_1, a_{N+1}) & \cdots & (f_\lambda, a_{N+1}) \end{pmatrix} = \Lambda_0 \begin{pmatrix} f_{11} & \cdots & f_{\lambda 1} \\ f_{12} & \cdots & f_{\lambda 2} \\ \vdots & & \vdots \\ f_{1N+1} & \cdots & f_{\lambda N+1} \end{pmatrix}, \tag{4.7}$$

where Λ_0 was defined as before. According to the condition that $\widetilde{f}_1(z_*) \wedge \widetilde{f}_2(z_*) \wedge \cdots \wedge \widetilde{f}_\lambda(z_*) \ne 0$, we know that the rank of matrix of the left of (4.7) is λ. Therefore, $f_1(z_*) \wedge f_2(z_*) \wedge \cdots \wedge f_\lambda(z_*) \ne 0$, which contradicts the assumption.

By the condition that there exist q integer numbers $l_1, l_2, \cdots, l_q$ satisfying $2 \le l_j \le \lambda$ $(1 \le j \le q)$ such that for any increasing sequence $1 \le i_1 < i_2 < \cdots < i_{l_j} \le \lambda$, $f_{i_1}(z) \wedge f_{i_2}(z) \wedge \cdots \wedge f_{i_{l_j}}(z) = 0$ for all $z \in A_j$ $(1 \le j \le q)$, and from the above discussion, we know $\widehat{f}_{i_1}(z) \wedge \widehat{f}_{1_2}(z) \wedge \cdots \wedge \widehat{f}_{i_{l_j}}(z) = 0$ for all $z \in A_j$ $(1 \le j \le q)$.

If $l_1 = \lambda$, then for (4.6) and all $z \in S$,

$$\mu_{\widehat{f}_1 \wedge \widehat{f}_2 \wedge \cdots \wedge \widehat{f}_{\lambda-1} \wedge W}(z) \ge \lambda - l_1 = 0.$$

If $l_1 \le \lambda - 1$, then the family $\{\widehat{f}_1, \widehat{f}_2, \cdots, \widehat{f}_{\lambda-1}, W\}$ is located in (l_1+1)-special position on S.

Applying Theorem 2.1, for all $z \in U_0 \cap S \setminus c_\lambda^{-1}(0)$, we have

$$\mu_{\widehat{f}_1 \wedge \widehat{f}_2 \wedge \cdots \wedge \widehat{f}_{\lambda-1} \wedge W}(z) \ge \lambda - l_1.$$

Through the discussion above, from (4.6), we know that for all $z \in U_0 \cap S \setminus c_\lambda^{-1}(0)$,

$$\mu_{\widehat{f}_1\wedge\widehat{f}_2\wedge\cdots\wedge\widehat{f}_\lambda}(z) \geq \nu_u(z) + \lambda - l_1 = \min_{1\leq i\leq\lambda}\{\nu^M_{(f_i,a_1),\leq m_1}(z)\} + \lambda - l_1.$$

By the assumption $\dim\{z \mid (f,a_{j_1})(z) = (f,a_{j_2})(z) = 0\} \leq n-2$ for any $1 \leq j_1 < j_2 \leq q$, we have

$$\begin{aligned}&\sum_{j=1}^{\lambda}\Big(\min_{1\leq i\leq\lambda}\{\nu_{(f_i,a_j),\leq m_j}(z_0)\} - \nu^1_{(f_1,a_j),\leq m_j}(z_0)\Big)\\ &+\sum_{j=1}^{q}(\lambda - l_1 + 1)\nu^1_{(f_1,a_j),\leq m_j}(z_0)\\ =&\min_{1\leq i\leq\lambda}\{\nu_{(f_i,a_1),\leq m_1}(z_0)\} + \lambda - l_1 \leq \mu_{\widehat{f}_1\wedge\widehat{f}_2\wedge\cdots\wedge\widehat{f}_\lambda}(z_0).\end{aligned}$$

For the case of $z_0 \in C$, without loss of generality, we may assume $z_0 \in A_{\lambda+1}$. Similarly, by the assumption, we know that the family $\{\widehat{f}_1, \widehat{f}_2, \cdots, \widehat{f}_\lambda\}$ is located in $l_{\lambda+1}$-special position on each irreducible component of A containing z_0. Applying Theorem 2.1, we have

$$\mu_{\widehat{f}_1\wedge\widehat{f}_2\wedge\cdots\wedge\widehat{f}_\lambda}(z) \geq \lambda - l_{\lambda+1} + 1$$

holds on each irreducible component of A containing z_0. Thus, by the assumption $\dim\{z \mid (f,a_{j_1})(z) = (f,a_{j_2})(z) = 0\} \leq n-2$ for any $1 \leq j_1 < j_2 \leq q$, we have

$$\begin{aligned}&\sum_{j=1}^{\lambda}\Big(\min_{1\leq i\leq\lambda}\{\nu_{(f_i,a_j),\leq m_j}(z_0)\} - \nu^1_{(f_1,a_j),\leq m_j}(z_0)\Big)\\ &+\sum_{j=1}^{q}(\lambda - l_{\lambda+1} + 1)\nu^1_{(f_1,a_j),\leq m_j}(z_0)\\ =&\ \lambda - l_{\lambda+1} + 1 \leq \mu_{\widehat{f}_1\wedge\widehat{f}_2\wedge\cdots\wedge\widehat{f}_\lambda}(z_0).\end{aligned}$$

Through the discussion above, we know that

$$\begin{aligned}&\sum_{j=1}^{\lambda}\Big(\min_{1\leq i\leq\lambda}\{\nu_{(f_i,a_j),\leq m_j}(z)\} - \nu^1_{(f_1,a_j),\leq m_j}(z)\Big)\\ &+\sum_{j=1}^{q}(\lambda - l_j + 1)\nu^1_{(f_1,a_j),\leq m_j}(z)\\ \leq&\ \mu_{\widehat{f}_1\wedge\widehat{f}_2\wedge\cdots\wedge\widehat{f}_\lambda}(z)\end{aligned}\tag{4.8}$$

holds for all $z \in \mathbb{C}^n \setminus \Big(A_0 \cup \bigcup_{i=1}^{\lambda} I(f_i) \cup \{z \mid a_1(z)\wedge a_2(z)\wedge\cdots\wedge a_\lambda(z) = 0\}\Big)$.

Noting the fact that

$$\min_{1\leq i\leq\lambda}\{\nu_{(f_i,a_j),\leq m_j}(z)\} \geq \sum_{i=1}^{\lambda}\nu^s_{(f_i,a_j),\leq m_j}(z) - (\lambda-1)s\nu^1_{(f_i,a_j),\leq m_j}(z),$$

then by applying Theorem 2.2 and (4.8), we have

$$
\begin{aligned}
&\sum_{j=1}^{\lambda}\Big(\sum_{i=1}^{\lambda}N^{s}_{(f_i,a_j),\le m_j}(r)-((\lambda-1)s+1)N^{1}_{(f_i,a_j),\le m_j}(r)\Big)\\
&\quad+\sum_{j=1}^{q}(\lambda-l_j+1)N^{1}_{(f_1,a_j),\le m_j}(r)\\
&\le\sum_{i=1}^{\lambda}T(r,f_i)+o\Big(\max_{1\le i\le\lambda}T(r,f_i)\Big).
\end{aligned}\tag{4.9}
$$

For each $1\le j\le q$, we set

$$N_j(r)=\sum_{i=1}^{\lambda}N^{s}_{(f_i,a_j),\le m_j}(r)-((\lambda-1)s+1)N^{1}_{(f_i,a_j),\le m_j}(r).\tag{4.10}$$

For each permutation $I=(j_1,j_2,\cdots,j_q)$ of $(1,2,\cdots,q)$, we define

$$E_I:=\{r\in[0,+\infty)\mid N_{j_1}(r)\ge N_{j_2}(r)\ge\cdots\ge N_{j_q}(r)\}.$$

And we have $\bigcup_I E_I=[0,+\infty)$. Noting that the number of permutations I of $(1,2,\cdots,q)$ is finite, then there exists a permutation I_0 such that the measure of E_{I_0} is infinite. Here, we can assume $I_0=(1,2,\cdots,q)$. Thus, $N_1(r)\ge N_2(r)\ge\cdots\ge N_q(r)$ on $r\in E_{I_0}$. At the beginning of the proof, we select the $N+1$ vectors $\{a_j\}_{j=1}^{N+1}$ as we need.

Set $T(r)=\sum_{i=1}^{\lambda}T(r,f_i)$. By the definition of $N_j(r)$ in (4.10) and the selection of permutation I_0 for (4.9), we have

$$
\begin{aligned}
T(r)&\ge\sum_{j=1}^{\lambda}N_j(r)+\sum_{j=1}^{q}(\lambda-l_j+1)N^{1}_{(f_1,a_j),\le m_j}(r)+o(T(r))\\
&\ge\frac{\lambda}{q}\sum_{j=1}^{q}N_j(r)+\sum_{j=1}^{q}(\lambda-l_j+1)N^{1}_{(f_1,a_j),\le m_j}(r)+o(T(r))\\
&=\sum_{j=1}^{q}\Big\{\frac{\lambda}{q}\sum_{i=1}^{\lambda}N^{s}_{(f_i,a_j),\le m_j}(r)+\Delta_jN^{1}_{(f_1,a_j),\le m_j}(r)\Big\}+o(T(r)),
\end{aligned}
$$

where $\Delta_j=(\lambda-l_j+1)-\frac{\lambda((\lambda-1)s+1)}{q}$. By the given condition, we know

$$
\begin{aligned}
&q(\lambda-l_j+1)\ge\lambda((\lambda-1)s+1), && 1\le j\le q,\\
&\nu^{M}_{(f_1,a_j),\le m_j}=\nu^{M}_{(f_2,a_j),\le m_j}=\cdots=\nu^{M}_{(f_\lambda,a_j),\le m_j}, && 1\le j\le q.
\end{aligned}
$$

Hence, we can get the further result as follows:

$$
\begin{aligned}
T(r)&\ge\sum_{i=1}^{\lambda}\sum_{j=1}^{q}\Big\{\frac{\lambda}{q}N^{s}_{(f_i,a_j),\le m_j}(r)+\frac{\Delta_j}{\lambda}N^{1}_{(f_i,a_j),\le m_j}(r)\Big\}+o(T(r))\\
&\ge\sum_{i=1}^{\lambda}\sum_{j=1}^{q}\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big)N^{s}_{(f_i,a_j),\le m_j}(r)+o(T(r)).
\end{aligned}
$$

In a similar way as the proof of Theorem 1.5, we can obtain

$$\begin{aligned}T(r) &\geq \sum_{i=1}^{\lambda}\sum_{j=1}^{q}\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big)N^s_{(f_i,a_j),\leq m_j}(r)+o(T(r))\\ &\geq \sum_{i=1}^{\lambda}\sum_{j=1}^{q}\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big)\Big(\frac{(m_j+1)N^s_{(f_i,a_j)}(r)-sT(r,f_i)}{m_j+1-s}\Big)+o(T(r)).\end{aligned}$$

Before applying Theorem 1.4, we verify the fact that

$$(2N-s+2)\max_{1\leq j\leq q}\Big\{\frac{m_j+1}{m_j+1-s}\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big)\Big\}<\sum_{j=1}^{q}\frac{m_j+1}{m_j+1-s}\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big).$$

In addition, we can see that $\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\leq\frac{\lambda}{\lambda+1}$. In fact,

$$\begin{aligned}\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}&\leq\frac{\lambda}{q}+\frac{1}{\lambda s}\Big(\lambda-1-\frac{\lambda((\lambda-1)s+1)}{q}\Big)\\ &=\frac{1}{q}\Big(1-\frac{1}{s}\Big)+\frac{1}{s}\Big(1-\frac{1}{\lambda}\Big).\end{aligned}$$

For $s=1$, $\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\leq\frac{\lambda}{\lambda+1}$ holds. For the case of $s\geq 2$, we have

$$\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}<\frac{1}{q}+\frac{1}{s}\Big(1-\frac{1}{\lambda}\Big)\leq\frac{1}{q}+\frac{1}{2}. \tag{4.11}$$

By the given assumption we know that $q\geq\frac{\lambda((\lambda-1)s+1)}{\lambda-1}$. Hence,

$$\frac{1}{q}\leq\frac{\lambda-1}{\lambda((\lambda-1)s+1)}l\leq\frac{\lambda-1}{\lambda(2(\lambda-1)+1)}\leq\frac{\lambda-1}{3\lambda}\leq\frac{\lambda-1}{2(\lambda+1)}=\frac{1}{2}-\frac{1}{\lambda+1}.$$

Therefore, by (4.11), we know that $\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\leq\frac{\lambda}{\lambda+1}$ holds. By the given assumption, we have

$$1<\sum_{j=1}^{q}\frac{q(\lambda-l_j+1)+\lambda(s-1)}{\lambda sq(m_j+1-s)}\Big(\frac{m_j+1}{2N-s+2}-s\Big).$$

Thus, we know that

$$(2N-s+2)s<m_j+1, \tag{4.12}$$

$$\begin{aligned}&(2N-s+2)+(2N-s+2)\sum_{j=1}^{q}\frac{s}{m_j+1-s}\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big)\\ &<\sum_{j=1}^{q}\frac{m_j+1}{m_j+1-s}\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big).\end{aligned} \tag{4.13}$$

By (4.12) and the fact $\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\leq\frac{\lambda}{\lambda+1}$, we have

$$\begin{aligned}&(2N-s+2)\max_{1\leq j\leq q}\Big\{\frac{m_j+1}{m_j+1-s}\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big)\Big\}\\ &\leq(2N-s+2)\frac{2N-s+2}{2N-s+1}\max_{1\leq j\leq q}\Big\{\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big)\Big\}\end{aligned}$$

$$\leq (2N-s+2)\frac{2N-s+2}{2N-s+1}\frac{\lambda}{\lambda+1}.$$

Note that $f(x)=\frac{x}{x-a}$ is a decreasing function of x ($>a$). Since $2N-s+2>\lambda+1$, by (4.13), we have

$$(2N-s+2)\max_{1\leq j\leq q}\Big\{\frac{m_j+1}{m_j+1-s}\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big)\Big\}<\sum_{j=1}^{q}\frac{m_j+1}{m_j+1-s}\Big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\Big).$$

Then by applying Theorem 1.4 and through some simple computation, we have

$$T(r)\geq\Big(\sum_{j=1}^{q}\frac{\big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\big)\big(\frac{m_j+1}{2N-s+2}-s\big)}{m_j+1-s}\Big)T(r)+o(T(r)),$$

which yields that

$$\begin{aligned}1&\geq\sum_{j=1}^{q}\frac{\big(\frac{\lambda}{q}+\frac{\Delta_j}{\lambda s}\big)\big(\frac{m_j+1}{2N-s+2}-s\big)}{m_j+1-s}\\&=\sum_{j=1}^{q}\frac{q(\lambda-l_j+1)+\lambda(s-1)}{\lambda sq(m_j+1-s)}\Big(\frac{m_j+1}{2N-s+2}-s\Big).\end{aligned}$$

It is a contradiction. Thus, we complete the proof of Theorem 1.6.

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