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# Existence and Uniqueness of Viscosity Solutions for Nonlinear Variational Inequalities Associated with Mixed Control

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**Abstract** The author investigates the nonlinear parabolic variational inequality derived from the mixed stochastic control problem on finite horizon. Supposing that some sufficiently smooth conditions hold, by the dynamic programming principle, the author builds the Hamilton-Jacobi-Bellman (HJB for short) variational inequality for the value function. The author also proves that the value function is the unique viscosity solution of the HJB variational inequality and gives an application to the quasi-variational inequality.

**Keywords** Optimal stopping, Mixed control, Variational inequality, Viscosity solution
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## 1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a given complete filtered probability space, where $\{\mathcal{F}_t; t \in [0, T], T < \infty\}$ satisfies the usual conditions. Let $W_t$ be an $n$-dimensional standard Brownian motion. Let $\mathcal{S}[0, T]$ be the set of all $\{\mathcal{F}_t\}_{0\leq t\leq T}$-stopping times taking values in $[0, T]$. For any $\tau_1, \tau_2 \in \mathcal{S}[0, T]$ with $\tau_1 \leq \tau_2$ almost surely, $\mathbb{P}\{\tau_1 < \tau_2\} > 0$.

For any $s \in [0, T)$ and $x \in \mathbb{R}^n$, consider the following stochastic differential equation (SDE for short):

$$\begin{cases} \mathrm{d}X_t = [b(t, X_t) + C_t]\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad t \in [s, T], \\ X_s = x, \end{cases} \tag{1.1}$$

where the mappings $b(t, x)$ and $\sigma(t, x)$ are two Lipschitz continuous functions and take value in $\mathbb{R}^n$ and $\mathbb{R}^n \otimes \mathbb{R}^n$, respectively. Let $\mathcal{A}$ denote the class of all $n$-dimensional $\mathcal{F}_t$-progressively measurable processes $C = (C_t)$ and there exists an $M > 0$ such that $\mathbb{E}\big[\int_0^T |C_s|^2\mathrm{d}s\big] < M$. Thus, SDE (1.1) has a unique strong solution $X_\cdot := X(\cdot; s, x)$. The cost functional is given by

$$\begin{aligned} &J(C(\cdot), \tau; s, x) \\ &= \mathbb{E}\Big[\int_s^\tau \mathrm{e}^{-\alpha(t-s)}\{f(t, X_t) + a^{-1}|C_t|^2\}\mathrm{d}t + \mathrm{e}^{-\alpha(\tau-s)}g(\tau, X_\tau)\Big], \quad \tau \in \mathcal{S}[s, T], \end{aligned} \tag{1.2}$$

where $a > 0$, mappings $f, g : [0, T] \times \mathbb{R}^n \to [0, \infty)$ are non-negative and satisfy proper conditions and $\alpha > 0$ is a discount rate. In this case, the value function $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is defined as

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follows:

$$V(s,x) = \inf_{(C(\cdot),\tau)\in\mathcal{A}\times\mathcal{S}[s,T]} J(C(\cdot),\tau;s,x). \tag{1.3}$$

We call $\overline{\tau} \in \mathcal{S}[s,T]$ is an optimal stopping time if the cost functional $J$ defined by (1.2) has attained its infimum value, and the smallest one is referred to as the smallest optimal stopping time.

The above optimal stopping problem over a finite time horizon can be reduced to the following variational inequality:

$$\begin{aligned}
&\frac{\partial V}{\partial t} - LV + f - \frac{a}{4}|DV|^2 \geq 0 \quad \text{in } [0,T)\times\mathbb{R}^n,\\
&V \leq g \quad \text{in } [0,T]\times\mathbb{R}^n,\\
&\Big(\frac{\partial V}{\partial t} - LV + f - \frac{a}{4}|DV|^2\Big)(V-g)^- = 0 \quad \text{in } [0,T)\times\mathbb{R}^n,\\
&V(T,x) = g(T,x) \quad \text{on } \mathbb{R}^n.
\end{aligned} \tag{1.4}$$

Here $L = L_0 + \alpha$, $L_0$ denotes the second order differential operator

$$L_0 = -\frac{1}{2}\mathrm{tr}(\sigma\sigma^* D^2) - bD.$$

Here $|\cdot|$ is the Euclidean norm, $\sigma^*$ is the transpose of $\sigma$, $x^- := \max(-x,0)$, $D := \big(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\big)$. The purpose of this paper is to prove the existence and uniqueness of the solution of the parabolic variational inequality (1.4) and then to characterize the solution $V$.

The theoretical results in the control of discrete stopping time are originated from Krylov [23], El Karoui [7], Bensoussan and Lions [3] and Morimoto [21–22]. Many scholars have introduced the method of variational inequality in order to solve the optimal stopping time problems (see [4, 9, 11–14]). Some classic results for variational inequalities can be found in [2, 10, 20]. Morimoto [24] investigated the elliptic variational inequality derived from the mixed stochastic control; under the without uniform ellipticity condition, he proved the existence and uniqueness of the viscosity solution for the elliptic variational inequality. In many scientific fields such as engineering and finance (see Øksendal [25], Shiryaev [27], Karazas and Shreve [16]), there is an optimal stopping time problem for a finite or an infinite time horizon for Itô diffusion processes. Pham [26] investigated the state equation driven by a combination of the Brownian motion and the compensated jump martingale random measure. He proved that the value function is continuous and is a viscosity solution of the integrodifferential variational inequality arising from the associated dynamic programming. Goreac and Serea [15] investigated that the value functions had been introduced via linear optimization problems on appropriate sets of probability measures. Both the lower and upper semicontinuous cases were considered. Then they proved that the value function is a generalized viscosity solution of the associated HJB system, respectively, of some variational inequality. Because the control state space of the above two papers is a compact metric separable space, the second-order differential operator of the Hamiltonian function in [26] is only linear growth with respect to gradient. However, we study variational inequalities that have the square growth of the gradient. Because the control state space of [15, 26] is a compact metric separable space, their second-order differential operator of the Hamiltonian functions are only linear growth with respect to gradient. But for the previous paper, we study the square of the gradient $D_xV$. We do not find appropriate functions ($b$ and $\sigma$) to rewrite (1.3) as in [26]. Moreover, our model and results have potential applications in contingent claims. We also note the use of some pure probability methods to study the optimal stopping time problems for general continuous-time stochastic processes. Without using the dynamic programming principle, the optimal stopping time is characterized by the Snell envelope, the super martingale, and so on. Assuming that the stochastic processes (see [17–19]) is not using the dynamic programming principle, there is no natural HJB equation.

The rest of the paper is organized as follows. In Section 2 we derive the backward partial differential variational inequality (BPDVI for short). In addition, we discuss the existence and uniqueness of the viscosity solution $V_\epsilon$ of the penalized problem. In Section 3 the definition of the viscosity solutions to BPDVI (1.4) is given, and we prove that $V_\epsilon$ converges to a unique viscosity solution of the variational inequality (1.4). In Section 4 we investigate that the quasi-variational inequality is derived from mixed impulse control.

## 2 Penalized Problem

In order to simplify the whole paper we take $a = 1$. Let $\mathcal{C} = \mathcal{C}([0,T]\times\mathbb{R}^n)$ denote the Banach space which consists of all bounded uniformly continuous functions $h$ on $[0,T]\times\mathbb{R}^n$ with norm $\|h\| = \sup\limits_{(t,x)\in[0,T]\times\mathbb{R}^n} |h(t,x)|$, and $\mathcal{C}_+ = \{h\in\mathcal{C} : h \ge 0\}$.

Throughout the paper, we need the following three assumptions.

($\mathrm{H}_1$) Mappings $b:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ and $\sigma:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n\otimes\mathbb{R}^n$ are Lipschitz contiuous and they satisfy the following condition: There exists a $\kappa>0$ such that

$$|b(t,x)-b(s,y)| + |\sigma(t,x)-\sigma(s,y)| \le \kappa[|x-y|+|t-s|] \quad \text{a.e. } t,s\in[0,T],\ \forall x,y\in\mathbb{R}^n.$$

($\mathrm{H}_2$)

$$\begin{aligned}\alpha > \nu := \sup\Big\{\mathrm{tr}\Big[&\frac{(\sigma(t,x)-\sigma(t,y))(\sigma(t,x)-\sigma(t,y))^*}{|x-y|^2}\Big]\\ &+\frac{2\langle x-y, b(t,x)-b(t,y)\rangle}{|x-y|^2} : t\in[0,T];\ x,y\in\mathbb{R}^n,\ x\ne y\Big\}.\end{aligned}$$

($\mathrm{H}_3$) $f,\ g\in\mathcal{C}_+$.

We will now derive the variational inequality (1.4) as follows. Firstly, for any $\tau\in\mathcal{S}[s,T]$, we get

$$V(s,x)\le J(C(\cdot),\tau;s,x) := \mathbb{E}\Big[\int_s^\tau \mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t + \mathrm{e}^{-\alpha(\tau-s)}g(\tau,X_\tau)\Big]. \tag{2.1}$$

Taking $\tau = s$, we have $V(s,x)\le g(s,x)$ for all $(s,x)\in[0,T]\times\mathbb{R}^n$. Secondly, $V$ is supposed to be smooth. By the dynamic programming principle for (2.1), we get

$$V(s,x)\le\mathbb{E}\Big[\int_s^{s+\delta}\mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha\delta}V(s+\delta,X_{s+\delta})\Big],\quad\forall\delta\ge 0.$$

From this principle, we have

$$\begin{aligned}&\lim_{\delta\downarrow 0}\frac{\mathbb{E}\left[\mathrm{e}^{-\alpha\delta}V(s+\delta,X_{s+\delta})-V(s,x)\right]}{\delta}\\ &=\frac{\partial V}{\partial s}(s,x)-LV(s,x)+\langle C_s, DV(s,x)\rangle\ge -f(s,x)-|C_s|^2\end{aligned}$$

for all $(s, x) \in [0, T] \times \mathbb{R}^n$.

Suppose that $(s, x) \in [0, T] \times \mathbb{R}^n$ is such that

$$V(s, x) < g(s, x), \tag{2.2}$$

and let $\tau_n \in \mathcal{S}[s, T]$ and $C^n(\cdot) \in \mathcal{A}$ be such that

$$\lim_{n \to +\infty} J(C^n(\cdot), \tau_n; s, x) = V(s, x). \tag{2.3}$$

We claim that there exists $s_0 > s$ such that

$$\tau_n \geq s_0 > s \tag{2.4}$$

for sufficiently large $n$. To see this, set $\delta_n = J(C^n(\cdot), \tau_n; s, x) - V(s, x)$. By assumption ($\mathrm{H}_3$), we have

$$\begin{aligned} V(s, x) + \delta_n \geq -\|f\| \int_s^{\tau_n} \mathrm{e}^{-\alpha(t-s)} \mathrm{d}t - \mathbb{E} \int_s^{\tau_n} \mathrm{e}^{-\alpha(t-s)} |C_t^0|^2 \mathrm{d}t \\ + g(\tau_n, X_{\tau_n}) \mathrm{e}^{-\alpha(\tau_n - s)}. \end{aligned}$$

If for some subsequence $\tau_n \to s$, the preceding would imply

$$V(s, x) \geq g(s, x),$$

a contradiction to (2.2), thus (2.4) holds. Note that for $u \in [s, s_0]$,

$$\begin{aligned} J(C^n(\cdot), \tau_n; s, x) = \mathbb{E}\Big[ \int_s^u \mathrm{e}^{-\alpha(t-s)} \{f(t, X_t) + |C_t^n|^2\} \mathrm{d}t \\ + \mathrm{e}^{-\alpha(u-s)} J(C^n(\cdot), \tau_n; u, X_u^{s,x}) \Big]. \end{aligned}$$

Since $u \in [s, s_0]$, according to definition of $V$, we get

$$\begin{aligned} J(C^n(\cdot), \tau_n; s, x) \geq \inf_{C(\cdot) \in \mathcal{A}_{[s,u]}} \mathbb{E}\Big[ \int_s^u \mathrm{e}^{-\alpha(t-s)} \{f(t, X_t) + |C_t|^2\} \mathrm{d}t \\ + \mathrm{e}^{-\alpha(u-s)} V(u, X_u^{s,x}) \Big]. \end{aligned}$$

Letting $n \to +\infty$, by (2.3) we have

$$V(s, x) \geq \inf_{C(\cdot) \in \mathcal{A}_{[s,u]}} \mathbb{E}\Big[ \int_s^u \mathrm{e}^{-\alpha(t-s)} \{f(t, X_t) + |C_t|^2\} \mathrm{d}t + \mathrm{e}^{-\alpha(u-s)} V(u, X_u^{s,x}) \Big].$$

If $V(s, x) < g(s, x)$ holds, we have

$$\Big( \frac{\partial V}{\partial t} - LV + f - \frac{1}{4}|DV|^2 \Big)(V - g)^- = 0,$$

which implies (1.4).

We hope that the solution of the variational inequality (1.4) can be approximated by the solution of the following penalized equation: For $\epsilon \in (0, 1]$,

$$\begin{cases} \dfrac{\partial V}{\partial t} - LV + f - \dfrac{1}{4}|DV|^2 - \dfrac{1}{\epsilon}(V - g)^+ = 0, & (t, x) \in [0, T) \times \mathbb{R}^n, \\ V(T, x) = g(T, x), & x \in \mathbb{R}^n, \end{cases} \tag{2.5}$$

which is equivalent to

$$\begin{cases}\dfrac{\partial V}{\partial t}-\Big(\alpha+\dfrac{1}{\epsilon}\Big)V-L_0V+f-\dfrac{1}{4}|DV|^2+\dfrac{1}{\epsilon}(V\wedge g)=0, \quad (t,x)\in[0,T)\times\mathbb{R}^n,\\ V(T,x)=g(T,x), \quad x\in\mathbb{R}^n.\end{cases} \tag{2.6}$$

Thus, if the solution $V_\epsilon$ of (2.6) exists, then it satisfies the following integral equation: For $\epsilon\in(0,1]$,

$$\begin{aligned} V(s,x)=\inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^T & \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\Big\{\Big(f+\frac{1}{\epsilon}(V\wedge g)\Big)(t,X_t)+|C_t|^2\Big\}\mathrm{d}t\\ &+\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}g(T,X_T)\Big].\end{aligned}\tag{2.7}$$

## 2.1 Existence

In this subsection, we prove the existence of the solution of (2.7) and prove that the solution of integral equation (2.7) is a viscosity solution of (2.6).

**Theorem 2.1** *Let the assumptions* $(\mathrm{H}_1),(\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *hold. Then* (2.7) *has a unique solution* $V\in\mathcal{C}_+$.

**Proof** It is clear that $\mathcal{C}_+$ is a closed subset of $\mathcal{C}$. We define

$$\begin{aligned}\mathbb{T}w(s,x):=\inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^T & \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\Big\{\Big(f+\frac{1}{\epsilon}(w\wedge g)\Big)(t,X_t)+|C_t|^2\Big\}\mathrm{d}t\\ &+\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}g(T,X_T)\Big] \quad \text{for } w\in\mathcal{C}.\end{aligned}\tag{2.8}$$

We will prove

$$\mathbb{T}:\mathcal{C}_+\to\mathcal{C}_+. \tag{2.9}$$

We can calculate that

$$\begin{aligned}0\le\mathbb{T}w(s,x)&\le\mathbb{E}\Big[\int_s^T\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\Big\{\Big(f+\frac{1}{\epsilon}g\Big)(t,\overline{X}_t)\Big\}\mathrm{d}t+\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}g(T,\overline{X}_T)\Big]\\ &\le\frac{\epsilon}{\alpha\varepsilon+1}\Big[\|f\|+\Big(\alpha+\frac{1}{\epsilon}\Big)\|g\|\Big], \quad w\in\mathcal{C}_+\end{aligned}$$

for the correspondence $\overline{X}_t$ to $C_t=0$. Moreover, without loss of generality, let $r<s$ hold and from (2.8) we have

$$\begin{aligned}&|\mathbb{T}w(s,x)-\mathbb{T}w(r,y)|\\ \le&\sup_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^T\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\Big\{|f(t,X_t)-f(t,Y_t)|+\frac{1}{\epsilon}(|w(t,X_t)-w(t,Y_t)|\\ &+|g(t,X_t)-g(t,Y_t)|)\Big\}\mathrm{d}t+\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}(g(T,X_T)-g(T,Y_T))\Big]\\ &+\inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_r^s\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-r)}\Big\{\Big(f+\frac{1}{\epsilon}(w\wedge g)\Big)(t,X_t)+|C_t|^2\Big\}\mathrm{d}t\Big]\\ \le&\,\mathrm{I}_f+\frac{1}{\epsilon}(\mathrm{I}_w+\mathrm{I}_g)+\sup_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}(g(T,X_T)-g(T,Y_T))\Big]\\ &+\mathbb{E}\Big[\int_r^s\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-r)}\Big\{\Big(f+\frac{1}{\epsilon}(w\wedge g)\Big)(t,X_t)+|C_t|^2\Big\}\mathrm{d}t\Big].\end{aligned}$$

Here $Y_t$ is the solution of (1.1) with initial value $Y_r = y$, and

$$\mathrm{I}_h = \sup_{C(\cdot)\in\mathcal{A}} \mathbb{E}\Big[\int_s^T \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}|h(t,X_t)-h(t,Y_t)|\mathrm{d}t\Big] \quad \text{for } h\in\mathcal{C}.$$

By ($\mathrm{H}_2$) we can select $\eta>0$ such that $-\alpha+\nu+\eta<0$. We apply Itô's formula to the function $|X_t-Y_t|^2\mathrm{e}^{(-\alpha+\eta)(t-s)}$, then we get

$$\mathbb{E}[|X_t-Y_t|^2\mathrm{e}^{(-\alpha+\eta)(t-s)}] \le \mathbb{E}|x-Y_s|^2. \tag{2.10}$$

By a simple calculation, we have $\mathbb{E}|x-Y_s|^2 \le C_{2,T}(|x-y|^2+|r-s|+\mathbb{E}\int_r^s |C_t|^2\mathrm{d}t)$. Furthermore, it is clear that there exists a constant $C_{\zeta,h}>0$, for any $\zeta>0$, such that

$$|h(t,x)-h(t,y)| \le \zeta + C_{\zeta,h}|x-y|, \quad t\in[0,T],\ x,y\in\mathbb{R}^n. \tag{2.11}$$

Then

$$\begin{aligned}\mathrm{I}_h &\le \sup_{C(\cdot)\in\mathcal{A}} \mathbb{E}\Big[\int_s^T \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}(\zeta+C_{\zeta,h}|X_t-Y_t|)\mathrm{d}t\Big] \\ &\le \frac{\zeta}{\alpha} + \frac{2\sqrt{C_{2,T}}C_{\zeta,h}}{\alpha+\eta}\Big(|x-y|+\sqrt{|r-s|}+\Big(\mathbb{E}\int_r^s |C_t|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big)\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&\sup_{C(\cdot)\in\mathcal{A}} \mathbb{E}[\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}(g(T,X_T)-g(T,Y_T))] \\ &\le \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})}\zeta + \sqrt{C_{2,T}}C_{\zeta,g}\mathrm{e}^{-\frac{\alpha+\eta+\frac{2}{\epsilon}}{2}(T-s)}\Big(|x-y|+\sqrt{|r-s|}+\Big(\mathbb{E}\int_r^s |C_t|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big).\end{aligned}$$

Therefore, letting $\delta\to 0$ and $\zeta\to 0$, we have

$$\begin{aligned}&\lim_{\delta\to 0}\sup_{|x-y|+|r-s|<\delta}\mathrm{I}_h = 0, \quad \lim_{\delta\to 0}\sup_{|x-y|+|r-s|<\delta}\mathbb{E}[\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}(g(T,X_T)-g(T,Y_T))] = 0, \\ &\lim_{\delta\to 0}\sup_{|r-s|<\delta}\mathbb{E}\Big[\int_r^s \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-r)}\Big\{\Big(f+\frac{1}{\epsilon}(w\wedge g)\Big)(t,X_t)\Big\}\mathrm{d}t\Big] = 0,\end{aligned} \tag{2.13}$$

and thus

$$\lim_{\delta\to 0}\sup_{|x-y|+|s-r|<\delta}|Tw(s,x)-Tw(r,y)| = 0, \tag{2.14}$$

which denotes (2.9).

Now, by (2.8) we obtain

$$\begin{aligned}|\mathbb{T}w_1(s,x)-\mathbb{T}w_2(s,x)| &\le \sup_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^T \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\Big(\frac{1}{\epsilon}|(w_1\wedge g-w_2\wedge g)(t,X_t)|\Big)\mathrm{d}t\Big] \\ &\le \sup_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^T \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\Big(\frac{1}{\epsilon}|w_1(t,X_t)-w_2(t,X_t)|\Big)\mathrm{d}t\Big] \\ &\le \frac{1}{\alpha\epsilon+1}\|w_1-w_2\|.\end{aligned}$$

Thus $\mathbb{T}$ is a contraction mapping and the proof is complete.

### 2.2 Viscosity solutions

In this subsection, we study the viscosity solution of the following penalized equation:

$$\begin{cases}\dfrac{\partial V}{\partial t}-LV+f-\dfrac{1}{4}|DV|^2-\dfrac{1}{\epsilon}(V-g)^+=0, & (t,x)\in[0,T)\times\mathbb{R}^n,\\ V(T,x)=g(T,x), \quad x\in\mathbb{R}^n.\end{cases} \tag{2.15}$$

Firstly, we introduce the definition of the viscosity solution for the quasilinear parabolic partial differential equation (2.15).

**Definition 2.1** *A function $w\in\mathcal{C}$ is called a viscosity subsolution of* (2.15) *if*

$$w(T,x)\le g(T,x), \quad \forall x\in\mathbb{R}^n, \tag{2.16}$$

*and for any $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)$, whenever $w-\varphi$ attains a local maximum at $(t,z)\in[0,T)\times\mathbb{R}^n$, we have*

$$-\varphi_t(t,z)+\alpha w(t,z)+L_0\varphi(t,z)-f(t,z)+\frac{1}{\epsilon}(w-g)^+(t,z)+\frac{1}{4}|D\varphi(t,z)|^2\le 0. \tag{2.17}$$

*A function $w\in\mathcal{C}$ is called a viscosity supersolution of* (2.15) *if in* (2.16)–(2.17) *the inequalities "$\le$" are changed to "$\ge$" and "local maximum" is changed to "local minimum". Further, if $w\in\mathcal{C}$ is both a viscosity subsolution and viscosity supersolution of* (2.15), *then it is called a viscosity solution of* (2.15).

The above definition has the following equivalent definition (see Fleming and Soner [8] and Crandall, Ishii and Lions [5]). We introduce the notions of superjet and subjet of a continuous function $w$ and let $\mathbb{S}^n$ be the set of $n\times n$ symmetric real matrices.

**Definition 2.2** *Let $w:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ be a continuous function and $(t,x)\in[0,T)\times\mathbb{R}^n$. We denote $J^{2,+}w(t,x)$ (the parabolic superjet of $w$ at $(t,x)$) the set of triples $(p,q,P)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ which are such that for all $(s,y)\in[0,T)\times\mathbb{R}^n$ in a neighborhood of $(t,x)$,*

$$\begin{aligned}w(s,y)\le{}& w(t,x)+p(s-t)+\langle q,y-x\rangle\\ &+\frac{1}{2}\langle P(y-x),(y-x)\rangle+o(|s-t|+|y-x|^2).\end{aligned}$$

*We similarly define $J^{2,-}w(t,x)$ (the parabolic subjet of $w$ at $(t,x)$) as the set of triples $(p,q,P)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ which are such that for all $(s,y)\in[0,T)\times\mathbb{R}^n$ in a neighborhood of $(t,x)$,*

$$\begin{aligned}w(s,y)\ge{}& w(t,x)+p(s-t)+\langle q,y-x\rangle\\ &+\frac{1}{2}\langle P(y-x),(y-x)\rangle+o(|s-t|+|y-x|^2).\end{aligned}$$

*Here, $r\to o(r)$ denotes any function such that $\lim\limits_{r\to 0}\frac{o(r)}{r}=0$.*

According to [6] and [8], Definition 2.1 is equivalent to the following Definition 2.3.

**Definition 2.3** *A function $w\in\mathcal{C}$ is called a viscosity solution of the variational inequality*

(1.4) *if the following inequalities* (2.18)–(2.19) *hold*:

$$\begin{cases} p - \alpha w + \dfrac{1}{2}\mathrm{tr}(\sigma\sigma^* P) + \langle b, q\rangle + f - \dfrac{1}{\epsilon}(w-g)^+ - \dfrac{1}{4}|q|^2 \geq 0, \\ \forall (p,q,P) \in J^{2,+}w(t,x),\ \forall (t,x) \in [0,T) \times \mathbb{R}^n, \\ w(T,x) \leq g(T,x), \quad \forall x \in \mathbb{R}^n, \end{cases} \tag{2.18}$$

$$\begin{cases} p - \alpha w + \dfrac{1}{2}\mathrm{tr}(\sigma\sigma^* P) + \langle b, q\rangle + f - \dfrac{1}{\epsilon}(w-g)^+ - \dfrac{1}{4}|q|^2 \leq 0, \\ \forall (p,q,P) \in J^{2,-}w(t,x),\ \forall (t,x) \in [0,T) \times \mathbb{R}^n, \\ w(T,x) \geq g(T,x), \quad \forall x \in \mathbb{R}^n. \end{cases} \tag{2.19}$$

We prove that $V$ is a viscosity solution of (2.15) and define $V_k \in \mathcal{C}$ by

$$\begin{aligned} V_k(s,x) := \inf \Big\{ \mathbb{E}\Big[ \int_s^T \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)} \{F(t,X_t) + |C_t|^2\} \mathrm{d}t \\ + \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)} g(T,X_T)\Big] : C(\cdot) \in \mathcal{A}_k \Big\} \end{aligned} \tag{2.20}$$

for every $k > 0$, where $\mathcal{A}_k = \{C(\cdot) \in \mathcal{A} : |C_t| \leq k,\ \forall t \in [0,T]\}$ and

$$F = f + \frac{1}{\epsilon} V \wedge g \in \mathcal{C}_+. \tag{2.21}$$

The following Lemma 2.1 is a classic result. The similar proof can be found in [28].

**Lemma 2.1** *Let the assumptions* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *hold. Then* $V_k$ *is a viscosity solution of*

$$\begin{cases} \dfrac{\partial V_k}{\partial t} - \Big(\alpha + \dfrac{1}{\epsilon}\Big) V_k - L_0 V_k + F + \min\limits_{|C| \leq k} (|C|^2 + \langle C, DV_k\rangle) = 0, \\ V_k(T,x) = g(T,x), \quad x \in \mathbb{R}^n. \end{cases} \tag{2.22}$$

**Lemma 2.2** *Let the assumptions* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *be satisfied. We get*

$$V_k \ \to V \ \textit{locally uniformly in } [0,T] \times \mathbb{R}^n.$$

**Proof** By (2.7) and (2.20), obviously there holds $V_k \geq V$. Using Dini's theorem, this is enough to show that

$$V_k(s,x) \ \downarrow \ V(s,x) \text{ as } k \to \infty \text{for each } (s,x). \tag{2.23}$$

Putting $C_t^k = C_t \chi_{\{|C_t| \leq k\}}$ for any $C(\cdot) \in \mathcal{A}$, $X_t^k$ denotes the solution of

$$\mathrm{d}X_t^k = [b(t,X_t^k) + C_t^k]\mathrm{d}t + \sigma(t,X_t^k)\mathrm{d}W_t, \quad X_s^k = x.$$

Application of Ito's formula and localized the stochastic integration, combined with the assumptions $(\mathrm{H}_1)$ and $(\mathrm{H}_2)$, for any $\theta \in \mathcal{S}[s,T]$ we have

$$\begin{aligned} &\mathbb{E}[\mathrm{e}^{-\alpha(\theta-s)}|X_\theta|^2] \\ \leq{}& |x|^2 + \mathbb{E}\Big[ \int_s^\theta \mathrm{e}^{-\alpha(t-s)} \{-\alpha|x|^2 + 2\langle x, (b(t,x) + C_t)\rangle + \mathrm{tr}(\sigma\sigma^*(t,x))\}\big|_{x=X_t} \mathrm{d}t \Big] \\ \leq{}& |x|^2 + \mathbb{E}\Big[ \int_s^\theta \mathrm{e}^{-\alpha(t-s)} \Big\{ -\frac{\alpha}{2}|X_t|^2 + \frac{4}{\alpha}|C_t|^2 + \beta_0(|X_t|^2 + 1) \Big\} \mathrm{d}t \Big] \\ <{}& \infty. \end{aligned} \tag{2.24}$$

Here $\beta_0$ is a positive constant and

$$
\begin{aligned}
&\mathbb{E}[\mathrm{e}^{-\alpha(\theta-s)}|X_\theta^k - X_\theta|^2] \\
\le\ & \mathbb{E}\Big[\int_s^\theta \mathrm{e}^{-\alpha(t-s)}\{(-\alpha+\nu)|X_t^k - X_t|^2 + 2\langle (X_t^k - X_t), (C_t^k - C_t)\rangle\}\mathrm{d}t\Big] \\
\le\ & \mathbb{E}\Big[\int_s^\theta \mathrm{e}^{-\alpha(t-s)}\{-\eta|X_t^k - X_t|^2 + 2\langle (X_t^k - X_t), (C_t^k - C_t)\rangle\}\mathrm{d}t\Big] \\
\le\ & \mathbb{E}\Big[\int_s^\theta \mathrm{e}^{-\alpha(t-s)}\Big\{-\frac{\eta}{2}|X_t^k - X_t|^2 + \frac{2}{\eta}|C_t^k - C_t|^2\Big\}\mathrm{d}t\Big].
\end{aligned}
$$

We define

$$
\begin{aligned}
V_C(s,x) := \mathbb{E}\Big[\int_s^T \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\{F(t,X_t) + |C_t|^2\}\mathrm{d}t + \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}g(T,X_T)\Big], \\
C(\cdot) \in \mathcal{A}_k.
\end{aligned}
\tag{2.25}
$$

According to (2.7) there exists $C_\zeta(\cdot) \in \mathcal{A}$ for any $\zeta > 0$, such that $V(s,x) + \zeta > V_{C_\zeta}(s,x)$. Then, by (2.11), (2.21) and (2.25), we obtain

$$
\begin{aligned}
|V_{C_\zeta}(s,x) - V_{C^k}(s,x)| \le\ & \mathbb{E}\Big[\int_s^T \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\{\zeta + C_{\zeta,F}|X_t - X_t^k| + |C_t|^2 - |C_t^k|^2\}\mathrm{d}t \\
& + \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}(\zeta + C_{\zeta,g}|X_T - X_T^k|)\Big] \\
\le\ & \Big(\frac{1}{\alpha}+1\Big)\zeta + C_{\zeta,F}\Big(\mathbb{E}\Big[\int_s^T \mathrm{e}^{-\alpha(t-s)}|X_t - X_t^k|^2\mathrm{d}t\Big]\Big)^{\frac{1}{2}} \\
& \Big(\int_s^T \mathrm{e}^{-\alpha(t-s)}\mathrm{d}t\Big)^{\frac{1}{2}} + \mathbb{E}\Big[\int_s^T \mathrm{e}^{-\alpha(t-s)}(|C_t|^2 - |C_t^k|^2)\mathrm{d}t\Big] \\
& + C_{\zeta,g}(\mathbb{E}[\mathrm{e}^{-2\alpha(T-s)}|X_T - X_T^k|^2])^{\frac{1}{2}} \\
\le\ & \Big(\frac{1}{\alpha}+3\Big)\zeta
\end{aligned}
$$

for a sufficiently large number $k$. Thus

$$
\begin{aligned}
V(s,x) + \zeta &\ge V_{C^k}(s,x) - [V_{C^k}(s,x) - V_{C_\zeta}(s,x)] \\
&\ge V_k(s,x) - \Big(\frac{1}{\alpha}+3\Big)\zeta.
\end{aligned}
$$

Making $k \to \infty$ and $\zeta \to 0$, we get (2.23).

**Theorem 2.2** *Let the assumptions* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *be satisfied. Then, the solution* $V$ *of* (2.7) *is a viscosity solution of* (2.15).

**Proof** According to Lemmas 2.1–2.2, we obtain the following conclusion by the stability result. Let $\varphi \in C^2([0,T]\times\mathbb{R}^n)$ and $V-\varphi$ attains a local maximum at $(t_0,x_0)$ such that

$$
(V-\varphi)(t_0,x_0) > (u-\varphi)(t,x), \quad \forall (t,x) \in ([0,T]\setminus\{t_0\}) \times (\overline{B}(x_0,\delta)\setminus\{x_0\}), \tag{2.26}
$$

where $\overline{B}(x_0,\delta)$ is the closed ball with radius $\delta$. By Lemma 2.2, $V_k - \varphi$ attains a local maximum at some $(t_k,x_k) \in [0,T]\times\overline{B}(x_0,\delta)$ and let $(t_k,x_k)$ be defined by

$$
(V_k-\varphi)(t_k,x_k) = \max_{[0,T]\times\overline{B}(x_0,\delta)} (V_k-\varphi).
$$

Since the sequence $(t_k, x_k)_{k\geq 1}$ is valued in the compact subset $[0,T]\times\overline{B}(x_0,\delta)$, we have $(t_k,x_k)\to(\overline{t},\overline{x})\in[0,T]\times\overline{B}(x_0,\delta)$, if necessary selecting a subsequence. Note that

$$V_k(t_k,x_k)-\varphi(t_k,x_k)>V_k(t,x)-\varphi(t,x),\quad (t,x)\in[0,T]\times\overline{B}(x_0,\delta).$$

Taking the limits, in view of Lemma 2.2 we have

$$V(\overline{t},\overline{x})-\varphi(\overline{t},\overline{x})\geq V(t,x)-\varphi(t,x),\quad (t,x)\in[0,T]\times\overline{B}(x_0,\delta).$$

Then, we must have $\overline{t}=t_0$ and $\overline{x}=x_0$.

By Lemma 2.1, we have

$$\begin{aligned}&\frac{\partial\varphi}{\partial t}(t_k,x_k)-\Big(\alpha+\frac{1}{\epsilon}\Big)V_k(t_k,x_k)-L_0\varphi(t_k,x_k)+F(t_k,x_k)\\&+\min_{|C|\leq k}(|C|^2+\langle C,D\varphi(t_k,x_k)\rangle)\geq 0.\end{aligned}$$

Consider

$$\min_{|C|\leq k}(|C|^2+\langle C,\xi\rangle)\ \to\min_{C}(|C|^2+\langle C,\xi\rangle)\quad\text{locally uniformly in }\mathbb{R}^n\text{ as }k\to\infty.$$

Letting $k\to\infty$, we have

$$\begin{aligned}&\frac{\partial\varphi}{\partial t}(t_0,x_0)-\Big(\alpha+\frac{1}{\epsilon}\Big)V(t_0,x_0)-L_0\varphi(t_0,x_0)+F(t_0,x_0)\\&+\min_{C}(|C|^2+\langle C,D\varphi(t_0,x_0)\rangle)\geq 0.\end{aligned}$$

According to the following relation

$$V\wedge g=V-(V-g)^+,$$

it is clear that $V$ satisfies (2.17). By a similar proof, we obtain that $V$ satisfies Definition 2.1.

### 2.3 Another representation of $V$

In this subsection, we will prove that the unique solution $V$ of (2.7) exists another representation,

$$\begin{aligned}V(s,x)=\inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^\tau&\mathrm{e}^{-\alpha(t-s)}\Big\{f(t,X_t)-\frac{1}{\epsilon}(V-g)^+(t,X_t)+|C_t|^2\Big\}\mathrm{d}t\\&+\mathrm{e}^{-\alpha(\tau-s)}V(\tau,X_\tau)\Big]\end{aligned}\tag{2.27}$$

for any $\tau\in\mathcal{S}[s,T]$. Let $H(t,x)=f(t,x)-\frac{1}{\epsilon}(V-g)^+(t,x)$ and note that

$$\begin{cases}\dfrac{\partial\xi}{\partial t}-L\xi+H(t,x)-\dfrac{1}{4}|D\xi|^2=0,\quad (t,x)\in[0,T)\times\mathbb{R}^n,\\ \xi(T,x)=g(T,x),\quad x\in\mathbb{R}^n.\end{cases}\tag{2.28}$$

Define

$$\xi(s,x):=\inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^T\mathrm{e}^{-\alpha(t-s)}\{H(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(T-s)}g(T,X_T)\Big],\tag{2.29}$$

which belongs to $\mathcal{C}$. Through a similar proof in Subsection 2.2, we obtain that $\xi$ meets the principle of dynamic programming

$$\xi(s,x)=\inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^\tau \mathrm{e}^{-\alpha(t-s)}\{H(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(\tau-s)}\xi(\tau,X_\tau)\Big],\quad \tau\in\mathcal{S}[s,T].$$

Thus, $\xi$ is a viscosity solution of (2.28).

**Theorem 2.3** *Let the assumptions* $(\mathrm{H}_1),(\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *be satisfied. We have* (2.27).

**Proof** From Theorem 2.2, the relationship (2.27) is the uniqueness of the viscosity solution (2.28). Divide the proof into the following steps. We claim

$$\xi_1\le\xi_2 \tag{2.30}$$

for two viscosity solutions $\xi_i\in\mathcal{C},\ i=1,2$ of (2.28).

Step 1 Assume that there exists $(\overline{t},\overline{x})\in[0,T)\times\mathbb{R}^n$ such that

$$\xi_1(\overline{t},\overline{x})-\xi_2(\overline{t},\overline{x})=\sup_{(t,x)\in[0,T]\times\mathbb{R}^n}(\xi_1(t,x)-\xi_2(t,x))>0.$$

It shows that

$$\xi_1(\overline{t},\overline{x})-\xi_2(\overline{t},\overline{x})\ge\delta \tag{2.31}$$

for some $\delta>0$. It is clear that $\overline{t}$ is not equal to $T$.

Define

$$\begin{aligned}\Phi_k(t,x;s,y):=&\,\xi_1(t,x)-\xi_2(s,y)-\frac{k}{2}(|x-y|^2+|t-s|^2)\\&-\frac{1}{k}(\psi(x)+\psi(y))+\frac{1}{k}(t+s)-\frac{2T}{k}.\end{aligned}\tag{2.32}$$

Here $\psi(x)=\log(1+|x|),\ k>0$. Since $\Phi_k$ is continuous and $\lim\limits_{|x|\vee|y|\to+\infty}\Phi_k(t,x;s,y)=-\infty$ uniformly in $t,s\in[0,T)$, there exists $(t_k,x_k;s_k,y_k)\in([0,T)\times\mathbb{R}^n)^2$ such that

$$\begin{aligned}\Phi_k(t_k,x_k;s_k,y_k)&=\sup_{([0,T)\times\mathbb{R}^n)^2}\Phi_k(t,x;s,y)\\&\ge\Phi_k(\overline{t},\overline{x};\overline{t},\overline{x})\\&=\xi_1(\overline{t},\overline{x})-\xi_2(\overline{t},\overline{x})-\frac{2}{k}\psi(\overline{x})-\frac{2}{k}(T-\overline{t})\\&\ge\delta-\frac{2}{k}\psi(\overline{x})-\frac{2}{k}(T-\overline{t})\\&\ge\frac{\delta}{2}\quad\text{for } k\ge k_0,\ \exists k_0>0.\end{aligned}\tag{2.33}$$

Thus

$$\begin{aligned}\frac{\delta}{2}\le&\,\xi_1(t_k,x_k)-\xi_2(s_k,y_k)-\frac{k}{2}(|x_k-y_k|^2+|t_k-s_k|^2)-\frac{1}{k}(\psi(x_k)+\psi(y_k))\\&-\frac{1}{k}(2T-(t_k+s_k))\le\xi_1(t_k,x_k)-\xi_2(s_k,y_k).\end{aligned}\tag{2.34}$$

Step 2 In view of the definition of $(t_k, x_k; s_k, y_k)$, we get

$$2\Phi_k(t_k, x_k; s_k, y_k) \geq \Phi_k(t_k, x_k; t_k, x_k) + \Phi_k(s_k, y_k; s_k, y_k),$$

or equivalently

$$\begin{aligned}
&2\Big[\xi_1(t_k, x_k) - \xi_2(s_k, y_k) - \frac{k}{2}(|x_k - y_k|^2 + |t_k - s_k|^2) \\
&\quad - \frac{1}{k}(\psi(x_k) + \psi(y_k)) - \frac{1}{k}(2T - (t_k + s_k))\Big] \\
&\geq \xi_1(t_k, x_k) - \xi_2(t_k, x_k) - \frac{2}{k}\psi(x_k) - \frac{2}{k}(T - t_k) \\
&\quad + \xi_1(s_k, y_k) - \xi_2(s_k, y_k) - \frac{2}{k}\psi(y_k) - \frac{2}{k}(T - s_k).
\end{aligned}$$

Therefore

$$k[|x_k - y_k|^2 + |t_k - s_k|^2] \leq \xi_1(t_k, x_k) - \xi_2(s_k, y_k) + \xi_2(t_k, x_k) - \xi_1(s_k, y_k) \leq C, \quad C > 0.$$

Hence

$$|x_k - y_k| + |t_k - s_k| \leq \Big(\frac{2C}{k}\Big)^{\frac{1}{2}}. \tag{2.35}$$

According to the uniform continuity of $\xi_i,\ i = 1, 2$, we have

$$\begin{aligned}
k[|x_k - y_k|^2 + |t_k - s_k|^2] \leq &\sup_{|x_k - y_k| + |t_k - s_k| \leq (\frac{2C}{k})^{\frac{1}{2}}} (|\xi_1(t_k, x_k) - \xi_1(s_k, y_k)| \\
&+ |\xi_2(t_k, x_k) - \xi_2(s_k, y_k)|) \to 0 \quad \text{as } k \to \infty.
\end{aligned} \tag{2.36}$$

Now we show that neither $t_k$ nor $s_k$ can converge to $T$ for all $k$. In fact, if $t_k = T,\ \forall k \geq 1$,

$$\begin{aligned}
\Phi_k(T, x_k; s_k, y_k) &\leq \xi_1(T, x_k) - \xi_2(T, x_k) + \xi_2(T, x_k) - \xi_2(s_k, y_k) \\
&\leq \xi_2(T, x_k) - \xi_2(T, y_k) + \xi_2(T, y_k) - \xi_2(s_k, y_k) \\
&\leq 2\zeta + C_{\zeta,\xi_2}|x_k - y_k| + C_{\zeta,\xi_2}|t_k - s_k|,
\end{aligned}$$

and we get a contradiction to (2.33) by choosing $k$ and $\zeta$ such that $2\zeta + C_{\zeta,\xi_2}|x_k - y_k| + C_{\zeta,\xi_2}|t_k - s_k| < \frac{\delta}{2}$. The proof that $s_k = T$ is similar.

Step 3 We need the following lemma. (Please refer to [8] for the proof of the lemma.)

**Lemma 2.3** (Ishii's Lemma) *Let $W$, $V$ be upper semicontinuous and lower semicontinuous, respectively, on $\overline{Q}$, where $Q$ is $([0, T) \times O)$, where $O$ is a locally compact subset of $\mathbb{R}^n$. Assume that $\phi$ is twice continuously differentiable, and $\Phi(t, x; s, y) = W(t, x) - V(s, y) - \phi(t, x; s, y)$ attains an interior maximum $(\overline{t}, \overline{x})$,$(\overline{s}, \overline{y}) \in [0, T) \times O$ satisfying*

$$\Phi(\overline{t}, \overline{x}; \overline{s}, \overline{y}) > \sup_{\partial[Q\times Q]} \Phi.$$

*Then for each $\theta > 0$ there exist symmetric matrices $A$ and $B$ satisfying*

$$\Big(\frac{\partial}{\partial t}\phi(\overline{t}, \overline{x}; \overline{s}, \overline{y}), D_x\phi(\overline{t}, \overline{x}; \overline{s}, \overline{y}), A\Big) \in \overline{J}^{2,+}W(\overline{t}, \overline{x}), \tag{2.37}$$

$$\Big(-\frac{\partial}{\partial s}\phi(\overline{t}, \overline{x}; \overline{s}, \overline{y}), -D_y\phi(\overline{t}, \overline{x}; \overline{s}, \overline{y}), B\Big) \in \overline{J}^{2,-}V(\overline{s}, \overline{y}), \tag{2.38}$$

*and*

$$-\Big(\frac{1}{\theta}+\|X\|\Big)I_{2n}\le\begin{pmatrix}A & 0\\ 0 & -B\end{pmatrix}\le X+\theta X^2,$$

where $I_{2n}$ is the $(2n\times 2n)$ identity matrix, $X=D^2\Phi(\overline{t},\overline{x};\overline{s},\overline{y})\in\mathbb{S}^n$ and the closure of $J^{2,+}$ is defined by

$$\begin{aligned}&\overline{J}^{2,+}W(t,x)\\&=\left\{(p,q,P)\in\mathbb{R}\times O\times\mathbb{S}^n\ \middle|\ \begin{array}{l}\exists(t_k,x_k,p_k,q_k,P_k)\in[0,T)\times Q\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n\\ \text{such that }(p_k,q_k,P_k)\in J^{2,+}W(t,x)\\ \text{and }(t_k,x_k,p_k,q_k,P_k)\\ \to(t,x,p,q,P)\ \text{ as }\ k\to\infty\end{array}\right\}.\end{aligned}\tag{2.39}$$

Similarly we can define the closure $\overline{J}^{2,-}$ of $J^{2,-}$. In particular, choosing $\theta=\frac{1}{k}$ and $\phi(t,x;s,y)=\frac{k}{2}(|x-y|^2+|t-s|^2)$ yield the elegant relations

$$-3k\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\le\begin{pmatrix}A & 0\\ 0 & -B\end{pmatrix}\le 3k\begin{pmatrix}I & -I\\ -I & I\end{pmatrix}.$$

Now we take

$$\begin{aligned}W(t,x)&=\xi_1(t,x)-\frac{1}{k}\psi(x)+\frac{1}{k}(t-T),\\ V(s,y)&=\xi_2(s,y)+\frac{1}{k}\psi(y)-\frac{1}{k}(s-T)\end{aligned}$$

and consider

$$\begin{aligned}J^{2,+}\xi_1(t,x)&=\Big\{(\overline{p},\overline{q},\overline{P})+\Big(-\frac{1}{k},\frac{1}{k}D\psi(x),\frac{1}{k}D^2\psi(x)\Big):(\overline{p},\overline{q},\overline{P})\in J^{2,+}W(t,x)\Big\},\\ J^{2,-}\xi_2(t,x)&=\Big\{(\overline{p},\overline{q},\overline{P})-\Big(-\frac{1}{k},\frac{1}{k}D\psi(y),\frac{1}{k}D^2\psi(y)\Big):(\overline{p},\overline{q},\overline{P})\in J^{2,+}V(s,y)\Big\}.\end{aligned}$$

Then from (2.37)–(2.38) combining with the definition of $\overline{J}^{2,+}\xi_1(t_k,x_k),\overline{J}^{2,-}\xi_1(s_k,y_k)$ we have

$$\begin{aligned}(p_1,q_1,P_1)&:=(k(t_k-s_k),k(x_k-y_k),A)\\&\qquad+\Big(-\frac{1}{k},\frac{1}{k}D\psi(x_k),\frac{1}{k}D^2\psi(x_k)\Big)\in\overline{J}^{2,+}\xi_1(t_k,x_k),\end{aligned}\tag{2.40}$$

$$\begin{aligned}(p_2,q_2,P_2)&:=(k(t_k-s_k),k(x_k-y_k),B)\\&\qquad-\Big(-\frac{1}{k},\frac{1}{k}D\psi(y_k),\frac{1}{k}D^2\psi(y_k)\Big)\in\overline{J}^{2,-}\xi_2(s_k,y_k).\end{aligned}\tag{2.41}$$

Step 4 By (2.28) and (2.41) we obtain

$$\begin{aligned}&-\alpha\xi_2(s_k,y_k)+k(t_k-s_k)+\frac{1}{k}+\frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s_k,y_k)P_2)\\&+\langle b(s_k,y_k),q_2\rangle+H(s_k,y_k)-\frac{1}{4}|q_2|^2\le 0.\end{aligned}\tag{2.42}$$

Also, by (2.28) and (2.40),

$$\begin{aligned}\alpha\xi_1(t_k,x_k)\le k(t_k-s_k)&-\frac{1}{k}+\frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t_k,x_k)P_1)\\&+\langle b(t_k,x_k),q_1\rangle+H(t_k,x_k)-\frac{1}{4}|q_1|^2.\end{aligned}\tag{2.43}$$

(2.42) is added to (2.43). Thus we obtain

$$\begin{aligned}\alpha(\xi_1(t_k,x_k)-\xi_2(s_k,y_k)) \le & -\frac{2}{k}+\frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t_k,x_k)A-\sigma\sigma^*(s_k,y_k)B)\\ & +\frac{\mathrm{tr}(\sigma\sigma^*(t_k,x_k)D^2\psi(x_k)+\sigma\sigma^*(s_k,y_k)D^2\psi(y_k))}{2k}\\ & +[b(t_k,x_k)q_1-b(s_k,y_k)q_2]+[H(t_k,x_k)-H(s_k,y_k)]\\ & -\frac{[|q_1|^2-|q_2|^2]}{4}\\ \equiv & \ \mathrm{I}_1+\frac{\mathrm{I}_2}{2}+\frac{\mathrm{I}_3}{2}+\mathrm{I}_4+\mathrm{I}_5-\frac{\mathrm{I}_6}{4}.\end{aligned}$$

Step 5 We claim that $\mathrm{I}_j \to 0$ as $k\to\infty$ $(j=1,2,\cdots,6)$, which shows a contradiction to (2.34).

It is clear that $\mathrm{I}_1\to 0$ as $k\to\infty$.

**Lemma 2.4** *Let $A$, $B$ satisfy $-3k\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\le\begin{pmatrix} A & 0\\ 0 & -B\end{pmatrix}\le 3k\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}$. Then for any two $n\times d$ matrices $D$, $C$, we have*

$$\mathrm{tr}(DD^*A-CC^*B)\le 3k\|D-C\|^2=3k\sum_{i=1}^n\sum_{j=1}^n(D-C)_{ij}^2.$$

According to Lemma 2.4, we have

$$\begin{aligned}\mathrm{I}_2&=\mathrm{tr}(\sigma\sigma^*(t_k,x_k)A-\sigma\sigma^*(s_k,y_k)B)\le 3k|\sigma(t_k,x_x)-\sigma(s_k,y_k)|^2\\ &\le 6\kappa^2k(|x_k-y_k|^2+|t_k-s_k|^2)\to 0\quad\text{as } k\to\infty.\end{aligned}$$

By a calculation, we have

$$|D\psi(x)|=\frac{1}{1+|x|},\quad |D^2\psi(x)|\le\frac{1}{(1+|x|)^2}.$$

Then

$$\begin{aligned}\mathrm{I}_3&\le\frac{1}{2k}[|\sigma\sigma^*(t_k,x_k)D^2\psi(x_k)|+|\sigma\sigma^*(s_k,y_k)D^2\psi(y_k)|]\\ &\le\frac{1}{2k}\Big[\frac{|\sigma(0,0)|^2+2\kappa^2(|x_k|^2+|t_k|^2)}{(1+|x_k|)^2}+\frac{|\sigma(0,0)|^2+2\kappa^2(|y_k|^2+|s_k|^2)}{(1+|y_k|)^2}\Big]\to 0\quad\text{as } k\to\infty.\end{aligned}$$

By the Lipschitz continuity of $b(t,x)$, we have

$$\begin{aligned}|\mathrm{I}_4|&\le k\kappa\Big[\frac{5}{4}|x_k-y_k|^2+|t_k-s_k|^2\Big]+\frac{|b(t_k,x_k)||D\psi(x_k)|+|b(s_k,y_k)||D\psi(y_k)|}{k}\\ &\le k\kappa\Big[\frac{5}{4}|x_k-y_k|^2+|t_k-s_k|^2\Big]+\frac{2\sup\limits_x\Big(\dfrac{|b(0,0)|+|x|+T}{1+|x|}\Big)}{k}\to 0\quad\text{as } k\to\infty.\end{aligned}$$

According to (2.35), we get

$$|\mathrm{I}_5|\le\sup_{|x-y|+|t-s|\le(\frac{2C}{k})^{\frac{1}{2}}}|H(t,x)-H(s,y)|\to 0.$$

Finally, by (2.36),

$$
\begin{aligned}
|\mathrm{I}_6| &\leq \frac{2}{k}|\langle k(x_k-y_k),(D\psi(x_k)+D\psi(y_k))\rangle| + \Big|\frac{1}{k}D\psi(x_k)\Big|^2 + \Big|\frac{1}{k}D\psi(y_k)\Big|^2 \\
&\leq \frac{4}{k}|k(x_k-y_k)| + \frac{2}{k^2} \to 0 \quad \text{as } k\to\infty.
\end{aligned}
$$

Thus the proof is completed.

## 3 Viscosity Solutions of Variational Inequalities

Consider the convergence of $V_\epsilon = V$ as $\epsilon = \epsilon_n = 2^{-n} \to 0$. Define

$$
R_\beta h(s,x) = \mathbb{E}\Big[\int_s^T \mathrm{e}^{-\beta(t-s)} h(t,\overline{X}_t)\mathrm{d}t + \frac{1}{\beta}\mathrm{e}^{-\beta(T-s)}h(T,\overline{X}_T)\Big], \quad \beta>0,\ s\in[0,T]. \tag{3.1}
$$

We introduce a class of functions:

$$
\mathcal{D} = \{R_\beta(\beta h) : h\in\mathcal{C},\ \beta>\alpha\}. \tag{3.2}
$$

Here $\overline{X}_t$ is the unique solution of the following equation:

$$
\mathrm{d}\overline{X}_t = b(t,\overline{X}_t)\mathrm{d}t + \sigma(t,\overline{X}_t)\mathrm{d}W_t, \quad \overline{X}_s = x. \tag{3.3}
$$

### 3.1 Limit of the penalized problem

**Lemma 3.1** *Let the assumptions* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$ *be satisfied. Then* $\mathcal{D}$ *is dense in* $\mathcal{C}$*.*

**Proof** We claim that

$$
\mathcal{D}\subset\mathcal{C}. \tag{3.4}
$$

Let $h\in\mathcal{C}$ be arbitrary. By a simple computation, we get $\|R_\beta(\beta h)\|\leq\|h\|$. By (2.10) we have

$$
\mathbb{E}[|\overline{X}_t-\overline{Y}_t|^2\mathrm{e}^{(-\beta+\eta)(t-s)}] \leq |x-y|^2
$$

for the solution $\overline{Y}_t$ of (3.3) with $Y_t=y$. According to the similar arguments as (2.12)–(2.13), we obtain

$$
\begin{aligned}
|R_\beta(\beta h)(s,x)-R_\beta(\beta h)(t,y)| &\leq \zeta + \frac{C\beta}{\beta+\eta}(|x-y|+\sqrt{|s-t|}) + \Big(1-\frac{2\beta}{\beta+\eta}\Big) \\
&\quad\times C\mathrm{e}^{-\frac{\beta+\eta}{2}(T-s)}(|x-y|+\sqrt{|s-t|}), \quad \forall\,\zeta>0.
\end{aligned}
$$

Thus letting $\delta\to0$ and $\zeta\to0$, we get

$$
\lim_{\delta\to0}\sup_{|x-y|+|s-t|<\delta}|R_\beta(\beta h)(s,x)-R_\beta(\beta h)(t,y)| = 0,
$$

thus (3.4) holds. By (2.11), (3.3) and $(\mathrm{H}_1)$, we have

$$
\begin{aligned}
\mathbb{E}[|h(t,\overline{X}_t)-h(s,x)|] &\leq \zeta + C_{\zeta,h}\mathbb{E}[|t-s|+|\overline{X}_t-x|] \\
&\leq \zeta + C_{\zeta,h}(t-s) + \overline{C}(t-s+\sqrt{t-s}).
\end{aligned}
$$

Here $\overline{C}=C_{\zeta,h}\widehat{C}$ and $\widehat{C}:=\sup\limits_{(t,x)\in[0,T]\times\mathbb{R}^n}|b(t,x)|+\sup\limits_{(t,x)\in[0,T]\times\mathbb{R}^n}|\sigma(t,x)|$. Thus, letting $\beta\to\infty$ and $\zeta\to 0$, we have

$$\begin{aligned}
&\|R_\beta(\beta h)-h\|\\
&=\sup_{(s,x)\in[0,T]\times\mathbb{R}^n}\mathbb{E}\Big[\mathrm{e}^{-\beta(T-s)}|h(T,\overline{X}_T)-h(s,x)|+\int_s^T\beta\mathrm{e}^{-\beta(t-s)}|h(t,\overline{X}_t)-h(s,x)|\mathrm{d}t\Big]\\
&\le 2\zeta+\mathrm{e}^{-\beta(T-s)}[(C_{\zeta,h}+\overline{C})(T-s)+\overline{C}\sqrt{T-s}]\\
&\quad+\int_s^T\beta\mathrm{e}^{-\beta(t-s)}[(C_{\zeta,h}+\overline{C})(t-s)+\overline{C}\sqrt{t-s}]\mathrm{d}t\\
&=2\zeta+\mathrm{e}^{-\beta(T-s)}[(C_{\zeta,h}+\overline{C})(T-s)+\overline{C}\sqrt{T-s}]\\
&\quad+\int_0^{\beta(T-s)}\mathrm{e}^{-u}\Big[\frac{(C_{\zeta,h}+\overline{C})u}{\beta}+\overline{C}\sqrt{\frac{u}{\beta}}\Big]\mathrm{d}u\to 0.
\end{aligned}$$

The proof is complete.

**Lemma 3.2** *Let the assumptions* $(\mathrm{H}_1),(\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *be satisfied and* $\widetilde{V}_\epsilon$ *be the solution of the integral equation* (2.7) *consistent with* $\widetilde{g}\in\mathcal{C}_+$. *Then we obtain*

$$\|V_\epsilon-\widetilde{V}_\epsilon\|\le\|g-\widetilde{g}\|. \tag{3.5}$$

**Proof** Let $h,\widetilde{h}\in\mathcal{C}$ be satisfied

$$\|h-\widetilde{h}\|\le\|g-\widetilde{g}\|.$$

Then it is obviously to get

$$\|h\wedge g-\widetilde{h}\wedge\widetilde{g}\|\le\|g-\widetilde{g}\|.$$

Then, by (2.8) we have

$$\begin{aligned}
|Th(s,x)-\widetilde{T}\widetilde{h}(s,x)|&\le\sup_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^T\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\frac{1}{\epsilon}|h\wedge g-\widetilde{h}\wedge\widetilde{g}|(t,X_t)\mathrm{d}t\\
&\quad+\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}|g-\widetilde{g}|(T,X_T)\Big]\\
&\le\Big[\frac{\epsilon}{\alpha\epsilon+1}+\Big(1-\frac{\epsilon}{\alpha\epsilon+1}\Big)\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}\Big]\|g-\widetilde{g}\|\\
&\le\|g-\widetilde{g}\|.
\end{aligned}$$

Here $\widetilde{T}$ represents $T$ with $\widetilde{g}$ substituting for $g$. But

$$\begin{aligned}
\|T0-\widetilde{T}0\|\le\sup_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^T\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\frac{1}{\epsilon}|g^- -\widetilde{g}^-|(t,X_t)\mathrm{d}t\\
+\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}|g-\widetilde{g}|(T,X_T)\Big]\le\|g-\widetilde{g}\|.
\end{aligned}$$

We take $h=T0$, $\widetilde{h}=\widetilde{T}0$, and get

$$\|T^20-\widetilde{T}^20\|\le\|g-\widetilde{g}\|$$

and then, by iteration,

$$\|T^n0-\widetilde{T}^n0\|\le\|g-\widetilde{g}\|,\quad n=1,2,\cdots.$$

Letting $n\to\infty$, according to Theorem 2.1 we have (3.5).

**Lemma 3.3** *Let the assumptions* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *be satisfied. We have*

$$\begin{aligned} V_\epsilon(s,x) = \inf_{C(\cdot)\in\mathcal{A}} \inf_{\tau} \mathbb{E}\Big[\int_s^\tau & \mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t \\ & + \mathrm{e}^{-\alpha(\tau-s)}\{g+(V_\epsilon-g)^+\}(\tau,X_\tau)\Big]. \end{aligned} \tag{3.6}$$

**Proof** By Theorem 2.3,

$$\begin{aligned} V_\epsilon(s,x) &= \inf_{C(\cdot)\in\mathcal{A}} \mathbb{E}\Big[\int_s^\tau \mathrm{e}^{-\alpha(t-s)}\Big\{f(t,X_t)-\frac{1}{\epsilon}(V_\epsilon-g)^+(t,X_t)+|C_t|^2\Big\}\mathrm{d}t \\ &\quad + \mathrm{e}^{-\alpha(\tau-s)}V_\epsilon(\tau,X_\tau)\Big] \\ &\le \inf_{C(\cdot)\in\mathcal{A}} \mathbb{E}\Big[\int_s^\tau \mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t + \mathrm{e}^{-\alpha(\tau-s)}(V_\epsilon\vee g)(\tau,X_\tau)\Big] \end{aligned}$$

for all $\tau\in\mathcal{S}[s,T]$. Taking $\tau=\theta:=\inf\{t: V_\epsilon(t,X_t)\ge g(t,X_t)\}$, we have

$$\mathrm{e}^{-\alpha(\theta-s)}V_\epsilon(\theta,X_\theta)=\mathrm{e}^{-\alpha(\theta-s)}g(\theta,X_\theta)=\mathrm{e}^{-\alpha(\theta-s)}(V_\epsilon\vee g)(\theta,X_\theta)$$

and

$$(V_\epsilon-g)(t,X_t)<0 \text{ for } t\in[s,\theta).$$

Then, we have

$$\begin{aligned} V_\epsilon(t,x) &= \inf_{C(\cdot)\in\mathcal{A}} \mathbb{E}\Big[\int_s^\theta \mathrm{e}^{-\alpha(t-s)}\Big\{f(t,X_t)-\frac{1}{\epsilon}(V_\epsilon-g)^+(t,X_t)+|C_t|^2\Big\}\mathrm{d}t \\ &\quad + \mathrm{e}^{-\alpha(\theta-s)}V_\epsilon(\theta,X_\theta)\Big] \\ &= \inf_{C(\cdot)\in\mathcal{A}} \mathbb{E}\Big[\int_s^\theta \mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t + \mathrm{e}^{-\alpha(\theta-s)}(V_\epsilon\vee g)(\theta,X_\theta)\Big], \end{aligned}$$

which completes the proof by applying $V_\epsilon\vee g=g+(V_\epsilon-g)^+$.

**Theorem 3.1** *Let the assumptions* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *be satisfied. We have*

$$V_{\epsilon_n}\to\overline{V}\in\mathcal{C}_+, \tag{3.7}$$

*where* $\epsilon_n=2^{-n}$.

**Proof** Let $g=R_\beta(\beta h)\in\mathcal{D}$ for some $h\in\mathcal{C}$. By the similar arguments as in Theorems 2.2–2.3, it is clear that $g$ is the unique viscosity solution of the following equation:

$$\begin{cases} \dfrac{\partial g}{\partial t}-\beta g-L_0 g+\beta h=0, & (t,x)\in[0,T)\times\mathbb{R}^n, \\ g(T,x)=h(T,x), & x\in\mathbb{R}^n, \end{cases}$$

or equivalently

$$\begin{cases} \dfrac{\partial g}{\partial t}-\Big(\alpha+\dfrac{1}{\epsilon}\Big)g-L_0 g+\beta\widetilde{h}+\dfrac{1}{\epsilon}g=0, & (t,x)\in[0,T)\times\mathbb{R}^n, \\ g(T,x)=h(T,x), & x\in\mathbb{R}^n. \end{cases}$$

Here $\beta\widetilde{h} = \beta h + (\alpha - \beta)g$. Thus we get $g = R_{\alpha+\frac{1}{\epsilon}}\big(\beta\widetilde{h} + \frac{1}{\epsilon}g\big)$. Therefore

$$\begin{aligned}
V_\epsilon - g &\le V_\epsilon - \inf_{C(\cdot)\in\mathcal{A}} \mathbb{E}\Big[\int_s^T \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\Big\{\beta\widetilde{h}(t,X_t) + \frac{1}{\epsilon}g(t,X_t) + |C_t|^2\Big\}\mathrm{d}t \\
&\quad + \frac{\epsilon}{\alpha\epsilon+1}\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}\Big(\beta\widetilde{h} + \frac{1}{\epsilon}g\Big)(T,X_T)\Big] \\
&\le \sup_{C(\cdot)\in\mathcal{A}} \mathbb{E}\Big[\int_s^T \mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(t-s)}\Big\{f - \beta\widetilde{h} + \frac{1}{\epsilon}(V_\epsilon \wedge g - g)\Big\}(t,X_t)\mathrm{d}t \\
&\quad + \frac{\epsilon}{\alpha\epsilon+1}\mathrm{e}^{-(\alpha+\frac{1}{\epsilon})(T-s)}(\alpha g - \beta\widetilde{h})(T,X_T)\Big] \\
&\le \epsilon(\|f - \beta\widetilde{h}\| + \|\alpha g - \beta\widetilde{h}\|).
\end{aligned} \tag{3.8}$$

Applying (3.6) to $V_{\epsilon_{n+1}}$ and $V_{\epsilon_n}$, by (3.8) we obtain

$$\begin{aligned}
|V_{\epsilon_{n+1}}(s,x) - V_{\epsilon_n}(s,x)| &\le \sup_{C(\cdot)}\sup_{\tau} \mathbb{E}[\mathrm{e}^{-\alpha(\tau-s)}|(V_{\epsilon_{n+1}} - g)^+ - (V_{\epsilon_n} - g)^+|(\tau,X_\tau)] \\
&\le (\epsilon_{n+1} + \epsilon_n)(\|f - \beta\widetilde{h}\| + \|\alpha g - \beta\widetilde{h}\|).
\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty}\|V_{\epsilon_{n+1}} - V_{\epsilon_n}\| \le \sum_{n=1}^{\infty}(\epsilon_{n+1} + \epsilon_n)(\|f - \beta\widetilde{h}\| + \|\alpha g - \beta\widetilde{h}\|) < \infty.$$

It shows that $\{V_{\epsilon_n}\}$ is a Cauchy sequence in $\mathcal{C}_+$, and we deduce (3.7).

Supposing that $g \in \mathcal{C}_+$, we can choose a convergent sequence $\{g_m\} \in \mathcal{D}$ converging to $g$. Let $V_\epsilon^m$ denote the solution of the integral equation (2.7) corresponding to $g_m$. By the above calculation, we have

$$V_{\epsilon_n}^m \to \overline{V}^m \in \mathcal{C}_+ \quad \text{as } n \to \infty. \tag{3.9}$$

Also, by Lemma 3.2,

$$\|V_{\epsilon_n}^m - V_{\epsilon_n}^{m'}\| \le \|g_m - g_{m'}\|.$$

Letting $n \to \infty$, we have

$$\|\overline{V}^m - \overline{V}^{m'}\| \le \|g_m - g_{m'}\|.$$

Thus $\{\overline{V}^m\}$ is a Cauchy sequence, and $\{\overline{V}^m\}$ converges to $\overline{V} \in \mathcal{C}_+$. Therefore

$$\begin{aligned}
\|V_{\epsilon_n} - \overline{V}\| &\le \|V_{\epsilon_n} - V_{\epsilon_n}^m\| + \|V_{\epsilon_n}^m - \overline{V}^m\| + \|\overline{V}^m - \overline{V}\| \\
&\le \|g - g_m\| + \|V_{\epsilon_n}^m - \overline{V}^m\| + \|\overline{V}^m - \overline{V}\|.
\end{aligned}$$

Letting $n \to \infty$ and then $m \to \infty$, we complete the proof.

Consider the following parabolic variational inequality:

$$\begin{aligned}
&\frac{\partial V}{\partial t} - LV + f - \frac{1}{4}|DV|^2 \ge 0 \quad \text{in } [0,T)\times\mathbb{R}^n, \\
&V \le g \quad \text{in } [0,T]\times\mathbb{R}^n, \\
&\Big(\frac{\partial V}{\partial t} - LV + f - \frac{1}{4}|DV|^2\Big)(V - g)^- = 0 \quad \text{in } [0,T)\times\mathbb{R}^n, \\
&V(T,x) = g(T,x) \quad \text{on } \mathbb{R}^n.
\end{aligned} \tag{3.10}$$

We introduce the following definition of the viscosity solution of the variational inequality.

**Definition 3.1** *A function $w \in \mathcal{C}$ is called a viscosity solution of the variational inequality* (3.10) *if the following claims hold*:

(*i*) *For any $z \in \mathbb{R}^n$, $w(T,z) = g(z)$.*

(*ii*) *For any $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$, whenever $w-\varphi$ has a local maximum at $(t,z) \in [0,T)\times\mathbb{R}^n$, we have*

$$\varphi_t(t,z) - \alpha w(t,z) - L_0\varphi(t,z) + f(t,z) - \frac{1}{4}|D\varphi(t,z)|^2 \geq 0,$$

*and*

$$w(t,x) \leq g(t,x), \quad \forall (t,x) \in [0,T]\times\mathbb{R}^n.$$

(*iii*) *For any $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$, whenever $w-\varphi$ has a local minimum at $(t,z) \in [0,T)\times\mathbb{R}^n$, we have*

$$\Big(\varphi_t(t,z) - \alpha w(t,z) - L_0\varphi(t,z) + f(t,z) - \frac{1}{4}|D\varphi(t,z)|^2\Big)(w-g)^-(t,z) \leq 0.$$

**Theorem 3.2** *Let the assumptions* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *be satisfied. Then the limit $\overline{V}$ of* (3.7) *is a viscosity solution of the variational inequality* (3.10).

**Proof** It is clear that $\overline{V}(T,z) = g(z)$ for any $z \in \mathbb{R}^n$. For any $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$, $\overline{V}-\varphi$ attains a local maximum at $(t,z)$, i.e.,

$$\overline{V}(t,z) - \varphi(t,z) > \overline{V}(u,x) - \varphi(u,x), \quad (u,x) \in \overline{B}(t,z;\delta),\ t \neq u,\ z \neq x,$$

where $\overline{B}(t,z;\delta) := \{(s,x) \in [0,T]\times\mathbb{R}^n \mid |t-s|^2 + |z-x|^2 \leq \delta\}$. The uniform convergence is applied in Theorem 3.1. We get $V_{\epsilon_n} - \varphi$ to attain a local maximum at $(u_n,x_n) \in \overline{B}(t,z;\delta)$. According to the same inference in Theorem 2.2, we have $(u_n,x_n) \to (t,z)$ and $V_{\epsilon_n}(u_n,x_n) - \varphi(u_n,x_n) \geq V_{\epsilon_n}(u,x) - \varphi(u,x)$, $\forall (u,x) \in \overline{B}(t,z;\delta)$.

Now, combining Theorem 2.2 and (2.17) we obtain

$$\begin{aligned}&\varphi_t(u_n,x_n) - \alpha V_{\epsilon_n}(u_n,x_n) - L_0\varphi(u_n,x_n) + f(u_n,x_n)\\ &- \frac{1}{\epsilon}(V_{\epsilon_n} - g)^+(u_n,x_n) - \frac{1}{4}|D\varphi(u_n,x_n)|^2 \geq 0.\end{aligned}$$

From the last inequality, we have

$$\varphi_t(u_n,x_n) - \alpha V_{\epsilon_n}(u_n,x_n) - L_0\varphi(u_n,x_n) + f(u_n,x_n) - \frac{1}{4}|D\varphi(u_n,x_n)|^2 \geq 0.$$

Letting $n \to \infty$, we get

$$\varphi_t(t,z) - \alpha\overline{V}(t,z) - L_0\varphi(t,z) + f(t,z) - \frac{1}{4}|D\varphi(t,z)|^2 \geq 0. \tag{3.11}$$

By (3.8) we have

$$(V_{\epsilon_n}^m - g_m)^+ \leq \epsilon_n(\|f - \beta\widetilde{h}_m\| + \|\alpha g_m - \beta\widetilde{h}_m\|).$$

Here $g_m = R_\beta(\beta h_m)$ and $\beta\widetilde{h}_m = \beta h_m + (\alpha-\beta)g_m$ for some $\widetilde{h}_m \in \mathcal{C}$. Letting $n \to \infty$, by (3.9) we have $\overline{V}^m \leq g_m$, and letting $m \to \infty$, we get

$$\overline{V} \leq g. \tag{3.12}$$

At last, let $\overline{V}-\varphi$ attain the minimizer at $(\overline{t},\overline{z})$, and let $V_{\epsilon_n}-\varphi$ attain the local minimum at $(\overline{u}_n,\overline{x}_n)$ with $(\overline{u}_n,\overline{x}_n)\to(\overline{t},\overline{z})$. Combining Definition 2.1 and Theorem 2.2, we obtain

$$\begin{aligned}&\varphi_t(\overline{u}_n,\overline{x}_n)-\alpha V_{\epsilon_n}(\overline{u}_n,\overline{x}_n)-L_0\varphi(\overline{u}_n,\overline{x}_n)+f(\overline{u}_n,\overline{x}_n)\\&-\frac{1}{\epsilon}(V_{\epsilon_n}-g)^+(\overline{u}_n,\overline{x}_n)-\frac{1}{4}|D\varphi(\overline{u}_n,\overline{x}_n)|^2\le 0.\end{aligned}$$

On both sides of the last inequality multiplied by $(V_{\epsilon_n}-g)^-$, we have

$$\begin{aligned}&\Big(\varphi_t(\overline{u}_n,\overline{x}_n)-\alpha V_{\epsilon_n}(\overline{u}_n,\overline{x}_n)-L_0\varphi(\overline{u}_n,\overline{x}_n)+f(\overline{u}_n,\overline{x}_n)\\&-\frac{1}{4}|D\varphi(\overline{u}_n,\overline{x}_n)|^2\Big)(V_{\epsilon_n}-g)^-(\overline{u}_n,\overline{x}_n)\le 0.\end{aligned}$$

Letting $n\to\infty$, we have

$$\Big(\varphi_t(\overline{t},\overline{z})-\alpha\overline{V}(\overline{t},\overline{z})-L_0\varphi(\overline{t},\overline{z})+f(\overline{t},\overline{z})-\frac{1}{4}|D\varphi(\overline{t},\overline{z})|^2\Big)(\overline{V}-g)^-(\overline{t},\overline{z})\le 0. \tag{3.13}$$

The claims (3.11)–(3.13) are proved. Thus $\overline{V}$ is a viscosity solution of the variational inequality (3.10).

### 3.2 Uniqueness

**Theorem 3.3** *Let the assumptions* $(\mathrm{H}_1),(\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *be satisfied and* $V_i\in\mathcal{C},\ i=1,2$ *be two viscosity solutions of* (3.10). *Then we have*

$$V_1=V_2.$$

**Proof** We will prove the following equation:

$$\begin{cases}\Big(p-\alpha V_2(t,z)+\dfrac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,z)P)+\langle b(t,z),q\rangle+f(t,z)-\dfrac{1}{4}|q|^2\Big)\\ \cdot(V_2-V_1)^-(t,z)\le 0,\\ \forall(p,q,P)\in\overline{J}^{2,-}V_2(t,x),\ \forall(t,z)\in[0,T]\times\mathbb{R}^n,\\ V_2(T,x)=g(T,x),\quad\forall x\in\mathbb{R}^n,\end{cases} \tag{3.14}$$

or equivalently

$$\begin{cases}\Big(\dfrac{\partial\varphi}{\partial t}(\overline{t},\overline{z})-\alpha V_2(\overline{t},\overline{z})-L_0\varphi(\overline{t},\overline{z})+f(\overline{t},\overline{z})-\dfrac{1}{4}|D\varphi(\overline{t},\overline{z})|^2\Big)(V_2-V_1)^-(\overline{t},\overline{z})\le 0,\\ V_2(T,x)=g(T,x),\quad\forall x\in\mathbb{R}^n.\end{cases} \tag{3.15}$$

Here $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)$ and $V_2-\varphi$ attains the minimizer at $(\overline{t},\overline{z})$. If $V_2\ge V_1$, then $(V_2-V_1)^-=0$. If $V_2<V_1$, then $V_2<V_1\le g$, and thus $(V_2-g)^->0$. By (3.13) we recall

$$\Big(\varphi_t(\overline{t},\overline{z})-\alpha V_2(\overline{t},\overline{z})-L_0\varphi(\overline{t},\overline{z})+f(\overline{t},\overline{z})-\frac{1}{4}|D\varphi(\overline{t},\overline{z})|^2\Big)(V_2-g)^-(\overline{t},\overline{z})\le 0, \tag{3.16}$$

and we conclude

$$\varphi_t(\overline{t},\overline{z})-\alpha V_2(\overline{t},\overline{z})-L_0\varphi(\overline{t},\overline{z})+f(\overline{t},\overline{z})-\frac{1}{4}|D\varphi(\overline{t},\overline{z})|^2\le 0.$$

Thus the proof of (3.15) is complete.

Now, by the same inference in Theorem 2.3 we prove Theorem 3.3. Assume that there exists $(\widetilde{t},\widetilde{x}) \in [0,T) \times \mathbb{R}^n$ such that

$$V_1(\widetilde{t},\widetilde{x}) - V_2(\widetilde{t},\widetilde{x}) = \sup_{(t,x)\in[0,T]\times\mathbb{R}^n} (V_1(t,x) - V_2(t,x)) > 0.$$

Thus, we have

$$V_1(\widetilde{t},\widetilde{x}) - V_2(\widetilde{t},\widetilde{x}) \geq \delta \tag{3.17}$$

for some $\delta > 0$. Define

$$\begin{aligned}\Phi_k(t,x;s,y) := {} & V_1(t,x) - V_2(s,y) - \frac{k}{2}(|x-y|^2 + |t-s|^2) \\ & - \frac{1}{k}(\psi(x) + \psi(y)) + \frac{1}{k}(t+s) - \frac{2T}{k}\end{aligned} \tag{3.18}$$

as in (2.32). Then $\Phi_k(t_k,x_k;s_k,y_k)$ has the maximum point $(t_k,x_k;s_k,y_k)$ with

$$\frac{\delta}{2} \leq V_1(t_k,x_k) - V_2(s_k,y_k) \tag{3.19}$$

for sufficiently large number $k$. As in Steps 2–3 of Theorem 2.3 we get

$$k[|x_k - y_k|^2 + |t_k - s_k|^2] \to 0 \quad \text{as } k \to \infty$$

and

$$\begin{aligned}(\widehat{p}_1,\widehat{q}_1,\widehat{P}_1) &:= (k(t_k - s_k), k(x_k - y_k), A) \\ &\qquad + \Big(-\frac{1}{k}, \frac{1}{k}D\psi(x_k), \frac{1}{k}D^2\psi(x_k)\Big) \in \overline{J}^{2,+}V_1(t_k,x_k) \\ (\widehat{p}_2,\widehat{q}_2,\widehat{P}_2) &:= (k(t_k - s_k), k(x_k - y_k), B) \\ &\qquad - \Big(-\frac{1}{k}, \frac{1}{k}D\psi(y_k), \frac{1}{k}D^2\psi(y_k)\Big) \in \overline{J}^{2,-}V_2(s_k,y_k).\end{aligned}$$

By (3.19) we have

$$\begin{aligned}V_1(s_k,y_k) - V_2(s_k,y_k) &\geq V_1(t_k,x_k) - V_2(s_k,y_k) - |V_1(t_k,x_k) - V_1(s_k,y_k)| \\ &\geq \frac{\delta}{2} - |V_1(s_k,y_k) - V_1(t_k,x_k)| \\ &\geq \frac{\delta}{4} \quad \text{for sufficiently large } k.\end{aligned}$$

It shows

$$(V_2(s_k,y_k) - V_1(s_k,y_k))^- > 0.$$

By (3.14), we get

$$\widehat{p}_2 - \alpha V_2(s_k,y_k) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s_k,y_k)\widehat{P}_2) + \langle b(s_k,y_k),\widehat{q}_2\rangle + f(s_k,y_k) - \frac{1}{4}|\widehat{q}_2|^2 \leq 0. \tag{3.20}$$

Also, by (3.11),

$$\alpha V_1(t_k,x_k)\le \widehat{p}_1+\frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t_k,x_k)\widehat{P}_1)+\langle b(t_k,x_k),\widehat{q}_1\rangle+f(t_k,x_k)-\frac{1}{4}|\widehat{q}_1|^2. \tag{3.21}$$

Thus, (3.20) and (3.21) have the similar relationships between (2.42) and (2.43). So, by the same calculations as in Steps 4–5 of Theorem 2.3, w have

$$\alpha(V_1(t_k,x_k)-V_2(s_k,y_k))\to 0\quad \text{as } k\to\infty.$$

This is contradictory to (3.19), which completes the proof.

### 3.3 A stochastic interpretation of $\overline{V}$

Firstly we introduce a stochastic interpretation of the viscosity solution $\overline{V}$ of (3.10).

**Theorem 3.4** *Let the assumptions* $(\mathrm{H}_1),(\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *be satisfied. Then we have*

$$\overline{V}(s,x)=\inf_{C(\cdot)\in\mathcal{A}}\inf_{\tau}\mathbb{E}\Big[\int_s^{\tau}\mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(\tau-s)}g(\tau,X_\tau)\Big]. \tag{3.22}$$

**Proof** Let $\widehat{V}$ denote the right-hand side of (3.22). By (2.27) we get

$$\begin{aligned}V_{\epsilon_n}(t,x)&=\inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^{\tau}\mathrm{e}^{-\alpha(t-s)}\Big\{f(t,X_t)-\frac{1}{\epsilon_n}(V-g)^+(t,X_t)+|C_t|^2\Big\}\mathrm{d}t\\&\quad+\mathrm{e}^{-\alpha(\tau-s)}V_{\epsilon_n}(\tau,X_\tau)\Big]\\&\le\inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^{\tau}\mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(\tau-s)}V_{\epsilon_n}(\tau,X_\tau)\Big].\end{aligned}$$

Letting $n\to\infty$, by (3.7) we get

$$\begin{aligned}\overline{V}(s,x)&\le\inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^{\tau}\mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(\tau-s)}\overline{V}(\tau,X_\tau)\Big]\\&\le\inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^{\tau}\mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(\tau-s)}g(\tau,X_\tau)\Big].\end{aligned}$$

This shows $\overline{V}\le\widehat{V}$. For the proof of reverse inequality, take any $C(\cdot)\in\mathcal{A}$ and let

$$R_m=\inf\Big\{t:\overline{V}(t,X_t)+\frac{1}{m}\ge g(t,X_t)\Big\}.$$

Since

$$\overline{V}(t,X_t)+\frac{1}{m}<g(t,X_t)\quad\text{on }\{t<R_m\},$$

we have

$$\begin{aligned}\mathbb{E}\Big[\int_s^{R_m}\mathrm{e}^{-\alpha(t-s)}(V_{\epsilon_n}-g)^+(t,X_t)\mathrm{d}t\Big]&\le\mathbb{E}\Big[\int_s^{R_m}\mathrm{e}^{-\alpha(t-s)}\Big(V_{\epsilon_n}-\Big(\overline{V}+\frac{1}{m}\Big)\Big)^+(t,X_t)\mathrm{d}t\Big]\\&\le\mathbb{E}\Big[\int_s^{R_m}\mathrm{e}^{-\alpha(t-s)}\Big(\|V_{\epsilon_n}-\overline{V}\|-\frac{1}{m}\Big)^+(t,X_t)\mathrm{d}t\Big]\\&=0\end{aligned}$$

for sufficiently large number $n$. Hence, by (2.27),

$$\begin{aligned}V_{\epsilon_n}(t,x) &= \inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^{R_m}\mathrm{e}^{-\alpha(t-s)}\Big\{f(t,X_t)-\frac{1}{\epsilon_n}(V_{\epsilon_n}-g)^+(t,X_t)+|C_t|^2\Big\}\mathrm{d}t\\&\quad+\mathrm{e}^{-\alpha(R_m-s)}V_{\epsilon_n}(R_m,X_{R_m})\Big]\\&\geq \inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^{R_m}\mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(R_m-s)}V_{\epsilon_n}(R_m,X_{R_m})\Big].\end{aligned}$$

Letting $n\to\infty$, applying the definition of $R_m$ and (3.7) we get

$$\begin{aligned}\overline{V}(s,x) &= \inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^{R_m}\mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(R_m-s)}\overline{V}(R_m,X_{R_m})\Big]\\&= \inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^{R_m}\mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(R_m-s)}\Big(g(R_m,X_{R_m})-\frac{1}{m}\Big)\Big]\\&\geq \inf_{C(\cdot)\in\mathcal{A}}\mathbb{E}\Big[\int_s^{R_m}\mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(R_m-s)}g(R_m,X_{R_m})\Big]-\frac{1}{m}\\&\geq \widehat{V}(s,x)-\frac{1}{m}.\end{aligned}$$

Letting $m\to\infty$, we have $\overline{V}(s,x)\geq\widehat{V}(s,x)$, hence the proof is finished.

# 4 Impulsive Control

## 4.1 Description of the problem

Consider an application of the variational inequality in the impulse control problem. Let a set $(\Theta,\gamma)$ be decision variables of impulse control

$$\begin{aligned}&\Theta=\{\theta_n\}_{n\geq1},\quad \theta_n\in\mathcal{S}[s,T]\uparrow T,\\&\gamma=\{\gamma_n\}_{n\geq1},\quad \gamma_n\in\mathbb{R}^n_+:\ \mathcal{F}_{\theta_n}\text{-measuable.}\end{aligned}$$

We consider the following state equation:

$$\begin{cases}\mathrm{d}\chi_t=[b(t,\chi_t)+C_t]\mathrm{d}t+\sigma(t,\chi_t)\mathrm{d}W_t,\quad t\in]\theta_n,\theta_{n+1}[,\\ \chi_s=x,\\ \chi_{\theta^n_+}=\chi_{\theta^n_-}+\gamma_n,\quad n=1,2,\cdots.\end{cases}\tag{4.1}$$

The notation $\chi_{\theta^n_-}$ stands for

$$\chi_{\theta^{n-1}_-}+\int_{\theta_{n-1}}^{\theta_n}[b(t,\chi_t)+C_t]\mathrm{d}t+\int_{\theta_{n-1}}^{\theta_n}\sigma(t,\chi_t)\mathrm{d}W_t$$

and $\chi_{\theta^n_+}:=\lim\limits_{t\to\theta^n_+}\chi_t$. The triplet $\beta=(C,\{\theta_n\},\{\gamma_n\})$ is the control. Taking $s\in[0,T)$, we define

$$\mathcal{K}[s,T]=\left\{\gamma(\cdot)=\sum_{n=1}^{\infty}\gamma_n\mathcal{X}_{[\theta_n,T]}(\cdot)\ \middle|\ \begin{array}{l}[s,T]\to\mathbb{R}^n_+,\ \theta_1\geq s,\ \theta_n\uparrow T,\\ \gamma_n\in\mathbb{R}^n_+,\ \forall n\geq1,\\ \sum\limits_{n=1}^{\infty}\mathrm{e}^{-\alpha(\theta_n-s)}\rho(\gamma_n)<\infty\end{array}\right\}.$$

Our objective is to minimize the cost functional

$$J(s,x,\beta)=\mathbb{E}\Big[\int_s^T \mathrm{e}^{-\alpha(t-s)}\{f(t,\chi_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(T-s)}g(T,\chi_T)$$
$$+\sum_{n=1}^{\infty}\mathrm{e}^{-\alpha(\theta_n-s)}\rho(\gamma_n)\Big],$$

where $f,g\in\mathcal{C}_+$ and the impulse cost $\rho$ is assumed to satisfy

$$\begin{cases}\rho(\xi)=k+\rho_0(\xi), \quad k>0 \text{ and } \rho_0\in\mathcal{C}(\mathbb{R}^n_+) \ with \ \rho_0(0)=0,\\ \rho_0(\xi+\widetilde{\xi})\le\rho_0(\xi)+\rho_0(\widetilde{\xi}), \quad \forall\xi,\widetilde{\xi}\in\mathbb{R}^n_+,\\ \rho_0(\xi)\le\rho_0(\widetilde{\xi}) \quad \text{if } \widetilde{\xi}-\xi\in\mathbb{R}^n_+,\\ \rho_0(\xi)\to+\infty \quad \text{as } |\xi|\to+\infty.\end{cases} \tag{4.2}$$

The value function of the present problem is

$$V(s,x):=\inf_{\beta\in\mathcal{A}\times\mathcal{K}[s,T]}J(s,x,\beta).$$

Now, we give the following quasi-variational inequality derived from the impulsive control problem

$$\begin{aligned}&\frac{\partial V}{\partial t}-LV+f-\frac{1}{4}|DV|^2\ge 0 && \text{in } [0,T)\times\mathbb{R}^n,\\ &V\le MV && \text{in } [0,T)\times\mathbb{R}^n,\\ &\Big(\frac{\partial V}{\partial t}-LV+f-\frac{1}{4}|DV|^2\Big)(V-MV)^-=0 && \text{in } [0,T)\times\mathbb{R}^n,\\ &V(T,x)=g(T,x) && \text{on } \mathbb{R}^n,\end{aligned} \tag{4.3}$$

where $MV(s,x):=\inf_{\gamma\in\mathbb{R}^n_+}[V(s,x+\gamma)+\rho(\gamma)]$.

### 4.2 Quasi-variational inequalities

In this subsection, we will prove the existence and uniqueness of the viscosity solution of the quasi-variational inequality (4.3).

We define

$$\begin{aligned}&Qw(s,x)\\ &:=\inf_{C(\cdot)\in\mathcal{A}}\inf_{\theta}\mathbb{E}\Big[\int_s^\theta \mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t+\mathrm{e}^{-\alpha(\theta-s)}Mw(\theta,X_\theta)\Big], \quad w\in\mathcal{C}_+.\end{aligned} \tag{4.4}$$

Here $X_t$ is as in (1.1) and $w(T,x)=g(T,x)$.

**Lemma 4.1** *Let the assumptions* $(\mathrm{H}_1),(\mathrm{H}_2)$ *and* (4.2) *be satisfied. For all* $w,\widetilde{w}\in\mathcal{C}_+$, *we have*

$$0\le Qw\le C(\|f\|+\|w\|), \tag{4.5}$$
$$Qw\in\mathcal{C}_+, \tag{4.6}$$
$$w\le\widetilde{w}\Rightarrow Qw\le Q\widetilde{w}, \tag{4.7}$$
$$Q(\mu w+(1-\mu)\widetilde{w})\ge\mu Qw+(1-\mu)Q\widetilde{w}, \quad \mu\in[0,1]. \tag{4.8}$$

**Proof** In [1], it is obvious that

$$\begin{aligned}&0 \le Mw \le \|w\| + k, \quad Mw \in \mathcal{C}_+,\\&\|Mw - M\widetilde{w}\| \le \|w - \widetilde{w}\|, \quad w \le \widetilde{w} \Rightarrow Mw \le M\widetilde{w},\\&M(\mu w + (1-\mu)\widetilde{w}) \ge \mu Mw + (1-\mu)M\widetilde{w}, \quad \mu \in [0,1].\end{aligned}$$

Thus (4.5) and (4.7) are clear. (4.8) is a simple inference of the concavity of $M$. By Theorems 3.2–3.4 we have (4.6).

**Lemma 4.2** *Let the assumptions* $(\mathrm{H}_1), (\mathrm{H}_2), (\mathrm{H}_3)$ *and* (4.2) *be satisfied and* $w, \widetilde{w} \in \mathcal{C}_+$ *satisfing* $w - \widetilde{w} \le \lambda w$ *for some* $\lambda \in [0,1]$*. Then we have*

$$Qw - Q\widetilde{w} \le \lambda(1-\mu)Qw, \quad \forall \mu \in \Big(0, \frac{k}{\|V^0\|} \wedge 1\Big), \tag{4.9}$$

*where*

$$\begin{aligned}V^0(s,x) := \inf_{C(\cdot)\in\mathcal{A}} \mathbb{E}\Big[&\int_s^T \mathrm{e}^{-\alpha(t-s)}\{f(t,X_t) + |C_t|^2\}\mathrm{d}t + \mathrm{e}^{-\alpha(T-s)}g(T,X_T)\\&+\sum_{n=1}^{\infty} \mathrm{e}^{-\alpha(\theta_n - s)}\rho(\gamma_n)\Big].\end{aligned}$$

**Proof** By (4.8) we obtain

$$Q((1-\lambda)w + \lambda 0) \ge (1-\lambda)Qw + \lambda Q0.$$

Since

$$(1-\lambda)w \le \widetilde{w},$$

we have

$$Q\widetilde{w} \ge (1-\lambda)Qw + \lambda Q0,$$

or equivalently

$$Qw - Q\widetilde{w} \le \lambda(Qw - Q0).$$

In view of (2.11)–(2.12), noting $V^0 \in \mathcal{C}_+$ and by (4.4), we get $Qw \le V^0$.

In order to finish the proof, we must also prove that (4.10) holds:

$$Q0 \ge \mu V^0, \quad \forall \mu \in \Big(0, \frac{k}{\|V^0\|} \wedge 1\Big). \tag{4.10}$$

By (4.2) we have $M0 = k$, and then

$$Q0 = \inf_{C(\cdot)\in\mathcal{A}} \inf_{\theta} \mathbb{E}\Big[\int_s^{\theta} \mathrm{e}^{-\alpha(t-s)}\{f(t,X_t) + |C_t|^2\}\mathrm{d}t + \mathrm{e}^{-\alpha(\theta-s)}k\Big].$$

Obviously, we have

$$\mu V^0(t,X_t) \le \mu\|V^0\| \le \frac{k}{\|V^0\|}\|V^0\| = k.$$

Therefore, we apply the dynamic programming principle to $V^0(s,x)$,

$$\begin{aligned} Q0 &\geq \inf_{C(\cdot)\in\mathcal{A}} \inf_{\theta} \mathbb{E}\Big[\int_s^\theta \mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t + \mathrm{e}^{-\alpha(\theta-s)}\mu V^0(\theta,X_\theta)\Big] \\ &\geq \mu \inf_{C(\cdot)\in\mathcal{A}} \inf_{\theta} \mathbb{E}\Big[\int_s^\theta \mathrm{e}^{-\alpha(t-s)}\{f(t,X_t)+|C_t|^2\}\mathrm{d}t + \mathrm{e}^{-\alpha(\theta-s)} V^0(\theta,X_\theta)\Big] \\ &= \mu V^0(s,x). \end{aligned}$$

This shows that (4.10) holds, and the proof is complete.

**Theorem 4.1** *Assuming that* $(\mathrm{H}_1), (\mathrm{H}_2), (\mathrm{H}_3)$ *and* (4.2) *are satisfied. Then there is only one viscosity solution* $\overline{V} \in \mathcal{C}_+$ *of the quasi-variational inequality* (4.3).

**Proof** Let $V^n = QV^{n-1} \in \mathcal{C}_+$. Obviously, we have

$$0 \leq V^1 = QV^0 \leq V^0$$

and then

$$0 \leq V^n \leq V^{n-1} \leq V^0.$$

Moreover, $V^0 - V^1 \leq V^0$. By (4.9) we have

$$QV^0 - QV^1 \leq (1-\mu)QV^0, \quad \forall \mu \in \Big(0, \frac{k}{\|V^0\|} \wedge 1\Big).$$

It shows that

$$V^1 - V^2 \leq (1-\mu)V^1.$$

By iteration, we have

$$V^n - V^{n+1} \leq (1-\mu)^n V^n \leq (1-\mu)^n V^0.$$

Hence we get

$$V^n \to \overline{V} \quad \text{in } \mathcal{C}_+.$$

By Theorems 3.2–3.4 we see that $V^n$ is a unique viscosity solution of the following quasi-variational inequality:

$$\begin{cases} \dfrac{\partial V^n}{\partial t} - LV^n + f - \dfrac{1}{4}|DV^n|^2 \geq 0 & \text{in } [0,T]\times\mathbb{R}^n, \\ V^n \leq MV^n & \text{in } [0,T]\times\mathbb{R}^n, \\ \Big(\dfrac{\partial V^n}{\partial t} - LV^n + f - \dfrac{1}{4}|DV^n|^2\Big)(V^n - MV^n)^- = 0 & \text{in } [0,T]\times\mathbb{R}^n, \\ V(T,x) = g(T,x) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.11}$$

Using the stability result of the viscosity solution in Theorem 2.2 and putting $n\to\infty$, we get that $\overline{V}$ is a viscosity solution of the quasi-variational inequality (4.3) in the sense of Definition 3.1 with $M\overline{V}$ substituting for g.

In order to obtain uniqueness, suppose that $V_i \in \mathcal{C}_+, i=1,2$ are two viscosity solutions of (4.3). Applying Theorems 3.3–3.4, we have

$$V_i = QV_i, \quad i = 1,2.$$

Obviously, we have $V_1 - V_2 \le V_1$. Appliying (4.9) and $\lambda = 1$, we have

$$QV_1 - QV_2 \le (1-\mu)QV_1, \quad \forall \mu \in \Big(0, \frac{k}{\|V^0\|} \wedge 1\Big).$$

Hence

$$V_1 - V_2 \le (1-\mu)V_1.$$

By iteration,

$$V_1 - V_2 \le (1-\mu)^n V_1, \quad n = 1, 2, \cdots .$$

Letting $n \to \infty$, we have $V_1 \le V_2$. The proof is complete.

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