Eventual Positivity of Hermitian Algebraic Functions and Associated Integral Operators*

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Abstract   Quillen proved that repeated multiplication of the standard sesquilinear form to a positive Hermitian bihomogeneous polynomial eventually results in a sum of Hermitian squares, which was the first Hermitian analogue of Hilbert's seventeenth problem in the nondegenerate case. Later Catlin-D'Angelo generalized this positivstellensatz of Quillen to the case of Hermitian algebraic functions on holomorphic line bundles over compact complex manifolds by proving the eventual positivity of an associated integral operator. The arguments of Catlin-D'Angelo involve subtle asymptotic estimates of the Bergman kernel. In this article, the authors give an elementary and geometric proof of the eventual positivity of this integral operator, thereby yielding another proof of the corresponding positivstellensatz.

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1 Introduction

A central topic in real algebraic geometry is Hilbert's seventeenth problem of representing a nonnegative form on $\mathbb{R}^n$ as a sum of squares of rational functions. An affirmative solution was first provided by Artin's positivstellensatz in 1927 [1]. Since then, related topics have continued to be widely studied from different viewpoints (see [6–8, 10, 11–12, 15–16] and the references therein).

For the corresponding problem in the Hermitian case, Quillen [11] and Catlin-D'Angelo [6] proved independently the following positivstellensatz: for any Hermitian bihomogeneous polynomial f on $\mathbb{C}^n$ which is positive on $\mathbb{C}^n \setminus \{0\}$, there exists $\ell_o > 0$ such that for any $\ell \geq \ell_o$, there are homogeneous holomorphic polynomials $g_1, \cdots, g_N$ on $\mathbb{C}^n$ satisfying

$$\Big(\sum_{i=1}^{n} |z_i|^2 \Big)^{\ell} \cdot f(z) = \sum_{j=1}^{N} |g_j(z)|^2, \tag{1.1}$$

where $z = (z_1, \cdots, z_n)$. Later, Catlin-D'Angelo generalized this positivstellensatz to the case of positive Hermitian algebraic functions on holomorphic line bundles over compact complex manifolds, and this was formulated as an isometric embedding theorem of the associated Hermitian metrics on the line bundles in [7] (see Section 2 for the two equivalent definitions of Hermitian algebraic functions and Catlin-D'Angelo's result stated as Theorem 2.1). For holomorphic line bundles L and E over a compact complex manifold X and positive Hermitian algebraic functions R and P on L and E respectively such that R satisfies the strong global Cauchy-Schwarz (SGCS) condition, Catlin-D'Angelo obtained their positivstellensatz by proving that, if m is a sufficiently large positive integer, then the associated integral operator $\mathbf{K}_{R^m P, \Omega}$ on $H^0(X, L^m \otimes E)$ is positive (see (3.1) and Section 2 for the definitions of $\mathbf{K}_{R^m P, \Omega}$ and SGCS respectively). Here Ω denotes the volume form on X induced from R. The arguments of Catlin-D'Angelo in [7] depend on Catlin's result [5] about perturbations of the Bergman kernel on the unit disk bundle associated to a negative line bundle.

In this article, we give an elementary and geometric proof of an asymptotic formula, which leads to the eventual positivity of the above integral operator. We state our main result as follows.

Theorem 1.1 *Let L and E be holomorphic line bundles over an n-dimensional compact complex manifold X. Suppose R and P are positive Hermitian algebraic functions on L and E respectively, such that R satisfies the strong global Cauchy-Schwarz condition. Then there exists a constant $C > 0$ such that for all $m \in \mathbb{N}$ and all $s \in H^0(X, L^m \otimes E)$, one has*

$$\left| \mathbf{K}_{R^m P, \Omega}(s, s) - \frac{\pi^n}{m^n} \|s\|_2^2 \right| \leq \frac{C}{m^{n+1}} \|s\|_2^2, \tag{1.2}$$

where Ω denotes the volume form on X induced from R, and $\|s\|_2$ denotes the L^2 norm of s with respect to $R^m P$ and Ω.

Theorem 1.1 would, in principle, also follow from [16, Section 5] (which depended on a result of Berman-Berndtsson-Sjötrand [2]). However, we have some difficulty in understanding some arguments in [16] (in particular, using the notation of [16, p. 313], it is possible that $\langle L_i^{(m)} a, a \rangle = 0$ for infinitely many m, which together with the last line of [16, p. 313], will imply that $\langle \mathcal{C}^{(m)} a, a \rangle = 0$, contradicting Theorem 1.1). As such, we are motivated to give an independent proof of the asymptotic formula in Theorem 1.1, which is the purpose of this article. See also [13] for a weaker version of (1.2) in the special case when X is a complex projective space.

In comparison to the approach of Catlin-D'Angelo [7] which depends on estimates on Bergman kernels, our proof of the above asymptotic formula is relatively elementary, direct and geometric, and does not depend on estimates on Bergman kernel. Roughly speaking, our approach is to consider separately the behaviour of the kernels of the integral operators $\mathbf{K}_{R^m P, \Omega}$ in tubular neighborhoods of the diagonal of the product manifold as well as that in the complementary region. To derive our desired estimates, we construct good approximants of the kernels of the integral operators in some carefully chosen tubular neighborhoods of the diagonal.

Some arguments in this article are not completely effective, for instance, Lemma 4.3 uses a continuity argument. After writing this article, we have found a way to make the arguments of this article effective, which will be discussed in a separate paper [14]. Still, we believe that this article is of independent interest as it illustrates how the diastatic function of Calabi [4] and the Bochner coordinate system [3] lead to an elementary proof of Theorem 1.1.

A Hermitian algebraic function Q on a holomorphic line bundle F over a compact complex manifold X is called a maximal sum of Hermitian squares (resp. a sum of Hermitian squares) if there exists a basis (resp. finite subset) $\{s_0, s_1 \cdots, s_N\}$ of $H^0(X, F)$ such that one has $Q(x, \overline{x}) = \sum_{i=0}^{N} s_i(x)\overline{s_i(x)}$ for all $x \in X$. Note that in this case, if Q is positive, then as in [7, Theorem 3], the associated map $\phi : X \to \mathbb{P}^N$ given by $\phi(x) = [s_0(x), \cdots, s_N(x)]$ is holomorphic and it induces an isometry between the Hermitian holomorphic line bundles (F^*, h_Q) and $(\mathcal{O}_{\mathbb{P}^N}(-1), h_N)$, i.e., $\phi^*\mathcal{O}_{\mathbb{P}^N}(-1) = F^*$ and $\phi^*h_N = h_Q$. Here h_N denotes the Hermitian metric on the universal line bundle $\mathcal{O}_{\mathbb{P}^N}(-1)$ over $\mathbb{P}^N$ induced by the polynomial $\sum_{i=0}^{N} |z_i|^2$, and h_Q denotes the Hermitian metric on F^* induced from Q. As is known in Catlin-D'Angelo [7] and Varolin [16], the eventual positivity of the integral operator in Theorem 1.1 leads to the following positivstellensatz.

Corollary 1.1 *Let X, L, E, R, P, n, C be as in Theorem 1.1. Then for each integer $m > \dfrac{C}{\pi^n}$ (resp. $m \geq \dfrac{C}{\pi^n}$), the Hermitian algebraic function $R^m P$ is a maximal sum of Hermitian squares (resp. a sum of Hermitian squares), and in particular, there exists some holomorphic map $\phi : X \to \mathbb{P}^N$ (with N depending on m) such that $((L^m \otimes E)^*, h_{R^m P}) = (\phi^*\mathcal{O}_{\mathbb{P}^N}(-1), \phi^*h_N)$.*

The organization of this paper is as follows. In Section 2, we cover some background material and introduce some notations. In Section 3, we give the construction of the approximants to the kernels of the integral operators being investigated. In Sections 4–6, we derive the desired estimates needed for the proof of Theorem 1.1.

2 Notation and Background Materials

In this section, we recall some background materials regarding Hermitian algebraic functions on holomorphic line bundles, which are taken from [7], [9] and [16]. As such, we will skip their proofs here and refer the reader to these references for their proofs.

Let X be an n-dimensional compact complex manifold, and let F be a holomorphic line bundle over X with the corresponding projection map denoted by $\pi : F \to X$. The dual holomorphic line bundle of F is denoted by F^*. The total space of F^* (denoted by the same symbol) is a complex manifold, and the complex conjugate manifold of F^* (resp. X) is denoted by $\overline{F^*}$ (resp. $\overline{X}$). Following [9] and [16], a Hermitian algebraic function Q on F is simply a Hermitian bilinear form on the complex vector space $H^0(X, F)^*$. One sees that with respect to any basis $\{s^\alpha\}$ of $H^0(X, F)$, there exists a corresponding Hermitian matrix $(C_{\alpha\overline{\beta}})$ such that,

for all $x, y \in X$, one has

$$Q(x, \overline{y}) = \sum_{\alpha, \beta} C_{\alpha\overline{\beta}} s^\alpha(x)\overline{s^\beta(y)}. \tag{2.1}$$

Let $\rho_1 : X \times \overline{X} \to X$ and $\rho_2 : X \times \overline{X} \to \overline{X}$ denote the projection maps onto the first and the second factor respectively, and consider the holomorphic line bundle $\rho_1^* F \otimes \rho_2^* \overline{F}$ over the complex manifold $X \times \overline{X}$, whose fiber at a point $(x, \overline{y}) \in X \times \overline{X}$ is naturally isomorphic to $F_x \otimes \overline{F_y}$. Here $F_x := \pi^{-1}(x)$ denotes the fiber of F at the point $x \in X$, etc. Then from (2.1), one sees that Q can be regarded as a global holomorphic section of $\rho_1^* F \otimes \rho_2^* \overline{F}$ (i.e., $Q \in H^0(X \otimes \overline{X}, \rho_1^* F \otimes \rho_2^* \overline{F})$) satisfying the condition $Q(x, \overline{y}) = \overline{Q(y, \overline{x})} \in F_x \otimes \overline{F_y}$ for all $x, y \in X$. Next, via pointwise evaluation, one obtains a function (denoted by the same symbol) $Q : F^* \times \overline{F^*} \to \mathbb{C}$ such that (i) Q is holomorphic on $F^* \times \overline{F^*}$, (ii) $Q(v, \overline{w}) = \overline{Q(w, \overline{v})}$ for all $v, w \in F^*$, and (iii) $Q(\lambda \cdot v, \overline{w}) = \lambda Q(v, \overline{w})$ for $\lambda \in \mathbb{C}$ and $v, w \in F^*$, where $\lambda \cdot v$ denotes scalar multiplication along fibers. From the above descriptions, one has

$$Q(v, \overline{w}) = \langle Q(x, \overline{y}), v \otimes \overline{w} \rangle \quad \text{for all } v \in F_x^*, \ w \in F_y^*, \ x, y \in X, \tag{2.2}$$

where $\langle \ , \ \rangle$ denotes the pointwise pairing between $\rho_1^* F \otimes \rho_2^* \overline{F}$ and its dual line bundle induced from that between F and F^*. Note that there is no confusion on which definition of Q is being used, as this is indicated by the object at which Q evaluates. We will often make our statement using only one of these two definitions, and leave the corresponding statement in terms of the other definition as an exercise to the reader.

As mentioned in the last section, we say that a Hermitian algebraic function Q is a sum of Hermitian squares (resp. a maximal sum of Hermitian squares) if the Hermitian matrix $\left(C_{\alpha\overline{\beta}} \right)$ in (2.1) with respect to one (and hence any) basis of $H^0(X, F)$ is positive semi-definite (resp. positive definite), or equivalently, there exists a finite subset (resp. a basis) $\{t^\alpha\}$ of $H^0(X, F)$ such that $Q(x, \overline{x}) = \sum_\alpha t^\alpha(x)\overline{t^\alpha(x)}$ for all $x \in X$. Also, the Hermitian algebraic function Q is said to be positive if $Q(v, \overline{v}) > 0$ for all $0 \neq v \in F^*$. If Q is positive, then Q induces a Hermitian metric h_Q on F^* given by $h_Q(v, w) = Q(v, \overline{w})$ for $v, w \in F_x^*$, $x \in X$. (In [7], such a Hermitian metric arising from a positive Hermitian algebraic function is called a globalizable metric.) We recall that the curvature form Θ_{h_Q} of the Hermitian metric h_Q is the $(1,1)$-form on X given locally as follows: On any open subset U of X and for any local non-vanishing holomorphic section s of $F^*|_U$, one has $\Theta_{h_Q}|_U = -\sqrt{-1}\partial\overline{\partial} \log h_Q(s, s)$. Again we denote by $\pi : F^* \to X$ the projection map. Following [16] again (and with origin in [7]), a positive Hermitian algebraic function Q on X is said to satisfy the strong global Cauchy-Schwarz (SGCS) condition if the following two conditions are satisfied:

(SGCS-1) $|Q(v, \overline{w})|^2 < Q(v, \overline{v})Q(w, \overline{w})$ for all non-zero $v, w \in F^*$ such that $\pi(v) \neq \pi(w)$. (Note that one always has $|Q(v, \overline{w})|^2 = Q(v, \overline{v})Q(w, \overline{w})$ whenever $\pi(v) = \pi(w)$.)

(SGCS-2) The $(1,1)$-form Θ_{h_Q} on X is negative definite.

Let Ω be a smooth Hermitian volume form on X, and let Q be a positive Hermitian algebraic function on F as before. We endow the vector space $H^0(X, F)$ with the L^2 Hermitian inner

product (induced from Q and Ω) given as follows. For $s, t \in H^0(X, F)$, one has

$$(s, t) := \int_X \langle s, t \rangle_Q(x)\, \Omega(x), \tag{2.3}$$

where $\langle s, t \rangle_Q(x) := \dfrac{s(x)\overline{t(x)}}{Q(x, \overline{x})}$ denotes the pointwise Hermitian pairing on F dual to (F^*, h_Q). Note that the quotient in (2.3) makes sense and is a scalar-valued function on X, since both the numerator $s(x)\overline{t(x)}$ and the denominator $Q(x, \overline{x})$ take values in $F_x \otimes \overline{F_x}$. For simplicity, we denote the associated L^2-norm (resp. pointwise norm) of s by $\|s\|_2$ (resp. $\|s(x)\|$, $x \in X$), i.e.,

$$\|s\|_2 = \sqrt{(s, s)} \quad \text{and} \quad \|s(x)\| = \sqrt{\langle s, s \rangle_Q(x)}. \tag{2.4}$$

Next one defines an integral operator $\mathbf{K}_{Q,\Omega}$ associated to Q and Ω and acting (as a Hermitian bilinear form) on the vector space $H^0(X, F)$ as follows. For $s, t \in H^0(X, F)$, we let

$$\mathbf{K}_{Q,\Omega}(s, t) := \iint_{X \times X} \frac{Q(x, \overline{y})s(y)\overline{t(x)}}{Q(x, \overline{x})Q(y, \overline{y})} \Omega(x)\Omega(y). \tag{2.5}$$

Note that as in (2.3), the quotient in the integrand in (2.5) makes sense as a scalar-valued function on $X \times X$. With respect to an orthonormal basis $\{s^\alpha\}$ of $H^0(X, F)$ for the Hermitian inner product in (2.3), it is easy to see that the Hermitian matrix $(C_{\alpha\overline{\beta}})$ associated to Q as given in (2.1) is simply given by $C_{\alpha\overline{\beta}} = \mathbf{K}_{Q,\Omega}(s^\beta, s^\alpha)$ for each α, β. It follows that Q is a sum of Hermitian squares (resp. a maximal sum of Hermitian squares) if and only if the integral operator $\mathbf{K}_{Q,\Omega}$ is positive semi-definite (resp. positive definite) in the sense that $\mathbf{K}_{Q,\Omega}(s, s) \geq 0$ (resp. $\mathbf{K}_{Q,\Omega}(s, s) > 0$) for all $0 \neq s \in H^0(X, F)$.

For a positive Hermitian algebraic function Q on X, we define the Cauchy-Schwarz function $\Psi_Q : X \times X \to \mathbb{R}$ associated to Q given by

$$\Psi_Q(x, y) := \frac{Q(x, \overline{y})Q(y, \overline{x})}{Q(x, \overline{x})Q(y, \overline{y})}, \quad x, y \in X. \tag{2.6}$$

From the positivity of Q and as in (2.5), one easily sees that Ψ_Q is a well-defined real-analytic function on $X \times X$. Note also that $\Psi_Q(x, y) = \Psi_Q(y, x)$ for all $x, y \in X$.

We remark that for two positive Hermitian algebraic functions Q_1 and Q_2 on two holomorphic line bundles F_1 and F_2 over X, the product $Q_1 Q_2$ (obtained by taking pointwise multiplication (resp. tensor product) when the Q_i's are taken as functions (resp. bundle-valued sections)) is a positive Hermitian algebraic function on $F_1 \otimes F_2$. Furthermore, one easily sees that

$$\Psi_{Q_1 Q_2}(x, y) = \Psi_{Q_1}(x, y) \cdot \Psi_{Q_2}(x, y) \quad \text{for all } x, y \in X. \tag{2.7}$$

For the remainder of this section, we will fix two holomorphic line bundles L and E over X. We also fix two positive Hermitian algebraic functions R and P on L and E respectively, such that R satisfies the SGCS condition. From (SGCS-2), X is endowed with a Kähler form ω and an associated Hermitian volume form Ω given by

$$\omega := -\Theta_{h_R} \quad \text{and} \quad \Omega := \frac{\omega^n}{n!}. \tag{2.8}$$

We recall the following result of Catlin-D'Angelo.

Theorem 2.1 (see [7]) *Let L and E be holomorphic line bundles over an n-dimensional compact complex manifold X. Suppose R and P are positive Hermitian algebraic functions on L and E respectively, such that R satisfies the SGCS condition. Then there exists $m_o \in \mathbb{N}$ such that for each integer $m \geq m_o$, $R^m P$ is a maximal sum of Hermitian squares (on the line bundle $L^m \otimes E$); in particular, there exists some holomorphic map $\phi : X \to \mathbb{P}^N$ (with N depending on m) such that $((L^m \otimes E)^*, h_{R^m P}) = (\phi^* \mathcal{O}_{\mathbb{P}^N}(-1), \phi^* h_N)$. Here h_N is as in Corollary* 1.1.

We remark that Catlin-D'Angelo obtained the above theorem by proving the positive-definiteness of the integral operators $\mathbf{K}_{R^m P, \Omega}$ for all sufficiently large m, where Ω is as in (2.8).

3 The Integral Operator and the Approximant

In this section, we are going to construct approximants to the kernels of the integral operators in Theorem 1.1. The Cauchy-Schwarz functions defined in (2.6) and the canonical coordinates (called Bochner coordinates in this article) associated to analytic Kähler metrics (as given in [3] and [4]) will play important roles in our construction.

Throughout this section and as in Theorem 1.1, we let L and E be holomorphic line bundles over an n-dimensional compact complex manifold X, and we let R and P be positive Hermitian algebraic functions on L and E respectively, such that R satisfies the SGCS condition. We recall from (2.8) the analytic Kähler form ω and the volume form Ω on X induced from R. Recall from Section 2 that for each $m \in \mathbb{N}$, $R^m P$ is a positive Hermitian algebraic function on the holomorphic line bundle $L^m \otimes E$, and one has an associated integral operator given by

$$\mathbf{K}_{R^m P, \Omega}(s, t) := \iint_{X \times X} \frac{R^m(x, \overline{y}) P(x, \overline{y}) s(y) \overline{t(x)}}{R^m(x, \overline{x}) P(x, \overline{x}) R^m(y, \overline{y}) P(y, \overline{y})} \Omega(x) \Omega(y) \tag{3.1}$$

for $s, t \in H^0(X, L^m \otimes E)$ (see (2.5)). We denote the diagonal of $X \times X$ by

$$\Delta_X := \{(x, x) \in X \times X \,\big|\, x \in X\} \cong X. \tag{3.2}$$

Consider the real-analytic subvarieties of $X \times X$ given by

$$Z_R := \{(x, y) \in X \times X \,\big|\, R(x, \overline{y}) = 0\}, \tag{3.3}$$

$$Z_P := \{(x, y) \in X \times X \,\big|\, P(x, \overline{y}) = 0\}. \tag{3.4}$$

From the positivity of R and P, one easily sees that $\Delta_X \cap (Z_R \cup Z_P) = \emptyset$. Recall from [4, p. 3], the diastatic function D_ω associated to the analytic Kähler form ω, which is defined on some open neighborhood of Δ_X in $X \times X$. In our present case where ω arises from R, D_ω actually extends to a function on $(X \times X) \setminus Z_R$ given by

$$D_\omega = -\log \Psi_R, \tag{3.5}$$

where $\Psi_R(x, y) = \frac{R(x, \overline{y}) R(y, \overline{x})}{R(x, \overline{x}) R(y, \overline{y})}$, $x, y \in X$, is the Cauchy-Schwarz function associated to R (see (2.6)). Later we will also need to consider the Cauchy-Schwarz function Ψ_P associated to P

given by

$$\Psi_P(x,y) = \frac{P(x,\overline{y})P(y,\overline{x})}{P(x,\overline{x})P(y,\overline{y})}, \quad x,y \in X. \tag{3.6}$$

Recall also from [4, p. 14], and [3, p. 181] that for any $x \in X$, there exists a canonical coordinate system $\widehat{z}$ centered at x given by an n-tuple of holomorphic coordinate functions $z = (z_1, z_2, \cdots, z_n) : B(x,r) \to \mathbb{C}^n$ such that $z(x) = 0$ and the power series expansion of the diastatic function $D_\omega(x, \cdot)$ takes the form

$$D_\omega(x,y) = |z(y)|^2 + \sum_{|\alpha|,|\beta|\geq 2} a_{\alpha\overline{\beta}}(\widehat{z})z(y)^\alpha \overline{z(y)^\beta}, \quad y \in B(x,r), \tag{3.7}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$ are multi-indices, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$, $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$, and the $a_{\alpha\overline{\beta}}(\widehat{z})$'s are Taylor coefficients of D_ω at x with respect to the coordinate system $\widehat{z}$, etc; furthermore,

$$B(x,r) := \{y \in X \,|\, |z(y)| < r\} \tag{3.8}$$

denotes the open coordinate ball (with respect to $\widehat{z}$) centered at x and of radius r. Note that the right-hand side of (3.7) does not possess any monomial term in $z^\alpha \overline{z}^\beta$ with $|\alpha| \leq 1$ or $|\beta| \leq 1$, except when $|\alpha| = |\beta| = 1$; in particular, it does not possess any monomial term of total degree $|\alpha| + |\beta| = 3$. For simplicity, any local coordinate system $\widehat{z}$ (with associated coordinate functions z) satisfying (3.7) will be called a Bochner coordinate at x. It was shown in [4, p. 14–15], that if $\widehat{z}$ (with associated coordinate functions z) is a Bochner coordinate at x, then a coordinate system $\widehat{z}'$ (with associated coordinate functions z') is a Bochner coordinate at x if and only if

$$z' = Uz \tag{3.9}$$

for some $U \in \mathbb{U}(n)$, where $\mathbb{U}(n)$ denotes the group of $n \times n$ unitary matrices. In particular, the coordinate ball $B(x,r)$ in (3.8) is a well-defined open subset of X independent of the choice of the Bochner coordinate $\widehat{z}$ at x, and we will simply call it the Bochner ball centered at x and of radius r. Then one easily sees from the arguments in [3, p. 181] on the existence of Bochner coordinates that for any $x_o \in X$, any Bochner coordinate $\widehat{z}$ at x_o (with associated coordinate functions $z : B(x_o, r) \to \mathbb{C}^n$), there exist some r' satisfying $0 < r' < r$ and open subsets $V \subset X$, $W \subset X \times X$, such that $x_o \in V$,

$$W = \bigcup_{x \in V} W_x, \tag{3.10}$$

where $W_x := \{x\} \times B(x,r') \cong B(x,r')$, and there exists a continuous function $\widetilde{z} : W \to \mathbb{C}^n$ such that the restriction $\widetilde{z}\big|_{W_x} : B(x,r') \to \mathbb{C}^n$ (under the identification in (3.10)) gives a Bochner coordinate at x for each $x \in V$, and $\widetilde{z}\big|_{W_{x_o}} = z$ on $B(x_o, r')$; furthermore, shrinking r' and V if necessary, we may assume that for all $x \in V$ and all Bochner coordinates $\widehat{z}$ at x, the associated coordinate functions z (which are necessarily of the form $U\widetilde{z}\big|_{W_x}$ for some $U \in \mathbb{U}(n)$ (see (3.9))) are defined on $B(x,r')$. Together with the compactness of X, it follows readily that there exists

some constant $r_1 > 0$ such that for all $x \in X$ and all Bochner coordinates $\widehat{z}$ at x, the associated coordinate functions are defined on $B(x, r_1)$. Furthermore, the Bochner coordinates form a principal $\mathbb{U}(n)$-bundle $p : \mathcal{G} \to X$ over X such that for each $x \in X$ and $\widehat{z} \in \mathcal{G}_x := p^{-1}(x)$, $\widehat{z}$ is a Bochner coordinate at x with associated coordinate functions $z : B(x, r_1) \to \mathbb{C}^n$, and for each $U \in \mathbb{U}(n)$, $U\widehat{z}$ is simply the Bochner coordinate at x with associated coordinate functions given by Uz. Throughout this article, we will fix a choice of the constant r_1, and same remark will apply to the other constants r_i's and C_j's defined later, unless stated otherwise.

Next we consider the power series expansions of the local expressions for Ψ_R, Ψ_P and Ω. Take a point $x \in X$ and a Bochner coordinate $\widehat{z}$ at x with associated holomorphic coordinate functions $z : B(x, r_1) \to \mathbb{C}^n$. From (3.5) and upon exponentiating the negative of both sides of (3.7), one easily sees that the power series expansion of Ψ_R with respect to $\widehat{z}$ takes the form

$$\Psi_R(x, y) = 1 - |z(y)|^2 + \sum_{|\alpha|, |\beta| \geq 2} \Psi_{R, \alpha\overline{\beta}}(\widehat{z}) z(y)^\alpha \overline{z(y)^\beta}, \quad y \in B(x, r_1). \tag{3.11}$$

Here as in (3.7), the $\Psi_{R, \alpha\overline{\beta}}(\widehat{z})$'s are the Taylor coefficients of Ψ_R at x with respect to the coordinate system $\widehat{z}$. Next we write

$$\omega = \frac{\sqrt{-1}}{2} \sum_{1 \leq i, j \leq n} \omega_{i\overline{j}}(\widehat{z}) \, \mathrm{d}z_i \wedge \mathrm{d}\overline{z_j} \quad \text{on } B(x, r_1). \tag{3.12}$$

where, for $1 \leq i, j \leq n$, $\omega_{i\overline{j}}(\widehat{z})$ denotes the (i, j)-th component of ω with respect to $\widehat{z}$. From (2.8) and (3.5), one has $\omega_{i\overline{j}}(\widehat{z}) = \partial_{z_i} \partial_{\overline{z_j}} D_\omega(x, \cdot)$. Together with (3.7), one easily sees that the power series expansion of $\omega_{i\overline{j}}(\widehat{z})$ takes the form

$$\omega_{i\overline{j}}(\widehat{z})(y) = \delta_{ij} + \sum_{|\alpha|, |\beta| \geq 1} \omega_{i\overline{j}, \alpha\overline{\beta}}(\widehat{z}) z(y)^\alpha \overline{z(y)^\beta}, \quad y \in B(x, r_1). \tag{3.13}$$

where δ_{ij} denotes the Kronecker symbol, and as before, the $\omega_{i\overline{j}, \alpha\overline{\beta}}(\widehat{z})$'s denote the Taylor coefficients of $\omega_{i\overline{j}}(\widehat{z})$ at x with respect to $\widehat{z}$. Next we let $\mathrm{d}V(\widehat{z})$ be the Euclidean volume form on $B(x, r_1)$ with respect to the Bochner coordinate $\widehat{z}$ at x given by

$$\mathrm{d}V(\widehat{z}) = \left(\frac{\sqrt{-1}}{2} \right)^n \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z_n}, \tag{3.14}$$

so that we may write

$$\Omega = \Omega(\widehat{z}) \, \mathrm{d}V(\widehat{z}) \quad \text{on } B(x, r_1), \tag{3.15}$$

where $\Omega(\widehat{z})$ is a positive real-analytic function. In fact, one easily sees that $\Omega(\widehat{z}) = \det(\omega_{i\overline{j}}(\widehat{z}))$ on $B(x, r_1)$. Together with (3.13), one easily sees that the power series expansion of $\Omega(\widehat{z})$ with respect to $\widehat{z}$ takes the form

$$\Omega(\widehat{z})(y) = 1 + \sum_{|\alpha|, |\beta| \geq 1} \Omega_{\alpha\overline{\beta}}(\widehat{z}) z(y)^\alpha \overline{z(y)^\beta}, \quad y \in B(x, r_1). \tag{3.16}$$

Here as before, the $\Omega_{\alpha\overline{\beta}}(\widehat{z})$'s are the Taylor coefficients of $\Omega(\widehat{z})$ at x with respect to $\widehat{z}$. As for Ψ_P, we consider its power series expansion with respect to $\widehat{z}$, which is valid on $B(x, r_2)$ for

some r_2 satisfying $0 < r_2 < r_1$, so that we have

$$\Psi_P(x,y) = \sum_{\alpha,\beta} \Psi_{P,\alpha\overline{\beta}}(\widehat{z}) z(y)^\alpha \overline{z(y)^\beta}, \quad y \in B(x,r_2). \tag{3.17}$$

By differentiating the expression of Ψ_P in (3.6), one easily sees that $\Psi_{P,\alpha\overline{\beta}}(\widehat{z}) = 1$ when $|\alpha| = |\beta| = 0$, and $\Psi_{P,\alpha\overline{\beta}}(\widehat{z}) = 0$ when exactly one of the two numbers $|\alpha|$, $|\beta|$ is 0. Thus, one may refine (3.17) as follows:

$$\Psi_P(x,y) = 1 + \sum_{|\alpha|,|\beta|\geq 1} \Psi_{P,\alpha\overline{\beta}}(\widehat{z}) z(y)^\alpha \overline{z(y)^\beta}, \quad y \in B(x,r_2). \tag{3.18}$$

Furthermore, it is easy to see that shrinking r_2 if necessary, we may choose (and will choose) r_2 so that (3.18) holds for all $x \in X$ and all $\widehat{z} \in \mathcal{G}_x$. Next we consider certain truncations of the power series expansions considered above. Take $x \in X$ and a Bochner coordinate $\widehat{z}$ at x with associated holomorphic coordinate functions $z : B(x,r_1) \to \mathbb{C}^n$ as before. With notation as in (3.11), we define a function $\Psi_{R,\leq 4}(\widehat{z}) : B(x,r_1) \to \mathbb{C}$ given by

$$\Psi_{R,\leq 4}(\widehat{z})(y) := 1 - |z(y)|^2 + \sum_{|\alpha|=|\beta|=2} \Psi_{R,\alpha\overline{\beta}}(\widehat{z}) z(y)^\alpha \overline{z(y)^\beta}, \quad y \in B(x,r_1). \tag{3.19}$$

In other words, $\Psi_{R,\leq 4}(\widehat{z})$ is obtained by taking the sum of the monomial terms of total degree $|\alpha| + |\beta| \leq 4$ in the power series expansion of $\Psi_R(x,\cdot)$ at x with respect to $\widehat{z}$. With notation as in (3.16) and (3.18), we similarly define

$$\Omega_{\leq 2}(\widehat{z})(y) := 1 + \sum_{|\alpha|=|\beta|=1} \Omega_{\alpha\overline{\beta}}(\widehat{z}) z(y)^\alpha \overline{z(y)^\beta}, \quad y \in B(x,r_1), \tag{3.20}$$

$$\Psi_{P,\leq 2}(\widehat{z})(y) := 1 + \sum_{|\alpha|=|\beta|=1} \Psi_{P,\alpha\overline{\beta}}(\widehat{z}) z(y)^\alpha \overline{z(y)^\beta}, \quad y \in B(x,r_2). \tag{3.21}$$

For an open subset $U \subset X$, a continuous section of $\mathcal{G}\big|_U$ is a continuous function $\sigma : U \to \mathcal{G}$ such that $\sigma(x) \in \mathcal{G}_x$ for all $x \in U$.

Lemma 3.1 *There exists an open subset $X' \subset X$ such that $\int_{X \setminus X'} \Omega = 0$ and the restriction $\mathcal{G}\big|_{X'}$ admits a continuous section.*

Proof Since X is compact, there exists a finite open cover $\{V_i\}_{1 \leq i \leq N}$ of X such that each $\mathcal{G}\big|_{V_i}$ is trivial, i.e., $\mathcal{G}\big|_{V_i}$ admits a continuous section σ_i for each $i = 1, \cdots, N$. Now, we let $U_1 := V_1$, and for each $2 \leq k \leq N$, we let $U_k := V_k \setminus \bigcup_{1 \leq i \leq k-1} \overline{V_i}$. Finally, we let $X' := \bigcup_{1 \leq i \leq N} U_i$, and let $\sigma : X' \to \mathcal{G}$ be given by $\sigma(x) := \sigma_i(x)$ for each $x \in U_i$, $i = 1, \cdots, N$. Then one easily checks that $\int_{X \setminus X'} \Omega = 0$ and σ is a continuous section of $\mathcal{G}\big|_{X'}$.

For each r satisfying $0 < r < r_1$, we let

$$W(r) := \{(x,y) \in X \times X \,|\, y \in B(x,r)\} \tag{3.22}$$

(see (3.8)), which is an open neighborhood of Δ_X in $X \times X$. For discussion in ensuing sections, we define a function $\rho : W(r_1) \to \mathbb{R}$ given by

$$\rho(x,y) := |z(y)|, \quad (x,y) \in W(r_1), \tag{3.23}$$

where $z = (z_1, \cdots, z_n)$ are the coordinate functions associated to a (and hence any) Bochner coordinate $\hat{z} \in \mathcal{G}_x$ (see (3.9)). It is easy to see that ρ is continuous on $W(r_1)$, but in general, ρ is not symmetric in x and y, i.e., $\rho(x, y) \neq \rho(y, x)$. Let Z_R and Z_P be as in (3.3) and (3.4). Since $\Delta_X \cap (Z_R \cup Z_P) = \emptyset$, it is easy to see that there exists a constant r_3 satisfying $0 < r_3 < r_1$ such that

$$W(r_3) \cap (Z_R \cup Z_P) = \emptyset. \tag{3.24}$$

We proceed to construct the approximant to the kernel associated to the integral operator in (3.1) (see (3.27) below). First we let

$$r_o := \min\{r_1, r_2, r_3\} > 0. \tag{3.25}$$

From now on and as given in Lemma 3.1, we fix an open subset $X' \subset X$ such that $\int_{X \setminus X'} \Omega = 0$ and a continuous section $\sigma : X' \to \mathcal{G}\big|_{X'}$, so that $\sigma(x) \in \mathcal{G}_x$ for each $x \in X'$. For $r > 0$, let

$$W'(r) := (X' \times X) \cap W(r) = \{(x, y) \in X' \times X \mid y \in B(x, r)\}. \tag{3.26}$$

For each $m \in \mathbb{N}$, we define a function $T_\sigma^{(m)} : W'(r_o) \to \mathbb{C}$ given by

$$T_\sigma^{(m)}(x, y) := \frac{(\Psi_{R, \leq 4}(\sigma(x))(y))^m \cdot \Psi_{P, \leq 2}(\sigma(x))(y) \cdot \Omega_{\leq 2}(\sigma(x))(y)}{\Omega(\sigma(x))(y)} \tag{3.27}$$

for $(x, y) \in W'(r_o)$ (see (3.15) and (3.19)–(3.21)). We remark that it follows readily from the continuity of σ and the associated continuous family of Bochner coordinates (cf. the construction of $\mathcal{G}$) that $T_\sigma^{(m)}$ is a continuous function on $W'(r_o)$. Note that for $(x, y) \in W(r_o)$, we have $R(y, \overline{x}) \neq 0$ and $P(y, \overline{x}) \neq 0$ (see (3.24)), and thus we have

$$\frac{R^m(x, \overline{y}) P(x, \overline{y}) s(y) \overline{s(x)}}{R^m(x, \overline{x}) P(x, \overline{x}) R^m(y, \overline{y}) P(y, \overline{y})} = \Psi_R^m(x, y) \Psi_P(x, y) \cdot \frac{s(y) \overline{s(x)}}{R^m(y, \overline{x}) P(y, \overline{x})} \tag{3.28}$$

for $(x, y) \in W(r_o)$ (see (3.5) and (3.6)). Also, since $\int_{X \setminus X'} \Omega = 0$, it follows that the value of the right-hand side of (3.1) remains unchanged if we replace the domain of integration there by $X' \times X$ (in lieu of $X \times X$). Together with (3.1) and (3.28), it follows that for each $m \in \mathbb{N}$, each $r > 0$ and each $s \in H^0(X, L^m \otimes E)$, if $r < r_o$, then

$$\mathbf{K}_{R^m P, \Omega}(s, s) - \frac{\pi^n}{m^n} \|s\|_2^2 = \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{3.29}$$

where

$$\mathrm{I} := \iint_{(x,y) \in W'(r)} \left(\Psi_R^m(x, y) \Psi_P(x, y) - T_\sigma^{(m)}(x, y) \right) \frac{s(y) \overline{s(x)}}{R^m(y, \overline{x}) P(y, \overline{x})} \Omega(y) \Omega(x),$$

$$\mathrm{II} := \iint_{(x,y) \in W'(r)} T_\sigma^{(m)}(x, y) \frac{s(y) \overline{s(x)}}{R^m(y, \overline{x}) P(y, \overline{x})} \Omega(y) \Omega(x) - \frac{\pi^n}{m^n} \|s\|_2^2, \tag{3.30}$$

$$\mathrm{III} := \iint_{(x,y) \in (X' \times X) \setminus W'(r)} \frac{R^m(x, \overline{y}) P(x, \overline{y}) s(y) \overline{s(x)}}{R^m(x, \overline{x}) P(x, \overline{x}) R^m(y, \overline{y}) P(y, \overline{y})} \Omega(y) \Omega(x).$$

In the ensuing sections, we will make suitable choices of $r = r(m)$ for each $m \in \mathbb{N}$, which will allow us to obtain desired estimates for I, II and III.

4 Estimation of I

In this section, we are going to estimate the integral I in (3.30).

Lemma 4.1 *There exist constants C_1, C_2, C_3, C_4, C_5, C_6, $r_4 > 0$ with $r_4 < r_o$ such that one has*

$$|\Psi_R(x,y) - \Psi_{R,\leq 4}(\widehat{z})(y)| \leq C_1 \rho(x,y)^5, \tag{4.1}$$

$$|\Psi_R(x,y) - (1 - \rho(x,y)^2)| \leq C_2 \rho(x,y)^4, \tag{4.2}$$

$$|\Psi_P(x,y) - \Psi_{P,\leq 2}(\widehat{z})(y)| \leq C_3 \rho(x,y)^3, \tag{4.3}$$

$$|\Psi_P(x,y) - 1| \leq C_4 \rho(x,y)^2, \tag{4.4}$$

$$|\Omega(\widehat{z})(y) - \Omega_{\leq 2}(\widehat{z})(y)| \leq C_5 \rho(x,y)^3, \tag{4.5}$$

$$|\Omega(\widehat{z})(y) - 1| \leq C_6 \rho(x,y)^2 \tag{4.6}$$

for all $(x,y) \in W(r_4)$ and all $\widehat{z} \in \mathcal{G}_x$. Here ρ and r_o is as in (3.23) and (3.25).

Proof To prove (4.1), we take an arbitrary point $x_o \in X$, and take an open subset V of X containing x_o such that $\mathcal{G}\big|_V$ is trivial, so that there exists a number $r_4' > 0$ and a continuous family of coordinate functions $\{z_{\widehat{z}}\}_{\widehat{z}\in\mathcal{G}}\big|_V$ such that for each $x \in V$ and each $\widehat{z} \in \mathcal{G}_x$, $z_{\widehat{z}} : B(x, r_4') \to \mathbb{C}^n$ are the coordinate functions associated to $\widehat{z}$. Then Ψ_R gives rise to a continuous family of real-analytic functions $\Psi_R(\widehat{z}) : B(x, r_4') \to \mathbb{R}$ (and given by a continuous family of power series expansions in the variables $z_{\widehat{z}}$) parametrized by $\widehat{z} \in \mathcal{G}\big|_V$ such that $\Psi_R(\widehat{z})(y) = \Psi_R(x,y)$ for all $x \in V$, $y \in B(x, r_4')$ and $\widehat{z} \in \mathcal{G}_x$. By polarization, we obtain for each $\Psi_R(\widehat{z})$ a holomorphic function $\widetilde{\Psi_R(\widehat{z})}$ on $B(x, r_4') \times \overline{B(x, r_4')}$ such that $\Psi_R(\widehat{z})(y) = \widetilde{\Psi_R(\widehat{z})}(y, \overline{y})$ for $y \in B(x, r_4')$. Here $\overline{B(x, r_4')}$ denotes the complex conjugate manifold of $B(x, r_4')$. Furthermore, it is clear that the $\widetilde{\Psi_R(\widehat{z})}$'s form a continuous family of functions parametrized by $\widehat{z} \in \mathcal{G}\big|_V$. Then it follows readily from standard theory for convergent power series of holomorphic functions (for the $\widetilde{\Psi_R(\widehat{z})}$'s) that for some $C_1, r_4 > 0$ satisfying $0 < r_4 < r_4'$, (4.1) holds for all $x \in V$, $y \in B(x, r_4)$ and $\widehat{z} \in \mathcal{G}\big|_V$, upon shrinking V if necessary. Together with the compactness of X, it follows that (4.1) holds for all $(x,y) \in W(r_4)$ and all $\widehat{z} \in \mathcal{G}_x$, upon shrinking r_4 and enlarging C_1 if necessary. The proofs of (4.2) to (4.6) are the same as that of (4.1), and thus they will be skipped. We just remark that the explicit expressions for the lower order terms in (3.11), (3.16), (3.18) are needed in the derivation of (4.2), (4.4) and (4.6) respectively.

Let X' and $\sigma : X' \to \mathcal{G}$ be as chosen in Section 3. Let $m \in \mathbb{N}$, r be a number satisfying $0 < r < r_o$, $s \in H^0(X, L^m \otimes E)$, and I be as in (3.30). Then one easily sees that

$$|\mathrm{I}| \leq \iint_{(x,y)\in W'(r)} |\Psi_R(x,y)^m \Psi_P(x,y) - T_\sigma^{(m)}(x,y)| \cdot \left|\frac{s(y)\overline{s(x)}}{R(y,\overline{x})^m P(y,\overline{x})}\right| \Omega(y)\Omega(x). \tag{4.7}$$

By (2.3)–(2.4) (with Q there given by $R^m P$), (3.5)–(3.6) and using the identities $R(y,\overline{x}) = \overline{R(x,\overline{y})}$, $P(y,\overline{x}) = \overline{P(x,\overline{y})}$, one easily sees that

$$\left|\frac{s(y)\overline{s(x)}}{R(y,\overline{x})^m P(y,\overline{x})}\right|^2 = \frac{\|s(x)\|^2 \|s(y)\|^2}{\Psi_R(x,y)^m \Psi_P(x,y)}. \tag{4.8}$$

Together with (4.7), one has

$$
|\mathrm{I}| \leq \iint_{(x,y) \in W'(r)} \left| 1 - \frac{T_\sigma^{(m)}(x,y)}{\Psi_R(x,y)^m \Psi_P(x,y)} \right| \Psi_R(x,y)^{\frac{m}{2}} \Psi_P(x,y)^{\frac{1}{2}}
$$
$$
\cdot \|s(x)\| \cdot \|s(y)\| \, \Omega(y)\Omega(x). \tag{4.9}
$$

Next we consider a pointwise estimate for part of the integrand in (4.9) as follows.

Lemma 4.2 *There exist constants C_7, $r_5 > 0$ with $r_5 < r_4$ such that, for all $m \in \mathbb{N}$ and all $(x,y) \in W'\left(\frac{r_5}{m^{\frac{1}{5}}}\right)$, one has*

$$
\left| 1 - \frac{T_\sigma^{(m)}(x,y)}{\Psi_R(x,y)^m \Psi_P(x,y)} \right| \leq C_7 (\rho(x,y)^3 + m\rho(x,y)^5). \tag{4.10}
$$

Here $\rho(x,y)$ is as in (3.23).

Proof Recall from (3.27) that, for $(x,y) \in W'(r_o)$,

$$
\frac{T_\sigma^{(m)}(x,y)}{\Psi_R(x,y)^m \Psi_P(x,y)} = \mathrm{A}(x,y) \cdot \mathrm{B}(x,y) \cdot \mathrm{C}(x,y)^m, \tag{4.11}
$$

where

$$
\begin{aligned}
\mathrm{A}(x,y) &:= \frac{\Psi_{P,\leq 2}(\sigma(x))(y)}{\Psi_P(x,y)}, \\
\mathrm{B}(x,y) &:= \frac{\Omega_{\leq 2}(\sigma(x))(y)}{\Omega(\widehat{z})(y)}, \\
\mathrm{C}(x,y) &:= \frac{\Psi_{R,\leq 4}(\sigma(x))(y)}{\Psi_R(x,y)}.
\end{aligned} \tag{4.12}
$$

Using the identity $1 - \mathrm{ABC}^m = (1 - \mathrm{A}) + \mathrm{A}(1 - \mathrm{B}) + \mathrm{AB}(1 - \mathrm{C}^m)$, one has

$$
\left| 1 - \frac{T_\sigma^{(m)}(x,y)}{\Psi_R(x,y)^m \Psi_P(x,y)} \right| \leq |1 - \mathrm{A}(x,y)| + |\mathrm{A}(x,y)| \, |1 - \mathrm{B}(x,y)|
$$
$$
+ |\mathrm{A}(x,y)| \, |\mathrm{B}(x,y)| \, |1 - \mathrm{C}(x,y)^m|. \tag{4.13}
$$

Let C_4 be as in (4.4). Now we choose $r_5 < r_4$ so that $1 - C_4 r_5^2 > \frac{1}{2}$. Then by (4.3)–(4.4), for all $(x,y) \in W'(r_5)$, one has

$$
|1 - \mathrm{A}(x,y)| \leq \frac{C_3 \rho(x,y)^3}{1 - C_4 \rho(x,y)^2} \leq \frac{C_3 \rho(x,y)^3}{1 - C_4 r_5^2} \leq C_8 \rho(x,y)^3, \tag{4.14}
$$

where $C_8 = 2C_3$. Similarly, using (4.1)–(4.2), (4.5)–(4.6), and shrinking r_5 if necessary, one easily sees that there exist constants $C_9, C_{10} > 0$ such that for all $(x,y) \in W'(r_5)$, one has

$$
|1 - \mathrm{B}(x,y)| \leq C_9 \rho(x,y)^3, \tag{4.15}
$$
$$
|1 - \mathrm{C}(x,y)| \leq C_{10} \rho(x,y)^5. \tag{4.16}
$$

By (4.3)–(4.4), and shrinking r_5 further if necessary, it is also clear that there exist constants $C_{11}, C_{12} > 0$ such that for all $(x,y) \in W'(r_5)$, one has

$$
|\mathrm{A}(x,y)| \leq C_{11}, \tag{4.17}
$$

$$|B(x,y)| \leq C_{12}. \tag{4.18}$$

Now let $m \in \mathbb{N}$ and $(x,y) \in W'\left(\frac{r_5}{m^{\frac{1}{5}}}\right)$ be given. By (4.16), one has $|C(x,y)| \leq 1 + \frac{C_{10}r_5^5}{m}$, so that for each $1 \leq k \leq m-1$,

$$|C(x,y)|^k \leq \left(1 + \frac{C_{10}r_5^5}{m}\right)^k \leq \left(1 + \frac{C_{10}r_5^5}{m}\right)^m \leq e^{C_{10}r_5^5}, \tag{4.19}$$

and thus

$$\begin{aligned}
|1 - C(x,y)^m| &= |1 - C(x,y)| \cdot |1 + C(x,y) + C(x,y)^2 + \cdots + C(x,y)^{m-1}| \\
&\leq C_{10}\rho(x,y)^5 \cdot m \cdot e^{C_{10}r_5^5} \\
&= C_{13} \cdot m\rho(x,y)^5, \quad \text{where } C_{13} := C_{10}e^{C_{10}r_5^5}.
\end{aligned} \tag{4.20}$$

Combining (4.13)–(4.15), (4.17)–(4.18) and (4.20), one has, for all $(x,y) \in W'\left(\frac{r_5}{m^{\frac{1}{5}}}\right)$,

$$\begin{aligned}
\left|1 - \frac{T_\sigma^{(m)}(x,y)}{\Psi_R(x,y)^m \Psi_P(x,y)}\right| &\leq C_8\rho(x,y)^3 + C_{11}C_9\rho(x,y)^3 + C_{11}C_{12}C_{13}m\rho(x,y)^5 \\
&= C_7(\rho(x,y)^3 + m\rho(x,y)^5),
\end{aligned} \tag{4.21}$$

where $C_7 := \max\{C_8 + C_{11}C_9, C_{11}C_{12}C_{13}\}$. This finishes the proof of the lemma.

Lemma 4.3 *There exists a constant $r_6 > 0$ with $r_6 < r_4$ such that, for all $(x,y) \in W(r_6)$, the quantity $\rho(y,x)$ is well-defined and $\rho(y,x) \leq 2\rho(x,y)$.*

Proof First it follows readily from the compactness of X, the construction of $\mathcal{G}$ and the definition of ρ that there exists a constant $r_7 > 0$ with $r_7 < r_4$ such that, for $(x,y) \in W(r_7)$, the quantity $\rho(y,x)$ is well-defined and satisfies $\rho(y,x) < r_4$. By shrinking r_7 if necessary and using (4.2), we may assume that

$$\frac{1}{2}\rho(x,y)^2 \leq |\Psi_R(x,y) - 1| \leq 2\rho(x,y)^2 \quad \text{for all } (x,y) \in W(r_7). \tag{4.22}$$

Repeating the above argument (with r_4 replaced by r_7), one sees that there exists a constant $r_6 > 0$ with $r_6 < r_7$ such that, for $(x,y) \in W(r_6)$, one has $\rho(y,x) < r_7$. Now let $(x,y) \in W(r_6)$ be given (so that $(y,x) \in W(r_7)$). Then using (4.22) but with the roles of x and y interchanged, one has

$$\frac{1}{2}\rho(y,x)^2 \leq |\Psi_R(y,x) - 1|. \tag{4.23}$$

Together with the identity $\Psi_R(x,y) = \Psi_R(y,x)$ and the second inequality in (4.22), one has $\frac{1}{2}\rho(y,x)^2 \leq 2\rho(x,y)^2$ and thus $\rho(y,x) \leq 2\rho(x,y)$.

For $\mathbb{C}^n$, we denote its Euclidean ball centered at 0 and of radius r and its Euclidean volume form by

$$B(r) := \{z \in \mathbb{C}^n \mid |z| < r\}, \tag{4.24}$$

$$dV(z) = \left(\frac{\sqrt{-1}}{2}\right)^n dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_n \wedge d\overline{z_n}. \tag{4.25}$$

Lemma 4.4 *There exist constants C_{14}, $r_8 > 0$ with $r_8 < r_4$ such that, for all $k, r \geq 0$ and all $x \in X$, if $r < r_8$, then*

$$\int_{y \in B(x,r)} \rho(x,y)^k \Omega(y) \leq C_{14} r^{2n+k}. \tag{4.26}$$

Proof One easily sees from (4.6) that there exist constants C_{15}, $r_8 > 0$ with $r_8 < r_4$ such that, for all $x \in X$, all $\widehat{z} \in \mathcal{G}_x$ and all $y \in B(x, r_8)$, one has $\Omega(\widehat{z})(y) \leq C_{15}$. Hence, in terms of the coordinate functions $z : B(x, r_4) \to \mathbb{C}^n$ associated to $\widehat{z}$, (4.24)–(4.25), one has, for each $k > 0$ and $0 < r < r_8$,

$$\begin{aligned}
\int_{y \in B(x,r)} \rho(x,y)^k \Omega(y) &\leq C_{15} \int_{z \in B(r)} |z|^k \, dV(z) \\
&= C_{15} \frac{2\pi^n r^{2n+k}}{(n-1)!(2n+k)} \\
&\leq C_{14} r^{2n+k}, \quad \text{where } C_{14} := 2\pi^n C_{15}.
\end{aligned} \tag{4.27}$$

Here the second line in (4.27) follows from a straightforward computation.

Proposition 4.1 *There exist constants C_{16}, $r_9 > 0$ such that, for all $m \in \mathbb{N}$, all $s \in H^0(X, L^m \otimes E)$ and all r satisfying $0 < r \leq \frac{r_9}{m^{\frac{n+2}{2n+5}}}$, one has*

$$|\mathrm{I}| \leq \frac{C_{16}}{m^{n+1}} \|s\|_2^2. \tag{4.28}$$

Here I *is as in* (3.30) *(with r there as above).*

Proof Recall from the (SGCS-1) condition for R that $|\Psi_R(x,y)| \leq 1$ for all $(x,y) \in X \times X$. One also easily sees from (4.4) that there exist constants C_{17}, $r_{10} > 0$ such that, for all $(x,y) \in W(r_o)$, one has $|\Psi_P(x,y)| \leq C_{17}$. Now, we let $r_9 := \min\{r_0, r_4, r_5, \frac{r_8}{2}, r_{10}\} > 0$, where the r_i's (and C_j's) are as chosen before. Let I be as in (3.30) with r there satisfying $0 < r \leq \frac{r_9}{m^{\frac{n+2}{2n+5}}}$. Note that for $n \geq 1$, one has $\frac{1}{m^{\frac{n+2}{2n+5}}} < \frac{1}{m^{\frac{1}{5}}}$. Hence from (4.9) and Lemma 4.2, one has

$$\begin{aligned}
|\mathrm{I}| &\leq C_7 C_{17} \iint_{(x,y) \in W'(r)} (\rho(x,y)^3 + m\rho(x,y)^5) \|s(x)\| \, \|s(y)\| \Omega(y)\Omega(x) \\
&= C_7 C_{17} (\mathrm{J}_3(r) + m \mathrm{J}_5(r)),
\end{aligned} \tag{4.29}$$

where, for $k > 0$ and $0 < r < r_0$,

$$\mathrm{J}_k(r) := \iint_{(x,y) \in W(r)} \rho(x,y)^k \|s(x)\| \, \|s(y)\| \Omega(y)\Omega(x). \tag{4.30}$$

By the Cauchy-Schwarz inequality, one has

$$|\mathrm{J}_k(r)| \leq (\mathrm{J}_{k,1}(r))^{\frac{1}{2}} (\mathrm{J}_{k,2}(r))^{\frac{1}{2}}, \tag{4.31}$$

where, as iterated integrals,

$$\mathrm{J}_{k,1}(r) := \int_{x \in X} \int_{y \in B(x,r)} \rho(x,y)^{2k} \|s(x)\|^2 \Omega(y)\Omega(x), \tag{4.32}$$

$$\mathrm{J}_{k,2}(r) := \int_{x \in X} \int_{y \in B(x,r)} \|s(y)\|^2 \Omega(y)\Omega(x). \tag{4.33}$$

By Lemma 4.4, one has

$$\mathrm{J}_{k,1}(r) = \int_{x \in X} \|s(x)\|^2 \int_{y \in B(x,r)} \rho(x,y)^{2k} \Omega(y)\Omega(x) \leq C_{14} r^{2n+2k} \|s\|_2^2. \tag{4.34}$$

For each $y \in X$, let $B'(y,r) := \{x \in X \mid \rho(x,y) < r\}$. Note that, for $r \leq r_9$, it follows from Lemma 4.3 that $B'(y,r) \subseteq B(y,2r) \subseteq B(y,r_8)$. Thus, upon interchanging the order of integration in (4.33), one has

$$\begin{aligned}
\mathrm{J}_{k,2}(r) &= \int_{y \in X} \|s(y)\|^2 \int_{x \in B'(y,r)} \Omega(x)\Omega(y) \\
&\leq \int_{y \in X} \|s(y)\|^2 \int_{x \in B(y,2r)} \Omega(x)\Omega(y) \\
&\leq C_{14}(2r)^{2n} \|s\|_2^2, \tag{4.35}
\end{aligned}$$

where the last line follows from Lemma 4.4. Combining (4.31) and (4.34)–(4.35), it follows that for $r \leq r_9$,

$$|\mathrm{J}_k(r)| \leq \left(C_{14} r^{2n+2k} \|s\|_2^2\right)^{\frac{1}{2}} \left(C_{14}(2r)^{2n} \|s\|_2^2\right)^{\frac{1}{2}} = C_{14} 2^n r^{2n+k} \|s\|_2^2. \tag{4.36}$$

From (4.29) and (4.36), one has

$$\begin{aligned}
|\mathrm{I}| &\leq C_7 C_{17}(C_{14} 2^n r^{2n+3} + m C_{14} 2^n r^{2n+5}) \|s\|_2^2 \\
&= C_{18}(r^{2n+3} + m r^{2n+5}) \|s\|_2^2, \quad \text{where } C_{18} := 2^n C_7 C_{17} C_{14}. \tag{4.37}
\end{aligned}$$

Now for each r satisfying $0 < r \leq \frac{r_9}{m^{\frac{n+2}{2n+5}}}$, one easily checks that $r^{2n+3} \leq \frac{r_9^{2n+3}}{m^{n+1}}$ and $mr^{2n+5} \leq \frac{r_9^{2n+5}}{m^{n+1}}$, and hence one has

$$|\mathrm{I}| \leq \frac{C_{16}}{m^{n+1}} \|s\|_2^2, \quad \text{where } C_{16} := C_{18}(r_9^{2n+3} + r_9^{2n+5}). \tag{4.38}$$

5 Estimation of II

In this section, we are going to estimate the expression II in (3.30). For $r > 0$, let $B(r)$ be as in (4.24), and denote its closure by $\overline{B(r)} := \{z \in \mathbb{C}^n \mid |z| \leq r\}$. Let q be an analytic function admitting a power series expansion $q(z) = \sum_{\alpha,\beta} q_{\alpha\overline{\beta}} z^\alpha \overline{z^\beta}$ on $\overline{B(r)}$ (here the notation is as in (3.7)). Then q is said to *have only quasi-diagonal terms* if $q_{\alpha\overline{\beta}} = 0$ whenever $|\alpha| \neq |\beta|$.

Lemma 5.1 *Let f be a holomorphic function admitting a power series expansion $f(z) = \sum_\alpha f_\alpha z^\alpha$ on $\overline{B(r)}$, and let q be an analytic function admitting a power series expansion $q(z) = \sum_{\alpha,\beta} q_{\alpha\overline{\beta}} z^\alpha \overline{z^\beta}$ on $\overline{B(r)}$. If q has only quasi-diagonal terms, then*

$$\int_{B(r)} f(z)q(z)\, \mathrm{d}V(z) = f(0) \int_{B(r)} q(z)\, \mathrm{d}V(z). \tag{5.1}$$

Here $\mathrm{d}V(z)$ is as in (4.25).

Proof First we recall that for multi-indices α and β, one has

$$\int_{B(r)} z^\alpha \overline{z}^\beta \mathrm{d}V(z) = 0 \quad \text{whenever } \alpha \neq \beta, \tag{5.2}$$

which can be verified easily by considering the change of variables given by $(z_1, \cdots, z_n) \to (\mathrm{e}^{\mathrm{i}\theta_1} z_1, \cdots, \mathrm{e}^{\mathrm{i}\theta_n} z_n)$, and then letting $\theta_1, \cdots, \theta_n$ vary. Let g be the function given by $g(z) = f(z) - f(0)$ for $z \in \overline{B(r)}$. Then $g(0) = 0$ and g is also a holomorphic function admitting a power series expansion $g(z) = \sum_{|\gamma|>0} g_\gamma z^\gamma$ on $\overline{B(r)}$, noting that $g_\gamma = g(0) = 0$ when $|\gamma| = 0$. Then one has

$$\int_{B(r)} g(z)q(z)\mathrm{d}V(z) = \sum_{|\gamma|>0} \sum_{\alpha,\beta} g_\gamma q_{\alpha\overline{\beta}} \int_{B(r)} z^{\alpha+\gamma}\overline{z}^\beta \mathrm{d}V(z)$$

$$= \sum_{|\gamma|>0} \sum_{\alpha} g_\gamma q_{\alpha,\overline{\alpha+\gamma}} \int_{B(r)} |z^{\alpha+\gamma}|^2 \mathrm{d}V(z), \tag{5.3}$$

where the last equality follows from (5.2). Since q has only quasi-diagonal terms, it follows that for each α and γ satisfying $|\gamma| > 0$, one has $|\alpha + \gamma| = |\alpha| + |\gamma| > |\alpha|$, and thus $q_{\alpha,\overline{\alpha+\gamma}} = 0$. Hence one has $\int_{B(r)} g(z)q(z)\,\mathrm{d}V(z) = 0$, which leads to (5.1) readily.

Lemma 5.2 *Notation as in* (4.24)–(4.25). *For each integer $m, k \geq 0$ and real number $a > 0$,*

$$\int_{z \in B(\frac{1}{\sqrt{a}})} |z|^{2k} \left(1 - a|z|^2\right)^m \mathrm{d}V(z) = \frac{\pi^n (n+k-1)! m!}{(n-1)!(m+k+n)! a^{n+k}}. \tag{5.4}$$

Proof We will skip the proof, which follows from a direct calcluation.

Let X' and $\sigma : X' \to \mathcal{G}$ be as chosen in Section 3. Let $m \in \mathbb{N}$, r be a number satisfying $0 < r < r_o$, $s \in H^0(X, L^m \otimes E)$, and II be as in (3.30). Rewriting the first term of II as an iterated integral, one has

$$\mathrm{II} = \int_{x \in X'} \Lambda(x)\Omega(x) - \frac{\pi^n}{m^n} \|s\|_2^2, \tag{5.5}$$

where, for each $x \in X'$,

$$\Lambda(x) := \int_{y \in B(x,r)} T_\sigma^{(m)}(x,y) \frac{s(y)\overline{s(x)}}{R(y,\overline{x})^m P(y,\overline{x})} \Omega(y). \tag{5.6}$$

Using (3.14)–(3.15) and (3.27), for $x \in X'$, one has

$$\Lambda(x) = \int_{y \in B(x,r)} \lambda(x,y) \frac{s(y)\overline{s(x)}}{R(y,\overline{x})^m P(y,\overline{x})} \, \mathrm{d}V(\sigma(x))(y), \tag{5.7}$$

where, for $(x, y) \in W'(r_o)$,

$$\lambda(x,y) := \Psi_{P,\leq 2}(\sigma(x))(y) \cdot \Omega_{\leq 2}(\sigma(x))(y) \cdot \left(\Psi_{R,\leq 4}(\sigma(x))(y)\right)^m. \tag{5.8}$$

For each $x \in X'$ and in terms of the coordinate functions $z : B(x, r_0) \to \mathbb{C}^n$ associated to $\sigma(x)$, it follows readily from (3.19)–(3.21) that $\lambda(x, y)$ is an analytic function in the variable $z = z(y)$

and has only quasi-diagonal terms. Also, the quotient $\frac{s(y)\overline{s(x)}}{R(y,\overline{x})^m P(y,\overline{x})}$ is a holomorphic function in the variable y. Thus, by Lemma 5.1, one has, for $x \in X'$,

$$\Lambda(x) = \|s(x)\|^2 \int_{y \in B(x,r)} \lambda(x,y) \, dV(\sigma(x))(y). \tag{5.9}$$

In a way similar to Lemma 4.2, we have the following pointwise estimate.

Lemma 5.3 *There exist constants C_{19}, $r_{10} > 0$ with $r_{10} < r_4$ such that, for all $m \in \mathbb{N}$ and all $(x,y) \in W'(r_{10})$, one has*

$$\left|\lambda(x,y) - (1 - \rho(x,y)^2)^m\right| \leq C_{19}(\rho(x,y)^2 + m\rho(x,y)^4)\left(1 - \frac{\rho(x,y)^2}{2}\right)^{m-1}. \tag{5.10}$$

Proof For $(x,y) \in W'(r_0)$, we rewrite (5.8) as

$$\lambda(x,y) = \mathcal{A}(x,y)\mathcal{B}(x,y)^m, \tag{5.11}$$

where

$$\mathcal{A}(x,y) := \Psi_{P,\leq 2}(\sigma(x))(y) \cdot \Omega_{\leq 2}(\sigma(x))(y),$$
$$\mathcal{B}(x,y) := \Psi_{R,\leq 4}(\sigma(x))(y).$$

It follows readily from Lemma 4.1 that there exist constants C_{20}, C_{21}, $r_{10} > 0$ with $r_{10} < \min\{r_4, 1\}$ such that, for all $(x,y) \in W'(r_{10})$, one has

$$|\mathcal{A}(x,y) - 1| \leq C_{20}\rho(x,y)^2, \tag{5.12}$$

$$\left|\mathcal{B}(x,y) - (1 - \rho(x,y)^2)\right| \leq C_{21}\rho(x,y)^4, \tag{5.13}$$

$$0 \leq \mathcal{B}(x,y) \leq 1 - \frac{\rho(x,y)^2}{2}. \tag{5.14}$$

(For example, (5.13) follows immediately from (4.1)–(4.2), while (5.14) follows from (5.13), upon shrinking r_{10} if necessary.) By (5.11), one has

$$\lambda(x,y) - (1 - \rho(x,y)^2)^m$$
$$= (\mathcal{A}(x,y) - 1)\mathcal{B}(x,y)^m + \mathcal{B}(x,y)^m - (1 - \rho(x,y)^2)^m$$
$$= (\mathcal{A}(x,y) - 1)\mathcal{B}(x,y)^m + [\mathcal{B}(x,y) - (1 - \rho(x,y)^2)] \cdot \sum_{j=0}^{m-1} \mathcal{B}(x,y)^j (1 - \rho(x,y)^2)^{m-1-j}. \tag{5.15}$$

For all $(x,y) \in W'(r_{10})$, from (5.14), one has

$$\sum_{j=0}^{m-1} \mathcal{B}(x,y)^j (1 - \rho(x,y)^2)^{m-1-j}$$
$$\leq \sum_{j=0}^{m-1} \left(1 - \frac{\rho(x,y)^2}{2}\right)^j (1 - \rho(x,y)^2)^{m-1-j}$$
$$\leq m\left(1 - \frac{\rho(x,y)^2}{2}\right)^{m-1}, \tag{5.16}$$

and together with (5.12), (5.14)–(5.15), one has

$$\left|\lambda(x,y) - (1 - \rho(x,y)^2)^m\right| \le C_{20}\rho(x,y)^2\left(1 - \frac{\rho(x,y)^2}{2}\right)^m + C_{21}\rho(x,y)^4 m\left(1 - \frac{\rho(x,y)^2}{2}\right)^{m-1}$$

$$\le C_{19}(\rho(x,y)^2 + m\rho(x,y)^4)\left(1 - \frac{\rho(x,y)^2}{2}\right)^{m-1},$$

where $C_{19} := \max\{C_{20}, C_{21}\}$, noting that $0 < 1 - \frac{\rho(x,y)^2}{2} < 1$.

Lemma 5.4 *There exist constants $C_{22}, r_{11} > 0$ such that, for all $m \in \mathbb{N}$, all $x \in X'$ and all r satisfying $\sqrt{\frac{(n+1)\log m}{m}} < r < r_{11}$, one has*

$$\left|\int_{y \in B(x,r)} \lambda(x,y)\, dV(\sigma(x))(y) - \frac{\pi^n}{m^n}\right| \le \frac{C_{22}}{m^{n+1}}. \tag{5.17}$$

Proof For $x \in X'$, $m \in \mathbb{N}$ and $r > 0$, we let

$$\eta(x,r) := \int_{y \in B(x,r)} \lambda(x,y)\, dV(\sigma(x))(y) - \frac{\pi^n}{m^n}$$

$$= \eta_1(x,r) + \eta_2(x,r), \tag{5.18}$$

where

$$\eta_1(x,r) := \int_{y \in B(x,r)} [\lambda(x,y) - (1 - \rho(x,y)^2)^m]\, dV(\sigma(x))(y), \tag{5.19}$$

$$\eta_2(x,r) := \int_{y \in B(x,r)} (1 - \rho(x,y)^2)^m\, dV(\sigma(x))(y) - \frac{\pi^n}{m^n}. \tag{5.20}$$

In terms of the coordinate functions $z : B(x, r_o) \to \mathbb{C}^n$ associated to $\sigma(x) \in \mathcal{G}_x$, and identifying $B(x,r)$ with $B(r)$ (following the notation in (4.24)–(4.25)), it follows readily from Lemma 5.3 that, if $0 < r < r_{10}$, then

$$|\eta_1(x,r)| \le C_{19}\int_{B(r)} (|z|^2 + m|z|^4)\left(1 - \frac{1}{2}|z|^2\right)^{m-1} dV(z). \tag{5.21}$$

From Lemma 5.2 (with $k = 1, 2$, $a = \frac{1}{2}$, and m replaced by $m-1$) and noting that $\frac{(m-1)!}{(m-1+n+k)!} \le \frac{1}{m^{n+k}}$, etc., one has, for $r < \sqrt{2}$,

$$|\eta_1(x,r)| \le \frac{C_{19}\pi^n n!(m-1)!2^{n+1}}{(n-1)!(m+n)!} + m\frac{C_{19}\pi^n(n+1)!(m-1)!2^{n+2}}{(n-1)!(m+n+1)!} \le \frac{C_{23}}{m^{n+1}}, \tag{5.22}$$

where $C_{23} := C_{19}\pi^n(n \cdot 2^{n+1} + n(n+1)2^{n+2})$. Similarly, for $r < 1$, one has

$$\eta_2(x,r) = \int_{B(r)} (1 - |z|^2)^m\, dV(z) - \frac{\pi^n}{m^n}$$

$$= \left(\int_{B(1)} (1 - |z|^2)^m\, dV(z) - \frac{\pi^n}{m^n}\right) - \int_{B(1)\setminus B(r)} (1 - |z|^2)^m\, dV(z). \tag{5.23}$$

By Lemma 5.2 again (with $k = 0$ and $a = 1$), one has

$$0 < \frac{\pi^n}{m^n} - \int_{B(1)} (1 - |z|^2)^m\, dV(z) = \frac{\pi^n}{m^n} - \frac{\pi^n m!}{(m+n)!} < \frac{\pi^n n(n+1)}{2m^{n+1}}, \tag{5.24}$$

where the last inequality can be obtained by substituting $x_k = \frac{k}{m+k}$ into the following generalization of Bernoulli's inequality (which follows from a straight-forward induction):

$$\prod_{k=1}^{n}(1 - x_k) \geq 1 - \sum_{k=1}^{n} x_k, \quad \text{if } 0 \leq x_1, \cdots, x_n \leq 1.$$

From pointwise consideration, one has

$$0 \leq \int_{B(1)\setminus B(r)} (1 - |z|^2)^m \, dV(z)$$
$$\leq (1 - r^2)^m \int_{B(1)\setminus B(r)} dV(z)$$
$$\leq (1 - r^2)^m \frac{\pi^n}{n!}. \tag{5.25}$$

Note that, if $0 < r < 1$, then, upon taking the natural logarithm,

$$(1 - r^2)^m < \frac{1}{m^{n+1}} \quad \Longleftrightarrow \quad -\log(1 - r^2) > \frac{(n+1)\log m}{m}. \tag{5.26}$$

Using the fact that $-\log(1 - t) > t$ for all $0 < t < 1$, one sees that both sides of (5.26) hold if $r > \sqrt{\frac{(n+1)\log m}{m}}$. For such r, it follows from (5.23)–(5.25) that

$$|\eta_2(x,r)| \leq \frac{\pi^n n(n+1)}{2m^{n+1}} + \frac{\pi^n}{n!} \frac{1}{m^{n+1}}. \tag{5.27}$$

Now we let $r_{11} = \min\{r_{10}, 1\} (> 0)$. Combining (5.18), (5.22) and (5.27), it follows that, if $\sqrt{\frac{(n+1)\log m}{m}} < r < r_{11}$, then

$$\left| \int_{y \in B(x,r)} \lambda(x,y) \, dV(\sigma(x))(y) - \frac{\pi^n}{m^n} \right| \leq \frac{C_{23}}{m^{n+1}} + \frac{\pi^n n(n+1)}{2m^{n+1}} + \frac{\pi^n}{n!} \frac{1}{m^{n+1}} = \frac{C_{22}}{m^{n+1}}, \tag{5.28}$$

where $C_{22} := C_{23} + \frac{\pi^n n(n+1)}{2} + \frac{\pi^n}{n!}$,

Proposition 5.1 *Let C_{22} and r_{11} be as in Lemma 5.4. Then, for all $m \in \mathbb{N}$, all $s \in H^0(X, L^m \otimes E)$ and all r satisfying $\sqrt{\frac{(n+1)\log m}{m}} < r < r_{11}$, one has*

$$|\mathrm{II}| \leq \frac{C_{22}}{m^{n+1}} \|s\|_2^2. \tag{5.29}$$

Here II *is as in* (3.30) *(with r there as above).*

Proof From (5.5) and (5.9), one easily sees that

$$\mathrm{II} = \int_{x \in X'} \|s(x)\|^2 \left[\int_{y \in B(x,r)} \lambda(x,y) \, dV(\sigma(x))(y) - \frac{\pi^n}{m^n} \right] \Omega(x). \tag{5.30}$$

Then, by Lemma 5.4, one immediately has

$$|\mathrm{II}| \leq \frac{C_{22}}{m^{n+1}} \int_{x \in X'} \|s(x)\|^2 \Omega(x) dx = \frac{C_{22}}{m^{n+1}} \|s\|_2^2. \tag{5.31}$$

6 Estimation of III and Proof of Theorem 1.1

Lemma 6.1 *There exists a constant r_{12} with $0 < r_{12} < r_4$ such that, for each real number r satisfying $0 < r < r_{12}$ and each $(x, y) \in (X \times X) \setminus W(r)$, one has $\Psi_R(x, y) \leq 1 - \frac{r^2}{2}$.*

Proof It follows readily from (4.2) that there exists a constant $r_{13} > 0$ with $r_{13} < r_4$ such that, for all $(x, y) \in W(r_{13})$, one has

$$\Psi_R(x, y) < 1 - \frac{1}{2}\rho(x, y)^2. \tag{6.1}$$

Then by (SGCS-1) for R, one has

$$\alpha := \sup_{(x,y)\in(X\times X)\setminus W(r_{13})} \Psi_R(x, y) < 1. \tag{6.2}$$

Now one has $r_{12} := \min\{\sqrt{2(1-\alpha)}, r_{13}\} > 0$, and let r be a number satisfying $0 < r < r_{12}$. Note that $(X\times X)\setminus W(r) = [(X\times X)\setminus W(R_{13})]\cup[W(r_{13})\setminus W(r)]$. For any $(x, y) \in X\times X\setminus W(r_{13})$, it follows from (6.2) and the definition of r_{12} that

$$\Psi_R(x, y) \leq \alpha \leq 1 - \frac{r_{12}^2}{2} < 1 - \frac{r^2}{2}. \tag{6.3}$$

On the other hand, if $(x, y) \in W_{r_{13}} \setminus W(r)$ (so that $\rho(x, y) \geq r$), one also has, from (6.1),

$$\Psi_R(x, y) \leq 1 - \frac{\rho(x, y)^2}{2} \leq 1 - \frac{r^2}{2}. \tag{6.4}$$

Hence, for all $(x, y) \in (X \times X) \setminus W(r)$, one has $\Psi_R(x, y) \leq 1 - \frac{r^2}{2}$.

Proposition 6.1 *Let r_{12} be as in Lemma 6.1. Then there exists a constant $C_{23} > 0$ such that, for all $m \in \mathbb{N}$, all $s \in H^0(X, L^m \otimes E)$ and all r satisfying $\sqrt{\frac{2(n+1)\log m}{m}} < r < r_{12}$, one has*

$$|\mathrm{III}| \leq \frac{C_{23}}{m^{n+1}}\|s\|_2^2. \tag{6.5}$$

Here III is as in (3.30) (with r there as above).

Proof From the definitions of Ψ_R, Ψ_P in (3.5)–(3.6) and similar to (4.9), one easily sees that the integral III in (3.30) satisfies

$$|\mathrm{III}| \leq \iint_{(X\times X)\setminus W(r)} \Psi_R(x, y)^m \Psi_P(x, y)\|s(x)\|\|s(y)\|\Omega(y)\Omega(x). \tag{6.6}$$

By Lemma 6.1, if $0 < r < r_{12}$, then one has, for all $(x, y) \in (X \times X) \setminus W(r)$ and $m \in \mathbb{N}$,

$$\Psi_R(x, y)^m \leq \left(1 - \frac{r^2}{2}\right)^m \leq \mathrm{e}^{-\frac{mr^2}{2}}, \tag{6.7}$$

where the last inequality follows from the fact that $0 \leq 1 - t \leq \mathrm{e}^{-t}$ for all $0 \leq t \leq 1$. By the compactness of the manifold X, there exists a constant $C_{24} > 0$ such that $0 \leq \Psi_P(x, y) \leq C_{24}$ for all $(x, y) \in X \times X$. Together with (6.6) and (6.7), for $0 < r < r_{12}$, one has

$$|\mathrm{III}| \leq C_{24}\mathrm{e}^{-\frac{mr^2}{2}} \iint_{(X\times X)\setminus W(r)} \|s(x)\|\|s(y)\|\Omega(y)\Omega(x) \leq C_{24}\mathrm{e}^{-\frac{mr^2}{2}}\mathrm{vol}_\Omega(X)\|s\|_2^2 \tag{6.8}$$

where the last inequality follows from the Cauchy-Schwarz inequality and $\mathrm{vol}_\Omega(X) := \int_X \Omega$. By taking the natural logarithm, one has

$$\mathrm{e}^{-\frac{mr^2}{2}} \leq \frac{1}{m^{n+1}} \quad \Longleftrightarrow \quad r \geq \sqrt{\frac{2(n+1)\log m}{m}}, \tag{6.9}$$

It follows that if $\sqrt{\frac{2(n+1)\log m}{m}} < r < r_{12}$, then one has

$$|\mathrm{III}| \leq \frac{C_{23}}{m^{n+1}}\|s\|_2^2, \quad \text{where } C_{23} := C_{24}\mathrm{vol}_\Omega(X). \tag{6.10}$$

Now we complete the proof of Theorem 1.1 as follows.

Proof of Theorem 1.1 Let C_{16}, r_9, C_{22}, r_{11}, C_{23}, r_{12} be as in Proposition 4.1, Proposition 5.1 and Proposition 6.1. Then it is easy to see that there exists $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$, one has

$$\sqrt{\frac{2(n+1)\log m}{m}} < \frac{r_9}{m^{\frac{n+2}{2n+5}}} < \min\{r_{11}, r_{12}\}. \tag{6.11}$$

Now, for each $m \geq m_0$, we choose a number $r(m)$ satisfying

$$\sqrt{\frac{2(n+1)\log m}{m}} \leq r(m) \leq \frac{r_9}{m^{\frac{n+2}{2n+5}}} \tag{6.12}$$

(in particular, $r(m)$ may be taken to be one of the two bounds). Then by (3.29)–(3.30), Proposition 4.1, Proposition 5.1 and Proposition 6.1, one sees that for all $m \geq m_0$ and all $s \in H^0(X, L^m \otimes E)$, one has, with the number r in I, II, III in (3.30) given by $r(m)$ in (6.12),

$$\left| \mathbf{K}_{R^m P, \Omega}(s,s) - \frac{\pi^n}{m^n}\|s\|_2^2 \right| \leq |\mathrm{I}| + |\mathrm{II}| + |\mathrm{III}|$$
$$\leq \frac{C_{16} + C_{22} + C_{23}}{m^{n+1}}\|s\|_2^2, \tag{6.13}$$

By the compactness of the manifold X, there exists a constant $C_{25} > 0$ such that, for each integer satisfying $1 \leq m < m_0$,

$$\left| \mathbf{K}_{R^m P, \Omega}(s,s) - \frac{\pi^n}{m^n}\|s\|_2^2 \right| \leq \frac{C_{25}}{m^{n+1}}\|s\|_2^2, \tag{6.14}$$

Thus by letting $C = \max\{C_{16} + C_{22} + C_{23}, C_{25}\} > 0$, one sees that (1.2) holds for all $m \in \mathbb{N}$, and the proof of Theorem 1.1 is completed.

Finally we remark that the deduction of Corollary 1.1 from Theorem 1.1 can be found in [7], and thus it will be skipped here.

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