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Composition Cesàro Operator on the Normal Weight Zygmund Space in High Dimensions*

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Abstract Let $n > 1$ and B be the unit ball in n dimensions complex space $\mathbf{C^n}$. Suppose that φ is a holomorphic self-map of B and $\psi \in H(B)$ with $\psi(0) = 0$. A kind of integral operator, composition Cesàro operator, is defined by

$$T_{\varphi,\psi}(f)(z) = \int_0^1 f[\varphi(tz)]R\psi(tz)\frac{\mathrm{d}t}{t}, \quad f \in H(B),\ z \in B.$$

In this paper, the authors characterize the conditions that the composition Cesàro operator $T_{\varphi,\psi}$ is bounded or compact on the normal weight Zygmund space $\mathcal{Z}_\mu(B)$. At the same time, the sufficient and necessary conditions for all cases are given.

Keywords Normal weight Zygmund space, Composition Cesàro operator, Boundedness and compactness
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1 Introduction

Let $\mathbf{C^n}$ be the Euclidean space of complex dimension n. For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbf{C^n}$, the inner product of z and w is denoted by

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}.$$

Let B denote the unit ball in $\mathbf{C^n}$. The class of all holomorphic functions on B is denoted by $H(B)$. For $f \in H(B)$, the complex gradient ∇f and the radial derivative Rf are defined by

$$\nabla f(z) = \Big(\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z)\Big), \quad Rf(z) = \langle \nabla f(z), \overline{z}\rangle = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

Definition 1.1 *A positive continuous function μ on $[0, 1)$ is called normal if there exist constants $0 < a \le b < \infty$ and $0 \le r_0 < 1$ such that*
(1) $\dfrac{\mu(r)}{(1-r)^a}$ *is decreasing on* $[r_0, 1)$; (2) $\dfrac{\mu(r)}{(1-r)^b}$ *is increasing on* $[r_0, 1)$.

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Such as, $\mu(r)=(1-r)^{\alpha}\big(\log\frac{\mathrm{e}}{1-r}\big)^{\beta}\big(\log\log\frac{\mathrm{e}^2}{1-r}\big)^{\gamma}$ ($\alpha>0$, β and γ real) and

$$\mu_1(r)=\begin{cases}\Big(\dfrac{(2n-2)!!}{(2n-1)!!}\Big)^{b-a}(1-r)^a, & 1-\dfrac{1}{n}\le r<1-\dfrac{1}{2}\Big(\dfrac{1}{n}+\dfrac{1}{n+1}\Big),\\ \Big(\dfrac{(2n)!!(n+1)}{(2n+1)!!}\Big)^{b-a}(1-r)^b, & 1-\dfrac{1}{2}\Big(\dfrac{1}{n}+\dfrac{1}{n+1}\Big)\le r<1-\dfrac{1}{n+1}\end{cases}$$

($n=1,2,\cdots,\ b>a>0$) are the normal functions.

Without loss of generality, let $r_0=0$ in this paper.

Let D be the disc in complex plane $\mathbf{C}$. If $f\in H(D)$ and $\sup\limits_{z\in D}(1-|z|^2)|f''(z)|<\infty$, then f is said to belong to the Zygmund space $\mathcal{Z}(D)$. In fact, the function $1-|z|^2$ may be regarded as a kind of weight function. Later, the space is called as the Zygmund type space $\mathcal{Z}^p(D)$ if the weight function $1-|z|^2$ is generalized to $(1-|z|^2)^p$ ($p>0$). In this paper, we generalize the weight function $1-|z|^2$ to the normal function $\mu(|z|)$, and generalize the variable from one complex variable to several complex variables.

Definition 1.2 *Let μ be a normal function on $[0,1)$. A function f is said to belong to the normal weight Zygmund space $\mathcal{Z}_\mu(B)$ if $f\in H(B)$ and*

$$||f||_\mu=\sup_{z\in B}\mu(|z|)\sum_{k=1}^n\sum_{j=1}^n\Big|\frac{\partial^2 f}{\partial z_j\partial z_k}(z)\Big|<\infty.$$

It is easy to prove that $\mathcal{Z}_\mu(B)$ is a Banach space under the norm

$$||f||_{\mathcal{Z}_\mu}=|f(0)|+\sum_{l=1}^n\Big|\frac{\partial f}{\partial z_l}(0)\Big|+||f||_\mu.$$

In particular, it is just the Zygmund space $\mathcal{Z}(B)$ when $\mu(r)=1-r^2$ or the Zygmund type space $\mathcal{Z}^p(B)$ when $\mu(r)=(1-r^2)^p$ $(0<p<\infty)$.

When $n>1$, we gave several equivalent norms of $\mathcal{Z}_\mu(B)$ in [1]. About various Zygmund type spaces, there have been a lot of work for examples see [1–27].

Definition 1.3 *Let μ be a normal function on $[0,1)$. $f\in H(B)$ is said to belong to the normal weight Bloch space $\mathcal{B}_\mu(B)$ if $f\in H(B)$ and*

$$||f||_{\mathcal{B}_\mu}=|f(0)|+\sup_{z\in B}\mu(|z|)|\nabla f(z)|<\infty.$$

In particular, it is just the Bloch space $\mathcal{B}(B)$ when $\mu(r)=1-r^2$.

In the complex plane, the Cesàro operator is defined by

$$C(f)(z)=\sum_{j=0}^\infty\Big(\frac{a_0+a_1+\cdots+a_j}{j+1}\Big)z^j,\quad \text{where } f(z)=\sum_{j=0}^\infty a_jz^j\in H(D).$$

It is known that $C(f)(z)=\frac{1}{z}\int_0^z f(t)\big(\log\frac{1}{1-t}\big)'\mathrm{d}t$. Therefore, the Cesàro operator $C(\cdot)$ is extended to the weighted Cesàro operator as follows:

$$T_g(f)(z)=\int_0^z f(t)g'(t)\mathrm{d}t,\quad f\in H(D),$$

where g is a given analytic function.

In several complex variables, the extended Cesàro operator is defined by

$$T_g(f)(z) = \int_0^1 f(tz)Rg(tz)\frac{\mathrm{d}t}{t}, \quad f \in H(B),$$

where g is a given holomorphic function on B with $g(0) = 0$.

No matter one complex variables or several complex variables, many mathematicians have done a lot of research on various Cesàro type operators. For example, see [2–4, 6–7, 9–10, 17, 24–25, 28–42]. In practical applications, we often encounter the combination of Cesàro type operator and composition operator. In this paper, we consider the following composition Cesàro type operator.

Definition 1.4 *Let $\varphi = (\varphi_1, \cdots, \varphi_n)$ be a holomorphic self-map of B and $\psi \in H(B)$ with $\psi(0) = 0$. The composition Cesàro type operator is defined by*

$$T_{\varphi,\psi}(f)(z) = \int_0^1 f[\varphi(tz)]R\psi(tz)\frac{\mathrm{d}t}{t}, \quad f \in H(B),\ z \in B.$$

If $\varphi(z) = z$, then $T_{\varphi,\psi}$ is just the extended Cesàro operator T_ψ. The purpose of this paper is to characterize the conditions that the composition Cesàro type operator $T_{\varphi,\psi}$ is bounded or compact on $\mathcal{Z}_\mu(B)$ when $n > 1$, and to give the sufficient and necessary conditions for all cases. Ultimately, this problem can be transformed into a kinds of weighted composition operator problem from the normal weight Zygmund space to the normal weight Bloch space in high dimensions. Many scholars have discussed similar problems (see [4, 16, 18, 26–27] etc.). However, so far, for abstract normal weight μ, especially in high dimensions, the sufficient and necessary conditions for $T_{\varphi,\psi}$ to be bounded or compact on $\mathcal{Z}_\mu(B)$ have not been given.

In this paper, we use the symbols c, c_1, c_2, c_3, c_4 to denote positive constants independent of variables $z,\ w$. But they may depend on some parameters or fixed values, with different values in different cases. We say that two quantities E and F are equivalent (denoted by "$E \asymp F$" in the following) if there exist two positive constants A_1 and A_2 such that $A_1E \le F \le A_2E$.

2 Some Lemmas

Let μ be a normal function on $[0, 1)$ and

$$\frac{1}{\sigma_\mu(t)} = \frac{1}{\mu(0)} + \int_0^t \frac{\mathrm{d}\rho}{\mu(\rho)\sqrt{1-\rho}}, \quad 0 \le t < 1.$$

For any $u \in \mathbf{C^n}$, let $G_0^\mu(u) = \dfrac{|u|^2}{\mu^2(0)}$. When $0 \ne z \in B$, let

$$G_z^\mu(u) = \frac{1}{\mu^2(|z|)}\Big\{\frac{\mu^2(|z|)}{\sigma_\mu^2(|z|)}\ |u|^2 + \Big(1 - \frac{\mu^2(|z|)}{\sigma_\mu^2(|z|)}\Big)\ \frac{|\langle z, u\rangle|^2}{|z|^2}\Big\}.$$

When $z \neq 0$, we may decompose u to $u = u_1 \frac{z}{|z|} + u_2 \xi$ with $\langle z, \xi \rangle = 0$ and $\xi \in \partial B$. By computation, it is clear that

$$u_1 = \frac{\langle u, z \rangle}{|z|}, \quad u_2 = \langle u, \xi \rangle, \quad |u|^2 = |u_1|^2 + |u_2|^2, \quad G_z^\mu(u) = \frac{|u_1|^2}{\mu^2(|z|)} + \frac{|u_2|^2}{\sigma_\mu^2(|z|)},$$

$$(1-t)^b \Big(1 + \int_0^t \frac{\mathrm{d}\tau}{(1-\tau)^{b+\frac{1}{2}}}\Big) \leq \frac{\mu(t)}{\sigma_\mu(t)} \leq (1-t)^a \Big(1 + \int_0^t \frac{\mathrm{d}\tau}{(1-\tau)^{a+\frac{1}{2}}}\Big). \tag{2.1}$$

Therefore, there is a constant $c > 0$ such that $\frac{1}{\sigma_\mu(t)} \leq \frac{c}{\mu(t)}$ for all $0 \leq t < 1$.

It is known that $\frac{a_1^{\frac{1}{2}} + a_2^{\frac{1}{2}}}{2} \leq (a_1 + a_2)^{\frac{1}{2}} \leq a_1^{\frac{1}{2}} + a_2^{\frac{1}{2}}$ for all $a_1 \geq 0$ and $a_2 \geq 0$. Therefore,

$$\sqrt{G_z^\mu(u)} \asymp \frac{|\langle u, z \rangle|}{|z|\mu(|z|)} + \frac{|\langle u, \xi \rangle|}{\sigma_\mu(|z|)}.$$

Further, by (2.1), there exists $\frac{1}{2} < t_0 < 1$ such that

$$\frac{1}{4}\Big(\frac{|\langle z, u \rangle|}{\mu(|z|)} + \frac{|u|}{\sigma_\mu(|z|)}\Big) \leq \sqrt{G_z^\mu(u)} \leq \frac{3}{2}\Big(\frac{|\langle z, u \rangle|}{\mu(|z|)} + \frac{|u|}{\sigma_\mu(|z|)}\Big) \quad \text{when} \quad t_0 < |z| < 1. \tag{2.2}$$

For more information on this metric, see [20–21, 43–44]. In order to prove the main results, we first give some lemmas.

Lemma 2.1 *Let μ be a normal function on $[0, 1)$ and $f \in H(B)$. Then the following conditions are equivalent*:

(1) $f \in \mathcal{Z}_\mu(B)$.

(2) $I_1 = |f(0)| + \sup_{z \in B} \mu(|z|)|R^{(2)} f(z)| < \infty$, *where* $R^{(2)} f = R(Rf)$.

(3) $I_2 = |f(0)| + \sup_{z \in B} \mu(|z|)\ |\nabla(Rf)(z)| < \infty$.

(4) $I_3 = |f(0)| + \sum_{j=1}^{n} \big|\frac{\partial f}{\partial z_j}(0)\big| + \sup_{z \in B} W_f^\mu(z) < \infty$, *where*

$$W_f^\mu(z) = \sup_{u \in \mathbf{C^n}\text{-}\{0\}} \sum_{l=1}^{n} \frac{|\langle \nabla(D_l f)(z), \overline{u} \rangle|}{\sqrt{G_z^\mu(u)}}, \quad D_l = \frac{\partial}{\partial z_l}.$$

Further, $I_1 \asymp I_2 \asymp I_3 \asymp ||f||_{\mathcal{Z}_\mu}$, and the controlling constants are independent of f. In particular, $I_1 \leq ||f||_{\mathcal{Z}_\mu}$.

Proof These results come from [1, Theorem 3.1] and [20, Lemma 2.1].

Lemma 2.2 *Let μ be a normal function on $[0, 1)$. If $f \in \mathcal{Z}_\mu(B)$, then*

$$|Rf(z)| \leq c\Big(\int_0^{|z|} \frac{1}{\mu(t)}\ \mathrm{d}t\Big)||f||_{\mathcal{Z}_\mu},$$

$$|\nabla f(z)| \leq c\Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\ \mathrm{d}t\Big)||f||_{\mathcal{Z}_\mu},$$

$$|f(z)| \leq c\Big\{1 + \int_0^{|z|} \Big(\int_0^\rho \frac{1}{\mu(t)}\ \mathrm{d}t\Big)\mathrm{d}\rho\Big\}||f||_{\mathcal{Z}_\mu}, \quad z \in B.$$

Proof These results comes from [2].

Lemma 2.3 *Let μ be a normal function on $[0,1)$ and*

$$g(\xi)=1+\sum_{s=1}^{\infty}2^s\ \xi^{n_s},\quad \xi\in D.$$

Then $g(r)$ is strictly increasing on $[0,1)$ and

$$\inf_{r\in[0,1)}\mu(r)g(r)=N_0>0,\quad \sup_{\xi\in D}\mu(|\xi|)|g(\xi)|=M_0<\infty,$$

where n_s is the integer part of $(1-r_s)^{-1}$, $r_0=0$, $\mu(r_s)=2^{-s}$ $(s=1,2,\cdots)$.

Proof These results come from [45, Theorem 1].

Lemma 2.4 *Let μ be a normal function on $[0,1)$. Suppose that k is a positive integer. Let $0<r_0<1$ be a fixed number. Then*

$$\begin{aligned}&\int_0^{|w|}\frac{\mathrm{d}t}{\mu(t)}\asymp\int_0^{|w|^k}\frac{\mathrm{d}t}{\mu(t)},\\ &\int_0^{|w|}\frac{\mathrm{d}t}{\sqrt{1-t}\mu(t)}\asymp\int_0^{|w|^k}\frac{\mathrm{d}t}{\sqrt{1-t}\mu(t)},\\ &\int_0^{|w|}\Big(\int_0^{\rho}\frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho\asymp\int_0^{|w|^k}\Big(\int_0^{\rho}\frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho\end{aligned}$$

when $r_0<|w|<1$.

Proof The first two results come from [19, Lemma 2.5]. Notice that

$$\begin{aligned}&\int_0^{|w|^k}\Big(\int_0^{\rho}\frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho\\ &=\int_0^{|w|}\Big(\int_0^{x}\frac{\mathrm{d}y}{\mu(|w|^{k-1}y)}\Big)|w|^{2k-2}\ \mathrm{d}x\\ &\geq\int_0^{|w|}\Big(\int_0^{x}\Big(\frac{1-y}{1-|w|^{k-1}y}\Big)^b\frac{\mathrm{d}y}{\mu(y)}\Big)|w|^{2k-2}\ \mathrm{d}x\\ &\geq\frac{r_0^{2k-2}}{k^b}\int_0^{|w|}\Big(\int_0^{x}\frac{\mathrm{d}y}{\mu(y)}\Big)\mathrm{d}x.\end{aligned}$$

This shows that the third result also holds.

Lemma 2.5 *Let μ be normal on $[0,1)$. If the sequence $\{f_j(z)\}$ is bounded on $\mathcal{Z}_\mu(B)$ and converges to 0 uniformly on any compact subset of B.*

(1) *If* $\int_0^1\frac{\mathrm{d}t}{\mu(t)}<\infty$, *then* $\lim\limits_{j\to\infty}\sup\limits_{z\in B}|\nabla f_j(z)|=0=\lim\limits_{j\to\infty}\sup\limits_{z\in B}|f_j(z)|$.

(2) *If* $\int_0^1\Big(\int_0^{\rho}\frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho<\infty$, *then* $\lim\limits_{j\to\infty}\sup\limits_{z\in B}|f_j(z)|=0$.

Proof These results comes from [2].

Lemma 2.6 *Let μ be normal on $[0,1)$ such that*

$$\int_0^1 \Big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho < \infty.$$

For $0 < r_0 < 1$ and $f \in \mathcal{Z}_\mu(B)$, if $|\nabla f(z)| \le m$ when $|z| \le r_0$, then there exists constant $c > 0$ such that

$$|\langle \nabla f(z), \overline{\xi}\rangle| \le m + c||f||_{\mathcal{Z}_\mu} \int_{r_0}^1 \Big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho$$

for all $r_0 < |z| < 1$, where $\xi \in \partial B$ with $\langle z, \xi\rangle = 0$.

Proof By a unitary transformation, we may let $z = (|z|, 0, \cdots, 0)$ with $|z| < 1$ and $\xi = (0, 1, 0, \cdots, 0)$. For fixed $0 \le \rho < 1$, we let $h(\eta) = D_1(Rf)(\rho, \eta, 0, \cdots, 0)$. If $f \in \mathcal{Z}_\mu(B)$, then by Lemma 2.1 we have

$$|h(z_2)| \le \frac{c_1||f||_{\mathcal{Z}_\mu}}{\mu(\sqrt{\rho^2 + |z_2|^2})} \le \frac{c_1||f||_{\mathcal{Z}_\mu}}{\mu\big(\sqrt{\frac{\rho^2+1}{2}}\big)} \le \frac{c_1 4^b ||f||_{\mathcal{Z}_\mu}}{\mu(\rho)}$$

for all $|z_2|^2 \le \dfrac{1-\rho^2}{2}$.

Therefore, for any $r_0 < |z| < 1$ and $0 \le t \le |z|$, we may obtain

$$\begin{aligned}
&|D_2(Rf)(t, 0, \cdots, 0) - D_2(Rf)(0, 0, \cdots, 0)| \\
&= \Big|\int_0^t h'(0)\ \mathrm{d}\rho\Big| \\
&= \frac{1}{2\pi}\Big|\int_0^t \Big(\int_{|w|=\sqrt{\frac{1-\rho^2}{2}}} \frac{h(w)\ dw}{w^2}\Big)\mathrm{d}\rho\Big| \\
&\le c_2||f||_{\mathcal{Z}_\mu} \int_0^t \frac{\mathrm{d}\rho}{\sqrt{1-\rho}\ \mu(\rho)}.
\end{aligned}$$

When $r_0 < |z| < 1$, we have

$$\begin{aligned}
|\langle \nabla f(z), \overline{\xi}\rangle| &= |D_2 f(|z|, 0, \cdots, 0)| \\
&= \frac{1}{|z|}\Big|r_0 D_2 f(r_0, 0, \cdots, 0) + \int_{r_0}^{|z|} D_2(Rf)(t, 0, \cdots, 0)\ \mathrm{d}t\Big| \\
&\le m + c||f||_{\mathcal{Z}_\mu} \int_{r_0}^1 \Big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho.
\end{aligned}$$

3 Boundedness of $\mathbf{T}_{\varphi,\psi}$

Theorem 3.1 *Let μ be a normal function on $[0,1)$. For $n > 1$, suppose that $\varphi = (\varphi_1, \cdots, \varphi_n)$ is a holomorphic self-map of B and $\psi \in H(B)$ with $\psi(0) = 0$. Then $T_{\varphi,\psi}$ is a bounded operator* *on $\mathcal{Z}_\mu(B)$ if and only if the following results hold:*

$$\sup_{z\in B}\mu(|z|)|R\psi(z)||\langle R\varphi(z),\varphi(z)\rangle|\int_0^{|\varphi(z)|}\frac{\mathrm{d}\rho}{\mu(\rho)}<\infty, \tag{3.1}$$

$$\sup_{z\in B}\mu(|z|)|R\psi(z)||R\varphi(z)|\Big\{1+\int_0^{|\varphi(z)|}\Big(\int_0^{\rho}\frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho\Big\}<\infty, \tag{3.2}$$

$$\sup_{z\in B}\mu(|z|)|R^{(2)}\psi(z)|\Big\{1+\int_0^{|\varphi(z)|}\Big(\int_0^{\rho}\frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho\Big\}<\infty, \tag{3.3}$$

where $R\varphi(z)=(R\varphi_1(z),\cdots,R\varphi_n(z))$.

Proof First, we prove sufficiency.

For any $f\in\mathcal{Z}_\mu(B)$, we have

$$\begin{aligned}&\langle(\nabla f)[\varphi(z)],\overline{R\varphi(z)}\rangle-\langle(\nabla f)(0),\overline{R\varphi(z)}\rangle\\ =&\sum_{l=1}^{n}R\varphi_l(z)\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}\{D_lf[t\varphi(z)]\}\ \mathrm{d}t\\ =&\sum_{l=1}^{n}\varphi_l(z)\int_0^1\langle\nabla(D_lf)[t\varphi(z)],\overline{R\varphi(z)}\rangle\ \mathrm{d}t.\end{aligned}$$

By Lemmas 2.1–2.2 and (2.2), we may obtain

$$\begin{aligned}&\mu(|z|)|R\psi(z)||\langle(\nabla f)[\varphi(z)],\overline{R\varphi(z)}\rangle|\\ \le&\{\mu(|z|)|R\psi(z)||R\varphi(z)|\}||f||_{\mathcal{Z}_\mu}\\ &+\ |\varphi(z)|\mu(|z|)|R\psi(z)|\int_0^1\Big(\sum_{l=1}^{n}|\langle\nabla(D_lf)[t\varphi(z)],\overline{R\varphi(z)}\rangle|\Big)\mathrm{d}t\\ \le&\{\mu(|z|)|R\psi(z)||R\varphi(z)|\}||f||_{\mathcal{Z}_\mu}\\ &+c_1|\varphi(z)|\mu(|z|)|R\psi(z)|\Big(\int_0^1\sqrt{G^{\mu}_{t\varphi(z)}[R\varphi(z)]}\ \mathrm{d}t\Big)||f||_{\mathcal{Z}_\mu}\\ \le&c_2\mu(|z|)|R\psi(z)|\Big\{|R\varphi(z)|+\int_0^{|\varphi(z)|}\Big(\frac{|\langle R\varphi(z),\varphi(z)\rangle|}{\mu(t)}+\frac{|R\varphi(z)|}{\sigma_\mu(t)}\Big)\mathrm{d}t\Big\}||f||_{\mathcal{Z}_\mu}.\end{aligned} \tag{3.4}$$

If (3.1)–(3.3) hold, then by Lemma 2.2 and (3.4) we have

$$\begin{aligned}&\mu(|z|)|R^{(2)}[T_{\varphi,\psi}(f)(z)]|=\mu(|z|)|R[f\circ\varphi(z)R\psi(z)]|\\ =&\mu(|z|)|R^{(2)}\psi(z)f[\varphi(z)]+R\psi(z)\langle(\nabla f)[\varphi(z)],\overline{R\varphi(z)}\rangle|\\ \le&\mu(|z|)|R^{(2)}\psi(z)|\Big\{1+\int_0^{|\varphi(z)|}\Big(\int_0^{\rho}\frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho\Big\}||f||_{\mathcal{Z}_\mu}\\ &+\ c\mu(|z|)|R\psi(z)|\Big\{|R\varphi(z)|+\int_0^{|\varphi(z)|}\Big(\frac{|\langle R\varphi(z),\varphi(z)\rangle|}{\mu(t)}+\frac{|R\varphi(z)|}{\sigma_\mu(t)}\Big)\mathrm{d}t\Big\}||f||_{\mathcal{Z}_\mu}\\ \le&c_1||f||_{\mathcal{Z}_\mu}.\end{aligned}$$

This shows that $T_{\varphi,\psi}$ is bounded on $\mathcal{Z}_\mu(B)$ by Lemma 2.1.

Conversely, if $T_{\varphi,\psi}$ is a bounded operator on $\mathcal{Z}_\mu(B)$, then $\psi \in \mathcal{Z}_\mu(B)$ by taking $f_0(z) = 1 \in \mathcal{Z}_\mu(B)$. At the same time, we have

$$\begin{aligned} &\mu(|z|)|R\psi(z)||R\varphi(z)| \\ \le& \sum_{l=1}^{n} \mu(|z|)|R\psi(z)R\varphi_l(z)| = \sum_{l=1}^{n} \mu(|z|)|R[\varphi_l(z)R\psi(z)] - \varphi_l(z)R^{(2)}\psi(z)| \\ \le& c\sum_{l=1}^{n} ||T_{\varphi,\psi}(f_{0,l})||_{\mathcal{Z}_\mu} + n||\psi||_{\mathcal{Z}_\mu} \end{aligned} \tag{3.5}$$

by taking $f_{0,l}(z) = z_l \in \mathcal{Z}_\mu(B)$ for any $l \in \{1, 2, \cdots, n\}$ and Lemma 2.1.

If there is always $|\varphi(z)| \le t_0$ (t_0 is the number in (2.2)), then (3.1)–(3.3) hold by (3.5) and $\psi \in \mathcal{Z}_\mu(B)$. If $||\varphi||_\infty = \sup\limits_{z\in B}|\varphi(z)| > t_0$, then for any $0 \ne w \in B$ with $|\varphi(w)| > t_0$ we take

$$f_w(z) = 2\int_0^{|\varphi(w)|^2\langle z,\varphi(w)\rangle} \Big(\int_0^\rho g(t)\ \mathrm{d}t\Big)\mathrm{d}\rho - \int_0^{\langle z,\varphi(w)\rangle^2} \Big(\int_0^\rho g(t)\ \mathrm{d}t\Big)\mathrm{d}\rho,$$

where g is the function in Lemma 2.3.

By Lemmas 2.3–2.4, it is clear that $(\nabla f_w)[\varphi(w)] = (0, 0, \cdots, 0)$ and

$$f_w[\varphi(w)] = \int_0^{|\varphi(w)|^4} \Big(\int_0^\rho g(t)\ \mathrm{d}t\Big)\mathrm{d}\rho \asymp \int_0^{|\varphi(w)|} \Big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho. \tag{3.6}$$

By Lemma 2.3 and the definitions of μ and g, we have

$$\begin{aligned} &\mu(|z|)|R^{(2)}f_w(z)| \\ =& \mu(|z|)\Big|2|\varphi(w)|^4\langle z,\varphi(w)\rangle^2 g(|\varphi(w)|^2\langle z,\varphi(w)\rangle) - 4\langle z,\varphi(w)\rangle^4 g(\langle z,\varphi(w)\rangle^2) \\ &+\ 2|\varphi(w)|^2\langle z,\varphi(w)\rangle \int_0^{|\varphi(w)|^2\langle z,\varphi(w)\rangle} g(\rho)\ \mathrm{d}\rho - 4\langle z,\varphi(w)\rangle^2 \int_0^{\langle z,\varphi(w)\rangle^2} g(\rho)\ \mathrm{d}\rho\Big| \\ \le& 6\mu(|z|)g(|z|) + 6\mu(|z|)\int_0^{|z|} g(\rho)\ \mathrm{d}\rho \le 12M_0. \end{aligned}$$

This shows that $||f_w||_{\mathcal{Z}_\mu} \le c$ by Lemma 2.1.

By the boundedness of $T_{\varphi,\psi}$, Lemma 2.1 and (3.6), we have

$$\begin{aligned} c||T_{\varphi,\psi}|| &\ge ||T_{\varphi,\psi}||||f_w||_{\mathcal{Z}_\mu} \ge ||T_{\varphi,\psi}(f_w)||_{\mathcal{Z}_\mu} \\ &\ge \mu(|w|)|R^{(2)}[T_{\varphi,\psi}(f_w)](w)| \\ &= \mu(|w|)|R^{(2)}\psi(w)f_w[\varphi(w)]| \\ &\ge c_1\mu(|w|)|R^{(2)}\psi(w)| \int_0^{|\varphi(w)|} \Big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho. \end{aligned} \tag{3.7}$$

(3.7) and $\psi \in \mathcal{Z}_\mu(B)$ show that (3.3) holds.

Similarly, if we take

$$f_w(z) = \int_0^{\langle z,\varphi(w)\rangle^2} \Big(\int_0^\rho g(t)\ \mathrm{d}t\Big)\mathrm{d}\rho - \int_0^{|\varphi(w)|^2\langle z,\varphi(w)\rangle} \Big(\int_0^\rho g(t)\ \mathrm{d}t\Big)\mathrm{d}\rho,$$

then we may obtain

$$\mu(|w|)|R\psi(w)||\langle R\varphi(w),\varphi(w)\rangle|\int_0^{|\varphi(w)|}\frac{\mathrm{d}\rho}{\mu(\rho)}\le c||T_{\varphi,\psi}||.$$

This shows that (3.1) holds.

We write $R\varphi(w)=\dfrac{u_1\varphi(w)}{|\varphi(w)|}+u_2\xi$, where $\langle\varphi(w),\xi\rangle=0$ with $\xi\in\partial B$. Take

$$f_w(z)=\langle z,\xi\rangle\int_0^{\langle z,\varphi(w)\rangle}\Big(\int_0^{\rho}\frac{g(t)\ \mathrm{d}t}{\sqrt{1-t}}\Big)\mathrm{d}\rho.$$

It is clear that $f_w[\varphi(w)]=0$ and

$$(\nabla f_w)[\varphi(w)]=\overline{\xi}\int_0^{|\varphi(w)|^2}\Big(\int_0^{\rho}\frac{g(t)\ \mathrm{d}t}{\sqrt{1-t}}\Big)\mathrm{d}\rho. \tag{3.8}$$

Since $|\langle z,\xi\rangle|^2+|\langle z,z_0\rangle|^2\le|z|^2<1$ and $|\varphi(w)|>t_0>\dfrac{1}{2}$, then

$$|\langle z,\xi\rangle|\le\frac{\sqrt{(|z|+|\langle z,z_0\rangle|)(|\varphi(w)||z|-|\langle z,\varphi(w)\rangle|)}}{\sqrt{|\varphi(w)|}}<2\sqrt{1-|\langle z,\varphi(w)\rangle|}. \tag{3.9}$$

Therefore, by Lemma 2.3 and (3.9), we have

$$\begin{aligned}\mu(|z|)|R^{(2)}f_w(z)|&=\mu(|z|)\Big|\langle z,\xi\rangle\int_0^{\langle z,\varphi(w)\rangle}\Big(\int_0^{\rho}\frac{g(t)\ \mathrm{d}t}{\sqrt{1-t}}\Big)\mathrm{d}\rho\\&\quad+\ 3\langle z,\xi\rangle\langle z,\varphi(w)\rangle\int_0^{\langle z,\varphi(w)\rangle}\frac{g(\rho)\ \mathrm{d}\rho}{\sqrt{1-\rho}}+\frac{\langle z,\xi\rangle\langle z,\varphi(w)\rangle^2g(\langle z,\varphi(w)\rangle)}{\sqrt{1-\langle z,\varphi(w)\rangle}}\Big|\\&\le 2\mu(|z|)\sqrt{1-|\langle z,\varphi(w)\rangle|}\int_0^{|\langle z,\varphi(w)\rangle|}\Big(\int_0^{\rho}\frac{g(t)\ \mathrm{d}t}{\sqrt{1-t}}\Big)\mathrm{d}\rho\\&\quad+\ 6\mu(|z|)\sqrt{1-|\langle z,\varphi(w)\rangle|}\int_0^{|\langle z,\varphi(w)\rangle|}\frac{g(\rho)\ \mathrm{d}\rho}{\sqrt{1-\rho}}\\&\quad+\ \frac{2\mu(|z|)\sqrt{1-|\langle z,\varphi(w)\rangle|}\ g(|\langle z,\varphi(w)\rangle|)}{\sqrt{|1-\langle z,\varphi(w)\rangle|}}\le 10M_0.\end{aligned}$$

This means that $||f_w||_{\mathcal{Z}_\mu}\le c$ by Lemma 2.1.

By the boundedness of $T_{\varphi,\psi}$ and (3.8), Lemmas 2.3–2.4, we have

$$\begin{aligned}c||T_{\varphi,\psi}||&\ge\mu(|w|)|R\psi(w)||\langle(\nabla f_w)[\varphi(w)],\overline{R\varphi(w)}\rangle|\\&=\mu(|w|)|R\psi(w)||\langle R\varphi(w),\xi\rangle|\int_0^{|\varphi(w)|^2}\Big(\int_0^{\rho}\frac{g(t)\ \mathrm{d}t}{\sqrt{1-t}}\Big)\mathrm{d}\rho\\&\ge c_1\mu(|w|)|R\psi(w)||u_2|\int_0^{|\varphi(w)|}\Big(\int_0^{\rho}\frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho\\&=c_1\mu(|w|)|R\psi(w)|\sqrt{|R\varphi(z)|^2-\frac{|\langle R\varphi(w),\varphi(w)\rangle|^2}{|\varphi(w)|^2}}\int_0^{|\varphi(w)|}\Big(\int_0^{\rho}\frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho\\&\ge c_1\mu(|w|)|R\psi(w)|\Big(|R\varphi(z)|-\frac{|\langle R\varphi(w),\varphi(w)\rangle|}{|\varphi(w)|}\Big)\int_0^{|\varphi(w)|}\Big(\int_0^{\rho}\frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho\end{aligned}$$

$$\Rightarrow \mu(|w|)|R\psi(w)||R\varphi(w)|\int_0^{|\varphi(w)|}\Big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho$$
$$\leq c_2||T_{\varphi,\psi}|| + c_3\mu(|w|)|R\psi(w)||\langle R\varphi(w),\varphi(w)\rangle|\int_0^{|\varphi(w)|}\frac{\mathrm{d}\rho}{\sigma_\mu(\rho)}. \tag{3.10}$$

By (2.1), (3.1), (3.5) and (3.10), this means that (3.2) holds.

The proof is completed.

Corollary 3.1 *Let μ be a normal function on $[0,1)$. For $n>1$, suppose $\psi\in H(B)$ with $\psi(0)=0$. Then the extended Cesàro operator T_ψ is a bounded operator on $\mathcal{Z}_\mu(B)$ if and only if*

$$\sup_{z\in B}\mu(|z|)|R\psi(z)|\int_0^{|z|}\frac{\mathrm{d}\rho}{\mu(\rho)}<\infty,$$
$$\sup_{z\in B}\mu(|z|)|R^{(2)}\psi(z)|\int_0^{|z|}\Big(\int_0^\rho\frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho<\infty.$$

Proof By (2.1), it is clear that

$$\int_0^\rho\frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\leq\frac{1}{\mu(\rho)}\quad\text{for all } 0\leq\rho<1.$$

Therefore, if $\varphi(z)=z$, then (3.2) is redundant in Theorem 3.1.

Note 3.1 In general, the above two conditions in Corollary 3.1 are not independent. Let a be the parameter in the definition of μ. If $|z|\to 1^-$, then we have

$$\int_0^{|z|}\frac{\mu(|z|)\,\mathrm{d}t}{\mu(t)}\asymp 1-|z|,\quad a>1,\qquad \int_0^{|z|}\Big(\int_0^\rho\frac{\mu(|z|)}{\mu(t)}\,\mathrm{d}t\Big)\mathrm{d}\rho\asymp(1-|z|)^2,\quad a>2.$$

This means that T_ψ is bounded on $\mathcal{Z}_\mu(B)$ if and only if $\psi\in\mathcal{B}(B)$ when $a>2$. Otherwise, it is clear that T_ψ is bounded on $\mathcal{Z}_\mu(B)$ if and only if $\psi\in\mathcal{Z}_\mu(B)$ when $\int_0^1\frac{\mathrm{d}t}{\mu(t)}<\infty$.

4 Compactness of $\mathbf{T}_{\varphi,\psi}$

Theorem 4.1 *Let μ be a normal function on $[0,1)$. For $n>1$, suppose that φ is a holomorphic self-map of B and $\psi\in H(B)$ with $\psi(0)=0$.*

(1) *If $||\varphi||_\infty<1$ or $\int_0^1\frac{\mathrm{d}t}{\mu(t)}<\infty$, then $T_{\varphi,\psi}$ is a compact operator on $\mathcal{Z}_\mu(B)$ if and only if $\psi\in\mathcal{Z}_\mu(B)$ and*

$$M=\sup_{z\in B}\mu(|z|)|R\psi(z)||R\varphi(z)|<\infty. \tag{4.1}$$

(2) *If $||\varphi||_\infty=1$ and $\int_0^1\big(\int_0^\rho\frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\big)\mathrm{d}\rho<\infty=\int_0^1\frac{\mathrm{d}t}{\mu(t)}$, then $T_{\varphi,\psi}$ is a compact operator on $\mathcal{Z}_\mu(B)$ if and only if $\psi\in\mathcal{Z}_\mu(B)$, (4.1) holds and*

$$\lim_{|\varphi(z)|\to 1^-}\mu(|z|)|R\psi(z)||\langle R\varphi(z),\varphi(z)\rangle|\int_0^{|\varphi(z)|}\frac{\mathrm{d}t}{\mu(t)}=0. \tag{4.2}$$

(3) *If* $||\varphi||_\infty = 1$ *and* $\int_0^1 \big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)}\big)\mathrm{d}\rho < \infty = \int_0^1 \big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\big)\mathrm{d}\rho$, *then* $T_{\varphi,\psi}$ *is a compact operator on* $\mathcal{Z}_\mu(B)$ *if and only if* $\psi \in \mathcal{Z}_\mu(B)$, (4.1)–(4.2) *hold and*

$$\lim_{|\varphi(z)|\to 1^-} \mu(|z|)|R\psi(z)||R\varphi(z)| \int_0^{|\varphi(z)|} \Big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho = 0. \tag{4.3}$$

(4) *If* $||\varphi||_\infty = 1$ *and* $\int_0^1 \big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)}\big)\mathrm{d}\rho = \infty$, *then* $T_{\varphi,\psi}$ *is a compact operator on* $\mathcal{Z}_\mu(B)$ *if and only if* $\psi \in \mathcal{Z}_\mu(B)$, (4.1)–(4.3) *hold and*

$$\lim_{|\varphi(z)|\to 1^-} \mu(|z|)|R^{(2)}\psi(z)| \int_0^{|\varphi(z)|} \Big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho = 0. \tag{4.4}$$

Proof First, we prove sufficiency.

Let $\{f_j(z)\}$ be a sequence which converges to 0 uniformly on any compact subset of B and $||f_j||_{\mathcal{Z}_\mu} \le 1$. Then $\{|\nabla f_j(z)|\}$ has the same uniformly convergence.

(1) (i) Case $||\varphi||_\infty < 1$.

If $\psi \in \mathcal{Z}_\mu(B)$ and (4.1) holds, then by Lemma 2.1 we have

$$\begin{aligned}||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu} &\asymp |T_{\varphi,\psi}(f_j)(0)| + \sup_{z\in B} \mu(|z|)|R^{(2)}[T_{\varphi,\psi}(f_j)](z)| \\ &\le \sup_{z\in B} \mu(|z|)|R^{(2)}\psi(z)||f_j[\varphi(z)]| + \sup_{z\in B} \mu(|z|)|R\psi(z)||\langle(\nabla f_j)[\varphi(z)], \overline{R\varphi(z)}\rangle| \\ &\le ||\psi||_{\mathcal{Z}_\mu} \sup_{|w|\le||\varphi||_\infty} |f_j(w)| + M \sup_{|w|\le||\varphi||_\infty} |\nabla f_j(w)| \to 0, \quad j\to\infty.\end{aligned}$$

(ii) Case $\int_0^1 \frac{\mathrm{d}t}{\mu(t)} < \infty$.

If $\psi \in \mathcal{Z}_\mu(B)$ and (4.1) holds, then by Lemmas 2.1 and 2.5 we have

$$\begin{aligned}||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu} &\le c||\psi||_{\mathcal{Z}_\mu} \sup_{z\in B} |f_j[\varphi(z)]| + cM \sup_{z\in B} |(\nabla f_j)[\varphi(z)]| \\ &\le c||\psi||_{\mathcal{Z}_\mu} \sup_{w\in B} |f_j(w)| + cM \sup_{w\in B} |\nabla f_j(w)| \to 0, \quad j\to\infty.\end{aligned}$$

(2) If (4.2) holds, then for any $\varepsilon > 0$, there exists $\frac{1}{2} < \delta < 1$ such that

$$\mu(|z|)|R\psi(z)||\langle R\varphi(z), \varphi(z)\rangle| \int_0^{|\varphi(z)|} \frac{\mathrm{d}t}{\mu(t)} < \varepsilon \quad \text{when } |\varphi(z)| > \delta. \tag{4.5}$$

By $\int_0^1 \big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\big)\mathrm{d}\rho < \infty$ and Lemma 2.6, there is a $\delta < r_0 < 1$ such that

$$|\langle(\nabla f_j)[\varphi(z)], \overline{\xi}\rangle| \le \sup_{|w|\le r_0} |\nabla f_j(w)| + c||f_j||_{\mathcal{Z}_\mu}\varepsilon \le \sup_{|w|\le r_0} |\nabla f_j(w)| + c\varepsilon, \tag{4.6}$$

where $\xi \in \partial B$ with $\langle\varphi(z), \xi\rangle = 0$.

If $\psi \in \mathcal{Z}_\mu(B)$ and (4.1)–(4.2) hold, then by Lemmas 2.1–2.2, (4.5)–(4.6) and

$$R\varphi(z) = \frac{\langle R\varphi(z), \varphi(z)\rangle}{|\varphi(z)|^2}\ \varphi(z) + \langle R\varphi(z), \xi\rangle\ \xi,$$

we have

$$\begin{aligned}||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu} &\le c||\psi||_{\mathcal{Z}_\mu}\sup_{z\in B}|f_j[\varphi(z)]| + c\sup_{z\in B}\mu(|z|)|R\psi(z)||\langle(\nabla f_j)[\varphi(z)],\overline{R\varphi(z)}\rangle|\\ &\le c||\psi||_{\mathcal{Z}_\mu}\sup_{z\in B}|f_j[\varphi(z)]| + cM\sup_{|w|\le\delta}|\nabla f_j(w)|\\ &\quad+\ c_1\sup_{|\varphi(z)|>\delta}\mu(|z|)|R\psi(z)||\langle R\varphi(z),\varphi(z)\rangle|\int_0^{|\varphi(z)|}\frac{\mathrm{d}t}{\mu(t)}\\ &\quad+cM\sup_{|\varphi(z)|>\delta}|\langle(\nabla f_j)[\varphi(z)],\overline{\xi}\rangle|\\ &\le c||\psi||_{\mathcal{Z}_\mu}\sup_{w\in B}|f_j(w)| + c_2M\sup_{|w|\le\delta}|\nabla f_j(w)| + (c_3M+c_1)\varepsilon.\end{aligned}$$

This shows that $\limsup\limits_{j\to\infty}||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu}\le(c_3M+c_1)\varepsilon$ by Lemma 2.5. Therefore, it implies that $\lim\limits_{j\to\infty}||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu}=0$ by the arbitrariness of ε.

(3) If $\psi\in\mathcal{Z}_\mu(B)$, (4.1)–(4.3) hold, then by the proof in (2), Lemma 2.5, (3.4) and

$$\begin{aligned}||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu} &\le c||\psi||_{\mathcal{Z}_\mu}\sup_{z\in B}|f_j[\varphi(z)]| + c_1M\sup_{|w|\le\delta}|\nabla f_j(w)|\\ &\quad+\ c_2\sup_{|\varphi(z)|>\delta}\mu(|z|)|R\psi(z)||\langle R\varphi(z),\varphi(z)\rangle|\int_0^{|\varphi(z)|}\frac{\mathrm{d}t}{\mu(t)}\\ &\quad+\ c_3\sup_{|\varphi(z)|>\delta}\mu(|z|)|R\psi(z)||R\varphi(z)|\int_0^{|\varphi(z)|}\Big(\int_0^\rho\frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho,\end{aligned}$$

we have $\lim\limits_{j\to\infty}||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu}=0$.

(4) If $\psi\in\mathcal{Z}_\mu(B)$, (4.1)–(4.4) hold, then by the method of proof in (2), (3.4) and

$$\begin{aligned}||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu} &\le c||\psi||_{\mathcal{Z}_\mu}\sup_{|w|\le\delta}|f_j(w)| + c_1M\sup_{|w|\le\delta}|\nabla f_j(w)|\\ &\quad+\ c_2\sup_{|\varphi(z)|>\delta}\mu(|z|)|R^{(2)}\psi(z)|\int_0^{|\varphi(z)|}\Big(\int_0^\rho\frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho\\ &\quad+\ c_3\sup_{|\varphi(z)|>\delta}\mu(|z|)|R\psi(z)||\langle R\varphi(z),\varphi(z)\rangle|\int_0^{|\varphi(z)|}\frac{\mathrm{d}t}{\mu(t)}\\ &\quad+\ c_4\sup_{|\varphi(z)|>\delta}\mu(|z|)|R\psi(z)||R\varphi(z)|\int_0^{|\varphi(z)|}\Big(\int_0^\rho\frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho,\end{aligned}$$

we have $\lim\limits_{j\to\infty}||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu}=0$.

In a word, we have $\lim\limits_{j\to\infty}||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu}=0$ for all cases. This means that $T_{\varphi,\psi}$ is a compact operator on $\mathcal{Z}_\mu(B)$ by the basic theory of functional analysis.

Conversely, if $T_{\varphi,\psi}$ is a compact operator on $\mathcal{Z}_\mu(B)$, then $T_{\varphi,\psi}$ is bounded on $\mathcal{Z}_\mu(B)$. By Theorem 3.1, it is clear that $\psi\in\mathcal{Z}_\mu(B)$ and (4.1) holds.

This means that (1) is true.

Let $\{z^j\}\subset B$ is a sequence with $\lim\limits_{j\to\infty}|\varphi(z^j)|=1$ and $|\varphi(z^j)|>t_0$ $(j=1,2,\cdots)$.

(2) We just need to prove that (4.2) holds. Let g be the function in Lemma 2.3. We choose function sequence as follows:

$$f_j(z)=\frac{\int_0^1\frac{F_j(\rho z)}{\rho}\,\mathrm{d}\rho}{\int_0^{|\varphi(z^j)|^2}g(t)\,\mathrm{d}t},\quad \text{where } F_j(z)=\Big(\int_0^{\langle z,\varphi(z^j)\rangle}g(t)\,\mathrm{d}t\Big)^2.$$

It is clear that $Rf_j(z)=\frac{F_j(z)}{\int_0^{|\varphi(z^j)|^2}g(t)\mathrm{d}t}$. Therefore, it is easy to prove that $||f_j||_{\mathcal{Z}_\mu}\le c$ and $\{f_j(z)\}$ converges to 0 uniformly on any compact subset of B by Lemmas 2.1 and 2.3. At the same time, we have

$$R[f_j\circ\varphi](z^j)=\frac{\langle R\varphi(z^j),\varphi(z^j)\rangle}{|\varphi(z^j)|^2}\int_0^{|\varphi(z^j)|^2}g(t)\,\mathrm{d}t. \tag{4.7}$$

By Lemma 2.1 and $\psi\in\mathcal{Z}_\mu(B)$, (4.7) and Lemmas 2.3–2.5, the compactness of $T_{\varphi,\psi}$, we have

$$\begin{aligned}0&\leftarrow ||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu}+||\psi||_{\mathcal{Z}_\mu}\sup_{w\in B}|f_j(w)|\\&\ge \mu(|z^j|)|R\psi(z^j)||R[f_j\circ\varphi](z^j)|\\&\ge c\mu(|z^j|)|R\psi(z^j)||\langle R\varphi(z^j),\varphi(z^j)\rangle|\int_0^{|\varphi(z^j)|}\frac{\mathrm{d}t}{\mu(t)},\quad j\to\infty.\end{aligned}$$

This shows that (4.2) holds.

(3) We just need to prove (4.3). Let $R\varphi(z^j)=\dfrac{u_1^j\varphi(z^j)}{|\varphi(z^j)|}+u_2^j\xi^j$ with $\langle\varphi(z^j),\xi^j\rangle=0$ and $\xi^j\in\partial B$ $(j=1,2,\cdots)$. We take function sequence

$$f_j(z)=\langle z,\xi^j\rangle\frac{\big\{\int_0^{\langle z,\varphi(z^j)\rangle}\big(\int_0^\rho\frac{g(t)\,\mathrm{d}t}{\sqrt{1-t}}\big)\mathrm{d}\rho\big\}^2}{\int_0^{|\varphi(z^j)|^2}\big(\int_0^\rho\frac{g(t)\,\mathrm{d}t}{\sqrt{1-t}}\big)\mathrm{d}\rho}.$$

It is easy to prove that $||f_j||_{\mathcal{Z}_\mu}\le c$ and $\{f_j(z)\}$ converges to 0 uniformly on any compact subset of B by (3.9), Lemmas 2.1 and 2.3–2.4. Otherwise, we have

$$R[f_j\circ\varphi](z^j)=\langle R\varphi(z^j),\xi^j\rangle\int_0^{|\varphi(z^j)|^2}\Big(\int_0^\rho\frac{g(t)\,\mathrm{d}t}{\sqrt{1-t}}\Big)\mathrm{d}\rho. \tag{4.8}$$

By Lemma 2.1 and (4.8), $\psi\in\mathcal{Z}_\mu(B)$ and Lemmas 2.3–2.5, the compactness of $T_{\varphi,\psi}$, we have

$$\lim_{|\varphi(z)|\to1^-}\mu(|z|)|R\psi(z)||\langle R\varphi(z),\xi\rangle|\int_0^{|\varphi(z)|}\Big(\int_0^\rho\frac{\mathrm{d}t}{\mu(t)\sqrt{1-t}}\Big)\mathrm{d}\rho=0 \tag{4.9}$$

with $\langle\varphi(z),\xi\rangle=0$ and $\xi\in\partial B$.

By (2.1), (4.2), (4.9) and $|R\varphi(z)|\asymp|\langle R\varphi(z),\varphi(z)\rangle|+|\langle R\varphi(z),\xi\rangle|$ $(|\varphi(z)|>t_0)$, it is clear that (4.3) holds.

(4) All that remains is to prove (4.4). We take function sequence

$$f_j(z)=\frac{\big\{\int_0^{\langle z,\varphi(z^j)\rangle}\big(\int_0^\rho g(t)\,\mathrm{d}t\big)\mathrm{d}\rho\big\}^2}{\int_0^{|\varphi(z^j)|^2}\big(\int_0^\rho g(t)\,\mathrm{d}t\big)\mathrm{d}\rho}.$$

Then $||f_j||_{\mathcal{Z}_\mu} \le c$ and $\{f_j(z)\}$ converges to 0 uniformly on any compact subset of B by simple calculation. At the same time, we have

$$R[f_j \circ \varphi](z^j) = 2\langle R\varphi(z^j), \varphi(z^j)\rangle \int_0^{|\varphi(z^j)|^2} g(t)\ \mathrm{d}t. \tag{4.10}$$

By Lemmas 2.1, 2.3 –2.4, $\psi \in \mathcal{Z}_\mu(B)$, (4.2), (4.10) and the compactness of $T_{\varphi,\psi}$, it is clear that

$$\begin{aligned} 0 &\leftarrow ||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu} + c_1\mu(|z^j|)|R\psi(z^j)||\langle R\varphi(z^j), \varphi(z^j)\rangle| \int_0^{|\varphi(z^j)|} \frac{\mathrm{d}t}{\mu(t)} \\ &\ge ||T_{\varphi,\psi}(f_j)||_{\mathcal{Z}_\mu} + 2\mu(|z^j|)|R\psi(z^j)||\langle R\varphi(z^j), \varphi(z^j)\rangle| \int_0^{|\varphi(z^j)|^2} g(\rho)\ \mathrm{d}\rho \\ &\ge \mu(|z^j|)|R^{(2)}\psi(z^j)| \int_0^{|\varphi(z^j)|^2} \Big(\int_0^\rho g(t)\ \mathrm{d}t\Big)\mathrm{d}\rho \\ &\ge c_2\mu(|z^j|)|R^{(2)}\psi(z^j)| \int_0^{|\varphi(z^j)|} \Big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho, \quad j \to \infty. \end{aligned}$$

This shows that (4.4) holds.

The proof is completed.

Corollary 4.1 *Let μ be normal on $[0,1)$. For $n > 1$, suppose $\psi \in H(B)$ with $\psi(0) = 0$.*

(1) *If $\int_0^1 \frac{\mathrm{d}t}{\mu(t)} < \infty$, then T_ψ is compact on $\mathcal{Z}_\mu(B)$ if and only if $\psi \in \mathcal{Z}_\mu(B)$.*

(2) *If $\int_0^1 \big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)}\big)\mathrm{d}\rho < \infty = \int_0^1 \frac{\mathrm{d}t}{\mu(t)}$, then T_ψ is a compact operator on $\mathcal{Z}_\mu(B)$ if and only if $\psi \in \mathcal{Z}_\mu(B)$ and*

$$\lim_{|z|\to 1^-} \mu(|z|)|R\psi(z)| \int_0^{|z|} \frac{\mathrm{d}t}{\mu(t)} = 0. \tag{4.11}$$

(3) *If $\int_0^1 \big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)}\big)\mathrm{d}\rho = \infty$, then T_ψ is compact on $\mathcal{Z}_\mu(B)$ if and only if* (4.11) *holds and*

$$\lim_{|z|\to 1^-} \mu(|z|)|R^{(2)}\psi(z)| \int_0^{|z|} \Big(\int_0^\rho \frac{\mathrm{d}t}{\mu(t)}\Big)\mathrm{d}\rho = 0. \tag{4.12}$$

Proof By taking $\varphi(z) = z$ in Theorem 4.1, it is easy to obtain these results. Otherwise, if (4.12) holds, then $\psi \in \mathcal{Z}_\mu(B)$.

Note 4.1 If $a > 2$, then T_ψ is a compact operator on $\mathcal{Z}_\mu(B)$ if and only if $\psi \in \mathcal{B}_0(B)$ (the little Bloch space on B).

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