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Gibbs Measure for the Higher Order Modified Camassa-Holm Equation*

Lin LIN[1] Wei YAN[2] Jinqiao DUAN[3]

Abstract This paper is devoted to constructing a globally rough solution for the higher order modified Camassa-Holm equation with randomization on initial data and periodic boundary condition. Motivated by the works of Thomann and Tzvetkov (Nonlinearity, 23 (2010), 2771–2791), Tzvetkov (Probab. Theory Relat. Fields, 146 (2010), 4679–4714), Burq, Thomann and Tzvetkov (Ann. Fac. Sci. Toulouse Math., 27 (2018), 527–597), the authors first construct the Borel measure of Gibbs type in the Sobolev spaces with lower regularity, and then establish the existence of global solution to the equation with the helps of Prokhorov compactness theorem, Skorokhod convergence theorem and Gibbs measure.

Keywords Higher-order modified Camassa-Holm equation, The randomization of the initial value, Gibbs measure, Global solution
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1 Introduction

In this paper, we consider the Gibbs measure and the Cauchy problem for the following periodic higher order modified Camassa-Holm equation under randomization on initial data:

$$\begin{cases} u_t + \partial_x^3 u + \dfrac{1}{2}\partial_x(u^2) + \partial_x(1-\partial_x^2)^{-1}\Big[u^2 + \dfrac{1}{2}(\partial_x u)^2\Big] = 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{S}^1 = [0,2\pi). \end{cases} \tag{1.1}$$

Omitting the second term in (1.1) yields the Camassa-Holm equation in nonlocal form

$$u_t + \frac{1}{2}\partial_x(u^2) + (1-\partial_x^2)^{-1}\partial_x\Big[u^2 + \frac{1}{2}(\partial_x u)^2\Big] = 0,$$

which was introduced by Camassa and Holm as a nonlinear model for water wave motion in shallow channels and we recommend the reader to refer to [8] for details. Many researchers have investigated the Cauchy problem for the Camassa-Holm equation, for instance, see [4, 8, 11–12, 26].

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[1]School of Arts and Sciences, Shanghai Dianji University, Shanghai 201306, China.
E-mail: linlin@sdju.edu.cn
[2]Corresponding author. School of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, Henan, China. E-mail: 011133@htu.edu.cn
[3]Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA.
E-mail: duan@iit.edu
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A Gibbs measure is an efficient tool to obtain a global solution of some dispersive equations by the continuation of local solution. It compensates the absence of conservation law in the Sobolev space with lower regularity to some extent. One can refer to [6–7, 9, 19, 28] and the references therein, in which the Gibbs measure and its applications were demonstrated. The invariant measure in this paper is obtained by the nonlinear flow generated by the concerned modified Camassa-Holm equation. Actually, the construction of invariant measures under nonlinear flows for many partial differential equations with random initial data was introduced by Lebowitz-Rose-Speer [17], and since then the Gibbs measure has attracted much research attention due to its special feature and applications. There are rich existing results on this topic, for examples, by using some new spaces and a suitable randomization, Burq and Tzvetkov [6–7] constructed the local and global strong solutions for a large set of initial data to the supercritical wave equation; Colliander and Oh [10] studied almost surely well-posedness for the nonlinear cubic Schrödinger equation with the randomization on the initial data. Besides, we name but a few more references like [2–3, 5, 13–14, 20–24] for interested readers.

For the Cauchy problem of (1.1) we are concerned with, many studies have carried out on the real line and in the periodic settings (see e.g. [1, 15–16, 18, 25, 27]). Different from existing results, we consider the Cauchy problem for the periodic modified Camassa-Holm equation with randomization on initial data in this paper. For this, inspired by [22–23], by using Prokhorov compactness theorem and Skorokhod convergence theorem, we construct the Borel measure of Gibbs type in the Sobolev spaces with lower regularity, and then establish the existence of global solution.

Next we clarify some notations used throughout this paper. Set

$$\langle n\rangle=\sqrt{n^2+1},\quad \mathscr{F}_xf(k)=\int_0^{2\pi}f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,\quad \|f\|_{H^s}=\|\langle k\rangle^s\mathscr{F}_xf(k)\|_{L^2},$$
$$X^{\frac12}(\mathbb{S}^1)=\bigcap_{\sigma<\frac12}H^\sigma(\mathbb{S}^1).$$

We denote the spectral projector by Π_N.

Let $(\Omega,\mathcal{F},\mathbf{p})$ be a probability space and $(g_n(\omega))_{n\in Z}$ be a sequence of independent complex normalized Gaussians, to be specific, $g_n\in\mathcal{N}_{\mathbb{C}}(0,1)$ with a form below:

$$g_n(\omega)=\frac{1}{\sqrt{2}}(h_n(\omega)+\mathrm{i}l_n(\omega)),\tag{1.2}$$

where $(h_n(\omega))_{n\in\mathbb{Z}}$, $(l_n(\omega))_{n\in\mathbb{Z}}$ are independent standard real Gaussians with the distribution $\mathcal{N}_{\mathbb{R}}(0,1)$.

Also we use

$$E_N=\mathrm{span}((\mathrm{e}^{\mathrm{i}nx})_{-N\le n\le N})$$

to denote the complex vector space and

$$\Pi_N\Big(\sum_{n\in\mathbb{Z}}c_n\mathrm{e}^{\mathrm{i}nx}\Big)=\sum_{n=-N}^{N}c_n\mathrm{e}^{\mathrm{i}nx}$$

to denote the projection operator on E_N. Setting $c_n=a_n+\mathrm{i}b_n$, we consider the probability measure $\mathrm{d}\theta_N$ on $\mathbb{R}^{2(2N+1)}$ defined as

$$\mathrm{d}\theta_N=C_N^{-1}\mathrm{e}^{-\pi\sum\limits_{n=-N}^{N}(\langle n\rangle^2+\langle n\rangle^4)(a_n^2+b_n^2)}\prod_{n=-N}^{N}\mathrm{d}a_n\mathrm{d}b_n,$$

where

$$C_N = \int_{\mathbb{R}^{2(2N+1)}} \mathrm{e}^{-\pi \sum\limits_{n=-N}^{N} (\langle n\rangle^2+\langle n\rangle^4)(a_n^2+b_n^2)} \prod_{n=-N}^{N} \mathrm{d}a_n \mathrm{d}b_n$$
$$= \Big(\prod_{n=-N}^{N} \frac{1}{\sqrt{\langle n\rangle^2+\langle n\rangle^4}} \Big)^2.$$

Then θ_N can be regarded as the distribution of the E_N valued random variable

$$\varphi_N(x,\omega) = \sum_{|n|\le N} \frac{g_n(\omega)}{2\sqrt{\pi}\langle n\rangle^2} \mathrm{e}^{\mathrm{i}nx},$$

where $(g_n)_{n=-N}^N$ are Gaussians in (1.2).

We denote by $L^2(\Omega; H^\sigma(\mathbb{S}^1))$ the Banach space of $H^\sigma(\mathbb{S}^1)$-valued functions on Ω. Obviously, (φ_N) is a Cauchy sequence in $L^2(\Omega; H^\sigma(\mathbb{S}^1))$ which leads to the limit

$$\varphi(x,\omega) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{2\sqrt{\pi}\langle n\rangle^2} \mathrm{e}^{\mathrm{i}nx} \in L^2(\Omega; H^\sigma(\mathbb{S}^1)).$$

In particular, $\varphi(\cdot,\omega) \in H_0^\sigma(\mathbb{S}^1)$ almost surely and the map $\omega \to \varphi(\cdot,\omega)$ is measurable from $(\Omega,\mathcal{A})$ to $(H_0^\sigma(\mathbb{S}^1),\mathcal{B})$, where $\mathcal{B}$ denotes the Borel sigma algebra of $H_0^\sigma(\mathbb{S}^1)$. Then, $\varphi(\cdot,\omega)$ defines a measure θ on $(H_0^\sigma(\mathbb{S}^1),\mathcal{B})$ and we define the measure $\theta = \mathbf{p}\circ\varphi^{-1}$ on $X^{\frac{1}{2}}(\mathbb{S}^1)$.

Let χ_R: $\mathbb{R} \to [0,1]$ be a continuous function with a compact support such that $\chi_R(x) = 1$ for $|x| \le R$. Denote $u_N = \Pi_N u$. Then we define the density $G_N(u)$ as

$$G_N(u) = \chi_R(\|u_N\|^2_{L^2(S^1)} + \|\partial_x u_N\|^2_{L^2(\mathbb{S}^1)} - \alpha_N)\mathrm{e}^{-\int_{\mathbb{S}^1}(u_N^3 + u_N(\partial_x u_N)^2)\mathrm{d}x}, \tag{1.3}$$

and the measure μ_N on $H^\sigma(\mathbb{S}^1)$ as

$$\mathrm{d}\mu_N(u) = G_N(u)\mathrm{d}\theta(u), \tag{1.4}$$

where

$$\alpha_N = \sum_{|n|\le N} \frac{1}{\langle n\rangle^4}.$$

To end this section, we show two main results in this paper.

Theorem 1.1 (Constructions of Gibbs Measure) *The sequence $G_N(u)$ converges in measure as $N\to\infty$ with respect to the measure θ. We denote by $G(u)$ the limit of* (1.3) *as $N\to\infty$ and define $\mathrm{d}\mu(u) = G(u)\mathrm{d}\theta(u)$. Then, for every $p\in[1,\infty)$, $G(u)\in L^p(\mathrm{d}\theta(u))$ and the sequence G_N converges to G in $L^p(\mathrm{d}\theta(u))$ as $N\to\infty$.*

Theorem 1.2 (Global Solution) *There exists a set Σ of full μ measure such that for every $u_0\in\Sigma$,* (1.1) *has a solution*

$$u \in C(\mathbb{R}; X^{\frac{1}{2}}(\mathbb{S}^1)).$$

The rest of the paper is organized as follows. In the next section, we give two important conservation laws and several elementary inequalities. In Section 3, the hypercontractivity properties of the Ornstein-Uhlenbeck semi-group are recalled and a Gibbs measure with support in $H^\sigma(\mathbb{S}^1)$ for any $\sigma < \frac{1}{2}$ is constructed, which is the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2 to construct the global solution of the concerned equation (1.1).

2 Conservation Laws and Some Inequalities

To begin with, we give two important conservation laws and some useful estimates in the following lemmas in this section.

Lemma 2.1 *Let u be a 2π periodic function which is the solution of* (1.1). *Then,* $F(u) = F(u_0)$, *where*

$$F(u) = \frac{1}{2}\int_{\mathbb{S}^1}((\partial_x u)^2 + (\partial_x^2 u)^2)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{S}^1}(u^3 + u(\partial_x u)^2)\mathrm{d}x.$$

Proof First note that

$$\frac{\mathrm{d}}{\mathrm{d}t}F(u) = \mathrm{I}_1 - \mathrm{I}_2,$$

where

$$\begin{aligned}\mathrm{I}_1(u) &= \int_{\mathbb{S}^1}\partial_x^2 u(\partial_t\partial_x^2 u - u_t)\mathrm{d}x,\\ \mathrm{I}_2(u) &= \int_{\mathbb{S}^1}u_t\Big(\frac{3}{2}u^2 + \frac{1}{2}(\partial_x u)^2 - \frac{1}{2}\partial_x^2(u^2)\Big)\mathrm{d}x.\end{aligned}$$

By acting $(1-\partial_x^2)$ on both sides of (1.1), we have

$$u_t - \partial_t\partial_x^2 u + \partial_x^3 u - \partial_x^5 u + \frac{1}{2}\partial_x(u^2) - \frac{1}{2}\partial_x^3(u^2) + \partial_x\Big(u^2 + \frac{1}{2}(\partial_x u)^2\Big) = 0, \tag{2.1}$$

which implies

$$\begin{aligned}\mathrm{I}_1 &= \int_{\mathbb{S}^1}\partial_x^2 u\Big[\partial_x^3 u - \partial_x^5 u + \frac{1}{2}\partial_x(u^2) - \frac{1}{2}\partial_x^3(u^2) + \partial_x\Big(u^2 + \frac{1}{2}(\partial_x u)^2\Big)\Big]\mathrm{d}x\\ &= -\int_{\mathbb{S}^1}\partial_x^3 u\Big(\frac{3}{2}u^2 + \frac{1}{2}(\partial_x u)^2 - \frac{1}{2}\partial_x^2(u^2)\Big)\mathrm{d}x.\end{aligned}\tag{2.2}$$

Inserting (1.1) into I_2 and integrating by parts, we conclude that

$$\begin{aligned}\mathrm{I}_2 &= -\int_{\mathbb{S}^1}\Big[\partial_x^3 u + \frac{1}{2}\partial_x(u^2) + \partial_x(1-\partial_x^2)^{-1}\Big(u^2 + \frac{1}{2}(\partial_x u)^2\Big)\Big]\Big(\frac{3}{2}u^2 + \frac{1}{2}(\partial_x u)^2 - \frac{1}{2}\partial_x^2(u^2)\Big)\mathrm{d}x\\ &= \mathrm{I}_1 - \frac{1}{4}\int_{\mathbb{S}^1}\partial_x(u^2)(\partial_x u)^2\mathrm{d}x - \int_{\mathbb{S}^1}\Big[\partial_x(1-\partial_x^2)^{-1}\Big(u^2 + \frac{1}{2}(\partial_x u)^2\Big)\Big]\Big(u^2 + \frac{1}{2}(\partial_x u)^2\Big)\mathrm{d}x\\ &\quad - \frac{1}{2}\int_{\mathbb{S}^1}\Big[\partial_x(1-\partial_x^2)^{-1}\Big(u^2 + \frac{1}{2}(\partial_x u)^2\Big)\Big](u^2 - \partial_x^2(u^2))\mathrm{d}x\\ &= \mathrm{I}_1 + \frac{1}{2}\int_{\mathbb{S}^1}u(\partial_x u)^3\mathrm{d}x - \frac{1}{2}\int_{\mathbb{S}^1}u(\partial_x u)^3\mathrm{d}x.\end{aligned}\tag{2.3}$$

Combining (2.2)–(2.3) into (2.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F(u) = 0.$$

This completes the proof of Lemma 2.1.

Lemma 2.2 *Let u be a 2π periodic function which is the solution of* (1.1). *Then,* $G(u) = G(u_0)$, *where*

$$G(u) = \int_{\mathbb{S}^1}(u^2 + (\partial_x u)^2)\mathrm{d}x.$$

Proof Multiplying by $2u$ on both sides of (2.1) and integrating by parts with respect to $[0,2\pi)$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^1}^2)=0,$$

which completes the proof of Lemma 2.2.

Lemma 2.3 *Let u be a 2π periodic function which is the solution of* (1.1). *Then*

$$u_t=-\partial_x(1-\partial_x^2)^{-1}\frac{\delta F(u)}{\delta u}. \tag{2.4}$$

Consequently, (1.1) *possesses the Hamiltonian structure due to the fact that $\partial_x(1-\partial_x^2)^{-1}$ is a Hamiltonian operator.*

Proof By an immediate computation, we have

$$\frac{\delta F(u)}{\delta u}=\partial_x^4u-\partial_x^2u+\frac{3}{2}u^2+\frac{1}{2}(\partial_xu)^2-\frac{1}{2}\partial_x^2(u^2). \tag{2.5}$$

Combining (2.5) with (1.1), we have (2.4) which leads to the desired result.

Lemma 2.4 *Let $N\in\mathbb{Z}$ and $\alpha>\frac{1}{2}$. Then there exists $C_\beta>0$ such that*

$$\sum_{n\in\mathbb{Z}}\frac{1}{\langle n\rangle^\alpha\langle n-N\rangle^\alpha}\le\frac{C_\beta}{\langle N\rangle^\beta}$$

for all $\beta<2\alpha-1$ when $\frac{1}{2}<\alpha\le1$ and $\beta=\alpha$ when $\alpha>1$.

For the proof of Lemma 2.4, we recommend the reader to refer to [5, Lemma 5.1].

Lemma 2.5 *Let $\sigma<\frac{1}{2}$. Then there exists $C>0$ such that*

$$\|\|u\|_{H^\sigma}\|_{L^p_\theta}\le C\sqrt{p}$$

for all $p\ge2$.

For the proof of Lemma 2.5, we recommend the reader to refer to [5, Lemma 4.4].

3 Proof of Theorem 1.1

First we have a statement as below and one can refer to [22, Proposition 2.4] for the details of the proof.

Lemma 3.1 (Wiener Chaos) *Let $d\ge1$, $c(n_1,\cdots,n_k)\in\mathbb{C}$ and $(g_n)_{1\le n\le d}\in N_{\mathbb{C}}(0,1)$ be complex L^2-normalized independent Gaussians. For $k\ge1$, denote by $A(k,d)=\{(n_1,\cdots,n_k)\in\{1,\cdots,d\}^k,\ n_1\le\cdots\le n_k\}$ and*

$$S_k(\omega)=\sum_{A(k,d)}c(n_1,\cdots,n_k)g_{n_1}(\omega)\cdots g_{n_k}(\omega).$$

Then for all $d\ge1$ and $p\ge2$,

$$\|S_k\|_{L^p(\Omega)}\le\sqrt{k+1}(p-1)^{\frac{k}{2}}\|S_k\|_{L^2(\Omega)}.$$

Define the function $f_N : H^\sigma(\mathbb{S}^1) \to \mathbb{R}$ by

$$f_N(u) = \int_{\mathbb{S}^1} u_N(x)(\partial_x u_N(x))^2 \mathrm{d}x.$$

Then we prove some lemmas in preparation to prove Theorem 1.1.

Lemma 3.2 *For the sequence $(f_N)_{N\geq 1}$ defined above, it is a Cauchy sequence in $L^2(H^\sigma(\mathbb{S}^1), \mathrm{d}\theta)$. In particular, there exists $C > 0$ such that for every $M > N \geq 1$,*

$$\|f_M(u) - f_N(u)\|_{L^2(H^\sigma(\mathbb{S}^1),\, \mathcal{B},\, \mathrm{d}\theta)} \leq \frac{C}{N}. \tag{3.1}$$

Moreover, for every $M > N \geq 1$ and $p \geq 2$,

$$\|f_M(u) - f_N(u)\|_{L^p(H^\sigma(\mathbb{S}^1),\, \mathcal{B},\, \mathrm{d}\theta)} \leq Cp^{\frac{3}{2}} N^{-1}. \tag{3.2}$$

Proof Note that

$$\partial_x \varphi_N(\omega) = \sum_{|n|\leq N} \mathrm{i}n \frac{g_n(\omega)}{2\sqrt{\pi}\langle n\rangle^2} \mathrm{e}^{\mathrm{i}nx},$$

which implies

$$(\partial_x \varphi_N(\omega))^2 = -\sum_{|n_1|,\, |n_2|\leq N} n_1 n_2 \frac{g_{n_1}(\omega) g_{n_2}(\omega)}{4\pi\langle n_1\rangle^2\langle n_2\rangle^2} \mathrm{e}^{\mathrm{i}(n_1+n_2)x}. \tag{3.3}$$

From (3.3), we have

$$\int_{\mathbb{S}^1} \varphi_N(\omega)(\partial_x \varphi_N(\omega))^2 \mathrm{d}x = -\sum_{A_N} n_1 n_2 \frac{g_{n_1}(\omega) g_{n_2}(\omega) g_{n_3}(\omega)}{8\pi^{\frac{3}{2}}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2}, \tag{3.4}$$

where

$$A_N = \{(n_1, n_2, n_3) \in \mathbb{Z}^3, |n_1|, |n_2|, |n_3| \leq N \text{ and } n_1 + n_2 + n_3 = 0\}.$$

We now split (3.4) into three parts as below

$$A_N = A_N^1 \cup A_N^2 \cup A_N^3,$$

where

$$\begin{aligned}
A_N^1 &= \{(n_1, n_2, n_3) \in A_N,\ n_1 \neq \pm n_2,\ n_1 \neq \pm n_3,\ n_2 \neq \pm n_3\},\\
A_N^2 &= \{(n, -n, 0), (n, 0, -n), (0, n, -n)\},\\
A_N^3 &= \{(n, n, -2n), (n, -2n, n), (-2n, n, n)\}.
\end{aligned}$$

Thus, we have

$$\int_{\mathbb{S}^1} \varphi_N(\omega)(\partial_x \varphi_N(\omega))^2 \mathrm{d}x = \sum_{j=1}^{3} f_N^j(u),$$

where

$$f_N^1(u) = -\sum_{A_N^1} n_1 n_2 \frac{g_{n_1}(\omega) g_{n_2}(\omega) g_{n_3}(\omega)}{8\pi^{\frac{3}{2}}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2},$$

$$f_N^2(u) = -\sum_{A_N^2} n_1 n_2 \frac{g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)}{8\pi^{\frac{3}{2}}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2},$$

$$f_N^3(u) = -\sum_{A_N^3} n_1 n_2 \frac{g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)}{8\pi^{\frac{3}{2}}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2}.$$

First, we show that there exists $C > 0$ such that for all $M > N \geq 1$,

$$\|f_M^1(u) - f_N^1(u)\|_{L^2(\Omega)} \leq \frac{C}{N^{\frac{3}{2}}}. \tag{3.5}$$

For $M > N \geq 1$, set

$$\begin{aligned}\Lambda(N,M) = \{&(n_1,n_2,n_3) \in \mathbb{Z}^3,\ n_1+n_2+n_3 = 0,\ n_1 \neq \pm n_2,\ n_1 \neq \pm n_3,\\ &n_2 \neq \pm n_3,\ |n_1|,|n_2|,|n_3| \leq M,\ \max(|n_1|,|n_2|,|n_3|) > N\}.\end{aligned}$$

Then we have

$$\begin{aligned}&\|f_M^1(u) - f_N^1(u)\|_{L^2(\Omega)}^2\\ &= \int_\Omega \sum_{\Lambda(N,M)} \frac{n_1n_2m_1m_2 g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)\overline{g}_{m_1}(\omega)\overline{g}_{m_2}(\omega)\overline{g}_{m_3}(\omega)}{64\pi^3\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\langle m_1\rangle^2\langle m_2\rangle^2\langle m_3\rangle^2}\mathrm{d}\mathbf{p}(\omega),\end{aligned} \tag{3.6}$$

where (n_1,n_2,n_3), $(m_1,m_2,m_3) \in \Lambda(N,M)$. Note that if (n_1,n_2,n_3) and (m_1,m_2,m_3) are two triples from $\Lambda(N,M)$ such that $\{n_1,n_2,n_3\} \neq \{m_1,m_2,m_3\}$, then

$$\int_\Omega g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)\overline{g}_{m_1}(\omega)\overline{g}_{m_2}(\omega)\overline{g}_{m_3}(\omega)\mathrm{d}\mathbf{p}(\omega) = 0.$$

We consider the following three cases:

(i) $(n_1,n_2,n_3) = (m_1,m_2,m_3)$ or $(n_1,n_2,n_3) = (m_1,m_3,m_2)$,

(ii) $(n_1,n_2,n_3) = (m_2,m_1,m_3)$ or $(n_1,n_2,n_3) = (m_2,m_3,m_1)$,

(iii) $(n_1,n_2,n_3) = (m_3,m_1,m_2)$ or $(n_1,n_2,n_3) = (m_3,m_2,m_1)$.

In fact, only the first case needs to be considered, and the other two cases can be proved in a similar way. For $(n_1,n_2,n_3) = (m_1,m_2,m_3)$, it follows from (3.6) that

$$\begin{aligned}&\|f_M^1(u) - f_N^1(u)\|_{L^2(\Omega)}^2\\ &= \int_\Omega \sum_{(n_1,n_2,n_3)\in\Lambda(N,M)} \frac{n_1^2n_2^2|g_{n_1}(\omega)|^2|g_{n_2}(\omega)|^2|g_{n_3}(\omega)|^2}{64\pi^3\langle n_1\rangle^4\langle n_2\rangle^4\langle n_3\rangle^4}\mathrm{d}\mathbf{p}(\omega)\\ &\leq C \sum_{(n_1,n_2,n_3)\in\Lambda(N,M)} \frac{n_1^2n_2^2}{\langle n_1\rangle^4\langle n_2\rangle^4\langle n_3\rangle^4}\\ &= \sum_{i=1}^3 \mathrm{I}_i,\end{aligned}$$

where

$$\mathrm{I}_1 = C\sum_{|n_1|\geq N} \frac{n_1^2n_2^2}{\langle n_1\rangle^4\langle n_2\rangle^4\langle n_1+n_2\rangle^4},$$

$$\mathrm{I}_2 = C\sum_{|n_2|\geq N} \frac{n_1^2n_2^2}{\langle n_1\rangle^4\langle n_2\rangle^4\langle n_1+n_2\rangle^4},$$

$$\mathrm{I}_3 = C \sum_{|n_1+n_2|\geq N} \frac{n_1^2 n_2^2}{\langle n_1\rangle^4 \langle n_2\rangle^4 \langle n_1+n_2\rangle^4}.$$

By the symmetry and Lemma 2.4, we further have

$$\begin{aligned}
\mathrm{I}_1 = \mathrm{I}_2 &\leq \frac{C}{N^2} \sum_{n_1\in\mathbb{Z},\, |n_2|\geq N} \frac{1}{\langle n_1\rangle^2 \langle n_1+n_2\rangle^2} \\
&= \frac{C}{N^2} \sum_{n_1\in\mathbb{Z},\, |n_2|\geq N} \frac{1}{\langle n_1\rangle^2 \langle n_1-(-n_2)\rangle^2} \\
&\leq \frac{C}{N^2} \sum_{|n_2|\geq N} \frac{1}{\langle n_2\rangle^2} \leq \frac{C}{N^3} \qquad (3.7)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{I}_3 &\leq \frac{C}{N^4} \sum_{n_1\in\mathbb{Z},\, |n_1+n_2|\geq N} \frac{n_1^2 n_2^2}{\langle n_1\rangle^4 \langle n_2\rangle^4} \leq \frac{C}{N^4} \sum_{n_1\in\mathbb{Z}, |n_1+n_2|\geq N} \frac{1}{\langle n_1\rangle^2 \langle n_2\rangle^2} \\
&= \frac{C}{N^4} \sum_{n_1\in\mathbb{Z},\, |n_1+n_2|\geq N} \frac{1}{\langle n_1\rangle^2 \langle n_1-(n_1+n_2)\rangle^2} \\
&\leq \frac{C}{N^4} \sum_{n_1\in\mathbb{Z},\, |n_1+n_2|\geq N} \frac{1}{\langle n_1+n_2\rangle^2} \leq \frac{C}{N^5}. \qquad (3.8)
\end{aligned}$$

Combining (3.7) with (3.8), we have (3.5).

Next we prove that there exists $C>0$ such that for all $M>N\geq 1$,

$$\|f_M^2(u) - f_N^2(u)\|_{L^2(\Omega)} \leq \frac{C}{N}. \qquad (3.9)$$

For this, we write

$$\begin{aligned}
f_N^2(u) &= \sum_{|n|\leq N} \frac{n^2 |g_n(\omega)|^2 g_0(\omega)}{8\pi^{\frac{3}{2}} \langle n\rangle^4} \\
&= \sum_{|n|\leq N} \frac{n^2 (|g_n(\omega)|^2 - 1) g_0(\omega)}{8\pi^{\frac{3}{2}} \langle n\rangle^4} + \sum_{|n|\leq N} \frac{n^2 g_0(\omega)}{8\pi^{\frac{3}{2}} \langle n\rangle^4} \\
&=: f_N^{21} + f_N^{22}.
\end{aligned}$$

For $M>N\geq 1$, it turns out that

$$\begin{aligned}
&\|f_M^{21}(u) - f_N^{21}(u)\|_{L^2(\Omega)}^2 \\
&= \int_\Omega \sum_{N<|n|\leq M} \frac{n^2 (|g_n(\omega)|^2 - 1) g_0(\omega)}{8\pi^{\frac{3}{2}} \langle n\rangle^4} \sum_{N<|m|\leq M} \frac{m^2 (|g_m(\omega)|^2 - 1) \overline{g}_0(\omega)}{8\pi^{\frac{3}{2}} \langle m\rangle^4} \mathrm{d}\mathbf{p}(\omega) \\
&= \int_\Omega \sum_{N<|n|\leq M} \frac{n^4 (|g_n(\omega)|^2 - 1)^2 |g_0(\omega)|^2}{64\pi^3 \langle n\rangle^8} \mathrm{d}\mathbf{p}(\omega) \\
&\leq \sum_{N<|n|\leq M} \frac{C}{\langle n\rangle^4} \leq \frac{C}{N^3} \qquad (3.10)
\end{aligned}$$

and

$$
\begin{aligned}
&\|f_M^{22}(u)-f_N^{22}(u)\|^2_{L^2(\Omega)}\\
&=\int_\Omega \sum_{N<|n|\le M}\frac{n^2 g_0(\omega)}{8\pi^{\frac{3}{2}}\langle n\rangle^4}\sum_{N<|m|\le M}\frac{m^2\overline{g}_0(\omega)}{8\pi^{\frac{3}{2}}\langle m\rangle^4}\mathrm{d}\mathbf{p}(\omega)\\
&=\int_\Omega \sum_{N<|n|,|m|\le M}\frac{n^2m^2|g_0(\omega)|^2}{64\pi^3\langle n\rangle^4\langle m\rangle^4}\mathrm{d}\mathbf{p}(\omega)\\
&\le \sum_{N<|n|,|m|\le M}\frac{C}{\langle n\rangle^2\langle m\rangle^2}\le\frac{C}{N^2}. \qquad (3.11)
\end{aligned}
$$

Combining (3.10) and (3.11), we deduce

$$
\|f_M^2(u)-f_N^2(u)\|_{L^2(\Omega)}\le\|f_M^{21}(u)-f_N^{21}(u)\|_{L^2(\Omega)}+\|f_M^{22}(u)-f_N^{22}(u)\|_{L^2(\Omega)}\le\frac{C}{N}.
$$

Finally we prove that there exists $C>0$ such that for every $M>N\ge1$,

$$
\|f_M^3(u)-f_N^3(u)\|_{L^2(\Omega)}\le\frac{C}{N^3}. \qquad (3.12)
$$

For $M>N\ge1$, it yields that

$$
\begin{aligned}
&\|f_M^3(u)-f_N^3(u)\|^2_{L^2(\Omega)}\\
&=C\int_\Omega\sum_{\frac{N}{2}<|n|\le\frac{M}{2}}\frac{n^4|g_n^2(\omega)|^2|g_{2n}(\omega)|^2}{\langle n\rangle^8\langle 2n\rangle^4}\mathrm{d}\mathbf{p}(\omega)\\
&\le C\sum_{\frac{N}{2}<|n|\le\frac{M}{2}}\frac{n^4\Big(\int_\Omega|g_n(\omega)|^6\mathrm{d}p(\omega)\Big)^{\frac{2}{3}}\Big(\int_\Omega|g_{2n}(\omega)|^6\mathrm{d}\mathbf{p}(\omega)\Big)^{\frac{1}{3}}}{\langle n\rangle^8\langle 2n\rangle^4}\\
&=C\sum_{\frac{N}{2}<|n|\le\frac{M}{2}}\frac{n^4\ 6^{\frac{2}{3}}\ 6^{\frac{1}{3}}}{\langle n\rangle^8\langle 2n\rangle^4}\le C\sum_{\frac{N}{2}<|n|\le\frac{M}{2}}\frac{1}{\langle n\rangle^4\langle 2n\rangle^4}\\
&\le\frac{C}{N^6},
\end{aligned}
$$

which implies (3.12).

Thus, combining (3.5), (3.9) with (3.12), we have (3.1). As for (3.2), it is a direct consequence of (3.1) and Lemma 3.1.

With Lemma 3.2, we immediately have the following corollary.

Corollary 3.1 *Denote by $f(u)\in L^2(H^\sigma(\mathbb{S}^1),\mathcal{B},\mathrm{d}\theta)$ the limit of $(f_N)_{N\ge1}$. Then the sequence $(f_N)_{N\ge1}$ converges in measure to f. In particular, for every $\epsilon>0$,*

$$
\lim_{N\to\infty}\theta(u\in H_0^\sigma(\mathbb{S}^1):|f(u)-f_N(u)|>\epsilon)=0.
$$

Proof The lemma immediately follows from an application of the Chebyshev inequality, so we leave out the proof here to save space.

We then study the limit of $\|u_N\|^2_{L^2(\mathbb{S}^1)}+\|\partial_x u_N\|^2_{L^2(\mathbb{S}^1)}-\alpha_N$ as $N\to\infty$. Define $r_N:H^\sigma(\mathbb{S}^1)\to R$ by

$$
r_N(u)=\|u_N\|^2_{L^2(\mathbb{S}^1)}+\|\partial_x u_N\|^2_{L^2(\mathbb{S}^1)}-\alpha_N. \qquad (3.13)
$$

Then the following statement can be obtained.

Lemma 3.3 *For the sequence $(r_N)_{N\geq 1}$ defined in* (3.13)*, it is a Cauchy sequence in $L^2(H_0^\sigma(\mathbb{S}^1), \mathcal{B}, \mathrm{d}\theta)$. In particular, there exists $C>0$ such that for every $M>N\geq 1$,*

$$\|r_M(u)-r_N(u)\|_{L^2(H_0^\sigma(\mathbb{S}^1),\mathcal{B},\mathrm{d}\theta)} \leq \frac{C}{N}. \tag{3.14}$$

Moreover, denote by $r(u)$ the limit of $r_N(u)$ in $L^2(H_0^\sigma(\mathbb{S}^1), \mathcal{B}, \mathrm{d}\theta)$, the sequence $(r_N)_{N\geq 1}$ converges to $r(u)$ in measure, i.e.,

$$\forall \epsilon>0, \quad \lim_{N\to\infty} \theta(u\in H_0^\sigma(\mathbb{S}^1) : |r(u)-r_N(u)|>\epsilon)=0.$$

Proof We write

$$\|\varphi_N\|^2_{L^2(\mathbb{S}^1)} = \frac{1}{2\pi}\int_0^{2\pi} \sum_{|n|\leq N} \frac{g_n(\omega)}{2\sqrt{\pi}\langle n\rangle^2}\mathrm{e}^{inx} \sum_{|m|\leq N}\frac{\overline{g}_m(\omega)}{2\sqrt{\pi}\langle m\rangle^2}\mathrm{e}^{-\mathrm{i}mx}\mathrm{d}x = \sum_{|n|\leq N}\frac{|g_n(\omega)|^2}{4\pi\langle n\rangle^4}, \tag{3.15}$$

$$\begin{aligned}\|\partial_x\varphi_N\|^2_{L^2(\mathbb{S}^1)} &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{|n|\leq N} \mathrm{i}n\frac{g_n(\omega)}{2\sqrt{\pi}\langle n\rangle^2}\mathrm{e}^{inx} \sum_{|m|\leq N}(-\mathrm{i}m)\frac{\overline{g}_m(\omega)}{2\sqrt{\pi}\langle m\rangle^2}\mathrm{e}^{-imx}\mathrm{d}x \\ &= \sum_{|n|\leq N}\frac{n^2|g_n(\omega)|^2}{4\pi\langle n\rangle^4}. \end{aligned}\tag{3.16}$$

Combining (3.15) with (3.16), we have

$$\begin{aligned}
&\|r_M-r_N\|^2_{L^2(H_0^\sigma(\mathbb{S}^1),\mathcal{B},\mathrm{d}\theta)} \\
&= \int_\Omega \big|\|\varphi_M\|^2_{L^2(\mathbb{S}^1)} + \|\partial_x\varphi_M\|^2_{L^2(\mathbb{S}^1)} - \alpha_M - \|\varphi_N\|^2_{L^2(\mathbb{S}^1)} - \|\partial_x\varphi_N\|^2_{L^2(\mathbb{S}^1)} + \alpha_N\big|^2\mathrm{d}\mathbf{p}(\omega) \\
&= \int_\Omega\Big|\sum_{N<|n|\leq M}\frac{(n^2+1)|g_n(\omega)|^2-4\pi}{4\pi\langle n\rangle^4}\Big|^2\mathrm{d}\mathbf{p}(\omega) \\
&\leq C\int_\Omega\Big|\sum_{N<|n|\leq M}\frac{|g_n(\omega)|^2-1}{\langle n\rangle^2}\Big|^2\mathrm{d}\mathbf{p}(\omega) + C\int_\Omega\Big|\sum_{N<|n|\leq M}\frac{|g_n(\omega)|^2-1}{\langle n\rangle^4}\Big|^2\mathrm{d}\mathbf{p}(\omega) \\
&\quad + \Big|\sum_{N<|n|\leq M}\frac{C}{\langle n\rangle^2}\Big|^2 + \Big|\sum_{N<|n|\leq M}\frac{C}{\langle n\rangle^4}\Big|^2 \\
&= \sum_{i=1}^4 J_i.
\end{aligned}$$

Set $R_n(\omega) = |g_n(\omega)|^2-1$. For $n\neq m$, since g_n and g_m are mutually independent and $E[|g_n(\omega)|^2]=1$, it follows that

$$E[R_n(\omega)R_m(\omega)] = E[R_n(\omega)]E[R_m(\omega)] = 0.$$

By an immediate computation, we have

$$\begin{aligned}
J_1 &= C\int_\Omega\sum_{N<|n|\leq M}\frac{|g_n(\omega)|^2-1}{\langle n\rangle^2}\sum_{N<|m|\leq M}\frac{|g_m(\omega)|^2-1}{\langle m\rangle^2}\mathrm{d}\mathbf{p}(\omega) \\
&= C\int_\Omega\sum_{N<|n|\leq M}\frac{(|g_n(\omega)|^2-1)^2}{\langle n\rangle^4}\mathrm{d}\mathbf{p}(\omega)
\end{aligned}$$

$$= C \sum_{N<|n|\le M} \frac{E|g_n(\omega)|^4 - 2E|g_n(\omega)|^2 + 1}{\langle n\rangle^4} \le \frac{C}{N^3}, \tag{3.17}$$

$$J_2 = C \int_\Omega \sum_{N<|n|\le M} \frac{|g_n(\omega)|^2 - 1}{\langle n\rangle^4} \sum_{N<|m|\le M} \frac{|g_m(\omega)|^2 - 1}{\langle m\rangle^4} \mathrm{d}\mathbf{p}(\omega)$$

$$= C \int_\Omega \sum_{N<|n|\le M} \frac{(|g_n(\omega)|^2 - 1)^2}{\langle n\rangle^8} \mathrm{d}\mathbf{p}(\omega)$$

$$= C \sum_{N<|n|\le M} \frac{E|g_n(\omega)|^4 - 2E|g_n(\omega)|^2 + 1}{\langle n\rangle^8} \le \frac{C}{N^7}. \tag{3.18}$$

Obviously,

$$J_3 \le \frac{C}{N^2}, \tag{3.19}$$

$$J_4 \le \frac{C}{N^6}. \tag{3.20}$$

From (3.17)–(3.20), we conclude

$$\|r_M(u) - r_N(u)\|_{L^2(H_0^\sigma(\mathbb{S}^1),\mathcal{B},\mathrm{d}\theta)} \le \frac{C}{N}.$$

This yields (3.14). Finally, the convergence of $(r_N)_{N\ge 1}$ in measure follows from the Chebyshev inequality, which puts an end of the proof of Lemma 3.3.

Lemma 3.4 *There exists $C > 0$ such that for every $M > N \ge 1$, $p > 2$,*

$$\|r_M(u) - r_N(u)\|_{L^p(H_0^\sigma(\mathbb{S}^1),\, \mathcal{B},\, \mathrm{d}\theta)} \le CpN^{-1}.$$

Proof Notice

$$\begin{aligned}
&\|r_M - r_N\|^p_{L^p(H_0^\sigma(\mathbb{S}^1),\mathcal{B},\mathrm{d}\theta)} \\
&= \Big(\frac{1}{2}\Big)^p \int_\Omega \Big| \sum_{N<|n|\le M} \frac{(n^2+1)|g_n(\omega)|^2 - 1}{\langle n\rangle^4} \Big|^p \mathrm{d}\mathbf{p}(\omega) \\
&\le C \int_\Omega \Big| \sum_{N<|n|\le M} \frac{|g_n(\omega)|^2 - 1}{\langle n\rangle^2} \Big|^p \mathrm{d}\mathbf{p}(\omega) + C \int_\Omega \Big| \sum_{N<|n|\le M} \frac{|g_n(\omega)|^2 - 1}{\langle n\rangle^4} \Big|^p \mathrm{d}\mathbf{p}(\omega) \\
&\quad + \Big| \sum_{N<|n|\le M} \frac{C}{\langle n\rangle^2} \Big|^p + \Big| \sum_{N<|n|\le M} \frac{C}{\langle n\rangle^4} \Big|^p.
\end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned}
&\|r_M - r_N\|_{L^p(H_0^\sigma(\mathbb{S}^1),\mathcal{B},\mathrm{d}\theta)} \\
&\le \|r_M - r_N\|_{L^2(H^\sigma(\mathbb{S}^1),\, \mathrm{d}\theta)} + \Big| \sum_{N<|n|\le M} \frac{C}{\langle n\rangle^2} \Big| + \Big| \sum_{N<|n|\le M} \frac{C}{\langle n\rangle^4} \Big| \\
&\le \frac{Cp}{N} + \frac{C}{N} + \frac{C}{N^3},
\end{aligned}$$

which implies

$$\|r_M - r_N\|_{L^p(H_0^\sigma(\mathbb{S}^1),\mathcal{B},\mathrm{d}\theta)} \le CpN^{-1}.$$

This puts an end of the proof for Lemma 3.4.

We are now able to define the function

$$G : H_0^\sigma(\mathbb{S}^1) \to \mathbb{R}$$

by

$$G(u) = \chi_R(r(u))\mathrm{e}^{-f(u)-\int_{\mathbb{S}^1} u^3 \mathrm{d}x}.$$

Then it is no risk to say that $G(u)$ is the limit in measure of $\chi_R(r_N(u))\mathrm{e}^{-f_N(u)-\int_{\mathbb{S}^1} u_N^3 \mathrm{d}x}$ as $N \to \infty$. Actually, based on [23, Corollary 4.4], Corollary 3.1 and Lemma 3.3, we have the convergence of $\|u_N\|_{L^3(\mathbb{S}^1)}$, $f_N(u)$ and $\chi_R(r_N(u))$ to $\|u\|_{L^3(\mathbb{S}^1)}$, $f(u)$ and $\chi_R(r(u))$ in the θ measure, respectively. Then by composition and multiplication of continuous functions, the conclusion that $G_N(u)$ converges to $G(u)$ in measure follows immediately.

Lemma 3.5 *For $1 \le p < \infty$, there exists $C > 0$ such that for every $N \ge 1$,*

$$\|G_N(u)\|_{L^p(\mathrm{d}\theta(u))} \le C.$$

Proof First recall that

$$G_N(u) = \chi_R(\|u_N\|^2_{L^2(\mathbb{S}^1)} + \|\partial_x u_N\|^2_{L^2(\mathbb{S}^1)} - \alpha_N)\mathrm{e}^{-\int_{\mathbb{S}^1}(u_N^3 + u_N(\partial_x u_N)^2)\mathrm{d}x}.$$

Thus it follows that

$$\begin{aligned}
\|G_N(u)\|_{L^p(\mathrm{d}\theta(u))} &\le \|\mathrm{e}^{-\int_{\mathbb{S}^1}(u_N^3 + (u_N\partial_x u_N)^2)\mathrm{d}x}\|_{L^p(\mathrm{d}\theta(u))} \\
&\le \|\mathrm{e}^{\int_{\mathbb{S}^1}|u_N|(u_N^2 + (\partial_x u_N)^2)\mathrm{d}x}\|_{L^p(\mathrm{d}\theta(u))} \\
&\le \|\mathrm{e}^{\|u_N\|_{L^\infty(\mathbb{S}^1)}\|u_N\|^2_{H^1(\mathbb{S}^1)}}\|_{L^p(\mathrm{d}\theta(u))} \\
&\le \|\mathrm{e}^{\|u_N\|^3_{H^1(\mathbb{S}^1)}}\|_{L^p(\mathrm{d}\theta(u))} \\
&\le \|\mathrm{e}^{\alpha_N + R}\|_{L^p(\mathrm{d}\theta(u))} \le C,
\end{aligned}$$

which ends the proof of Lemma 3.5.

Now we are well prepared to prove Theorem 1.1.

Consider the sequence $(G_N(u))_{N\ge 1}$. Noticing the convergence of $G_N(u)$ to $G(u)$ in measure, we know that there exists a subsequence N_k such that $G_{N_k}(u) \to G(u)$, θ a.s. Note that Lemma 3.5 implies that there exists a constant $C > 0$ such that $\|G_{N_K}(u)\|_{L^p(\mathrm{d}\theta(u))} \le C$, $\forall k \in N$. Thus, by Fatou lemma, we have

$$\int_{H_0^\sigma(\mathbb{S}^1)} |G(u)|^p \mathrm{d}\theta(u) \le \liminf_{k\to\infty} \int_{H_0^\sigma(\mathbb{S}^1)} |G_{N_k}(u)|^p \mathrm{d}\theta(u) \le C.$$

4 Proof of Theorem 1.2

The proof of Theorem 1.2 is divided into Lemmas 4.1–4.3 in this section.

We first state some useful notation. Setting $h_N(u_N, \partial_x u_N) = \frac{1}{2}u_N^2 + (1-\partial_x^2)^{-1}(u_N^2 + \frac{1}{2}(\partial_x u_N)^2)$, we consider the approximation of (1.1):

$$\begin{cases}
u_t + \partial_x^3 u + \Pi_N \partial_x(h_N(u_N, \partial_x u_N)) = 0, \\
u(x,0) = \displaystyle\sum_{n\in Z} \frac{g_n(\omega)}{2\sqrt{\pi}\langle n\rangle^2}\mathrm{e}^{\mathrm{i}nx}.
\end{cases} \tag{4.1}$$

Set $\Pi^0 = 1 - \Pi_0$, the orthogonal projection on 0-mean functions. We then prove the following lemma inspired by [23, Lemma 5.1].

Lemma 4.1 *For $\sigma \le \frac{1}{2}-2\epsilon$, $h_N(u_N,\partial_x u_N)$ is a Cauchy sequence in $L^p(X^{\frac{1}{2}}(\mathbb{S}^1),\mathcal{B},\mathrm{d}\theta,H_0^\sigma(\mathbb{S}^1))$. More precisely, for all $p\ge 2$, there exist $\eta>0$ and $C>0$ such that for all $1\le M<N$,*

$$\int_{X^{\frac{1}{2}}(\mathbb{S}^1)} \|\Pi^0(h_N)-\Pi^0(h_M)\|^p_{H^\sigma(\mathbb{S}^1)}\mathrm{d}\theta(u)\le \frac{C}{M^\epsilon}.$$

If we denote its limit by $\Pi^0 h(u,\partial_x u)$, we further have

$$\partial_x h(u,\partial_x u)=\partial_x(\Pi^0 h(u,\partial_x u)).$$

Proof According to the definition of the measure θ, we know

$$\begin{aligned}&\int_{X^{\frac{1}{2}}(\mathbb{S}^1)} \|\Pi^0 h_N(u_N,\partial_x u_N)-\Pi^0 h_M(u_M,\partial_x u_M)\|^p_{H^\sigma(\mathbb{S}^1)}\mathrm{d}\theta(u)\\ &=\int_\Omega \|\Pi^0 h_N(\varphi_N,\partial_x\varphi_N)-\Pi^0 h_M(\varphi_M,\partial_x\varphi_M)\|^p_{H^\sigma(\mathbb{S}^1)}\mathrm{d}\mathbf{p}.\end{aligned}$$

Also, by definition of φ_N, we have

$$\int_{\mathbb{S}^1}\Pi^0(h_N(u_N,\partial_x u_N)-h_M(u_M,\partial_x u_M))\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x=\sum_{\Omega^k_{M,N}}\frac{g_{n_1}g_{n_2}}{\prod\limits_{j=1}^2\langle n_j\rangle^2}\frac{n^2-n_1n_2-1}{2(1+n^2)},$$

where

$$\Omega^k_{M,N}=\{(n_1,n_2)\in\mathbb{Z}^2 \mid k=n_1+n_2,\ n_1\ne -n_2,\ M<\max\{|n_1|,|n_2|\}\le N\}.$$

Since $(g_n)_{n\in\mathbb{Z}^*}$ is a sequence of mutually independent complex normalised Gaussian, we arrive at

$$\begin{aligned}&\|\mathscr{F}_x\Pi^0(h_N(u_N,\partial_x u_N)-h_M(u_M,\partial_x u_M))\|^2_{L^2(\Omega)}\\ &=\int_\Omega\sum_{n_1,n_2\in\Omega^k_{M,N}}\sum_{m_1,m_2\in\Omega^k_{M,N}}\frac{g_{n_1}g_{n_2}}{\prod\limits_{j=1}^2\langle n_j\rangle^2}\frac{n^2-n_1n_2-1}{2(1+n^2)}\frac{\overline{g_{m_1}g_{m_2}}}{\prod\limits_{j=1}^2\langle m_j\rangle^2}\frac{m^2-m_1m_2-1}{2(1+m^2)}\mathrm{d}\mathbf{p}\\ &\le C\int_\Omega\sum_{n_1,n_2\in\Omega^k_{M,N}}\Big(\frac{1}{\prod\limits_{j=1}^2\langle n_j\rangle^2}\frac{n^2-n_1n_2-1}{2(1+n^2)}\Big)^2\\ &\le C\sum_{n_1,n_2\in\Omega^k_{M,N}}\frac{C}{\prod\limits_{j=1}^2\langle n_j\rangle^2}\\ &\le C\sum_{|n|>M}\frac{1}{|n||n-k|}\\ &\le\frac{C}{M^\epsilon\langle k\rangle^{2-\epsilon}}.\end{aligned}$$

Consequently, due to $\sigma\le\frac{1}{2}-2\epsilon$, we obtain

$$\|\mathscr{F}_x\Pi^0(h_N(u_N,\partial_x u_N)-h_M(u_M,\partial_x u_M))\|^2_{L^2(\Omega;H^\sigma(\mathbb{S}^1))}\le\frac{C}{M^\epsilon}\sum_{k\in\mathbb{Z}}\langle k\rangle^{-2+\epsilon+2\sigma}\le\frac{C}{M^\epsilon}.$$

This completes the proof of Lemma 4.1.

Next we consider the probability measure μ_N defined by (1.4). Define the measure ν_N on $C([-T,T];X^{\frac{1}{2}})$ as the image measure of μ_N by the map

$$X^{\frac{1}{2}} \to C([-T,T];X^{\frac{1}{2}}),$$
$$v \mapsto \Phi_N(t)v,$$

where Φ_N is the flow generated by (4.1). Moreover, for any measurable $F : C([-T,T];X^{\frac{1}{2}}) \to \mathbb{R}$,

$$\int_{C([-T,T];X^{\frac{1}{2}})} F(u)\mathrm{d}\nu_N(u) = \int_{X^{\frac{1}{2}}} F(\Phi_N(t)(v))\mathrm{d}\mu_N(v). \tag{4.2}$$

Then we are able to prove the following estimates.

Lemma 4.2 *For $\sigma \le \frac{1}{2} - \epsilon$, there exists $C > 0$ such that*

$$\|\|u\|_{L^p_T H^\sigma}\|_{L^p_{\nu_N}} \le C\sqrt{p} \tag{4.3}$$

and

$$\|\|u_t\|_{L^p_T H^{\sigma-3}}\|_{L^p_{\nu_N}} \le C \tag{4.4}$$

for all $N \ge 1$ and $p \ge 2$.

Proof The bound (4.3) is obtained thanks to Lemmas 2.5 and 3.5. We now turn to (4.4). From (4.1), it follows that

$$u_t = -\partial_x^3 u - \Pi_N \partial_x (h_N(u_N, \partial_x u_N)).$$

Thus

$$\|\|u_t\|_{L^p_T H^{\sigma-3}}\|_{L^p_{\nu_N}} \le \|\|u\|_{L^p_T H^\sigma}\|_{L^p_{\nu_N}} + \|\Pi^0 h_N(u_N, \partial_x u_N)\|_{L^p_{\nu_N} L^p_T H^\sigma}.$$

Obviously, by using Lemma 4.1 and (4.2), we have

$$\begin{aligned}
&\|\Pi^0 h_N(u_N, \partial_x u_N)\|^p_{L^p_{\nu_N} L^p_T H^\sigma} = \int_{C([-T,T];X^{\frac{1}{2}})} \|\Pi^0 h_N(u_N, \partial_x u_N)\|^p_{L^p_T H^\sigma} \mathrm{d}\nu_N \\
&= \int_{X^{\frac{1}{2}}} \|\Pi^0 h_N(\Phi_N(t)v, \partial_x \Phi_N(t)v)\|^p_{L^p_T H^\sigma} \mathrm{d}\mu_N(v) \\
&= \int_{X^{\frac{1}{2}}} \|\Pi^0 h_N(v, \partial_x v)\|^p_{L^p_T H^\sigma} \mathrm{d}\mu_N(v) \\
&= 2T \int_{X^{\frac{1}{2}}} \|\Pi^0 h_N(v, \partial_x v)\|^p_{H^\sigma} G_N(v)\mathrm{d}\theta(v) \\
&\le C_T \Big(\int_{X^{\frac{1}{2}}} \|\Pi^0 h_N(v, \partial_x v)\|^{2p}_{H^\sigma} \mathrm{d}\theta(v) \Big)^{\frac{1}{2}} \|G_N(v)\|_{L^2_\theta} \le C,
\end{aligned}$$

which ends the proof of Lemma 4.2.

By Lemma 4.2, similar to [5, Proposition 4.11], we can prove the following lemma.

Lemma 4.3 *For $T > 0$ and $\sigma \le \frac{1}{2} - \epsilon$, the family of measures*

$$\nu_N = \mathscr{L}_{C_T H^\sigma}(u_N(t); t \in [-T,T])_{N \ge 1}$$

is tight in $C([-T,T]; H^\sigma(\mathbb{S}^1))$.

Finally, we are well prepared to prove Theorem 1.2. Based on Lemma 4.3, Prokhorov theorem can be used, i.e., for each $T>0$, there exists a sub-sequence ν_{N_k} and a limit measure ν on the space $C([-T,T];X^{\frac{1}{2}}(\mathbb{S}^1))$ such that $\nu_{N_k}\to\nu$ weakly on $C([-T,T];H^\sigma(\mathbb{S}^1))$ for all $\sigma\le\frac{1}{2}-\epsilon$. Thanks to the Skorokhod theorem, there exists a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{P}})$, a sequence of random variables $(\widetilde{u}_{N_k})$ and a random variable u with values in $C([-T,T];X^{\frac{1}{2}}(\mathbb{S}^1))$ so that

$$\mathscr{L}(\widetilde{u}_{N_k}:t\in[-T,T])=\mathscr{L}(u_{N_k}:t\in[-T,T])=\nu_{N_k},\quad \mathscr{L}(\widetilde{u}:t\in[-T,T])=\nu,$$

and for all $\sigma\le\frac{1}{2}-\epsilon$, $\widetilde{u}_{N_k}\to\widetilde{u}$, $\widetilde{\mathbf{p}}$-a.s. in $C([-T,T];H^\sigma(\mathbb{S}^1))$. Consequently, we have $\mathscr{L}_{X^{\frac{1}{2}}(\mathbb{S}^1)}(\widetilde{u}_{N_k}(t))=\mathscr{L}_{X^{\frac{1}{2}}(\mathbb{S}^1)}(u_{N_k}(t))=\mu_{N_k}$ for all $t\in[-T,T]$ and $k\ge1$. Hence $\mathscr{L}_{X^{\frac{1}{2}}(\mathbb{S}^1)}(u(t))$ $=\mu$. On the other hand, note that $\widetilde{u}_{N_k}$ satisfies the following equation $\widetilde{\mathbf{p}}$-a.s.

$$\widetilde{u}_{N_k t}+\partial_x^3\widetilde{u}_{N_k}+\Pi_{N_k}\partial_x(h_{N_k}(\Pi_{N_k}\widetilde{u}_{N_k},\partial_x\Pi_{N_k}\widetilde{u}_{N_k}))=0. \tag{4.5}$$

We can see that all terms of (4.5) converges as $k\to\infty$, among which the convergence of the nonlinear terms is derived from [5, Lemma 5.6]. Therefore, it follows that, for $\sigma\le\frac{1}{2}-\epsilon$, there exists $C>0$ such that

$$\Pi^0 h_{N_k}(\Pi_{N_k}\widetilde{u}_{N_k},\partial_x\Pi_{N_k}\widetilde{u}_{N_k})\to\Pi^0 h(\widetilde{u},\partial_x\widetilde{u})\ \text{ a.s.}$$

in $L^2([-T,T];H^\sigma(\mathbb{S}^1))$ as $k\to\infty$. This finishes the proof of Theorem 1.2.

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