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A Quasilinear System Related with the Asymptotic Equation of the Nematic Liquid Crystal's Director Field*

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Abstract In this paper, the author studies the local existence of strong solutions and their possible blow-up in time for a quasilinear system describing the interaction of a short wave induced by an electron field with a long wave representing an extension of the motion of the director field in a nematic liquid crystal's asymptotic model introduced in [Saxton, R. A., Dynamic instability of the liquid crystal director. In: Current Progress in Hyperbolic Systems (Lindquist, W. B., ed.), Contemp. Math., Vol.100, Amer. Math. Soc., Providence, RI, 1989, pp.325–330] and [Hunter, J. K. and Saxton, R. A., Dynamics of director fields, SIAM J. Appl. Math., 51, 1991, 1498–1521] and studied in [Hunter, J. K. and Zheng, Y., On a nonlinear hyperbolic variational equation I, Arch. Rat. Mech. Anal., 129, 1995, 305–353], [Hunter, J. K. and Zheng, Y., On a nonlinear hyperbolic variational equation II, Arch. Rat. Mech. Anal., 129, 1995, 355–383] and in [Zhang, P. and Zheng, Y., On oscillation of an asymptotic equation of a nonlinear variational wave equation, Asymptotic Anal., 18, 1998, 307–327] and, more recently, in [Bressan, A., Zhang, P. and Zheng, Y., Asymptotic variational wave equations, Arch. Rat. Mech. Anal., 183, 2007, 163–185].

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1 Introduction and Main Results

Motivated by the study of an asymptotic equation for the director field in a simplified model of a nematic liquid crystal, introduced by Saxton, R. A. and Hunter, J. K. in [10, 14] and studied by Hunter, J. K., Zheng, Y. and Zhang, P. in [11–12, 15], we study the interaction with a short wave induced by an electron beam (cf. also [3]).

This interaction can be described by the following coupled time dependent system of the Benney type (cf. [2, 5–8] for some previous examples):

$$\begin{cases} \mathrm{i}u_t + u_{xx} = \alpha|u|^2u + bvu, \\ v_t + \frac{1}{2}(v^2)_x = ah(x)\int_0^x (v_x)^2\mathrm{d}y - b(|u|^2r)_x \end{cases} \tag{1.1}$$

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for $x \geq 0$ and $t \geq 0$, where $\alpha > 0$, $b \in \mathbb{R}$ and $a \in \mathbb{R}$ are given constants, $h \in C^\infty([0, +\infty)$ is a non-negative function with compact support, $\mathrm{i} = \sqrt{-1}$, $u(x,t) \in \mathbb{C}$ represents the short wave (electron field) and $v(x,t) \in \mathbb{R}$ represents the long wave (the director field in [10], where $b = 0$, $a = \frac{1}{2}$ and $h(x) \equiv 1$).

We will consider the IBV (initial and boundary value) problem for (1.1) with null boundary condition and with given initial data

$$u(x,0) = u_0(x) \in H_0^1(\mathbb{R}_+) \cap H^4(\mathbb{R}_+), \quad v(x,0) = v_0(x) \in H_0^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+). \tag{1.2}$$

In [15] the equation

$$v_t + \frac{1}{2}(v^2)_x = a \int_0^x (v_x)^2 \mathrm{d}y, \quad x \geq 0,\ t \geq 0 \tag{1.3}$$

is studied with $a = \frac{1}{2}$. By setting $w = v_x$, we derive

$$w_t + v w_x = (a-1) w^2 \tag{1.4}$$

and with $a = \frac{1}{2}$, $a_1 = a - 1 = -\frac{1}{2}$, $w_0(x) = w(x,0) \geq 0$, $w_0 \in C_c^\infty(\mathbb{R}_+)$, they obtain by the method of characteristics, with

$$\begin{cases} \dfrac{\mathrm{d}\phi_t(x)}{\mathrm{d}t} = v(t, \phi_t(x)) = \displaystyle\int_0^{\phi_t(x)} w(t,y)\mathrm{d}y, \\ \phi_0(x) = x, \end{cases}$$

the solution

$$w(t,x) = \frac{2 w_0(\phi_t^{-1}(x))}{2 + w_0(\phi_t^{-1}(x)) t},$$

and so

$$v(t,x) = \int_0^x \frac{2 w_0(\phi_t^{-1}(y))}{2 + w_0(\phi_t^{-1}(y)) t} \mathrm{d}y \geq 0.$$

Replacing $a = \frac{1}{2}$ by $a < 1$ and so $a_1 = a - 1 < 0$, we derive, by the same method,

$$v(t,x) = \int_0^x \frac{w_0(\phi_t^{-1}(y))}{1 - a_1 w_0(\phi_t^{-1}(y)) t} \mathrm{d}y \geq 0. \tag{1.5}$$

In Section 2, we study the local in time existence (and uniqueness) of a strong solution for the problem (1.1)–(1.2) with

$$h \in C^\infty([0, +\infty)), \quad h \geq 0 \text{ and with compact support},$$

by applying a variant of T. Kato's theorem in [13], after making a suitable transformation of the system to avoid some limited smoothing properties of the Schrödinger kernel and following the ideas introduced in [7–8]. We will prove the following result:

Theorem 1.1 *Assume that* (u_0, v_0) *verifies* (1.2). *Then there exist* $T > 0$ *and a unique strong solution* (u, v) *of the IBV problem* (1.1)–(1.2) *with*

$$(u,v) \in (C^j([0,T]; H^{4-2j}(\mathbb{R}_+)) \times C^j([0,T]; H^{3-j}(\mathbb{R}_+)) \cap (C^j([0,T]; H_0^1(\mathbb{R}_+)))^2, \quad j = 0, 1.$$

In Section 3, we start by deducing some identities for the flow (1.1) and, with some additional requirements on the initial data, and in particular for $v \geq 0$ and $a < 0$, we derive a blow-up in time result for the local strong solution of the IBV problem (1.1)–(1.2) obtained in Theorem 1.1. We will apply a virial technique developed in the seminal work of R. T. Glassey (cf. [9]) concerning nonlinear Schrödinger equations (see [8] for a related result and [10–12], for blow-up results concerning the equation (1.3)). The function

$$t \to \int x^2 |u(x,t)|^2 \mathrm{d}x, \quad \text{where} \int = \int_{\mathbb{R}_+}$$

used in [9] for the Schrödinger equation will be replaced by

$$\phi(t) = \frac{1}{2} \int x^2 |u|^2 \mathrm{d}x + \int_0^t \int x v^2 \mathrm{d}x \mathrm{d}\tau + c_0 \int_0^t \int x |u|^2 \mathrm{d}x \mathrm{d}\tau \tag{1.6}$$

for a convenient constant $c_0 > 0$ depending of α, b and the initial data (u_0, v_0) verifying

$$M(0) = \int v_0^2 \mathrm{d}x - 2 \operatorname{Im} \int u_0 \overline{u}_{0x} \mathrm{d}x < 0. \tag{1.7}$$

We will prove the following result:

Theorem 1.2 *Under the hypothesis of Theorem 1.1, let us assume* $a<0$, $b>0$, $xu_0, x^{\frac{1}{2}} v_0 \in L^2(\mathbb{R}_+)$, $u_0 \in H_0^2(\mathbb{R}_+)$ *and* (1.7). *With*

$$E(0) = \frac{1}{2} \int |u_{0x}|^2 \mathrm{d}x + \frac{\alpha}{4} \int |u_0|^4 \mathrm{d}x + \frac{b}{2} \int v_0 |u_0|^2 \mathrm{d}x + \frac{1}{12} \int v_0^3 \mathrm{d}x, \tag{1.8}$$

let c_0 *be a positive constant such that*

$$c_0 > \frac{b^2}{\alpha}, \quad 8E(0) + c_0 M(0) < 0. \tag{1.9}$$

Then there is no global in time solution (u, v) *for the IBV problem* (1.1)–(1.2) *such that* $v(x,t) \geq 0$ *for* $x \geq 0$ *and* $t \geq 0$.

2 Local Existence and Uniqueness

Following [7–8], for $u_0 \in H_0^1(\mathbb{R}_+) \cap H^4(\mathbb{R}_+)$ and $v_0 \in H_0^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$, let us take (u, v) as a possible solution in $\mathbb{R}_+ \times [0, T)$, $T > 0$, of the IBV problem (1.1)–(1.2) and make the following formal computations.

By setting $F = u_t$, we derive from (1.1) that

$$\mathrm{i}F + u_{xx} - u = \alpha |u|^2 u + bvu - u$$

and so, with $\Delta = \frac{\partial^2}{\partial x^2}$,

$$u = (\Delta - 1)^{-1} (\alpha |u|^2 u + u(bv - 1) - \mathrm{i}F). \tag{2.1}$$

Differentiating the first equation in (1.1) with respect to t, we obtain

$$\mathrm{i}F_t + F_{xx} = 2\alpha |u|^2 F + \alpha u^2 \overline{F} + bFv + buv_t,$$

and using the second equation in (1.2), we get

$$\mathrm{i}F_t + F_{xx} = 2\alpha|u|^2F + \alpha u^2\overline{F} + bFv - b^2u(|u|^2)_x - \frac{b}{2}u(v^2)_x + abuh\int_0^x (v_x)^2\mathrm{d}y.$$

These formal computations suggest us to consider the following IBV problem with null boundary condition:

$$\begin{cases} \mathrm{i}F_t + F_{xx} = 2\alpha|u|^2F + \alpha u^2\overline{F} + bFv - b^2u\big(|\widetilde{u}|^2\big)_x \\ \qquad\qquad - \dfrac{b}{2}u(v^2)_x + abuh\displaystyle\int_0^x (v_x)^2\mathrm{d}y, \\ v_t + \dfrac{1}{2}(v^2)_x = ah\displaystyle\int_0^x (v_x)^2\mathrm{d}y - b(|\widetilde{u}|^2)_x, \end{cases} \tag{2.2}$$

$$F(x,0) = F_0(x) \in H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+), \qquad v(x,0) = v_0(x) \in H^3(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+), \tag{2.3}$$

where u and $\widetilde{u}$ are given by

$$\begin{cases} u(x,t) = u_0(x) + \displaystyle\int_0^t F(x,\tau)\mathrm{d}\tau, \\ \widetilde{u}(x,t) = (\Delta - 1)^{-1}(\alpha|u|^2u + u(bv-1) - \mathrm{i}F). \end{cases} \tag{2.4}$$

The regularization provided by the operator $(\Delta-1)^{-1}$ implies $\widetilde{u} \in H^4 \cap H_0^1$ and this prevents the derivative losing in the right hand side of the first equation in (2.2).

The following lemma will be proved using a variant of the T. Kato's result, Theorem 6 in [13], on quasilinear systems.

Lemma 2.1 *Let* $(F_0, v_0) \in (H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+))^2$. *Then, there exist* $T > 0$ *and a unique strong solution* (F, v) *of the IBV problem* (2.2), *with*

$$(F,v) \in (C^j([0,T]; H^{2-2j}(\mathbb{R}_+)) \times C^j([0,T]; H^{3-j}(\mathbb{R}_+))) \cap (C([0,T]; H_0^1(\mathbb{R}_+)))^2, \quad j = 0, 1.$$

This lemma implies Theorem 1.1. Indeed, if (F, v) is a solution of the IBV problem (2.2) we obtain $u_t = F$ and $u(x,0) = u_0(x)$. We derive

$$\begin{aligned} &(\mathrm{i}u_t + u_{xx})_t \\ &= \mathrm{i}F_t + F_{xx} \\ &= 2\alpha|u|^2F + \alpha u^2\overline{F} + bFv - b^2u(|\widetilde{u}|^2)_x - \frac{b}{2}u(v^2)_x + abuh\int_0^x (v_x)^2\mathrm{d}y \\ &= 2\alpha|u|^2u_t + \alpha u^2\overline{u}_t + bu_tv + buv_t. \end{aligned}$$

Hence,

$$(\mathrm{i}u_t + u_{xx} - \alpha|u|^2u - bvu)_t = 0$$

and so

$$\mathrm{i}u_t + u_{xx} - \alpha|u|^2u - bvu = \phi_0(x),$$

where

$$\phi_0 = \mathrm{i}F_0 + (u_0)_{xx} - \alpha|u_0|^2u_0 - bv_0u_0.$$

If we set

$$F_0 = \mathrm{i}((u_0)_{xx} - \alpha|u_0|^2 u_0 - bv_0 u_0),$$

we obtain $\phi_0 = 0$ and (u, v) satisfies the first equation in (1.1). In addition,

$$u = (\Delta - 1)^{-1}(\alpha|u|^2 u + u(bv - 1) - \mathrm{i}u_t) \tag{2.5}$$

and so $\widetilde{u} = u$ and (u, v) satisfies the second equation in (1.1). Finally, we observe that $u_t = F \in C([0,T]; (H^2 \cap H_0^1)(\mathbb{R}_+))$ and so by (2.5) we obtain $u \in C([0,T]; (H^4 \cap H_0^1)(\mathbb{R}_+))$.

We now pass to sketch the proof of Lemma 2.1. In order to apply a variant of the Theorem 6 in [13], we follow the ideas developed in [7] and introduce the new real variables $F_1 = \operatorname{Re} F$, $F_2 = \operatorname{Im} F$, $u_1 = \operatorname{Re} u$, $u_2 = \operatorname{Im} u$, $U = (F_1, F_2, v)$, $F_{10} = \operatorname{Re} F_0$, $F_{20} = \operatorname{Im} F_0$. The IBV problem (2.2)–(2.3) can be written as follows:

$$\begin{cases} U_t + A(U)U = g(t, U), \\ U(x, 0) = (F_{10}, F_{20}, v_0) \in (H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+))^2, \end{cases} \tag{2.6}$$

where

$$A(U) = \begin{pmatrix} 0 & \Delta & 0 \\ -\Delta & 0 & 0 \\ 0 & 0 & v\frac{\partial}{\partial x} \end{pmatrix}$$

and

$$g(t,U) = \begin{bmatrix} 2\alpha|u|^2 F_2 - \alpha(u_1^2 - u_2^2)F_2 + 2\alpha u_1 u_2 F_1 + bvF_2 - b^2 u_2(|\widetilde{u}|^2)_x - bu_2 vv_x + abu_2 h\displaystyle\int_0^x (v_x)^2 \mathrm{d}y \\ 2\alpha|u|^2 F_1 - \alpha(u_1^2 - u_2^2)F_1 + 2\alpha u_1 u_2 F_2 - bvF_1 + b^2 u_1(|\widetilde{u}|^2)_x + bu_1 vv_x - abu_1 h\displaystyle\int_0^x (v_x)^2 \mathrm{d}y \\ ah\displaystyle\int_0^x (v_x)^2 \mathrm{d}y - b\big(|\widetilde{u}|^2\big)_x \end{bmatrix}.$$

Note that $g(t, U)$ is nonlocal.

We now set $X = (L^2(\mathbb{R}_+))^2 \times H^1(\mathbb{R}_+)$, $Y = ((H^2 \cap H_0^1)(\mathbb{R}_+))^2 \times (H^3 \cap H_0^1)(\mathbb{R}_+)$, and introduce the isomorphism $S = I - \Delta\colon Y \to X$. Moreover, $A\colon U = (F_1, F_2, v) \in W \to G(X, 1, \beta)$, where W is an open ball in Y centered at the origin and with radius R, and $G(X, 1, \beta)$ denotes the set of all linear operators D in X, such that $-D$ generates a C_0-semigroup $\{\mathrm{e}^{-tD}\}$ with $\|\mathrm{e}^{-tD}\| \le \mathrm{e}^{\beta t}$, $t \in [0, +\infty)$, $\beta \le \frac{1}{2} \sup_{x\in\mathbb{R}_+} |v_x(x)| \le cR$, $(F_1, F_2, v) \in W$. For fixed $T > 0$, it is easy to see that $\|g(t, U)\|_Y \le \lambda$, for $t \in [0, T]$ and $U \in C\big([0, T]; W\big)$. Now, with $B_0(v) \in \mathcal{L}(H^1)$, v in a ball W_1 in $(H^3 \cap H_0^1)(\mathbb{R}_+)$, defined by (8.7) in [13]

$$B_0(v) = -v_{xx}\frac{\partial}{\partial x}(1 - \Delta)^{-1} - vv_x\frac{\partial^2}{\partial x^2}(1 - \Delta)^{-1}$$

(which verifies $[S, A(v)]S^{-1} = B_0(v)$, following (8.7) in [13]), we introduce an operator $B(U) \in \mathcal{L}(X)$, $U = (F_1, F_2, v) \in W$, defined by

$$B(U) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_0(v) \end{bmatrix}.$$

In [13, §8], Kato proved that, for $v \in W_1$, $(1-\Delta)(vv_x)(1-\Delta)^{-1} = v\frac{\partial v}{\partial x} + B_0(v)$, and hence

$$SA(U)S^{-1} = A(U) + B(U), \quad \text{for } U \in W.$$

Moreover, for each pair (U, U^*), $U = (F_1, F_2, v)$, $U^* = (F_1^*, F_2^*, v^*)$ in W, it is not hard to prove that, for each $T' \leq T$,

$$\|g(t,U) - g(t,U^*)\|_{L^1(0,T';X)} \leq c(T') \sup_{0 \leq t \leq T'} \|U(t) - U^*(t)\|_X,$$

where $c(T')$ is a continuous increasing function such that $c(0) = 0$.

For example, it is easy to obtain, denoting by $\|\cdot\|_p$ the L^p norm,

$$\|h[(v_x)^2 - (v_x^*)^2]\|_2 \leq c\|v_x + v_x^*\|_\infty \|v_x - v_x^*\|_2 \leq c(R)\|v - v^*\|_{H^1},$$

$$\Big\|h' \int_0^x [(v_x)^2 - (v_x^*)^2]\mathrm{d}y\Big\|_2 \leq c\|h'\|_2 \int_0^\infty |v_x + v_x^*||v_x - v_x^*|\mathrm{d}x \leq c(R)\|v - v^*\|_{H^1},$$

and similar estimates for the remainder terms (cf. [7] for details). Finally, it is also easy to prove that

$$\|A(U) - A(U^*)\|_{\mathcal{L}(Y,X)} \leq c_1\|U - U^*\|_X,$$

c_1 not depending on $t \in [0,T]$, and this achieves the proof of Lemma 2.1.

3 A Blow-up Result

We start with the proof of some important identities for the strong solutions of the IBV problem (1.1)–(1.2).

Proposition 3.1 *Let (u,v) be a local solution of the IBV problem* (1.1)–(1.2) *under the conditions obtained in Theorem 1.1. Then we have that, for $t \in [0,T]$,*

$$\int |u(x,t)|^2\mathrm{d}x = \int |u_0(x)|^2\mathrm{d}x, \tag{3.1}$$

$$E(t) - \frac{a}{4}\int_0^t \int v^2 h\Big(\int_0^x (v_x)^2\mathrm{d}y\Big)\mathrm{d}x\mathrm{d}\tau - \frac{ab}{2}\int_0^t \int |u|^2 h\Big(\int_0^x (u_x)^2\mathrm{d}y\Big)\mathrm{d}x\mathrm{d}\tau = E(0), \tag{3.2}$$

where

$$E(t) = \frac{1}{2}\int |u_x|^2\mathrm{d}x + \frac{\alpha}{4}\int |u|^4\mathrm{d}x + \frac{b}{2}\int v|u|^2\mathrm{d}x + \frac{1}{12}\int v^3\mathrm{d}x \tag{3.3}$$

and, assuming that $u_0 \in H_0^2(\mathbb{R}_+)$,

$$M(t) - 2a\int_0^t \int vh\Big(\int_0^x (v_x)^2\mathrm{d}y\Big)\mathrm{d}x\mathrm{d}\tau = M(0), \tag{3.4}$$

where

$$M(t) = \int v^2\mathrm{d}x - 2\,\mathrm{Im}\int u\overline{u}_x\mathrm{d}x. \tag{3.5}$$

Proof The first identity is trivially obtained by multiplying the first equation in (1.1) by $\overline{u}$, integrating and taking the imaginary part to obtain $\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\mathrm{d}x = 0$.

Now, we derive by (1.1),

$$\mathrm{i}u_t\overline{u}_t + u_{xx}\overline{u}_t = \alpha|u|^2 u\overline{u}_t + bv\overline{u}_t$$

and so, taking the real part and integrating, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|u_x|^2\mathrm{d}x+\frac{\alpha}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int|u|^4\mathrm{d}x+\frac{b}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int v|u|^2\mathrm{d}x-\frac{b}{2}\int\frac{\partial v}{\partial t}|u|^2\mathrm{d}x=0 \tag{3.6}$$

and

$$-\frac{b}{2}\int\frac{\partial v}{\partial t}|u|^2\mathrm{d}x=\frac{b}{4}\int|u|^2(v^2)_x\mathrm{d}x-\frac{ab}{2}\int|u|^2h\Big(\int_0^x(v_x)^2\mathrm{d}y\Big)\mathrm{d}x.$$

Moreover,

$$\begin{aligned}&\frac{b}{4}\int|u|^2(v^2)_x\mathrm{d}x\\&=-\frac{b}{4}\int(|u|^2)_xv^2\mathrm{d}x\\&=\frac{1}{4}\int v^2\Big[v_t+\frac{1}{2}(v^2)_x-ah\Big(\int_0^x(v_x)^2\mathrm{d}y\Big)\Big]\mathrm{d}x\\&=\frac{1}{12}\frac{\mathrm{d}}{\mathrm{d}t}\int v^3\mathrm{d}x-\frac{a}{4}\int v^2h\Big(\int_0^x(v_x)^2\mathrm{d}y\Big)\mathrm{d}x.\end{aligned}$$

Hence, we derive by (3.6),

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)-\frac{a}{4}\int v^2h\Big(\int_0^x(v_x)^2\mathrm{d}y\Big)\mathrm{d}x-\frac{ab}{2}\int|u|^2h\Big(\int_0^x(v_x)^2\mathrm{d}y\Big)\mathrm{d}x=0$$

and so we obtain (3.2) by integrating in t. Similarly, we derive

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int v^2\mathrm{d}x\\&=\int v\Big(-\frac{1}{2}(v^2)_x+ah\int_0^x(v_x)^2\mathrm{d}y-b(|u|^2)_x\Big)\mathrm{d}x\\&=a\int vh\Big(\int_0^x(v_x)^2\mathrm{d}y\Big)\mathrm{d}x+b\int v_x|u|^2\mathrm{d}x\end{aligned} \tag{3.7}$$

and, assuming that $u_0\in H_0^2(\mathbb{R}_+)$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\int u\overline{u}_x\mathrm{d}x\\&=2\operatorname{Im}\int u_t\overline{u}_x\mathrm{d}x=-2\operatorname{Re}\int \mathrm{i}u_t\overline{u}_x\mathrm{d}x\\&=-2\operatorname{Re}\int(-u_{xx}+\alpha|u|^2u+bvu)\overline{u}_x\mathrm{d}x\\&=-b\int v(|u|^2)_x\mathrm{d}x=b\int v_x|u|^2\mathrm{d}x.\end{aligned}$$

Hence, by (3.7) we derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int v^2\mathrm{d}x-\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\int u\overline{u}_x\mathrm{d}x=a\int vh\Big(\int_0^x(v_x)^2\mathrm{d}y\Big)\mathrm{d}x$$

and so we obtain (3.4) by integrating in t.

To prove the blow-up result Theorem 1.2, we begin to establish the following lemma.

Lemma 3.1 *Under the assumptions of Theorem 1.2, the function* $t \to \int x^2|u|^2\mathrm{d}x$ *belongs to* $C^2([0,T])$*, the functions* $t \to \int xv^2\mathrm{d}x$ *and* $t \to \int x|u|^2\mathrm{d}x$ *belong to* $C^1([0,T])$ *and, for* $c_0 > 0$*,*

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big[\frac{1}{2}\int x^2|u|^2\mathrm{d}x \int_0^t \int xv^2\mathrm{d}x\mathrm{d}\tau + c_0\int_0^t\int x|u|^2\mathrm{d}x\mathrm{d}\tau\Big]\\ =&\,8E(t) - \alpha\int |u|^4\mathrm{d}x - 2\int v|u|^2\mathrm{d}x - c_0\int v^2\mathrm{d}x\\ &+ 2a\int xvh\Big(\int_0^x (v_x)^2\mathrm{d}y\Big)\mathrm{d}x + c_0 M(t).\end{aligned} \tag{3.8}$$

Proof Using the multiplier $\mu_\varepsilon(x) = e^{-\varepsilon x}$ and letting $\varepsilon \to 0^+$, it is easy to justify (see [4], [1] for similar arguments) the following formal computations:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int x^2|u|^2\mathrm{d}x = -\operatorname{Im}\int x^2\overline{u}\frac{\partial^2 u}{\partial x^2}\mathrm{d}x = 2\operatorname{Im}\int x\frac{\partial u}{\partial x}\overline{u}\mathrm{d}x$$

and (by (1.1))

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int x^2|u|^2\mathrm{d}x = 2\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\int x\frac{\partial u}{\partial x}\overline{u}\mathrm{d}x\\ =&\,2\operatorname{Im}\int x\frac{\partial^2 u}{\partial x\partial t}\overline{u}\mathrm{d}x + 2\operatorname{Im}\int x\frac{\partial u}{\partial x}\frac{\partial \overline{u}}{\partial t}\mathrm{d}x\\ =&-2\operatorname{Im}\int \frac{\partial u}{\partial t}\overline{u}\mathrm{d}x - 2\operatorname{Im}\int x\frac{\partial u}{\partial t}\frac{\partial \overline{u}}{\partial x}\mathrm{d}x + 2\operatorname{Im}\int x\frac{\partial u}{\partial x}\frac{\partial \overline{u}}{\partial t}\mathrm{d}x\\ =&-2\operatorname{Im}\int \frac{\partial u}{\partial t}\overline{u}\mathrm{d}x - 4\operatorname{Im}\int x\frac{\partial u}{\partial t}\frac{\partial \overline{u}}{\partial x}\mathrm{d}x\\ =&-2\operatorname{Re}\int \frac{\partial^2 u}{\partial x^2}\overline{u}\mathrm{d}x + 2\alpha\int |u|^4\mathrm{d}x + 2b\int v|u|^2\mathrm{d}x\\ &-4\operatorname{Re}\int x\frac{\partial \overline{u}}{\partial x}\Big(\frac{\partial^2 u}{\partial x^2} - \alpha|u|^2u - bvu\Big)\mathrm{d}x\\ =&\,2\int \Big|\frac{\partial u}{\partial x}\Big|^2\mathrm{d}x + 2\alpha\int |u|^4dx + 2b\int v|u|^2\mathrm{d}x\\ &-2\int x\frac{\partial}{\partial x}|u_x|^2\mathrm{d}x + \alpha\int x\frac{\partial}{\partial x}|u|^4\mathrm{d}x + 2b\int xv\frac{\partial}{\partial x}|u|^2\mathrm{d}x\\ =&\,4\int \Big|\frac{\partial u}{\partial x}\Big|^2\mathrm{d}x + \alpha\int |u|^4\mathrm{d}x + 2b\int v|u|^2\mathrm{d}x\\ &-2\int xv\Big[v_t + \frac{1}{2}(v^2)_x - ah\Big(\int_0^x (v_x)^2dy\Big)\Big]\mathrm{d}x\\ =&\,4\int \Big|\frac{\partial u}{\partial x}\Big|^2\mathrm{d}x + \alpha\int |u|^4\mathrm{d}x + 2b\int v|u|^2\mathrm{d}x\\ &-\frac{\mathrm{d}}{\mathrm{d}t}\int xv^2\mathrm{d}x - \int xv(v^2)_x\mathrm{d}x + 2a\int xvh\Big(\int_0^x (v_x)^2\mathrm{d}y\Big)\mathrm{d}x\\ =&\,4\int \Big|\frac{\partial u}{\partial x}\Big|^2\mathrm{d}x + \alpha\int |u|^4\mathrm{d}x + 2b\int v|u|^2\mathrm{d}x\\ &-\frac{\mathrm{d}}{\mathrm{d}t}\int xv^2\mathrm{d}x + \frac{2}{3}\int v^3\mathrm{d}x + 2a\int xvh\Big(\int_0^x (v_x)^2\mathrm{d}y\Big)\mathrm{d}x.\end{aligned} \tag{3.9}$$

We also have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int x|u|^2\mathrm{d}x = -2\,\mathrm{Im}\int u\frac{\partial \overline{u}}{\partial x}\mathrm{d}x = M(t) - \int v^2\mathrm{d}x \tag{3.10}$$

by (3.5).

Hence, by (3.4)–(3.5) and with $c_0 > 0$ and $E(t)$ defined by (3.3), we derive, for

$$\phi(t) = \frac{1}{2}\int x^2|u|^2\mathrm{d}x + \int_0^t\int xv^2\mathrm{d}x\mathrm{d}\tau + c_0\int_0^t\int x|u|^2\mathrm{d}x\mathrm{d}\tau \tag{3.11}$$

that $\phi \in C^2([0,T])$ and

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi(t) =& 8E(t) - \alpha\int |u|^4\mathrm{d}x - 2b\int v|u|^2\mathrm{d}x - c_0\int v^2\mathrm{d}x\\ &+ 2a\int xvh\Big(\int_0^x (v_x)^2\mathrm{d}y\Big)\mathrm{d}x + c_0M(t),\end{aligned}$$

and Lemma 3.1 is proved.

We can now prove Theorem 1.2.

Proof of Theorem 1.2 Assuming $v(x,t) \geq 0$ for $x \geq 0$ and $t \geq 0$ and the other hypothesis of the theorem, namely $a < 0$ and c_0, $E(0)$, $M(0)$ verifying (1.9), it is easy to verify that $\phi(t) \geq 0$, $\phi(0) > 0$ and

$$\frac{\mathrm{d}^2}{\mathrm{d}^2}\phi(t) \leq 8E(t) + c_0M(t) \leq 8E(0) + c_0M(0) < 0.$$

This leads to a contradiction when t tends to infinity and so the theorem is proved (classical virial argument).

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