Chin. Ann. Math. Ser. B
42(2), 2021, 173–198
DOI: 10.1007/s11401-021-0252-y

Chinese Annals of Mathematics, Series B

Metrics with Positive Scalar Curvature at Infinity and Localization Algebra*

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Abstract In this paper, the authors give a new proof of Block and Weinberger's Bochner vanishing theorem built on direct computations in the K-theory of the localization algebra.

Keywords Positive scalar curvature at infinity, K-theory of C^*-algebras, Higher index theory
2020 MR Subject Classification 19K56, 53C21

1 Introduction

Let M be an n dimensional complete Riemannian manifold with metric g. The scalar curvature, $k : M \to \mathbb{R}$, of g is the function satisfying

$$\mathrm{Vol}(B_M(p,\epsilon)) = \mathrm{Vol}(B_{\mathbb{R}^n}(0,\epsilon))\Big(1 - \frac{k(p)}{6(n+2)}\epsilon^2 + \cdots\Big)$$

for all sufficiently small $\epsilon > 0$, where p is any point of M, $B_M(p,\epsilon)$ is the open ball with center p and radius ϵ, and $B_{\mathbb{R}^n}(0,\epsilon)$ is the open ball in $\mathbb{R}_n$ centered at the origin with radius ϵ. We say that g is a uniformly positive scalar curvature metric if there exists $k_0 \leq 0$ such that $k(p) \geq k_0 > 0$ for all $p \in M$.

One of the most important applications of the Atiyah-Singer index theorem (see [1]) in Riemannian geometry is on the study of manifolds of positive scalar curvature (see [8]). Under the assumption that M is a compact spin manifold, in [9], Lichnerowicz established the formula

$$D^2 = \nabla^*\nabla + \frac{k}{4}, \tag{1.1}$$

where D is the Dirac operator. This implies that if there exists a positive scalar curvature metric on a compact spin manifold, then the Fredholm index of D on this manifold vanishes since D is invertible.

The higher index of D is an obstruction to the existence of the uniformly positive scalar curvature metrics on a closed manifold. Let Γ be the fundamental group of the compact spin manifold M. The higher index of the Dirac operator on M is defined to be a K theoretical class in $K_*(C_r^*(\Gamma))$, the K-theory of the reduced group C^*-algebra of Γ. With the aid of the higher

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*This work was supported by the National Natural Science Foundation of China (Nos. 11811530291, 11831006, 11771092).

index, it was shown that an enlargeable manifold cannot carry any positive scalar curvature metric (see [3]).

The index theoretic method can be generalized to study the positive scalar curvature metric outside a compact set of noncompact manifolds. In [4], Gromov and Lawson developed a relative index theorem to show the nonexistence of uniformly positive scalar curvature metric for a large class of noncompact manifolds. Roe studied the problem of the existence of positive scalar curvature metric outside a compact set on a complete manifold in terms of the K-theory of the Roe algebra and the coarse index (see [10–11]).

Let M be an n-dimensional complete noncompact manifold with fundamental group Γ. In [2], Block and Weinberger proved a Bochner type vanishing theorem. More precisely, if M carries a uniformly positive scalar curvature metric off a compact set M_1, and N is an $(n-1)$-dimensional submanifold of $M\backslash M_1$, then the higher index of the Dirac operator on N, viewed as a K-theory class in $K_{n-1}(C_r^*(\Gamma))$, vanishes. Note that there is no assumption on whether N bears any positive scalar curvature metric. Block and Weinberger gave this Bochner type vanishing theorem to illustrate an obstruction to the existence of the uniformly positive scalar curvature metrics on $M\backslash M_1$ by the higher index of the Dirac operator on N. This in turn helped them to prove that the arithmetic manifold with $\mathbb{Q}$-rank 1 or 2, carries no uniformly positive scalar curvature metric.

Block and Weinberger proved their Bochner type vanishing theorem by a KK-theory approach, which is a powerful tool but also a method that is famous for its great difficulty. The localization algebra approach arises for the reason that on the one hand usually it is as useful as KK-theory approach is, on the other hand it would greatly simplified the KK-theory method. In this paper, we take a localization algebra approach to provide a much more concise proof of their theorem.

Theorem 1.1 (Block and Weinberger) *Let (M,g) be an n-dimensional complete noncompact Riemannian spin manifold with fundamental group Γ. Assume that g is a metric of bounded geometry. Let $f : M \to \mathbb{R}$ be a proper smooth function. If the scalar curvature k of g satisfies $k \geq k_0 > 0$ off a compact set M_1, then the higher index of the Dirac operator on $N = f^{-1}(t_0) \neq \emptyset$, where t_0 is a sufficiently large positive regular value of f, is trivial in $K_{n-1}(C_r^*(\Gamma))$.*

Our proof is built on the definition of the equivariant coarse index introduced in [10] and the K-theory of the equivariant localization algebra introduced in [17]. The advantage of Roe's equivariant coarse index is that it can be represented by a geometrically defined operator. As shown in [17], the K-theory of the equivariant localization algebra is a homology theory, and satisfies a Mayer-Vietoris sequence. Our new proof is a direct computation of the equivariant coarse index of the Dirac operator via the Mayer-Vietoris sequence of the K-theory of the equivariant localization algebras. Our work is inspired by [11, 15–16]. Nevertheless, our method in this paper can also be adapted to simplify the proof of the main result of [16].

This paper is organized as follows. In Section 2, we recall the definition of the K-homology class and the equivariant coarse index of the Dirac operator in terms of the K-theory of equivariant localization algebra and equivariant Roe algebra. In Sections 3–4 we carry out some computations about the K-homology class and the equivariant coarse index of the Dirac operator respectively. In Section 5, we state and prove our main result.

2 Preliminaries

In this section, we first recall the definition of the equivariant Roe algebra and the equivariant localization algebra. The Mayer-Vietoris sequence of the K-theory of the localization algebra is also briefly reviewed. We then recall the construction of the equivariant coarse index and the K-homology class of Dirac operator.

2.1 Equivariant Roe algebra and equivariant localization algebra

In this subsection, we recall the definitions of the equivariant Roe algebra and the equivariant localization algebra, and review the Mayer-Vietoris sequence of the K-theory of the equivariant localization algebra. We refer the readers to [10, 14, 17] for more details.

Suppose that X is a proper metric space, i.e., every closed ball in X is compact. An X-module is a separable Hilbert space equipped with a $*$-representation of $C_0(X)$, the algebra of all continuous functions on X which vanish at infinity. An X-module is called nondegenerate if the $*$-representation of $C_0(X)$ is nondegenerate. An X-module is said to be standard if no nonzero function in $C_0(X)$ acts as a compact operator.

Definition 2.1 (see [10, 16]) *Let H_X be an X-module and T be a bounded linear operator acting on H_X.*

(1) *The support of T is defined to be*

$$\{(x, y) \in X \times X : hTg \neq 0,\ \forall\, h, g \in C_0(X),\ h(x) \neq 0,\ g(y) \neq 0\}$$

and is denoted by $\mathrm{Supp}(T)$*.*

(2) *Propagation (T), called the propagation of T, is defined to be*

$$\sup\{d(x, y) \mid (x, y) \in \mathrm{Supp}(T)\}.$$

(3) *The operator T is said to be locally compact if hT and Th are compact for all $h \in C_0(X)$.*

(4) *The operator T is said to be pseudo-local if $hT - Th$ is compact for all $h \in C_0(X)$.*

Definition 2.2 (see [16–17]) *Let H_X be a standard nondegenerate X-module and $\mathcal{B}(H)$ be the set of all bounded linear operators on H_X.*

(1) *The Roe algebra of X, denoted by $C^*(X)$, is the C^*-algebra generated by all locally compact operators with finite propagation in $\mathcal{B}(H)$.*

(2) *The localization algebra $C_L^*(X)$ is the C^*-algebra generated by all bounded and uniformly norm-continuous functions $h : [0, \infty) \to C^*(X)$ such that*

$$\mathrm{propagation}(h(t)) \to 0, \quad \textit{as } t \to \infty.$$

(3) *The $*$-homomorphism*

$$\mathrm{ev} : C_L^*(X) \to C^*(X), \quad \mathrm{ev}(h) = h(0)$$

is called evaluation map. The group homomorphism induced by ev :

$$\mathrm{ev}_* : K_*(C_L^*(X)) \to K_*(C^*(X))$$

is called assembly map.

Assume that there is a countable discrete group G acting freely and properly on X by isometries.

A G-X-module is an X-module equipped with a unitary representation of G which is compatible with the $C_0(X)$-representation, that is,

$$\pi(g)\phi(h) = \phi(g.h)\pi(g), \quad \forall\, h \in C_0(X),\ g \in G,$$

where ϕ (resp. π) is the $C_0(X)$ (resp. G) -representation on H_X and $g.h(x) = h(g^{-1}x)$.

Definition 2.3 (see [16]) *Let X be a locally compact metric space with a proper and free isometric action of G. If H_X is a G-X-module, denote by $\mathbb{C}[X]^G$ the $*$-algebra of all G-invariant locally compact operators with finite propagation in $\mathcal{B}(H_X)$, and define $C^*(X)^G$ to be the completion of $\mathbb{C}[X]^G$ in $\mathcal{B}(H_X)$.*

Similarly, we can also define the G-equivariant localization algebra $C_L^*(X)^G$.

Remark 2.1 The Roe algebra $C^*(X) = C^*(X, H_X)$ does not depend on the choice of the standard nondegenerate X-module H_X up to isomorphism (see [5, 14]). The same holds for $C_L^*(X)$ and their G-invariant versions.

For simplicity, in the following, we refer to the equivariant Roe and the equivariant localization algebra simply as Roe and localization algebra, respectively.

Now let us turn to the Mayer-Vietoris sequence of the K-theory of the localization algebra introduced by Yu in [17].

If $X = Y \cup Z$ where Y and Z are G-invariant closed subsets of X, we have

$$\begin{tikzcd}
K_0(C_L^*(Y\cap Z)^G) \arrow[r] & K_0(C_L^*(Y)^G) \oplus K_0(C_L^*(Z)^G) \arrow[r] & K_0(C_L^*(X)^G) \arrow[d, "\partial_0"] \\
K_1(C_L^*(X)^G) \arrow[u, "\partial_1"] & K_1(C_L^*(Y)^G) \oplus K_1(C_L^*(Z)^G) \arrow[l] & K_1(C_L^*(Y\cap Z)^G), \arrow[l]
\end{tikzcd}$$

where ∂_0 and ∂_1 are defined as follows (see [14]).

(1) Let P, Q be idempotents in $(C_L^*(X)^G)^+ \otimes M_n(\mathbb{C})$ representing the K-theory class $[P] - [Q] \in K_0(C_L^*(X)^G)$, where $(C_L^*(X)^G)^+$ is the unitization of $C_L^*(X)^G$ and $M_n(\mathbb{C})$ is the complex matrix of order n. Then

$$\partial_0([P] - [Q]) = [\exp(-2\pi \mathrm{i}\chi_Y P\chi_Y) \cdot \exp(2\pi \mathrm{i}\chi_Y Q\chi_Y)],$$

where χ_Y is the characteristic function of Y.

(2) Let U be an invertible element in $(C_L^*(X)^G)^+ \otimes M_n(\mathbb{C})$ representing the K-theory class $[U] \in K_1(C_L^*(X)^G)$, where $(C_L^*(X)^G)^+$ is the unitization of $C_L^*(X)^G$ and $M_n(\mathbb{C})$ is the complex matrix of order n. Take

$$W = \begin{pmatrix} 1 & \chi_Y U\chi_Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\chi_Y U^{-1}\chi_Y & 1 \end{pmatrix} \begin{pmatrix} 1 & \chi_Y U\chi_Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then

$$\partial_1[U] = \left[W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^{-1}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right],$$

where χ_Y is the characteristic function of Y on X.

Here ∂_0 and ∂_1 are usually called the connecting maps.

Recall that two $*$-homomorphisms $\varphi_0,\ \varphi_1 : A \to B$ between C^*-algebras are called homotopic if there is $(\varphi_t)_{0\le t\le 1}$ such that φ_t is $*$-homomorphism from A to B for every $0 \le t \le 1$ and

$$[0,1] \to A, \quad t \mapsto \varphi_t(a)$$

is continuous for every $a \in A$. In this case, $\varphi_{0*} = \varphi_{1*} : K_*(A) \to K_*(B)$ (see [13]).

Proposition 2.1 *Assume that X is a proper metric space and G is a countable discrete group acting on X by isometries.*

(1) *For every $s \ge 0$,*

$$\Phi_s : C_L^*(X)^G \to C_L^*(X)^G, \quad h \mapsto \Phi_s(h), \quad \Phi_s(h)(t) = h(t+s)$$

is a $*$-homomorphism.

(2) *For every $s \ge 0$, Φ_s is homotopic to identity.*

(3) *For every $s \ge 0$,*

$$\Phi_{s*} = \mathrm{id} : K_*(C_L^*(X)^G) \to K_*(C_L^*(X)^G).$$

Here we explain how this proposition can bring a convenience to our computation in the K-theory of the localization algebra. If $x = [h] \in K_*(C_L^*(X)^G)$ where $h \in C_L^*(X)^G$, $\sup\limits_{t\ge 0}\{\text{propagation}(h(t))\} < \infty$ and

$$\text{propagation}(h(t)) \to 0, \quad t \to \infty,$$

then for every $\epsilon > 0$, there exists $t_0 > 0$ such that

$$\sup_{t\ge t_0}\{\text{propagation}(h(t))\} < \epsilon,$$

which means

$$\sup_{t\ge 0}\{\text{propagation}(\Phi_{t_0}(h)(t))\} < \epsilon.$$

Based on this proposition, $x = [h] = [\Phi_{t_0}(h)]$. So we can get a representative element h' for x such that $\sup\limits_{t\ge 0}\{\text{propagation}(h'(t))\}$ is small enough.

2.2 *K*-homology class and equivariant coarse index of Dirac operator

In this subsection, we recall the construction of the K-homology class and the equivariant coarse index of the Dirac operator on a spin manifold.

Let M be a complete Riemannian spin manifold, and $\pi_H : M_H \to M$ be an H Galois covering space of M, where H is a discrete group. Let S be a spinor bundle on M and D_M be the Dirac operator on M. One can lift S and D_M to the bundle S_H and operator D_{M_H} on M_H respectively.

A smooth non-decreasing function $\chi : (-\infty, +\infty) \to [-1, 1]$ is called a normalizing function if it satisfies that

(1) the function χ is odd with $\chi(x) > 0$ for $x > 0$;

(2) $\lim \chi(x) = 1$, as $x \to +\infty$;

(3) the distributional Fourier transform $\widehat{\chi}$ has compact support.

For the existence of such functions, see [5]. Let χ be a normalizing function. Then $\chi(D_{M_H})$ is a self-adjoint and pseudo-local bounded operator with finite propagation (see [14, 8.2] and [5, 10.6]).

For every $n \in \mathbb{Z}_+$, we can choose an H-invariant locally finite open cover $\{U_n^j\}_{j\in\mathbb{Z}_+}$ of M satisfying $\operatorname{diam}(U_n^j) < \frac{1}{n}$ for all $j \in \mathbb{Z}_+$, and an H-invariant smooth partition of unity $\{\phi_n^j\}_{j\in\mathbb{Z}_+}$ subordinate to $\{U_n^j\}_{j\in\mathbb{Z}_+}$.

Define

$$F = \int_{-\infty}^{+\infty} \widehat{\chi}(s)\mathrm{e}^{\mathrm{i}sTD_{M_H}}\,\mathrm{d}s,$$

where T is any fixed positive number. Set $F_0 = F$ and

$$\begin{aligned} F_n &= \sum_{j=1}^{+\infty} \sqrt{\phi_n^j}F\sqrt{\phi_n^j}, \quad \forall\, n \in \mathbb{Z}_+,\\ F_t &= (1-t+n)F_n + (t-n)F_{n+1}, \quad t \in [n, n+1]. \end{aligned}$$

The K-homology class of D_M is originally defined in the K-homology group $K_*(M) \cong K_*^H(M_H)$ (see [5, 7]). However, in [17], Yu proved that $K_*(M) \cong K_*^H(M_H) \cong K_*(C_L^*(M_H)^H)$ (see [14] also). We adopt Yu's approach to consider the K-homology class of D_M which lies in $K_*(C_L^*(M_H)^H)$.

Now let us recall the definition of the K-homology class $[D_M]$ defined by Yu [17, 19]. We refer the reader to [16] for more details.

When M is odd dimensional, the K-homology class $[D_M]$ of the Dirac operator D_M is defined by

$$[D_M] = [\mathrm{e}^{2\pi\mathrm{i}\frac{F_t+1}{2}}] \in K_1(C_L^*(M_H)^H).$$

When M is even dimensional, since D is an odd operator and χ is an odd function, there exists the following decomposition

$$F_t = \begin{pmatrix} 0 & V_t \\ U_t & 0 \end{pmatrix}$$

with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading on the spinor bundle S. Define

$$W_t = \begin{pmatrix} 1 & U_t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -V_t & 1 \end{pmatrix}\begin{pmatrix} 1 & U_t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$P_t = W_t\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}W_t^{-1} = \begin{pmatrix} U_tV_t + U_tV_t(1-U_tV_t) & (2-U_tV_t)U_t(1-V_tU_t) \\ V_t(1-U_tV_t) & (1-V_tU_t)^2 \end{pmatrix}.$$

Then the K-homology class $[D_M]$ of the Dirac operator D_M is defined by

$$[D_M] = [P_t] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(C_L^*(M_H)^H).$$

It can be summarized as follows,

$$[D_M] = \begin{cases} [\mathrm{e}^{2\pi\mathrm{i}\frac{F_t+1}{2}}] \in K_1(C_L^*(M_H)^H) & \text{if } \dim M \text{ is odd},\\ [P_t] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(C_L^*(M_H)^H) & \text{if } \dim M \text{ is even}. \end{cases} \tag{2.1}$$

Since the homotopy equivalent elements define the same K-theory class, the definition of the K-homology class of the Dirac operator D_M is independent of the choice of the function χ and the number T.

Recall that the evaluation map (see [17])

$$\mathrm{ev}: C_L^*(M_H)^H \to C^*(M_H)^H, \quad \mathrm{ev}(h) = h(0)$$

induces a homomorphism of groups

$$\mathrm{ev}_*: K_*(C_L^*(M_H)^H) \to K_*(C^*(M_H)^H).$$

The equivariant coarse index of the Dirac operator D_M is defined to be

$$\mathrm{Ind}\, D_M = \mathrm{ev}_*[D_M] = \begin{cases} [\mathrm{e}^{2\pi\mathrm{i}\frac{F+1}{2}}] \in K_1(C^*(M_H)^H) & \text{if } \dim M \text{ is odd,} \\ [P_0] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(C^*(M_H)^H) & \text{if } \dim M \text{ is even.} \end{cases}$$

For this evaluation map definition of the equivariant coarse index, see [14].

For simplicity, in the following we call the equivariant coarse index as coarse index.

3 The *K*-Homology Class of Dirac Operator under the Connecting Map

In this section, we compute the image of the K-homology class of D_M under the connecting map in Mayer-Vietoris sequence of the K-theory of the localization algebras.

Let M be an n-dimensional Riemannian spin manifold with fundamental group Γ, and $\pi: \widetilde{M} \to M$ be the universal covering map. Let

$$f: M \to \mathbb{R}$$

be any smooth proper function.

In the following, we set

(1) $M_{f\le r} = f^{-1}((-\infty, r])$;

(2) $M_{f\ge r} = f^{-1}([r, \infty))$;

(3) $M_{r_1\le f\le r_2} = f^{-1}([r_1, r_2])$ for $r_1 \le r_2$;

(4) $\widetilde{A} = \pi^{-1}(A)$ for every subset A in M.

By Sard's theorem, there are a lot of $t \in \mathbb{R}$ such that $f(t)$ are regular values. Choose a regular value t_0 satisfying $f^{-1}(t_0) \ne \emptyset$. Then $N = f^{-1}(t_0)$ is a compact $(n-1)$-dimensional submanifold of M.

According to the Mayer-Vietoris sequence of the K-theory of the localization algebras, for $r_1 < t_0 < r_2$ and $s_1 < 0 < s_2$, we have the following Figure 1, and this leads to the following six-term exact sequences

$$\begin{tikzcd}
K_0(C_L^*(\widetilde{M}_{r_1\le f\le r_2})^\Gamma) \arrow[r] & K_0(C_L^*(\widetilde{M}_{f\ge r_1)})^\Gamma) \oplus K_0(C_L^*(\widetilde{M}_{f\le r_2})^\Gamma) \arrow[r] & K_0(C_L^*(\widetilde{M})^\Gamma) \arrow[d, "\partial_0"] \\
K_1(C_L^*(\widetilde{M})^\Gamma) \arrow[u, "\partial_1"] & K_1(C_L^*(\widetilde{M}_{f\ge r_1})^\Gamma) \oplus K_1(C_L^*(\widetilde{M}_{f\le r_2})^\Gamma) \arrow[l] & K_1(C_L^*(\widetilde{M}_{r_1\le f\le r_2})^\Gamma) \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
K_0(C_L^*(\widetilde{N}\times[s_1,s_2])^\Gamma) \arrow[r] & \begin{matrix} K_0(C_L^*(\widetilde{N}\times(-\infty,s_2])^\Gamma) \\ \oplus \\ K_0(C_L^*(\widetilde{N}\times[s_1,+\infty))^\Gamma) \end{matrix} \arrow[r] & K_0(C_L^*(\widetilde{N}\times\mathbb{R})^\Gamma) \arrow[d, "\partial_0'"] \\
K_1(C_L^*(\widetilde{N}\times\mathbb{R})^\Gamma) \arrow[u, "\partial_1'"] & \begin{matrix} K_1(C_L^*(\widetilde{N}\times(-\infty,s_2])^\Gamma) \\ \oplus \\ K_1(C_L^*(\widetilde{N}\times[s_1,+\infty))^\Gamma) \end{matrix} \arrow[l] & K_1(C_L^*(\widetilde{N}\times[s_1,s_2])^\Gamma). \arrow[l]
\end{tikzcd}$$

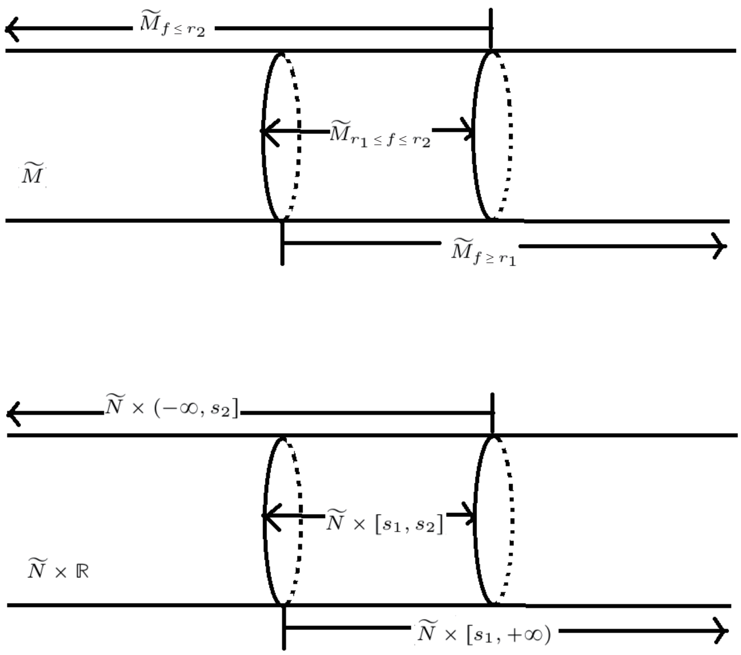

Figure 1

Recall that we assumed f to be a proper and smooth function. Consequently, there exist real numbers $a<t_0$, $b>t_0$, such that for all t, $a\leq t\leq b$, the tangent map is a rank one linear map, which implies that every $t\in[a,b]$ is regular value. Thus every $\widetilde{M}_{r_1\leq f\leq r_2}$, where $a<r_1<t_0<r_2<b$, is a tubular neighborhood of $\widetilde{N}$ and $\widetilde{M}_{a\leq f\leq b}$ is diffeomorphic to $\widetilde{N}\times[-r,r]$ for some $r>0$ (see [6, Theorem 3.3 in Chapter 1 and Theorem 5.2 in Chapter 4] and [12, Theorem 9 in Chapter 2]). Then for $a<r_1<t_0<r_2<b,\ s_1<0<s_2$, we have

$$\begin{tikzcd}
K_n(C_L^*(\widetilde{M})^\Gamma) \arrow[r, "\partial_n"] & K_{n-1}(C_L^*(\widetilde{M}_{r_1\leq f\leq r_2})^\Gamma) \arrow[r, "\alpha", "\cong"'] \arrow[d, "l"', "\cong"] & K_{n-1}(C_L^*(\widetilde{N})^\Gamma) \\
K_n(C_L^*(\widetilde{N}\times\mathbb{R})^\Gamma) \arrow[r, "\partial_n'"] & K_{n-1}(C_L^*(\widetilde{N}\times[s_1,s_2])^\Gamma), \arrow[ur, "\cong", "\beta"'] &
\end{tikzcd}$$

where l is the isomorphism induced by the diffeomorphism between $\widetilde{M}_{r_1\leq f\leq r_2}$ and $\widetilde{N}\times[s_1,s_2]$, β^{-1} is the isomorphism induced by the embedding of $\widetilde{N}$ into $\widetilde{N}\times\{0\}$ which is a strongly

Lipschitz homotopy equivalence (for the definition of strongly Lipschitz homotopy equivalence, see [17]), and $\alpha = \beta \circ l$.

Theorem 3.1 *Let ∂_n, ∂'_n be the connecting maps defined above. We have*
(1) $\alpha \circ \partial_n[D_M] = \beta \circ \partial'_n[D_{N\times\mathbb{R}}]$,
(2) $\alpha \circ \partial_n[D_M] = [D_N]$,
where D_M, $D_{N\times\mathbb{R}}$ and D_N are the Dirac operators on M, $N \times \mathbb{R}$ and N respectively.

Proof (1) Choose r_1, r_2, r'_2, s_1, s_2, s'_2 such that $a < r_1 < t_0 < r'_2 < r_2 < b$,$-r < s_1 < 0 < s'_2 < s_2 < b$ and

$$\begin{aligned} N_c(\widetilde{M}_{f\geq r'_2}) &\subset \widetilde{M}_{f\geq r_1}, \\ N_c(\widetilde{M}_{r_1\leq f\leq r_2}) &\subset \widetilde{M}_{a\leq f\leq b}, \\ N_c(\widetilde{N}\times[s_1,s_2]) &\subset \widetilde{N}\times[-r,r], \\ N_c(\widetilde{N}\times[s'_2,+\infty)) &\subset \widetilde{N}\times[s_1,+\infty), \\ \{x\in\widetilde{M}_{f\geq r_1} : d(x,\widetilde{M}_{r_1\leq f\leq r'_2})\leq c\} &\subset \widetilde{M}_{r_1\leq f\leq r_2}, \\ \{x\in\widetilde{N}\times[s_1,+\infty) : d(x,\widetilde{N}\times[s_1,s'_2])\leq c\} &\subset \widetilde{N}\times[s_1,s_2], \end{aligned}$$

where c is a positive number. Consider the above commutative diagram about r_1, r_2, s_1, s_2.

For every $n \in \mathbb{Z}_+$, choose a Γ-invariant locally finite open cover $\{U_n^j\}_{j\in\mathbb{Z}_+}$ of $\widetilde{M}$ satisfying $\mathrm{diam}(U_n^j) < \frac{1}{n}$ for all $j \in \mathbb{Z}_+$ and a Γ-invariant smooth partition of unity $\{\phi_n^j\}_{j\in\mathbb{Z}_+}$ subordinate to $\{U_n^j\}_{j\in\mathbb{Z}_+}$.

Similarly, for every $n \in \mathbb{Z}_+$, we can choose a Γ-invariant locally finite open cover $\{V_n^j\}_{j\in\mathbb{Z}_+}$ of $\widetilde{N}\times\mathbb{R}$ and a Γ-invariant smooth partition of unity $\{\varphi_n^j\}_{j\in\mathbb{Z}_+}$ subordinate to $\{V_n^j\}_{j\in\mathbb{Z}_+}$ satisfying $\mathrm{diam}(V_n^j) < \frac{1}{n}$.

Let $\epsilon_0 = \frac{1}{800000\pi \mathrm{e}^{4\pi}}$. Let $K \in \mathbb{N}$ be an integer such that

$$\sup_{x\in[-100,100]} \Big|\sum_{k=0}^{K} \frac{(2\pi\mathrm{i}x)^k}{k!} - \mathrm{e}^{2\pi\mathrm{i}x}\Big| < \epsilon_0,$$
$$\Big|\sum_{k=1}^{K} \frac{(2\pi\mathrm{i})^k}{k!}\Big| < \epsilon_0.$$

We can choose χ properly such that its distributional Fourier transform $\widehat{\chi}$ is supported on $\big(-\frac{c'}{2}, \frac{c'}{2}\big)$ where $c' = \frac{c}{100K}$.

Define

$$F = \int_{-\infty}^{+\infty} \widehat{\chi}(s)\mathrm{e}^{\mathrm{i}sD_{\widetilde{M}}}\mathrm{d}s,$$
$$F' = \int_{-\infty}^{+\infty} \widehat{\chi}(s)\mathrm{e}^{\mathrm{i}sD_{\widetilde{N}\times\mathbb{R}}}\mathrm{d}s.$$

Here F_t and F'_t are similarly defined as in Subsection 2.2. We now prove the theorem according to the parity of the dimension n.

Odd case When n is odd,

$$[D_M] = [\mathrm{e}^{2\pi\mathrm{i}\frac{F_t+1}{2}}],$$

$$[D_{N\times\mathbb{R}}]=[\mathrm{e}^{2\pi\mathrm{i}\frac{F_t'+1}{2}}].$$

We need the following lemma.

Lemma 3.1 *$[D_M]$ and $[D_{N\times\mathbb{R}}]$ have representative elements $(u_t)_{t\geq 0}$ and $(u_t')_{t\geq 0}$ satisfying*

$$\begin{aligned}&\sup_{t\geq 0}\{\text{propagation}(u_t)\}<\infty,\\ &\lim_{t\to\infty}\text{propagation}(u_t)=0,\\ &\sup_{t\geq 0}\{\text{propagation}(u_t')\}<\infty,\\ &\lim_{t\to\infty}\text{propagation}(u_t')=0.\end{aligned}$$

Proof Let $\epsilon_0=\frac{1}{800000\pi\mathrm{e}^{4\pi}}$ and K be the integer defined above such that

$$\sup_{x\in[-100,100]}\Big|\sum_{k=0}^{K}\frac{(2\pi\mathrm{i}x)^k}{k!}-\mathrm{e}^{2\pi\mathrm{i}x}\Big|<\epsilon_0,$$
$$\Big|\sum_{k=1}^{K}\frac{(2\pi\mathrm{i})^k}{k!}\Big|<\epsilon_0.$$

For $t\in[0,\infty)$, let $P_t=\frac{F_t+1}{2}$ and

$$\begin{aligned}u_t&=1+\sum_{k=1}^{K}\Big(\frac{(-2\pi\mathrm{i})^k}{k!}(P_t^k-P_t)\Big)\\ &=1+\Big(\sum_{k=2}^{K}\sum_{j=0}^{k-2}\frac{(2\pi\mathrm{i})^k}{k!}P_t^j(P_t^2-P_t)\Big).\end{aligned}$$

Note that for all $t\geq 0$, we have $\|F_t\|\leq 10\|F_0\|\leq 10$ (see [18, Lemma 2.6] and [16, Lemma 3.5]). Direct computation shows that $u_t\in C_L^*(\widetilde{M})^\Gamma$ and $||u_t-\mathrm{e}^{2\pi\mathrm{i}P_t}||<\frac{1}{2}$. Therefore, $[D_M]=[u_t]$. It is also clear to see

$$\begin{aligned}&\sup_{t\geq 0}\{\text{propagation}(u_t)\}<\infty,\\ &\lim_{t\to\infty}\text{propagation}(u_t)=0.\end{aligned}$$

Thus u_t is the desired representative element for $[D_M]$.

The argument for $[D_{N\times\mathbb{R}}]$ is similar. The proof is completed.

According to Lemma 3.1 and Proposition 2.1, we can find representative elements $(u_t)_{t\geq 0}$ and $(u_t')_{t\geq 0}$ for $[D_M]$ and $[D_{N\times\mathbb{R}}]$ respectively such that

$$\begin{aligned}&\sup_{t\geq 0}\{\text{propagation}(u_t)\}<Kc',\\ &\sup_{t\geq 0}\{\text{propagation}(u_t')\}<Kc'.\end{aligned}$$

Define

$$U_t=\chi_{\widetilde{M}_{f\geq r_1}}u_t\chi_{\widetilde{M}_{f\geq r_1}},$$

$$\begin{aligned}
V_t &= \chi_{\widetilde{M}_{f\geq r_1}} u_t^* \chi_{\widetilde{M}_{f\geq r_1}},\\
P_t &= \begin{pmatrix} U_tV_t + U_tV_t(1-U_tV_t) & (U_tV_tU_t - 2U_t)(V_tU_t - 1)\\ V_t(1-U_tV_t) & (V_tU_t-1)^2 \end{pmatrix},\\
U_t' &= \chi_{\widetilde{N}\times[s_1,\infty)} u_t' \chi_{\widetilde{N}\times[s_1,\infty)},\\
V_t' &= \chi_{\widetilde{N}\times[s_1,\infty)} u_t'^* \chi_{\widetilde{N}\times[s_1,\infty)},\\
P_t' &= \begin{pmatrix} U_t'V_t' + U_t'V_t'(1-U_t'V_t') & (U_t'V_t'U_t' - 2U_t')(V_t'U_t' - 1)\\ V_t'(1-U_t'V_t') & (V_t'U_t'-1)^2 \end{pmatrix}.
\end{aligned}$$

By the definition of the connecting map, we have

$$\begin{aligned}
\partial_1[D_M] &= [P_t] - \left[\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\right],\\
\partial_1'[D_{N\times\mathbb{R}}] &= [P_t'] - \left[\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\right].
\end{aligned}$$

For all $h \in L^2(\widetilde{M}_{f\geq r_2'})$, by the estimate on the finite propagation of u_t, one can obtain

$$\begin{aligned}
&\|V_tU_th - h\|\\
&= \|\chi_{\widetilde{M}_{f\geq r_1}} u_t^* \chi_{\widetilde{M}_{f\geq r_1}} \chi_{\widetilde{M}_{f\geq r_1}} u_t \chi_{\widetilde{M}_{f\geq r_1}} h - h\|\\
&= \|\chi_{\widetilde{M}_{f\geq r_1}} u_t^* \chi_{\widetilde{M}_{f\geq r_1}} u_t h - h\|\\
&= \|\chi_{\widetilde{M}_{f\geq r_1}} u_t^* u_t h - h\|\\
&= \|u_t^* u_t h - h\|\\
&\leq 100\epsilon_0\|h\|.
\end{aligned}$$

For the same reason, we have

$$\|U_tV_th - h\| \leq 100\epsilon_0\|h\|, \quad \forall\, h \in L^2(\widetilde{M}_{f\geq r_2'})$$

for all $h \in L^2(\widetilde{M}_{f\geq r_2'})$. Similarly, we have

$$\|V_t'U_t'h' - h'\| \leq 100\epsilon_0\|h'\|, \quad \|U_t'V_t'h' - h'\| \leq 100\epsilon_0\|h'\|$$

for all $h' \in L^2(\widetilde{N} \times [s_2', \infty))$. These estimates imply that the norm of both

$$P_t - \chi_{\widetilde{M}_{r_1\leq f\leq r_2}} P_t \chi_{\widetilde{M}_{r_1\leq f\leq r_2}} - \chi_{\widetilde{M}_{f\geq r_2}} \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} \chi_{\widetilde{M}_{f\geq r_2}}$$

and

$$P_t' - \chi_{\widetilde{N}\times[s_1,s_2]} P_t' \chi_{\widetilde{N}\times[s_1,s_2]} - \chi_{\widetilde{N}\times[s_2,\infty)} \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} \chi_{\widetilde{N}\times[s_2,\infty)}$$

are less than $500\epsilon_0$. Thus we have

$$\begin{aligned}
\partial_1[D_M] &= [\chi_{\widetilde{M}_{r_1\leq f\leq r_2}} P_t \chi_{\widetilde{M}_{r_1\leq f\leq r_2}}] - \left[\begin{pmatrix} \chi_{\widetilde{M}_{r_1\leq f\leq r_2}} & 0\\ 0 & 0\end{pmatrix}\right],\\
\partial_1'[D_{N\times\mathbb{R}}] &= [\chi_{\widetilde{N}\times[s_1,s_2]} P_t' \chi_{\widetilde{N}\times[s_1,s_2]}] - \left[\begin{pmatrix} \chi_{\widetilde{N}\times[s_1,s_2]} & 0\\ 0 & 0\end{pmatrix}\right].
\end{aligned}$$

Since $\widetilde{M}_{a\le f\le b}$ is diffeomorphic to $\widetilde{N}\times[-r,r]$ for some r, the Dirac operator $D_{\widetilde{M}}$ on $\widetilde{M}_{a\le f\le b}$ is induced from $D_{\widetilde{N}\times\mathbb{R}}$ on $\widetilde{N}\times[-r,r]$ by the diffeomorphism. According to a standard "energy estimates" argument in partial differential equations (see [5, Propositions 10.3.1, 10.3.5]), the operator $\chi(D_{\widetilde{M}})$ restricted on $\widetilde{M}_{a+2c'\le f\le b-2c'}$ coincides with the operator $\chi(D_{\widetilde{N}\times\mathbb{R}})$ restricted on $\widetilde{N}\times[-r+2c',r-2c']$. As a result, we have that

$$\chi_{\widetilde{M}_{r_1\le f\le r_2}}P_t\chi_{\widetilde{M}_{r_1\le f\le r_2}}$$

coincides with

$$\chi_{\widetilde{N}\times[s_1,s_2]}P'_t\chi_{\widetilde{N}\times[s_1,s_2]}.$$

Hence

$$l\circ\partial_1[D_M]=\partial'_1[D_{N\times\mathbb{R}}].$$

Consequently,

$$\alpha\circ\partial_1[D_M]=\beta\circ l\circ\partial_1[D_M]=\beta\circ\partial'_1[D_{N\times\mathbb{R}}].$$

Even case The even case is parallel to the odd case above. We will leave out some details. Notice that

$$F_t=\begin{pmatrix}0&U_t\\V_t&0\end{pmatrix}.$$

We have

$$[D_M]=\left[W_t\begin{pmatrix}1&0\\0&0\end{pmatrix}W_t^{-1}\right]-\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]\in K_0(C_L^*(\widetilde{M})^\Gamma),$$

where $W_t=\begin{pmatrix}1&U_t\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-V_t&1\end{pmatrix}\begin{pmatrix}1&U_t\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$.

From the definition of the connecting map, we have

$$\begin{aligned}\partial_0[D_M]&=\partial_0\left([P_t]-\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]\right)\\&=\left[\exp\left(-2\pi\mathrm{i}\chi_{\widetilde{M}_{f\ge r_1}}P_t\,\chi_{\widetilde{M}_{f\ge r_1}}\right)\cdot\exp\left(2\pi\mathrm{i}\chi_{\widetilde{M}_{f\ge r_1}}\begin{pmatrix}1&0\\0&0\end{pmatrix}\chi_{\widetilde{M}_{f\ge r_1}}\right)\right]\\&=[\exp\left(-2\pi\mathrm{i}\chi_{\widetilde{M}_{f\ge r_1}}P_t\,\chi_{\widetilde{M}_{f\ge r_1}}\right)],\end{aligned}$$

where $P_t{=}W_t\begin{pmatrix}1&0\\0&0\end{pmatrix}W_t^{-1}$.

Similarly, we have

$$F'_t=\begin{pmatrix}0&U'_t\\V'_t&0\end{pmatrix},$$

and

$$[D_{N\times\mathbb{R}}]=\left[W'_t\begin{pmatrix}1&0\\0&0\end{pmatrix}W_t'^{-1}\right]-\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]\in K_0(C_L^*(\widetilde{N}\times\mathbb{R})^\Gamma),$$

where $W'_t=\begin{pmatrix}1&U'_t\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-V'_t&1\end{pmatrix}\begin{pmatrix}1&U'_t\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$.

By the definition of the connecting map, we have

$$\begin{aligned}\partial'_0[D_{N\times\mathbb{R}}]&=\partial'_0\left([P'_t]-\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]\right)\\&=[\exp\left(-2\pi\mathrm{i}\chi_{\widetilde{N}\times[s_1,+\infty)}P'_t\chi_{\widetilde{N}\times[s_1,+\infty)}\right)]\\&\quad-\left[\exp\left(-2\pi\mathrm{i}\chi_{\widetilde{N}\times[s_1,+\infty)}\begin{pmatrix}1&0\\0&0\end{pmatrix}\chi_{\widetilde{N}\times[s_1,+\infty)}\right)\right]\end{aligned}$$

$$= [\exp(-2\pi \mathrm{i} \chi_{\widetilde{N}\times[s_1,+\infty)} P'_t \chi_{\widetilde{N}\times[s_1,+\infty)})],$$

where $P'_t = W'_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W'^{-1}_t$.

According to Proposition 2.1, we can suppose that the propagation of P_t is less than $6c'$ for all $t \in [0,\infty)$. As in Lemma 3.1, approximate

$$\exp(-2\pi \mathrm{i} \chi_{\widetilde{M}_{f\geq r_1}} P_t \; \chi_{\widetilde{M}_{f\geq r_1}})$$

by

$$u_t := 1 + \sum_{k=1}^{K} \frac{(-2\pi \mathrm{i})^k}{k!} ((\chi_{\widetilde{M}_{f\geq r_1}} P_t \; \chi_{\widetilde{M}_{f\geq r_1}})^k - (\chi_{\widetilde{M}_{f\geq r_1}} P_t \; \chi_{\widetilde{M}_{f\geq r_1}})),$$

where K is as above. For every $h \in L^2(\widetilde{M}_{f\geq r'_2})$, there is

$$P_t \; \chi_{\widetilde{M}_{f\geq r_1}} h = P_t \; h \in L^2(\widetilde{M}_{f\geq r_1})$$

because of the relationship between r_1 and r'_2. Hence for $h \in L^2(\widetilde{M}_{f\geq r'_2})$,

$$\begin{aligned}(\chi_{\widetilde{M}_{f\geq r_1}} P_t \; \chi_{\widetilde{M}_{f\geq r_1}})^2 h &= (\chi_{\widetilde{M}_{f\geq r_1}} P_t \; \chi_{\widetilde{M}_{f\geq r_1}}) P_t h \\ &= (\chi_{\widetilde{M}_{f\geq r_1}} P_t^2) h \\ &= (\chi_{\widetilde{M}_{f\geq r_1}} P_t \; \chi_{\widetilde{M}_{f\geq r_1}}) h.\end{aligned}$$

Since

$$\begin{aligned}\exp(-2\pi \mathrm{i} \chi_{\widetilde{M}_{f\geq r_1}} P_t \; \chi_{\widetilde{M}_{f\geq r_1}}) h &= \sum_{k=0}^{+\infty} \frac{1}{k!} (-2\pi \mathrm{i} \chi_{\widetilde{M}_{f\geq r_1}} P_t \; \chi_{\widetilde{M}_{f\geq r_1}})^k h \\ &= h + (\exp(-2\pi \mathrm{i}) - 1)(\chi_{\widetilde{M}_{f\geq r_1}} P_t \; \chi_{\widetilde{M}_{f\geq r_1}}) h \\ &= h,\end{aligned}$$

we have

$$\|u_t(h) - h\| \leq 100\epsilon_0 \|h\|, \quad \forall\, h \in L^2(\widetilde{M}_{f\geq r'_2}).$$

This implies that

$$\partial_0[D_M] = [u_t] = [\chi_{\widetilde{M}_{r_1\leq f\leq r_2}} u_t \chi_{\widetilde{M}_{r_1\leq f\leq r_2}}].$$

Similarly, one can show that

$$\partial'_0[D_{N\times\mathbb{R}}] = [\chi_{\widetilde{N}\times[s_1,s_2]} u'_t \chi_{\widetilde{N}\times[s_1,s_2]}],$$

where

$$u'_t = 1 + \sum_{k=1}^{K} \frac{(-2\pi \mathrm{i})^k}{k!} ((\chi_{\widetilde{N}\times[s_1,+\infty)} P'_t \; \chi_{\widetilde{N}\times[s_1,+\infty)})^k - (\chi_{\widetilde{N}\times[s_1,+\infty)} P'_t \; \chi_{\widetilde{N}\times[s_1,+\infty)})).$$

Since $\widetilde{M}_{a\leq f\leq b}$ is diffeomorphic to $\widetilde{N} \times [-r, r]$ for some r, the Dirac operator $D_{\widetilde{M}}$ on $\widetilde{M}_{a\leq f\leq b}$ clearly coincides with $D_{\widetilde{N}\times\mathbb{R}}$ on $\widetilde{N} \times [-r, r]$. By the same reason as above, we have that $\chi(D_{\widetilde{M}})$ on $\widetilde{M}_{a+2c'\leq f\leq b-2c'}$ coincides with $\chi(D_{\widetilde{N}\times\mathbb{R}})$ on $\widetilde{N} \times [-r + 2c', r - 2c']$, which implies that

$$\chi_{\widetilde{M}_{r_1\leq f\leq r_2}} u_t \chi_{\widetilde{M}_{r_1\leq f\leq r_2}}$$

coincides with

$$\chi_{\widetilde{M}_{r_1\le f\le r_2}} u_t' \chi_{\widetilde{M}_{r_1\le f\le r_2}}.$$

As a result,

$$l\circ\partial_0[D_M]=\partial_0'[D_{N\times\mathbb{R}}].$$

Consequently,

$$\alpha\circ\partial_0[D_M]=\beta\circ l\circ\partial_0[D_M]=\beta\circ\partial_0'[D_{N\times\mathbb{R}}].$$

(2) Thanks to the product formula

$$[D_{N\times\mathbb{R}}]=[D_N]\otimes\Big[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\Big],$$

we have $\beta\circ\partial_n'[D_{N\times\mathbb{R}}]=[D_N]$. Then we get

$$\alpha\circ\partial_n[D_M]=[D_N].$$

This completes the proof.

4 The Representative Class of the Coarse Index of Dirac Operator

Denote the Riemannian metric on M by g and the spinor bundle over $\widetilde{M}$ by S. Suppose that M_1 is a compact subset of M and the scalar curvature k of g has a positive lower bound $k_0>0$ on $M_2=M\backslash M_1$. In this section, we show that the coarse index of D_M can be represented by an operator supported on the pull back covering set of a compact subset of M.

With the notations $M_{f\le r}$, $M_{f\ge r}$ and $M_{r_1\le f\le r_2}$ as defined in Section 3, we consider $f(x)=d(x,M_1)$ in this section, where $x\in M$ and d is the metric on M induced by g. For abreviation, we use $M_{\le r}$, $M_{\ge r}$ and $M_{[r_1,r_2]}$ to denote $M_{f\le r}$, $M_{f\ge r}$ and $M_{r_1\le f\le r_2}$ respectively.

Based on the assumptions above, we would prove the following theorem in this section.

Theorem 4.1 *There exists a positive number Λ such that*

(1) *when the dimension of M, n, is odd, the coarse index of D_M can be represented by an invertible operator which is equal to I while restricted on $L^2(\widetilde{M}_{\ge\Lambda},S)$.*

(2) *when the dimension of M, n, is even, the coarse index of D_M can be represented by a formal difference of idempotents which is equal to*

$$0=\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

while restricted on $L^2(\widetilde{M}_{\ge\Lambda},S)$.

To prove Theorem 4.1, we choose χ and T for the definition of the coarse index of D_M as in Section 2. Define

$$F=\int_{-\infty}^{+\infty}\widehat{\chi}(s)\mathrm{e}^{\mathrm{i}sTD_{\widetilde{M}}}\,\mathrm{d}s,$$

where T is a certain number such that $|\chi^2(s)-1|<\epsilon_0$ for $s>\frac{T\sqrt{k_0}}{200}$ and $\epsilon_0=\frac{1}{800000\pi\mathrm{e}^{4\pi}}$.

Lemma 4.1 *The propagation of the operator F is less than or equal to $T\kappa$. Moreover, we have $F^2-1\in C^*(\widetilde{M})^\Gamma$.*

Proof Let us prove that the propagation of F is finite.

First, it follows from the assumption that $\widehat{\chi}$ has compact support that

$$F = \int_{-\infty}^{+\infty} \widehat{\chi}(t)\mathrm{e}^{\mathrm{i}tTD}\mathrm{d}t = \int_{-\frac{\kappa}{2}}^{\frac{\kappa}{2}} \widehat{\chi}(t)\mathrm{e}^{\mathrm{i}tTD}\mathrm{d}t,$$

where κ is a certain constant satisfying $\operatorname{supp}(\widehat{\chi}) \subset \left[-\frac{\kappa}{2}, \frac{\kappa}{2}\right]$. This leads to the estimate that $tT \in \left[-\frac{\kappa T}{2}, \frac{\kappa T}{2}\right]$.

Notice that the Dirac operator D is indeed an essentially self-adjoint differential operator on S. (Every symmetric differential operator on a compact manifold without boundary is essentially self-adjoint, see [5, 10.2]. The self-adjoint extension of D will also be denoted by D.) It follows from [14, Chapter 8] that

$$\text{propagation}(\mathrm{e}^{\mathrm{i}tD}) \leq\ C_D \cdot |t|,$$

where

$$C_D = \sup_{x\in M, \xi \in T_x^* M, \|\xi\|=1} \|\sigma_D(x,\xi)\|$$

with $\sigma_D(x,\xi)$ the principal symbol of D. Since

$$D = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i},$$

where $\{e_i\}$ is an orthonormal basis of T_xM, we have

$$\sigma_D(x,\xi) = \sum_{i=1}^{n} a_i e_i, \quad \xi = \sum_{i=1}^{n} a_i \xi_i,$$

where $\{\xi_i\}$ is the dual basis of $\{e_i\}$ in T_x^*M. This leads to the conclusion that C_D equals 1. Furthermore we have

$$\text{propagation}(\mathrm{e}^{\mathrm{i}tD}) \leq\ |t|.$$

Combined with the fact that χ is a bounded Borel function with the distributional Fourier transform $\widehat{\chi}$ supported on $[-\frac{\kappa}{2}, \frac{\kappa}{2}]$, we have that the propagation of F is less than or equal to $T\kappa$ (see [5]).

Let us show that $F^2 - 1$ is locally compact. In fact, from the assumption that $\lim \chi(s) = 1$, as $s \to +\infty$, we have $\chi^2 - 1 \in C_0(\mathbb{R})$, which leads to the conclusion that the operator $\chi^2(D) - 1 = (\chi^2 - 1)(D)$ is locally compact.

These complete the proof that $F^2 - 1 \in C^*(\widetilde{M})^\Gamma$.

Remark 4.1 From this lemma, we can assume that the operator F defined above has propagation small enough by choosing a proper χ with κ sufficiently small.

In the following two subsections, we prove Theorem 4.1 case by case according to the parity of the dimension of M.

4.1 The dimension of M is odd

In this subsection, we assume that M is of odd dimension and prove Theorem 4.1(1).

Since the intersection of $\widetilde{M}_{\leq 200T\kappa}$ and $\widetilde{M}_{\geq 200T\kappa}$ is a zero measure set, we have the decomposition

$$L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\leq 200T\kappa}, S) \oplus L^2(\widetilde{M}_{\geq 200T\kappa}, S),$$

where κ and T are two numbers satisfying

(1) the distributional Fourier transform $\widehat{\chi}$ is supported on $\big[-\frac{\kappa}{2}, \frac{\kappa}{2}\big]$;

(2) $|\chi^2(s) - 1| < \epsilon_0$ for $s > \frac{T\sqrt{k_0}}{200}$.

This leads to the expression

$$F^2 - 1 = \begin{pmatrix} A_{11} - 1 & A_{12} \\ A_{21} & A_{22} - 1 \end{pmatrix}$$

with respect to the decomposition above.

Lemma 4.2 *We have*

$$\|A_{12}\| < \epsilon_0, \quad \|A_{21}\| < \epsilon_0, \quad \|A_{22} - 1\| < \epsilon_0.$$

Proof It follows from the fact that χ is a real-valued function and F is self-adjoint, we have $A_{12} = A_{21}^*$, and it is sufficient to show that $\|A_{12}\| < \epsilon_0$ and $\|A_{22} - 1\| < \epsilon_0$.

Since

$$(F^2 - 1)(f) = F(F(f)) - f$$

for all f in $L^2(\widetilde{M}_{\geq 200T\kappa}, S)$. It follows from Lemma 4.1 that the propagation of F is less than or equal to $T\kappa$ that

$$F(f) \in L^2(\widetilde{M}_{\geq 199T\kappa}, S), \quad F^2(f) \in L^2(\widetilde{M}_{\geq 198T\kappa}, S),$$

and it turns out that

$$F^2(f) - f \in L^2(\widetilde{M}_{\geq 100T\kappa}, S).$$

Recall the formula

$$D_{\widetilde{M}}^2 = \nabla^*\nabla + \frac{k}{4}.$$

Thus the restriction of $D_{\widetilde{M}}^2$ to $\widetilde{M}_2$ is bounded below by $\frac{k_0}{4}$, which implies that $D_{\widetilde{M}}|_{\widetilde{M}_2}$ has a self-adjoint extension on $L^2(\widetilde{M}_2, S)$, denoted by $D_{\widetilde{M}_2}$, with $D_{\widetilde{M}_2}^2$ still bounded below by $\frac{k_0}{4}$.

It follows that

$$\mathrm{e}^{\mathrm{i}sD_{\widetilde{M}}}|_{\widetilde{M}_{\geq 200T\kappa}} = \mathrm{e}^{\mathrm{i}sD_{\widetilde{M}_2}}|_{\widetilde{M}_{\geq 200T\kappa}}, \quad \forall\, s \leq 100T\kappa.$$

Therefore we have,

$$F(f) = \Big(\int_{-\infty}^{+\infty} \widehat{\chi}(s)\mathrm{e}^{\mathrm{i}sTD_{\widetilde{M}}}\mathrm{d}s\Big)(f) = \Big(\int_{-\infty}^{+\infty} \widehat{\chi}(s)\mathrm{e}^{\mathrm{i}sTD_{\widetilde{M}_2}}\mathrm{d}s\Big)(f),$$

$$F^2(f) = \Big(\int_{-\infty}^{+\infty} \widehat{\chi}(s)\mathrm{e}^{\mathrm{i}sTD_{\widetilde{M}}}\mathrm{d}s\Big)^2(f) = \Big(\int_{-\infty}^{+\infty} \widehat{\chi}(s)\mathrm{e}^{\mathrm{i}sTD_{\widetilde{M}_2}}\mathrm{d}s\Big)^2(f) = \chi^2(TD_{\widetilde{M}_2})(f).$$

Note that $\widehat{\chi}$ is supported on $[-\frac{\kappa}{2}, \frac{\kappa}{2}]$ and that

$$|\chi^2(s) - 1| < \epsilon_0$$

for all s, with $|s| > \frac{T\sqrt{k_0}}{2} > \frac{T\sqrt{k_0}}{200}$. Thus we have

$$\|\chi^2(TD_{\widetilde{M}_2}) - 1\| \le |\chi^2(s) - 1| < \epsilon_0,$$

which leads to

$$\|F^2 - 1\| < \epsilon_0.$$

The lemma then follows from the fact that

$$\sqrt{\|A_{12}(f)\|^2 + \|(A_{22}-1)(f)\|^2} = \Big\|\Big(\int_{-\infty}^{+\infty} \widehat{\chi}(s)\mathrm{e}^{\mathrm{i}sTD_{\widetilde{M}}}\mathrm{d}s\Big)^2(f) - f\Big\|$$

for all $f \in L^2(\widetilde{M}_{\ge 200T\kappa}, S)$.

Now, let us define

$$P = \frac{F+1}{2}$$

and

$$f_N(x) = \sum_{n=0}^{N} \frac{(2\pi\mathrm{i})^n}{n!}x^n, \quad x \in \mathbb{R}.$$

We use the polynomials $f_N(x)$ to approximate the function $\mathrm{e}^{2\pi\mathrm{i}x}$. Choose a sufficiently large positive integer N such that

(1) $|\mathrm{e}^{2\pi\mathrm{i}x} - f_N(x)| < \epsilon_0$ for all $x \in [-100, 100]$;

(2) $\Big|\sum\limits_{n=1}^{N} \frac{(2\pi\mathrm{i})^n}{n!}\Big| < \epsilon_0$.

A routine computation shows that

$$\begin{aligned} f_N(P) &= \sum_{n=0}^{N} \frac{(2\pi\mathrm{i})^n}{n!}P^n \\ &= 1 + \Big(\sum_{n=1}^{N} \frac{(2\pi\mathrm{i})^n}{n!}(P^n - P)\Big) + \Big(\sum_{n=1}^{N} \frac{(2\pi\mathrm{i})^n}{n!}P\Big) \\ &= 1 + \Big(\sum_{n=1}^{N} \frac{(2\pi\mathrm{i})^n}{n!}P\Big) + \Big(\sum_{n=2}^{N}\sum_{j=0}^{n-2} \frac{(2\pi\mathrm{i})^n}{n!}P^j(P^2 - P)\Big). \end{aligned}$$

Under the decomposition

$$L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\le 200T\kappa}, S) \oplus L^2(\widetilde{M}_{\ge 200T\kappa}, S),$$

there is

$$P^2 - P = \frac{F^2 - 1}{4} = \frac{1}{4}\begin{pmatrix} A_{11} - 1 & A_{12} \\ A_{21} & A_{22} - 1 \end{pmatrix}.$$

Define

$$E = \frac{1}{4}\begin{pmatrix} A_{11} - 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Lemma 4.2 then implies that

$$\|E - (P^2 - P)\| \leq 3\epsilon_0.$$

Define

$$u = 1 + \Big(\sum_{n=2}^{N}\sum_{j=0}^{n-2}\frac{(2\pi\mathrm{i})^n}{n!}P^j\Big)E.$$

Lemma 4.3 *The operator u is invertible.*

Proof We have

$$\|P\| \leq \frac{\|F\|+1}{2} \leq \frac{1+1}{2} = 1,$$

since $F = \chi(TD)$ and χ is a function bounded by 1.

Thanks to the triangle inequality, we have

$$\Big\|\sum_{n=2}^{N}\sum_{j=0}^{n-2}\frac{(2\pi\mathrm{i})^n}{n!}P^j\Big\| \leq 4\pi\cdot\mathrm{e}^{4\pi}.$$

Recall the definition $\epsilon_0 = \frac{1}{800000\pi\mathrm{e}^{4\pi}} < \frac{1}{400\pi\mathrm{e}^{4\pi}1000\|\mathrm{e}^{-2\pi\mathrm{i}P}\|}$. It follows that

$$\begin{aligned}\|u - f_N(P)\| &\leq \Big\|\sum_{n=1}^{N}\frac{(2\pi\mathrm{i})^n}{n!}P\Big\| + \Big\|\Big(\sum_{n=2}^{N}\sum_{j=0}^{n-2}\frac{(2\pi\mathrm{i})^n}{n!}P^j\Big)(E-(P^2-P))\Big\| \\ &< \epsilon_0 + 4\pi\cdot\mathrm{e}^{4\pi}\cdot 3\epsilon_0 \\ &< \frac{1}{300\cdot\|\mathrm{e}^{-2\pi\mathrm{i}P}\|} + \frac{1}{300\cdot\|\mathrm{e}^{-2\pi\mathrm{i}P}\|} \\ &< \frac{2}{300\cdot\|\mathrm{e}^{-2\pi\mathrm{i}P}\|}.\end{aligned}$$

Thus

$$\|u - \mathrm{e}^{2\pi\mathrm{i}P}\| \leq \|u - f_N(P)\| + \|f_N(P) - \mathrm{e}^{2\pi\mathrm{i}P}\| < \frac{1}{100\cdot\|\mathrm{e}^{-2\pi\mathrm{i}P}\|}.$$

It follows that u is invertible.

Moreover, we have $u \in (C^*(\widetilde{M})^\Gamma)^+$. In fact, we have

$$\begin{aligned}u &= 1 + \Big(\sum_{n=1}^{N}\frac{(2\pi\mathrm{i})^n}{n!}(P^n - P)\Big)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \\ &= 1 + \Big(\sum_{n=1}^{N}\frac{(2\pi\mathrm{i})^n}{n!}(P^2 - P)\sum_{j=0}^{n-2}P^j\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\Big),\end{aligned}$$

and

$$(P^2 - P) = \frac{1}{4}(F^2 - 1) \in C^*(\widetilde{M}).$$

By the estimates above, it is clear to see that u represents the coarse index of D_M.

Set $\Lambda = 200T\kappa + 10NT\kappa$. Consider the following decomposition

$$L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\leq \Lambda}, S) \oplus L^2(\widetilde{M}_{\geq \Lambda}, S).$$

Theorem 4.2 *The invertible operator u preserves the decomposition $L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\leq \Lambda}, S) \oplus L^2(\widetilde{M}_{\geq \Lambda}, S)$. That is, if we write*

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

with respect to the decomposition $L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\leq \Lambda}, S) \oplus L^2(\widetilde{M}_{\geq \Lambda}, S)$, then

$$u_{12} = 0, \quad u_{21} = 0,$$

and

$$u = \begin{pmatrix} u_{11} & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof Decompose $L^2(\widetilde{M}_{\leq \Lambda}, S)$ as

$$L^2(\widetilde{M}_{\leq 200T\kappa}, S) \oplus L^2(\widetilde{M}_{[200T\kappa, \Lambda]}, S).$$

For all $f \in L^2(\widetilde{M}_{[200T\kappa, \Lambda]}, S)$, since

$$E(f) = 0,$$

the support of $E(f)$ is contained in $\widetilde{M}_{\leq 200T\kappa}$.

Since

$$\text{propagation}(P) \leq \text{propagation}(F) \leq T\kappa,$$

we have

$$\text{propagation}(u) \leq NT\kappa.$$

Now it is apparent that if $f \in L^2(\widetilde{M}_{\leq \Lambda}, S)$, the support of $u(f)$ is contained in $\widetilde{M}_{\leq \Lambda}$. It follows that $u_{21} = 0$.

In the meantime, we have $u_{12} = 0$ and $u_{22} = 1$ due to the fact that for $h \in L^2(\widetilde{M}_{\geq 200T\kappa}, S)$,

$$E(h) = 0.$$

This completes the proof.

Theorem 4.1(1) then immediately follows from Theorem 4.2.

4.2 The dimension of M is even

In this subsection, we focus on Theorem 4.1(2).

When the dimension of M is even, the Dirac operator $D = D_{\widetilde{M}}$ is an odd operator with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading on the spinor bundle S, and so is

$$F = \int_{-\infty}^{+\infty} \widehat{\chi}(s)\mathrm{e}^{\mathrm{i}sTD}\mathrm{d}s.$$

Set

$$F = \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}$$

with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading. Since $F^2 - 1$ is locally compact and

$$F^2 = \begin{bmatrix} VU & 0 \\ 0 & UV \end{bmatrix},$$

we have $VU - 1 \in C^*(\widetilde{M})^\Gamma$, $UV - 1 \in C^*(\widetilde{M})^\Gamma$.

By similar arguments as the odd case, set

$$L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\leq 200T\kappa}, S) \oplus L^2(\widetilde{M}_{\geq 200T\kappa}, S).$$

With respect to this decomposition, suppose

$$UV = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$
$$VU = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Similarly we can show that

$$\|B_{12}\| \leq \epsilon_0, \quad \|B_{21}\| \leq \epsilon_0, \quad \|B_{22} - 1\| \leq \epsilon_0,$$
$$\|C_{12}\| \leq \epsilon_0, \quad \|C_{21}\| \leq \epsilon_0, \quad \|C_{22} - 1\| \leq \epsilon_0.$$

Define

$$W = \begin{bmatrix} 1 & U \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -V & 1 \end{bmatrix} \begin{bmatrix} 1 & U \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$p = W \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W^{-1} = \begin{bmatrix} UV + UV(1 - UV) & (2 - UV)U(1 - VU) \\ V(1 - UV) & (1 - VU)^2 \end{bmatrix}.$$

Set

$$Z_1 = \begin{pmatrix} 1 - B_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 1 - C_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Immediately, we can see that

$$\|Z_1 - (1 - UV)\| \leq 3\epsilon_0, \quad \|Z_2 - (1 - VU)\| \leq 3\epsilon_0.$$

Define

$$q = \begin{bmatrix} 1 - Z_1^2 & (2 - UV)UZ_2 \\ VZ_1 & Z_2^2 \end{bmatrix}$$

with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading.

Lemma 4.4 *We have*

$$\|q^2 - q\| \leq 20(10 + 20\epsilon_0)\epsilon_0.$$

Proof Due to the triangle inequality, we have

$$\|q - q^2\| \leq \|q - p\| + \|p - p^2\| + \|p^2 - q^2\|.$$

Since p is an idempotent by definition, we have

$$\|p - p^2\| = 0.$$

By the definition of q and the estimate of Z_1 and Z_2 from the estimate of B_{11} and C_{11}, we obtain that

$$\|q - p\| \leq 20\epsilon_0.$$

Moreover, one can see that

$$\begin{aligned}\|p^2 - q^2\| &\leq \|p^2 - pq\| + \|pq - q^2\| \\ &\leq (\|p\| + \|q\|) \cdot \|p - q\| \\ &\leq 20(\|p\| + \|q\|)\epsilon_0.\end{aligned}$$

Thus we have

$$\begin{aligned}\|q - q^2\| &\leq \|q - p\| + \|p - p^2\| + \|p^2 - q^2\| \\ &\leq 20(1 + \|p\| + \|q\|)\epsilon_0 \\ &\leq 20(10 + 20\epsilon_0)\epsilon_0.\end{aligned}$$

By the definition of ϵ_0, we can see that $20(10+20\epsilon_0)\epsilon_0 < \frac{1}{4}$, and that p and q represent the same K-theory class.

Set $\Lambda = 200T\kappa + 10T\kappa$ and take a new decomposition

$$L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\leq\Lambda}, S) \oplus L^2(\widetilde{M}_{\geq\Lambda}, S).$$

Theorem 4.3 *The operator q preserves the decomposition $L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\leq\Lambda}, S) \oplus L^2(\widetilde{M}_{\geq\Lambda}, S)$. Moreover, we have*

$$q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading on $L^2(\widetilde{M}_{\geq\Lambda}, S)$.

Proof For the ease of notation, with respect to the decomposition $L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\leq\Lambda}, S) \oplus L^2(\widetilde{M}_{\geq\Lambda}, S)$, set

$$q = \begin{bmatrix} 1 - Z_1^2 & (2 - UV)UZ_2 \\ VZ_1 & Z_2^2 \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix},$$

where

$$
\begin{aligned}
q_{11} &= 1 - Z_1^2 = 1 - \begin{pmatrix} 1-B_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-B_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2B_{11}-B_{11}^2 & 0 \\ 0 & 1 \end{pmatrix}, \\
q_{22} &= Z_2^2 = \begin{pmatrix} 1-C_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-C_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (1-C_{11})^2 & 0 \\ 0 & 0 \end{pmatrix}, \\
q_{12} &= (2-UV)UZ_2 = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \\
q_{21} &= VZ_1 = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}.
\end{aligned}
$$

This leads to the conclusion that q_{11} and q_{22} preserve the decomposition.

Now, let us turn to

$$q_{12} = (2-UV)UZ_2 = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.$$

Recall that

$$\text{propagation}(U) \le \text{propagation}(F) \le T\kappa$$

and

$$\text{propagation}(V) \le \text{propagation}(F) \le T\kappa.$$

A routine computation gives the following estimate

$$\text{propagation}(2U - UVU) \le 3T\kappa.$$

Note that

$$L^2(\widetilde{M}_{\le\Lambda}, S) = L^2(\widetilde{M}_{\le 200T\kappa}, S) \oplus L^2(\widetilde{M}_{[200T\kappa,\Lambda]}, S)$$

and

$$Z_2 = \begin{pmatrix} 1-C_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\le 200T\kappa}, S) \oplus L^2(\widetilde{M}_{\ge 200T\kappa}, S)$. We have

$$Z_2(h) = 0$$

for all h in $L^2(\widetilde{M}_{[200T\kappa,\Lambda]}, S)$. It follows that if $f \in \widetilde{M}_{\le\Lambda}$, then the support of $Z_2(f)$ is contained in $\widetilde{M}_{\le 200T\kappa}$. Thus it is apparent from the estimation of propagation above that the support of $(2-UV)UZ_2(f)$ is contained in $\widetilde{M}_{\le\Lambda}$ for all $f \in L^2(\widetilde{M}_{\le\Lambda}, S)$, which follows that $D_{21} = 0$.

Furthermore, the definition of Z_2 leads to

$$Z_2(h) = 0, \quad \forall\, h \in L^2(\widetilde{M}_{\ge\Lambda}, S),$$

which implies that $D_{12} = 0$ and $D_{22} = 0$.

This completes the proof that q_{12} preserves the decomposition.

Now, let us consider

$$q_{21} = VZ_1 = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}.$$

Similarly, as what we had previously, we have

$$L^2(\widetilde{M}_{\leq \Lambda}, S) = L^2(\widetilde{M}_{\leq 200T\kappa}, S) \oplus L^2(\widetilde{M}_{[200T\kappa,\Lambda]}, S).$$

Note that

$$Z_1 = \begin{pmatrix} 1 - B_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\leq 200T\kappa}, S) \oplus L^2(\widetilde{M}_{\geq 200T\kappa}, S)$. We have

$$Z_1(h) = 0$$

for all h in $L^2(\widetilde{M}_{[200T\kappa,\Lambda]}, S)$. It is straightforward that if $f \in \widetilde{M}_{\leq \Lambda}$, then the support of $Z_1(f)$ is contained in $\widetilde{M}_{\leq 200T\kappa}$.

Thus it follows from the estimate

$$\text{propagation}(V) \leq T\kappa$$

that the support of $VZ_1(f)$ is contained in $\widetilde{M}_{\leq \Lambda}$ for all $f \in L^2(\widetilde{M}_{\leq \Lambda}, S)$. Hence $H_{21} = 0$.

Moreover, it follows from

$$Z_1(h) = 0, \quad \forall\, h \in L^2(\widetilde{M}_{\geq \Lambda}, S)$$

that $H_{12} = 0$ and $H_{22} = 0$.

This completes the proof that q_{21} preserves the decomposition.

Summarizing the above discussions we have proved that q preserves the decomposition

$$L^2(\widetilde{M}, S) = L^2(\widetilde{M}_{\leq \Lambda}, S) \oplus L^2(\widetilde{M}_{\geq \Lambda}, S).$$

Thus we have

$$\begin{aligned} q &= \begin{bmatrix} 1 - Z_1^2 & (2 - UV)UZ_2 \\ VZ_1 & Z_2^2 \end{bmatrix} \\ &= \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \\ &= \begin{bmatrix} \begin{pmatrix} 2B_{11} - B_{11}^2 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} H_{11} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} (1 - C_{11})^2 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}. \end{aligned}$$

Clearly, we have

$$q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

on $L^2(\widetilde{M}_{\geq \Lambda}, S)$.

The even case of Theorem 4.1 is then an immediate consequence of Theorem 4.3.

5 Main Result

In this section, we shall give a proof of the following Bochner type vanishing theorem which was first obtained by Block and Weinberger in [2].

Theorem 5.1 (Block and Weinberger) *Let (M, g) be an n-dimensional complete noncompact Riemannian spin manifold with fundamental group Γ, where g is a metric of bounded geometry. Let $\pi : \widetilde{M} \to M$ be the universal covering map. Let $f : M \to \mathbb{R}$ be a proper smooth function. If the scalar curvature k of g satisfies $k \geq k_0 > 0$ off a compact set M_1, then the coarse index of Dirac operator on $N = f^{-1}(t_0) \neq \emptyset$ is trivial in $K_{n-1}(C^*(\widetilde{N})^\Gamma)$ where t_0 is a sufficiently large positive regular value of f such that N lies in $M_{\geq\Lambda}$ (Λ is the number defined in Theorem* 4.1).

Before proving Theorem 5.1, we mention that since N is compact, under the isomorphism

$$K_{n-1}(C^*(\widetilde{N})^\Gamma) \cong K_{n-1}(C_r^*(\Gamma)),$$

the coarse index of D_N is equal to the higher index of D_N, considered in [2].

Proof of Theorem 5.1 Recall that the coarse index of D_N is defined as

$$\operatorname{Ind} D_N = \mathrm{ev}_*[D_N].$$

Take r_1, r_2, s_1, s_2 as in Theorem 3.1, satisfying $s_2 > 0 > s_1$, $r_2 > t_0 > r_1$ and $M_{f\geq r_1} \subseteq M_{\geq\Lambda}$.

We have the following commutative diagram:

$$\begin{tikzcd}
K_n(C_L^*(\widetilde{M})^\Gamma) \arrow[r, "\partial_n"] & K_{n-1}(C_L^*(\widetilde{M}_{r_1\leq f\leq r_2})^\Gamma) \arrow[r, "\alpha", "\cong"'] \arrow[d, "l"', "\cong"] & K_{n-1}(C_L^*(\widetilde{N})^\Gamma) \\
K_n(C_L^*(\widetilde{N}\times\mathbb{R})^\Gamma) \arrow[r, "\partial_n'"] & K_{n-1}(C_L^*(\widetilde{N}\times[s_1,s_2])^\Gamma), \arrow[ur, "\cong", "\beta"'] &
\end{tikzcd}$$

where l is the isomorphism induced by the diffeomorphism between $\widetilde{M}_{r_1\leq f\leq r_2}$ and $\widetilde{N} \times [s_1, s_2]$, and β^{-1} is the isomorphism induced by the embedding of $\widetilde{N}$ into $\widetilde{N} \times \{0\}$ which is a strongly Lipschitz homotopy equivalence (for the definition of strongly Lipschitz homotopy equivalence, see [17]) and $\alpha = \beta \circ l$.

It is shown in Theorem 3.1 that

(1) $\alpha \circ \partial_n[D_M] = \beta \circ \partial_n'[D_{N\times\mathbb{R}}]$,

(2) $\alpha \circ \partial_n[D_M] = [D_N]$.

The composition of the diffeomorphism between $\widetilde{M}_{r_1\leq f\leq r_2}$ and $\widetilde{N}\times[s_1, s_2]$ and the projection from $\widetilde{N} \times [s_1, s_2]$ to $\widetilde{N}$, which is a coarse map, induce a map in K-theory

$$\overline{\alpha} : K_{n-1}(C^*(\widetilde{M}_{r_1\leq f\leq r_2})^\Gamma) \to K_{n-1}(C^*(\widetilde{N})^\Gamma).$$

By the naturality of the assembly map, the following commutative diagram holds:

$$\begin{tikzcd}
K_{n-1}(C_L^*(\widetilde{M}_{r_1\leq f\leq r_2})^\Gamma) \arrow[r, "\alpha"] \arrow[d, "\mathrm{ev}_*"'] & K_{n-1}(C_L^*(\widetilde{N})^\Gamma) \arrow[d, "\mathrm{ev}_*"] \\
K_{n-1}(C^*(\widetilde{M}_{r_1\leq f\leq r_2})^\Gamma) \arrow[r, "\overline{\alpha}"] & K_{n-1}(C^*(\widetilde{N})^\Gamma).
\end{tikzcd}$$

Now we discuss case by case according to the parity of the dimension of M.

Odd case

$$\begin{aligned}\operatorname{Ind} D_N &= \mathrm{ev}_*([D_N]) \\ &= \mathrm{ev}_*(\alpha \circ \partial_1[D_M]).\end{aligned}$$

According to Theorem 4.1, we can find a representative element $(u_t)_{t\geq 0}$ for $[D_M]$ such that

$$u_0 = 1 \text{ when restricted to } \; L^2(\widetilde{N}_{\geq\Lambda}, S).$$

Let

$$\begin{aligned}
U_t'' &= \chi_{\widetilde{M}_{f\geq r_1}} u_t \chi_{\widetilde{M}_{f\geq r_1}}, \\
V_t'' &= \chi_{\widetilde{M}_{f\geq r_1}} u_t^{-1} \chi_{\widetilde{M}_{f\geq r_1}}, \\
P_t'' &= \begin{pmatrix} U_t''V_t'' + U_t''V_t''(1 - U_t''V_t'') & (U_t''V_t''U_t'' - 2U_t'')(V_t''U_t'' - 1) \\ V_t''(1 - U_t''V_t'') & (V_t''U_t'' - 1)^2 \end{pmatrix}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\operatorname{Ind} D_N &= \mathrm{ev}_* \circ \alpha \circ \partial_1[u_t] \\
&= \mathrm{ev}_* \circ \alpha \left([P_t''] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right) \\
&= \overline{\alpha} \circ \mathrm{ev}_* \left([P_t''] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right) \\
&= \overline{\alpha} \left([P_0''] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right) \\
&= \overline{\alpha} \left(\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right) \\
&= 0 \in K_0(C^*(\widetilde{N})^\Gamma).
\end{aligned}$$

Even case

$$\begin{aligned}
\operatorname{Ind} D_N &= \mathrm{ev}_*([D_N]) \\
&= \mathrm{ev}_*(\alpha \circ \partial_0[D_M]) \\
&= \mathrm{ev}_* \circ \alpha \circ \partial_0 \left([(P_t)] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right) \\
&= \mathrm{ev}_* \circ \alpha([\exp(-2\pi \mathrm{i} \chi_{\widetilde{M}_{f\geq r_1}} P_t \, \chi_{\widetilde{M}_{f\geq r_1}})]) \\
&= \overline{\alpha} \circ \mathrm{ev}_*([\exp(-2\pi \mathrm{i} \chi_{\widetilde{M}_{f\geq r_1}} P_t \, \chi_{\widetilde{M}_{f\geq r_1}})]).
\end{aligned}$$

According to Theorem 4.1, we can choose $(P_t)_{t\geq 0}$ properly such that

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ when restricted to } \; L^2(\widetilde{N}_{\geq\Lambda}, S).$$

Then we have

$$\begin{aligned}
\operatorname{Ind} D_N &= \overline{\alpha} \circ \mathrm{ev}_*([\exp(-2\pi \mathrm{i} \chi_{\widetilde{M}_{f\geq r_1}} P_t \, \chi_{\widetilde{M}_{f\geq r_1}})]) \\
&= \overline{\alpha}([\exp(-2\pi \mathrm{i} \chi_{\widetilde{M}_{f\geq r_1}} P_0 \, \chi_{\widetilde{M}_{f\geq r_1}})])
\end{aligned}$$

$$
\begin{aligned}
&= \overline{\alpha}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \in K_1(C^*(\widetilde{N})^\Gamma).
\end{aligned}
$$

Acknowledgements The authors are grateful to Prof. Xiaoman Chen, Prof. Shengzhi Xu, Prof. Xiang Tang and Yi-Jun Yao for their guidance, and they also want to thank Prof. Zhizhang Xie and Prof. Guoliang Yu for their helpful comments.

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