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Existence of the Eigenvalues for the Cone Degenerate p-Laplacian*

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Abstract The present paper is concerned with the eigenvalue problem for cone degenerate p-Laplacian. First the authors introduce the corresponding weighted Sobolev spaces with important inequalities and embedding properties. Then by adapting Lusternik-Schnirelman theory, they prove the existence of infinity many eigenvalues and eigenfunctions. Finally, the asymptotic behavior of the eigenvalues is given.

Keywords Quasi-linear, Degenerate operator, Variational methods
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1 Introduction and Main Results

Write $\mathbb{B} = [0,1) \times X$ as a local model of stretched cone-manifold (i.e., manifold with conical singularities) with dimension $N \geq 3$. Here $X \subset S^{N-1}$ is a bounded set in the unit sphere of $\mathbb{R}^N$, and $x' = (x_2, \cdots, x_N) \in X$. Let $\operatorname{int}\mathbb{B}$ be the interior of $\mathbb{B}$ and $\partial\mathbb{B} := \{0\} \times X$ be the boundary of $\mathbb{B}$. The cone degenerate p-Laplacian is defined as follows:

$$-\Delta_{p\mathbb{B}} := -x_1^{-p}\operatorname{div}_{\mathbb{B}}(|\nabla_{\mathbb{B}} \cdot|^{p-2}\nabla_{\mathbb{B}}\cdot), \tag{1.1}$$

where $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_N})$ is cone gradient operator and for $F = (F_{x_1}, F_{x_2}, \cdots, F_{x_N})$, the cone divergence operator $\operatorname{div}_{\mathbb{B}}$ is defined by $\operatorname{div}_{\mathbb{B}} F = \nabla_{\mathbb{B}} \cdot F = x_1\partial_{x_1}F_{x_1} + \partial_{x_2}F_{x_2} + \cdots + \partial_{x_N}F_{x_N}$.

The present paper is devoted to the following Dirichlet eigenvalue problem for the cone degenerate p-Laplacian, i.e.,

$$\begin{cases} -\Delta_{p\mathbb{B}}\, u := -x_1^{-p}\operatorname{div}_{\mathbb{B}}(|\nabla_{\mathbb{B}}u|^{p-2}\nabla_{\mathbb{B}}u) = \lambda|u|^{p-2}u & \text{in } \operatorname{int}\mathbb{B}, \\ u = 0 & \text{on } \partial\mathbb{B}, \end{cases} \tag{1.2}$$

where $\lambda > 0$, $2 < p < N$. We call the problem (1.2) to be typical Dirichlet eigenvalue problem, because if (u, λ) is a solution of (1.2), $(\alpha u, \lambda)$ is also a solution for all $\alpha \in \mathbb{R}$. Hence it is different from the following problem

$$\begin{cases} -\Delta_{p\mathbb{B}}\, u = \lambda|u|^{q-2}u & \text{in } \operatorname{int}\mathbb{B}, \\ u = 0 & \text{on } \partial\mathbb{B} \end{cases} \tag{1.3}$$

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with $p \neq q$. In fact, the problem (1.3) with different homogeneity of the right hand side preserves a curve of solution, namely, if $u \neq 0$ verifies the problem (1.3) with $\lambda = 1$, then for $\alpha > 0$, αu is a solution of (1.3) with $\lambda = \alpha^{p-q}$. That is why we need different ways to construct Palais-Smale sequence in these two problems. The existence of multiple solutions for problem (1.3) has been studied in [9].

Here we are looking for non-trivial solutions $(u, \lambda) \in \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}) \times \mathbb{R}_+$ with $u \neq 0$, which verify the problem (1.2) in the following weak sense (the definition of $\mathcal{H}^{1,\gamma}_{p,0}(\mathbb{B})$, please see Definition 2.4 below), we say $u \in \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B})$ is a weak solution, if

$$\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^{p-2} \nabla_{\mathbb{B}} u \cdot \nabla_{\mathbb{B}} \varphi \frac{\mathrm{d}x_1}{x_1} \mathrm{d}x' = \lambda \int_{\mathbb{B}} x_1^p |u|^{p-2} u \varphi \frac{\mathrm{d}x_1}{x_1} \mathrm{d}x' \tag{1.4}$$

holds for any $\varphi \in C_0^\infty(\operatorname{int} \mathbb{B})$. The weak solutions to the eigenvalue problem (1.2) are critical points of the following energy functional

$$J(u) = \frac{1}{p} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \frac{\mathrm{d}x_1}{x_1} \mathrm{d}x' - \frac{\lambda}{p} \int_{\mathbb{B}} x_1^p |u|^p \frac{\mathrm{d}x_1}{x_1} \mathrm{d}x'. \tag{1.5}$$

Then we have the following results.

Theorem 1.1 *For $2 < p < N$, the Dirichet eigenvalue problem* (1.2) *processes infinitely many non-trivial weak solutions $\{(u_k, \lambda_k)\}_{k \geq 1}$ in the sense of* (1.4) *in $\mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}) \times \mathbb{R}_+$.*

Here if the eigenvalues $\lambda_k \neq \lambda_j$, then the corresponding eigenfunctions are not equivalent, i.e., $u_k \neq u_j$. Furthermore, the limit of the sequence of eigenvalues $\{\lambda_k\}_{k \geq 1}$ is infinity.

Corollary 1.1 *The eigenvalues λ_k for the problem* (1.2) *turn to infinity as $k \to \infty$.*

The existence of solutions to nonlinear elliptic equations involving the p-Laplacian

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

have been widely studied, see [2, 4–5, 12], etc. The motivation for the cone degenerate p-Laplacian (1.1) comes from the calculus on manifolds with conical singularities, as follows. A finite dimensional manifold B with conical singularities is a topological space with a finite subset $B_0 = \{b_1, \cdots, b_M\} \subset B$ of conical singularities, having the two following properties:

1. $B \setminus B_0$ is a C^∞ manifold.
2. Every $b \in B_0$ has an open neighborhood U in B, such that there is a homeomorphism

$$\varphi : U \to X^\Delta$$

and φ restricts a diffeomorphism

$$\varphi' : U \setminus \{b\} \to X^\wedge.$$

Here X is a bounded subset of the unit sphere S^{N-1} of $\mathbb{R}^N$, and set

$$X^\Delta = \overline{\mathbb{R}}_+ \times X / (\{0\} \times X).$$

This local model is interpreted as a cone with the base X. Since the analysis is formulated off the singularity, it makes sense to pass to

$$X^\wedge = \mathbb{R}_+ \times X,$$

which is the open stretched cone with the base X. Here we take the simplest case that

$$\mathbb{B} = [0,1) \times X, \quad \partial\mathbb{B} = \{0\} \times X.$$

The typical linear differential operators on a manifold with conical singularities are called Fuchs type, if the operators in a neighborhood of $x_1 = 0$ are of the following form

$$A = x_1^{-m} \sum_{k=0}^{m} a_k(x_1)\Big(- x_1 \frac{\partial}{\partial x_1}\Big)^k \tag{1.6}$$

with the coefficients $a_k(x_1) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{m-k}(X))$. More examples of this kind of operators are expressed in [18]. Furthermore, in [11, 13, 15, 19] and references therein, one can find more information about operators on manifolds with singularites.

This paper is organized as follows. In Section 2, some preliminaries are given here, including the definitions and properties of weighted Sobolev spaces, such as inequalities and embedding properties, more details can be found in [6–8, 10]. Afterward, we introduce the idea of genus as a tool to give the categories of the sets in a convenient way, see [16] for more information. In Section 3, by adapting the idea of Lusternik-Schnirelman theory in [1], we prove the main result Theorem 1.1. After the deformation result achieved, the existence of eigenvalues and eigenfunctions of the present problem (1.2) is obtained by applying the min-max argument. Finally, the asymptotic behavior of the sequence of eigenvalues is given in Section 4.

2 Preliminaries

In order to express the weak solutions for Dirichlet problem (1.2), we need the adequate distribution spaces. To define the weighted Sobolev spaces on the stretched cone $\mathbb{B}$, we first introduce the weighted Sobolev spaces and weighted L^p spaces on $\mathbb{R}^N_+$.

Definition 2.1 *For the weight data $\gamma \in \mathbb{R}$, we say $u(x) \in L^\gamma_p(\mathbb{R}^N_+)$ for $x \in \mathbb{R}^N_+ := \mathbb{R}_+ \times \mathbb{R}^{N-1}$, if $u \in \mathcal{D}'(\mathbb{R}^N_+)$ and*

$$\|u\|_{L^\gamma_p(\mathbb{R}^N_+)} = \Big(\int_{\mathbb{R}_+} \int_{\mathbb{R}^{N-1}} |x_1^{\frac{N}{p}-\gamma} u(x)|^p \mathrm{d}\sigma \Big)^{\frac{1}{p}} < +\infty$$

hold here and after we simplify the notation as $\mathrm{d}\sigma = \frac{\mathrm{d}x_1}{x_1}\mathrm{d}x'$.

Definition 2.2 *For $m \in \mathbb{N}$ and $\gamma \in \mathbb{R}$, the spaces*

$$\mathcal{H}^{m,\gamma}_p(\mathbb{R}^N_+) := \{u \in \mathcal{D}'(\mathbb{R}^N_+) : (x_1\partial_{x_1})^\alpha \partial^\beta_{x'} u \in L^\gamma_p(\mathbb{R}^N_+)\}$$

for arbitrary $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^{N-1}$ and $\alpha + |\beta| \le m$. Moreover, let $\mathcal{H}^{m,\gamma}_{p,0}(\mathbb{R}^N_+)$ denote the closure of $C^\infty_0(\mathbb{R}^N_+)$ in $\mathcal{H}^{m,\gamma}_p(\mathbb{R}^N_+)$.

According to [18, Section 2.1], we can generalize the definitions of the weighted Sobolev spaces on $\mathbb{R}^N_+$ to $X^\wedge$.

Definition 2.3 *Let $\mathcal{U} = \{U_1, \cdots, U_M\}$ be an open covering of X by coordinate neighborhoods. Fix a subordinate partition of unity $\{\varphi_1, \cdots, \varphi_M\}$ and charts $\chi_j : U_j \to \mathbb{R}^{N-1}$, $j = 1, \cdots, M$, then $u(x) \in \mathcal{H}^{m,\gamma}_p(X^\wedge)$ if and only if $u \in \mathcal{D}'(X^\wedge)$ and satisfies the following*

$$\|u\|_{\mathcal{H}^{m,\gamma}_p(X^\wedge)} = \Big\{ \sum_{j=1}^{M} \| \, (1 \times \chi^*_j)^{-1} \varphi_j u \, \|^p_{\mathcal{H}^{s,\gamma}_p(\mathbb{R}^N_+)} \Big\}^{\frac{1}{p}} < +\infty.$$

Here $1 \times \chi_j^* : C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^{N-1}) \to C_0^\infty(\mathbb{R}_+ \times U_j)$ *is the pull-back function with respect to* $1 \times \chi_j : \mathbb{R}_+ \times U_j \to \mathbb{R}_+ \times \mathbb{R}^{N-1}$. *Moreover, we denote* $\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge)$ *as the closure of* $C_0^\infty(X^\wedge)$ *in* $\mathcal{H}_p^{m,\gamma}(X^\wedge)$.

Definition 2.4 *Let* $W_{\rm loc}^{m,p}(\operatorname{int}\mathbb{B})$ *denote the classical local Sobolev space (here* $\operatorname{int}\mathbb{B}$ *is the interior of* $\mathbb{B}$*). For* $1 \le p < \infty$, $m \in \mathbb{N}$ *and the weighted data* $\gamma \in \mathbb{R}$, $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ *denotes the subspace of all* $u \in W_{\rm loc}^{m,p}(\operatorname{int}\mathbb{B})$, *such that*

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \{u \in W_{\rm loc}^{m,p}(\operatorname{int}\mathbb{B}) \mid \omega u \in \mathcal{H}_p^{m,\gamma}(X^\wedge)\}$$

for any cut-off function ω *supported by a collar neighborhood of* $[0,1) \times \partial\mathbb{B}$. *Moreover, the subspace* $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ *of* $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ *can be defined by the following deformation*

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) := [\omega]\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge) + [1-\omega]W_0^{m,p}(\operatorname{int}\mathbb{B}),$$

where the cut-off functions ω *are defined as before, and* $W_0^{m,p}(\operatorname{int}\mathbb{B})$ *denotes the closure of* $C_0^\infty(\operatorname{int}\mathbb{B})$ *in Sobolev space* $W^{m,p}(\widetilde{X})$ *when* $\widetilde{X}$ *is a closed compact* C^∞ *manifold of dimension* N *containing* $\mathbb{B}$ *as a submanifold with boundary. Also, we have*

$$L_p^\gamma(\mathbb{B}) := \mathcal{H}_p^{0,\gamma}(\mathbb{B}), \quad x_1^{\gamma_1}\mathcal{H}_p^{m,\gamma_2}(\mathbb{B}) = \mathcal{H}_p^{m,\gamma_1+\gamma_2}(\mathbb{B}).$$

For the proof of the main result, the following inequalities and embeddings are necessary.

Proposition 2.1 (Cone Type Poincaré Inequality) *Let* $1 \le p < \infty$ *and* $\gamma \in \mathbb{R}$. *If* $u(x) \in \mathcal{H}_{p,0}^{1,\gamma}(\mathbb{B})$, *then*

$$\|u(x)\|_{L_p^\gamma(\mathbb{B})} \le c\|\nabla_{\mathbb{B}} u(x)\|_{L_p^\gamma(\mathbb{B})}, \tag{2.1}$$

where the constant c *depends only on* $\mathbb{B}$ *and* p.

Proof Follow the same process of [7, Theorem 2.5].

Remark 2.1 The cone type Poincaré inequality implies that the norm $\|u\|_{\mathcal{H}_{p,0}^{1,\gamma}(\mathbb{B})}$ is equivalent to the norm $\|\nabla_{\mathbb{B}} u\|_{L_p^\gamma(\mathbb{B})}$.

Lemma 2.1 *For* $1 < p_2 < N$ *and* $1 \le p_1 < p_2^* = \frac{Np_2}{N-p_2}$, *the embedding*

$$\mathcal{H}_{p_2,0}^{1,\gamma_2}(\mathbb{B}) \hookrightarrow L_{p_1}^{\gamma_1}(\mathbb{B})$$

is compact, provided that $\frac{N}{p_1} - \gamma_1 > \frac{N}{p_2} - \gamma_2$.

Proof According to Definition 2.4, we write

$$\mathcal{H}_{p_2,0}^{1,\gamma_2}(\mathbb{B}) := [\omega]\mathcal{H}_{p_2,0}^{1,\gamma_2}(X^\wedge) + [1-\omega]W_0^{1,p_2}(\operatorname{int}\mathbb{B}),$$
$$\mathcal{H}_{p_1,0}^{0,\gamma_1}(\mathbb{B}) = [\omega]\mathcal{H}_{p_1,0}^{0,\gamma_1}(X^\wedge) + [1-\omega]W_0^{0,p_1}(\operatorname{int}\mathbb{B}),$$

and observe that the embedding $\mathcal{H}_{p_1,0}^{0,\gamma_1}(\mathbb{B}) \hookrightarrow L_{p_1}^{\gamma_1}(\mathbb{B})$ is continuous. To verify this result, we employ the classical compact embedding (see details in [14, Chapter 5]) as follows

$$[1-\omega]W_0^{1,p_2}(\operatorname{int}\mathbb{B}) \hookrightarrow [1-\omega]W_0^{0,p_1}(\operatorname{int}\mathbb{B})$$

is compact for $1 \le p_1 < p_2^*$. It remains to prove that the embedding

$$[\omega]\mathcal{H}_{p_2,0}^{1,\gamma_2}(X^\wedge) \hookrightarrow [\omega]\mathcal{H}_{p_1,0}^{0,\gamma_1}(X^\wedge)$$

is compact.

To this end, we introduce a map as follows. Set $1 \le q < \infty$. For any $v(x) \in \mathcal{H}^{m,\gamma}_{q,0}(X^\wedge)$, we define

$$(S_{\frac{N}{q},\gamma} v)(x_1, x') = \mathrm{e}^{-r(\frac{N}{q}-\gamma)} v(\mathrm{e}^{-r}, x').$$

Then $S_{\frac{N}{q},\gamma}$ induces an isomorphism as follows:

$$S_{\frac{N}{q},\gamma} : [\omega]\mathcal{H}^{m,\gamma}_{q,0}(X^\wedge) \to [\widetilde{\omega}]W^{m,q}_0(\mathbb{R} \times X)$$

with $\widetilde{\omega}(r) = \omega(\mathrm{e}^{-r})$, where $W^{m,q}_0(\mathbb{R} \times X)$ is the classical Sobolev space.

For $u_1(x) \in \mathcal{H}^{0,\gamma_1}_{p_1,0}(X^\wedge)$, one has

$$S_{\frac{N}{p_1},\gamma_1}(\omega(x_1)u_1(x)) = \omega(\mathrm{e}^{-r})\mathrm{e}^{-r(\frac{N}{p_1}-\gamma_1)} u_1(\mathrm{e}^{-r}, x'),$$

and it induces an isomorphism

$$S_{\frac{N}{p_1},\gamma_1} : [\omega]\mathcal{H}^{0,\gamma_1}_{p_1,0}(X^\wedge) \to [\widetilde{\omega}]W^{0,p_1}_0(\mathbb{R} \times X)$$

On the other hand, for $u_2(x) \in \mathcal{H}^{1,\gamma_2}_{p_2,0}(X^\wedge)$,

$$\begin{aligned} S_{\frac{N}{p_1},\gamma_1}(\omega(x_1)u_2(x)) &= \omega(\mathrm{e}^{-r})\mathrm{e}^{-r(\frac{N}{p_1}-\gamma_1)} u_2(\mathrm{e}^{-r}, x') \\ &= \mathrm{e}^{-r((\frac{N}{p_1}-\gamma_1)-(\frac{N}{p_2}-\gamma_2))} \omega(\mathrm{e}^{-r})\mathrm{e}^{-r(\frac{N}{p_2}-\gamma_2)} u_2(\mathrm{e}^{-r}, x'), \end{aligned}$$

and it also induces an isomorphism

$$S_{\frac{N}{p_1},\gamma_1} : [\omega]\mathcal{H}^{1,\gamma_2}_{p_2,0}(X^\wedge) \to [\widetilde{\omega}]\mathrm{e}^{-r\delta} W^{1,p_2}_0(\mathbb{R} \times X)$$

with $\delta := \big(\frac{N}{p_1} - \gamma_1\big) - \big(\frac{N}{p_2} - \gamma_2\big) > 0$. The following embedding

$$[\widetilde{\omega}]\mathrm{e}^{-r\delta} W^{1,p_2}_0(\mathbb{R} \times X) \hookrightarrow [\widetilde{\omega}]W^{0,p_1}_0(\mathbb{R} \times X)$$

is compact, since the function $\varphi(r) = \mathrm{e}^{-r\delta} \cdot r^s$ and all derivatives in r are uniformly bounded on supp $\widetilde{\omega}$ for every $s > 0$. This completes the proof.

Remark 2.2 With the same idea, for $1 < p_2 < N$ and $1 \le p_1 < p_2^*$, the embedding

$$\mathcal{H}^{1,\gamma_2}_{p_2,0}(\mathbb{B}) \hookrightarrow L^{\gamma_1}_{p_1}(\mathbb{B})$$

is continuous, provided that $\frac{N}{p_1} - \gamma_1 \ge \frac{N}{p_2} - \gamma_2$.

Now we verify the following Breizis-Lieb type result in the weighted Sobolev spaces.

Lemma 2.2 (Breizis-Lieb Type Result) *Let $1 \le p < \infty$ and $\{u_k\} \subset L^{\gamma}_p(\mathbb{B})$. If the following conditions are satisfied*

(i) *$\{u_k\}$ is bounded in $L^{\gamma}_p(\mathbb{B})$,*

(ii) *$u_k \to u$ a.e. in $\operatorname{int}\mathbb{B}$ as $k \to \infty$,*

then

$$\lim_{k\to\infty}\big(\|u_k\|^p_{L^{\gamma}_p(\mathbb{B})} - \|u_k - u\|^p_{L^{\gamma}_p(\mathbb{B})}\big) = \|u\|^p_{L^{\gamma}_p(\mathbb{B})}. \tag{2.2}$$

Proof Due to Fatou lemma, it yields

$$\begin{aligned}\|u\|^p_{L_p^\gamma} &= \int |x_1^{\frac{N}{p}-\gamma}u|^p \mathrm{d}\sigma\\ &\le \liminf_{k\to\infty}\int |x_1^{\frac{N}{p}-\gamma}u_k|^p \mathrm{d}\sigma\\ &= \liminf_{k\to\infty}\|u_k\|^p_{L_p^\gamma} < \infty.\end{aligned}$$

For simplicity, we set here $\widetilde{u}_k = x_1^{\frac{N}{p}-\gamma}u_k$ and $\widetilde{u} = x_1^{\frac{N}{p}-\gamma}u$. Since $p>1$, $j(t)=t^p$ is convex. For any fixed $\varepsilon>0$, there exists a constant c_ε, such that

$$||\widetilde{u}_k - \widetilde{u} + \widetilde{u}|^p + |\widetilde{u}_k - \widetilde{u}|^p| \le \varepsilon|\widetilde{u}_k - \widetilde{u}|^p + c_\varepsilon|\widetilde{u}|^p,$$

and then

$$||\widetilde{u}_k - \widetilde{u} + \widetilde{u}|^p - |\widetilde{u}_k - \widetilde{u}|^p - |\widetilde{u}|^p| \le \varepsilon|\widetilde{u}_k - \widetilde{u}|^p + (1+c_\varepsilon)|\widetilde{u}|^p.$$

Therefore, we obtain that

$$f_k^\varepsilon := (||\widetilde{u}_k|^p - |\widetilde{u}_k - \widetilde{u}|^p - |\widetilde{u}|^p| - \varepsilon|\widetilde{u}_k - \widetilde{u}|^p)^+ \le (1+c_\varepsilon)|\widetilde{u}|^p.$$

Then Lebesgue dominate theorem induces

$$\lim_{k\to\infty}\int_{\mathbb{B}} f_k^\varepsilon(x)\mathrm{d}\sigma = \int_{\mathbb{B}}\lim_{k\to\infty} f_k^\varepsilon(x)\mathrm{d}\sigma = 0.$$

Since

$$||x_1^{\frac{N}{p}-\gamma}u_k|^p - |x_1^{\frac{N}{p}-\gamma}u_k - x_1^{\frac{N}{p}-\gamma}u|^p - |x_1^{\frac{N}{p}-\gamma}u|^p| \le f_k^\varepsilon + \varepsilon|x_1^{\frac{N}{p}-\gamma}u_k - x_1^{\frac{N}{p}-\gamma}u|^p,$$

it follows that

$$\limsup_{k\to\infty}\int_{\mathbb{B}} ||x_1^{\frac{N}{p}-\gamma}u_k|^p - |x_1^{\frac{N}{p}-\gamma}(u_k-u)|^p - |x_1^{\frac{N}{p}-\gamma}u|^p|\mathrm{d}\sigma \le c\varepsilon$$

where

$$c := \sup\int_{\mathbb{B}} |x_1^{\frac{N}{p}-\gamma}(u_k-u)|^p\mathrm{d}\sigma.$$

Let $\varepsilon\to 0$, then it verifies the result.

For investigating the existence of solutions to the Dirichlet problem (1.2), some important concepts in variational methods are presented in the following. Let E be a Banach space. Define the class in E as

$$\Sigma(E) = \{A\subset E \mid A \text{ is closed, and } A=-A\}.$$

Definition 2.5 *For $A\in\Sigma(E)$, define the genus of A, denoted by $\gamma(A)$, as*

$$\gamma(A) = \begin{cases} 0, & \text{if } A=\emptyset,\\ \infty, & \text{if } \{m\in\mathbb{N}_+; \exists\, h\in C(A,\mathbb{R}^m\setminus\{0\}), h(-x)=-h(x)\}=\emptyset,\\ \inf\{m\in\mathbb{N}_+; \exists\, h\in C(A,\mathbb{R}^m\setminus\{0\}), h(-x)=-h(x)\}. \end{cases}$$

Proposition 2.2 *Let $A,B\in\Sigma(E)$, the genus γ possesses the following properties*
(1) If $\psi\in C(A,B)$ is odd, then $\gamma(A)\le\gamma(B)$.
(2) If $\psi\in C(A,B)$ is an odd homeomorphism, then $\gamma(A)=\gamma(B)=\gamma(\psi(A))$.
(3) If $A\subset B$, then $\gamma(A)\le\gamma(B)$.

(4) *If* $\gamma(B) < \infty$, $\gamma(\overline{A-B}) \geq \gamma(A) - \gamma(B)$.
(5) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
(6) *If* S^{n-1} *is the sphere in* $\mathbb{R}^n$, *then* $\gamma(S^{n-1}) = n$.
(7) *If* A *is compact, then* $\gamma(A) < \infty$.
(8) *If* A *is compact, there exists* $\delta > 0$ *such that for* $N_\delta(A) = \{x \in X : d(x, A) < \delta\}$, *we have* $\gamma(A) = \gamma(N_\delta(A))$.

Proof The proof can be found in [16, Section 3].

3 Proof of Theorem 1.1

The idea of Lusternik-Schnirelman theory in [1] is adapted here for the proof. Consider the following two operators

$$B(u) = \frac{1}{p}\int_{\mathbb{B}} x_1^p |u|^p \mathrm{d}\sigma : \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B}) \to \mathbb{R} \tag{3.1}$$

and

$$b(u) = x_1^p |u|^{p-2}u : \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B}) \to \mathcal{H}_p^{-1,-\frac{N}{p}}(\mathbb{B}), \tag{3.2}$$

where $\mathcal{H}_p^{-1,-\frac{N}{p}}(\mathbb{B})$ is the dual space of $\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$ with the norm as follows

$$\|g\|_{\mathcal{H}_p^{-1,-\frac{N}{p}}} = \sup_{\varphi \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}} \frac{|\langle g, \varphi\rangle|}{\|\varphi\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}}.$$

Lemma 3.1 *We have the following properties of the above two operators.*
(i) *The operator* b *defined in* (3.2) *is odd, compact and uniformly continuous on bounded sets.*
(ii) *The operator* B *defined in* (3.1) *is even and compact.*

Proof It is obvious that B is even and b is odd. First we verify the uniformly continuity of b on bounded set. Let u_1, u_0 be in a bounded set in $\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, and set $\delta := u_1 - u_0 \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, then for any $\varphi \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, we have that

$$\begin{aligned}
&|\langle b(u_1) - b(u_0), \varphi\rangle| \\
={}& \Big|\int_{\mathbb{B}} (x_1^p |u_1|^{p-2}u_1 - x_1^p |u_0|^{p-2}u_0)\varphi \mathrm{d}\sigma\Big| \\
={}& \Big|\int_{\mathbb{B}} x_1^p (|u_0 + \delta|^{p-2}(u_0 + \delta) - |u_0|^{p-2}u_0)\varphi \mathrm{d}\sigma\Big|,
\end{aligned}$$

where

$$\begin{aligned}
&|u_0 + \delta|^{p-2}(u_0 + \delta) - |u_0|^{p-2}u_0 \\
={}& \Big|\sum_{l=1}^{p-2} C_{p-2}^l u_0^{p-2-l}\delta^l + u_0^{p-2}\Big|(u_0 + \delta) - |u_0|^{p-2}u_0 \\
\leq{}& \Big|\sum_{l=1}^{p-2} C_{p-2}^l u_0^{p-1-l}\delta^l\Big| + \Big|\sum_{l=1}^{p-2} C_{p-2}^l u_0^{p-2-l}\delta^{l+1}\Big| + |u_0^{p-2}\delta| \\
\leq{}& C\sum_{l=1}^{p-1} |u_0^{p-1-l}\delta^l|.
\end{aligned}$$

Hence, it implies that

$$|\langle b(u_1)-b(u_0),\varphi\rangle| = C\sum_{l=1}^{p-1}\int_B |x_1^p u_0^{p-1-l}\delta\varphi|\mathrm{d}\sigma.$$

Set $p_1=\frac{p}{p-1-l}$, $p_2=\frac{p}{l}$, $p_3=p$ and choose $\gamma_1=\gamma_2=\gamma_3=\frac{N}{p}-1$ such that $\big(\frac{N}{p}-\gamma_1\big)(p-1-l)+\big(\frac{N}{p}-\gamma_2\big)l+\big(\frac{N}{p}-\gamma_3\big)=p$ with $\frac{N}{p}-\gamma_i>0$ for $i=1,2,3$, then by Hölder inequality, we have

$$\begin{aligned}&|\langle b(u_1)-b(u_0),\varphi\rangle|\\ &\le C\sum_{l=1}^{p-1}\Big(\int_{\mathbb{B}}|x_1^{\frac{N}{p}-\gamma_1}u_0|^{(p-1-l)p_1}\mathrm{d}\sigma\Big)^{\frac{1}{p_1}}\Big(\int_{\mathbb{B}}|x_1^{\frac{N}{p}-\gamma_2}\delta|^{l\,p_2}\mathrm{d}\sigma\Big)^{\frac{1}{p_2}}\Big(\int_{\mathbb{B}}|x_1^{\frac{N}{p}-\gamma_3}\varphi|^{p_3}\mathrm{d}\sigma\Big)^{\frac{1}{p_3}}\\ &=C\sum_{l=1}^{p-1}\|u_0\|^{p-1-l}_{L_p^{\gamma_1}}\|\delta\|^{l}_{L_p^{\gamma_2}}\|\varphi\|_{L_p^{\gamma_3}}.\end{aligned}$$

According to Lemma 2.1 and the conditions $\frac{N}{p}-\gamma_i>0$ for $i=1,2,3$, we have

$$|\langle b(u_1)-b(u_0),\varphi\rangle|\le C\Big(\sum_{l=1}^{p-1}\|u_0\|^{p-1-l}_{\mathcal{H}^{1,\frac{N}{p}}_{p,0}}\|\delta\|^{l}_{\mathcal{H}^{1,\frac{N}{p}}_{p,0}}\Big)\|\varphi\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,0}}.$$

Due to the assumption that u_1,u_2 are in a bounded set and $\delta=u_1-u_0$, we have

$$\|b(u_1)-b(u_2)\|_{\mathcal{H}_p^{-1,-\frac{N}{p}}}:=\sup_{\varphi\in\mathcal{H}^{1,\frac{N}{p}}_{p,0}}\frac{|\langle b(u_1)-b(u_0),\varphi\rangle|}{\|\varphi\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,0}}}\le C\sum_{l=1}^{p-1}\|u_1-u_0\|^{l}_{\mathcal{H}^{1,\frac{N}{p}}_{p,0}},$$

which verifies the uniformly continuity of b in bounded sets.

Now we show that b is a compact operator. For $\{u_k\}$ bounded in $\mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B})$, there exists a subsequence of $\{u_k\}$ (here and after the subsequence is denoted by the same notation) such that

$$u_k\rightharpoonup u\quad\text{in }\mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B})\text{ as }k\to\infty,$$

and by Lemma 2.1,

$$u_k\to u\quad\text{in }L_p^{\gamma_1}(\mathbb{B})\text{ as }k\to\infty$$

for choosing a proper γ_1 such that $\frac{N}{p}-\gamma_1>0$. As a consequence of convergence in $L_p^{\gamma_1}(\mathbb{B})$, we claim that there exists a subsequence holding that

$$x_1^{\frac{N}{p}-\gamma_1}u_k\to x_1^{\frac{N}{p}-\gamma_1}u\quad\text{a.e. in int}\,\mathbb{B}.\tag{3.3}$$

In fact, there is a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that

$$\|u_{k_{j+1}}-u_{k_j}\|_{L_p^{\gamma_1}}\le\frac{1}{2^j},\quad j=1,2,\cdots.$$

Let $x_1^{\frac{N}{p}-\gamma_1}v_k=\sum_{j=1}^{k}|x_1^{\frac{N}{p}-\gamma_1}u_{k_{j+1}}-x_1^{\frac{N}{p}-\gamma_1}u_{k_j}|$, then by Minkowski inequality, we get

$$\|v_k\|_{L_p^{\gamma_1}}\le\sum_{j=1}^{k}\|u_{k_{j+1}}-u_{k_j}\|_{L_p^{\gamma_1}}\le 1.$$

We set $x_1^{\frac{N}{p}-\gamma_1}v(x) = \lim_{k\to\infty} x_1^{\frac{N}{p}-\gamma_1}v_k(x)$. By Fatou lemma, it follows that

$$\int_{\mathbb{B}} |x_1^{\frac{N}{p}-\gamma_1}v(x)|^p \mathrm{d}\sigma \le \liminf_{k\to\infty} \int_{\mathbb{B}} |x_1^{\frac{N}{p}-\gamma_1}v_k(x)|^p \mathrm{d}\sigma \le 1.$$

The absolutely convergence implies that

$$x_1^{\frac{N}{p}-\gamma_1}u_{k_1} + \sum_{j=1}^{k}(x_1^{\frac{N}{p}-\gamma_1}u_{k_{j+1}} - x_1^{\frac{N}{p}-\gamma_1}u_{k_j}) \to x_1^{\frac{N}{p}-\gamma_1}u(x) \quad \text{a.e in } \operatorname{int}\mathbb{B},$$

which verifies the claim (3.3).

Then for any $v \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$ and $0<\gamma<p$, it implies that

$$\begin{aligned}&|\langle b(u_k) - b(u), v\rangle| \\ \le& \Big(\int_{\mathbb{B}} |x^{\gamma}(|u_k|^{p-2}u_k - |u|^{p-2}u)|^{\frac{p}{p-1}}\mathrm{d}\sigma\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{B}} |x_1^{p-\gamma}v|^p \mathrm{d}\sigma\Big)^{\frac{1}{p}} \\ :=& I_1 \cdot I_2.\end{aligned}$$

Together with Lemma 2.1, by taking $\frac{N}{p} - \gamma_2 = p - \gamma > 0$, we have

$$I_2 = \|v\|_{L_p^{\gamma_2}} \le C\|v\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}.$$

Due to $x_1^{\frac{N}{p}-\gamma_1}u_k \to x_1^{\frac{N}{p}-\gamma_1}u$ a.e in $\operatorname{int}\mathbb{B}$ as $k\to\infty$, we apply Lebesgue dominate convergence theory to I_1, and obtain that $I_1 \to 0$ as $k\to\infty$. This implies that

$$b(u_k) \to b(u) \quad \text{in } \mathcal{H}_p^{-1,-\frac{N}{p}}(\mathbb{B}) \text{ as } k\to\infty.$$

For the compactness of B, we take a bounded sequence $\{u_k\}$ in $\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, then, as before, up to a subsequence, we have

$$u_k \to u \quad \text{in } L_p^{\gamma_1}(\mathbb{B}) \text{ as } k\to\infty,$$

here taking $\gamma_1 = \frac{N}{p} - 1$. Then

$$B(u_k) = \frac{1}{p}\int_{\mathbb{B}} x_1^p|u|^p \mathrm{d}\sigma = \frac{1}{p}\int_{\mathbb{B}} |x_1 u|^p \mathrm{d}\sigma = \frac{1}{p}\|u_k\|^p_{L_p^{\frac{N}{p}-1}} \to \frac{1}{p}\|u\|^p_{L_p^{\frac{N}{p}-1}} = B(u) \text{ as } k\to\infty.$$

The main idea of the proof is to obtain the critical points of $B(u)$ on the manifold

$$M = \Big\{u \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B}) \Big| \frac{1}{p}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma = \alpha\Big\}, \tag{3.4}$$

here $\alpha>0$ is fixed. For each $u \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B}) \setminus \{0\}$, we can find $\lambda(u) > 0$ such that $\lambda(u)u \in M$ in the following way

$$\lambda(u) = \Big(\frac{p\alpha}{\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma}\Big)^{\frac{1}{p}}. \tag{3.5}$$

Hence $\lambda : \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})\backslash\{0\} \to (0,+\infty)$. It is obvious that $\lambda(u)$ is uniformly continuous on manifold M. By direct computation, the derivative of λ is as follows

$$\langle \lambda'(u), \varphi\rangle = -(p\alpha)^{\frac{1}{p}}\Big(\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma\Big)^{-\frac{p+1}{p}} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^{p-2}\nabla_{\mathbb{B}} u\cdot\nabla_{\mathbb{B}}\varphi \mathrm{d}\sigma \tag{3.6}$$

for any $\varphi \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$. Therefore, $\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^{p-2}\nabla_{\mathbb{B}} u\cdot\nabla_{\mathbb{B}}\varphi \mathrm{d}\sigma = 0$ implies $\langle \lambda'(u), \varphi\rangle = 0$.

Lemma 3.2 *The functional $\lambda'(\cdot)$ is uniformly continuous on M.*

Proof Indeed, let u_1, u_0 be in M, and set $u_1 - u_0 =: \delta \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, then for any $\varphi \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, we have

$$\begin{aligned}
&|\langle \lambda'(u_1) - \lambda'(u_0), \varphi\rangle| \\
&= C\Big|\Big(\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_1|^p \mathrm{d}\sigma\Big)^{-\frac{p+1}{p}} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_1|^{p-2}\nabla_{\mathbb{B}} u_1\cdot\nabla_{\mathbb{B}}\varphi \mathrm{d}\sigma \\
&\quad - \Big(\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_0|^p \mathrm{d}\sigma\Big)^{-\frac{p+1}{p}} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_0|^{p-2}\nabla_{\mathbb{B}} u_0\cdot\nabla_{\mathbb{B}}\varphi \mathrm{d}\sigma\Big| \\
&= C\Big|\int_{\mathbb{B}} (|\nabla_{\mathbb{B}} u_1|^{p-2}\nabla_{\mathbb{B}} u_1 - |\nabla_{\mathbb{B}} u_0|^{p-2}\nabla_{\mathbb{B}} u_0)\cdot\nabla_{\mathbb{B}}\varphi \mathrm{d}\sigma\Big|,
\end{aligned}$$

here, as in Lemma 3.1, we have

$$\begin{aligned}
&|\nabla_{\mathbb{B}} u_1|^{p-2}\nabla_{\mathbb{B}} u_1 - |\nabla_{\mathbb{B}} u_0|^{p-2}\nabla_{\mathbb{B}} u_0 \\
&= |\nabla_{\mathbb{B}} u_0 + \nabla_{\mathbb{B}}\delta|^{p-2}(\nabla_{\mathbb{B}} u_0 + \nabla_{\mathbb{B}}\delta) - |\nabla_{\mathbb{B}} u_0|^{p-2}\nabla_{\mathbb{B}} u_0 \\
&\le C\sum_{l=1}^{p-1} |\nabla_{\mathbb{B}} u_0|^{p-1-l}|\nabla_{\mathbb{B}}\delta|^l.
\end{aligned}$$

Hence, by setting $p_1 = \frac{p}{p-1-l}$, $p_2 = \frac{p}{l}$ and $p_3 = p$, it implies that

$$\begin{aligned}
&|\langle \lambda'(u_1) - \lambda'(u_0), \varphi\rangle| \\
&\le C\sum_{l=1}^{p-1}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_0|^{p-1-l}|\nabla_{\mathbb{B}}\delta|^l|\nabla_B\varphi|\mathrm{d}\sigma \\
&\le C\sum_{l=1}^{p-1}\Big(\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_0|^p \mathrm{d}\sigma\Big)^{\frac{1}{p_1}}\Big(\int_{\mathbb{B}} |\nabla_{\mathbb{B}}\delta|^p \mathrm{d}\sigma\Big)^{\frac{1}{p_2}}\Big(\int_{\mathbb{B}} |\nabla_{\mathbb{B}}\varphi|^p \mathrm{d}\sigma\Big)^{\frac{1}{p}} \\
&= C\sum_{l=1}^{p-1} \|u_0\|^{p-1-l}_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}} \|\delta\|^l_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}} \|\varphi\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}},
\end{aligned}$$

which leads to the uniformly continuity of $\lambda'(\cdot)$ on M, i.e.,

$$\|\lambda'(u_1) - \lambda'(u_0)\|_{\mathcal{H}_p^{-1,-\frac{N}{p}}} \le C\sum_{l=1}^{p-1}\|u_1 - u_0\|^l_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}.$$

The next step is to construct a flow on M related to the functional $B(u)$ and the corresponding deformation result allows us to apply the min-max theory, see [17]. Let $D(u)$ denote the derivative of $B(\lambda(u)u)$ for $u \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B}) \setminus \{0\}$, then we have

$$D(u) = \frac{p\alpha}{\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma}\Big(b(u) - \frac{\langle b(u), u\rangle}{\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma}(-x_1^p\Delta_{p\mathbb{B}} u)\Big) \in \mathcal{H}_p^{-1,-\frac{N}{p}}(\mathbb{B}),$$

i.e., for any $v \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$,

$$\langle D(u), v\rangle = \frac{p\alpha}{\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma}\Big(\langle b(u), v\rangle - \frac{\langle b(u), u\rangle}{\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^{p-2}\nabla_{\mathbb{B}} u\cdot\nabla_{\mathbb{B}} v \mathrm{d}\sigma\Big).$$

If $u \in M$, then

$$D(u) = b(u) - \frac{\langle b(u), u\rangle}{\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma}(-x_1^p \Delta_{p\mathbb{B}} u).$$

We claim that $D(u)$ is uniformly continuous on M. Since $b(u)$ and $-x_1^p\Delta_{p\mathbb{B}}u$ are uniformly continuous on M as proved in Lemmas 3.1–3.2, it is sufficient to verify that $\langle b(u), u\rangle$ hold this property on M. In fact, let $u_1, u_0 \in M$, and set $\delta := u_1 - u_0 \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, then we have

$$\begin{aligned}
&|\langle b(u_1), u_1\rangle - \langle b(u_0), u_0\rangle|\\
=& \int_{\mathbb{B}} x_1^p(|u_0+\delta|^p - |u_0|^p)\mathrm{d}\sigma\\
\le& C\sum_{l=1}^{p}\int_{\mathbb{B}} x_1^p |u_0|^{p-l}\delta^l \mathrm{d}\sigma = C\sum_{l=1}^{p}\int_{\mathbb{B}} |x_1^{\frac{N}{p}-\gamma_1}u_0|^{p-l}|x_1^{\frac{N}{p}-\gamma_2}\delta|^l\mathrm{d}\sigma\\
\le& C\sum_{l=1}^{p}\Big(\int_{\mathbb{B}} |x_1^{\frac{N}{p}-\gamma_1}u_0|^{(p-l)p_1}\mathrm{d}\sigma\Big)^{\frac{1}{p_1}}\Big(\int_{\mathbb{B}} |x_1^{\frac{N}{p}-\gamma_2}\delta|^{lp_2}\mathrm{d}\sigma\Big)^{\frac{1}{p_2}}\\
=& C\sum_{l=1}^{p}\|u_0\|_{L_p^{\gamma_1}}^{p-l}\|\delta\|_{L_p^{\gamma_2}}^{l} \le C\sum_{l=1}^{p}\|u_0\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}^{p-1}\|u_1-u_0\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}^{l} \le C\sum_{l=1}^{p}\|u_1-u_0\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}^{l},
\end{aligned}$$

where the compact embedding Lemma 2.1 is employed, and $p_1 = \frac{p}{p-l}$, $p_2 = \frac{p}{l}$, the proper γ_1 and γ_2 are chosen here, such that $\big(\frac{N}{p}-\gamma_1\big)(p-l) + \big(\frac{N}{p}-\gamma_2\big)l = p$ with $\frac{N}{p}-\gamma_i > 0$ for $i = 1, 2$.

Recall the definition of duality map.

Definition 3.1 *Let E be normed vector space, E^* be the dual space of E. We set for every $x_0 \in E$,*

$$\mathcal{J}(x_0) = \{f_0 \in E^*; \|f_0\|_{E^*} = \|x_0\|_E \text{ and } \langle f_0, x_0\rangle = \|x_0\|^2\}.$$

The map $x_0 \mapsto \mathcal{J}(x_0)$ is called the duality map from E into E^*.

According to the information of duality map in [3, Chapter 1], here we define the duality map

$$\mathcal{J} : \mathcal{H}_p^{-1,-\frac{N}{p}}(\mathbb{B}) \to \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$$

for all $f \in \mathcal{H}_p^{-1,-\frac{N}{p}}(\mathbb{B})$, such that $\mathcal{J}$ verifies

(i) $\|\mathcal{J}(f)\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})} = \|f\|_{\mathcal{H}_p^{-1,-\frac{N}{p}}(\mathbb{B})}$,

(ii) $\langle f, \mathcal{J}(f)\rangle = \|f\|^2_{\mathcal{H}_p^{-1,-\frac{N}{p}}(\mathbb{B})}$,

(iii) $\mathcal{J}(\cdot)$ is uniformly continuous on bounded sets.

For each $u \in M$, we define the tangent component as follows

$$T(u) = \mathcal{J}(D(u)) - \frac{\langle -x_1^p\Delta_{p\mathbb{B}}u, \mathcal{J}(D(u))\rangle}{\langle -x_1^p\Delta_{p\mathbb{B}}u, u\rangle}u,$$

such that

$$T : M \to \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$$

and

$$\langle -x_1^p \Delta_{p\mathbb{B}} u, T(u)\rangle = 0$$

hold, which implies that if $u \in M$, then

$$\langle \lambda'(u), T(u)\rangle = 0.$$

Lemma 3.3 *The tangent component $T(u)$ processes the following properties*
(i) *$T(u)$ is odd,*
(ii) *$T(u)$ is uniformly continuous on M,*
(iii) *$T(u)$ is bounded on M.*

Proof According to the definition of duality map and the fact that $D(u)$ is odd, we arrive that $T(u)$ is odd. Since both $D(\cdot)$ and $\mathcal{J}(\cdot)$ are uniformly continuous on bounded sets, one can deduce that $T(u)$ is uniformly continuous on M by applying the very similar procedure as in Lemma 3.2.

On the manifold M, the norm of $T(u)$ is estimated as follows:

$$\|T(u)\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}} \le \|\mathcal{J}(D(u))\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}} + \frac{|\langle -x_1^p \Delta_{p\mathbb{B}} u, \mathcal{J}(D(u))\rangle|}{|\langle -x_1^p \Delta_{p\mathbb{B}} u, u\rangle|}\|u\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}} := I_1 + I_2.$$

By applying Hölder inequality, we obtain that

$$\mathrm{I}_1 = \|\mathcal{J}(D(u))\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}} = \|D(u)\|_{\mathcal{H}_{p}^{-1,-\frac{N}{p}}} \le C\|u\|^{p-1}_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}$$

and

$$I_2 \le \frac{\|u\|^{p-1}_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}\|\mathcal{J}(Du)\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}}{\|u\|^{p}_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}}\|u\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}} = \|\mathcal{J}(Du)\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}} = \|D(u)\|_{\mathcal{H}_{p}^{-1,-\frac{N}{p}}} \le C\|u\|^{p-1}_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}}.$$

Then we have

$$\|T(u)\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}} \le C\|u\|^{p-1}_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}},$$

which implies that $T(u)$ is bounded on M.

For all $u \in M$, there exist $\gamma_0 > 0$ and $t_0 > 0$ such that for all $(u,t) \in M \times [-t_0, t_0]$, it holds

$$\|u + tT(u)\|_{\mathcal{H}_{p,0}^{1,\frac{N}{p}}} \ge \gamma_0 > 0.$$

As a consequence, we define the flow

$$\sigma(u,t) : M \times [-t_0, t_0] \to M \tag{3.7}$$

by

$$(u,t) \mapsto \sigma(u,t) = \lambda(u + tT(u))(u + tT(u)).$$

Then $\sigma(u,t)$ verifies the following properties:
(i) $\sigma(u,t)$ is odd with respect to u for fixed t,
(ii) $\sigma(u,t)$ is uniformly continuous with respect to u on M,
(iii) $\sigma(u,0) = u$ for $u \in M$.

Indeed, it is obvious that the properties (i) and (iii) of $\sigma(u,t)$ hold. The uniformly continuity of $\sigma(u,t)$ can be induced from the uniformly continuity of both $\lambda(\cdot)$ and $T(\cdot)$.

In order to obtain the deformation result, we first discover the relation between the functional $B(u)$ and the flow $\sigma(u,t)$ on M.

Lemma 3.4 *Let $\sigma(u,t)$ be defined in (3.7). Then there exists*

$$r : M \times [-t_0, t_0] \to \mathbb{R}$$

such that

$$\lim_{\tau\to 0} r(u,\tau) = 0$$

uniformly on M and

$$B(\sigma(u,t)) - B(u) = \int_0^t (\|D(u)\|^2_{\mathcal{H}_p^{-1,-\frac{N}{p}}} + r(u,s))\mathrm{d}s$$

for all $u \in M$ and $t \in [-t_0, t_0]$.

Proof Since $\sigma(u,0) = u$, $B(u) = B(\sigma(u,0))$. By the definitions of functional B in (3.1) and the operator b in (3.2), we have that for any $v \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$,

$$\langle B'(u), v\rangle = \langle b(u), v\rangle.$$

Hence,

$$B(\sigma(u,t)) - B(u) = \int_0^t \langle b(\sigma(u,s)), \partial_s \sigma(u,s)\rangle \mathrm{d}s.$$

Due to the fact that $\langle \lambda'(u), T(u)\rangle = 0$ and $\lambda(u) = 1$ on M, one can derive

$$\begin{aligned}
&\partial_s \sigma(u,s)\\
&= \partial_s(\lambda(u + sT(u))(u + sT(u)))\\
&= \langle \lambda'(u + sT(u)), T(u)\rangle (u + sT(u)) + \lambda(u + sT(u))T(u)\\
&= \langle \lambda'(u + sT(u)) - \lambda'(u), T(u)\rangle (u + sT(u)) + (\lambda(u + sT(u)) - \lambda(u))T(u) + T(u)\\
&:= R(u,s) + T(u),
\end{aligned}$$

where

$$R(u,s) = \langle \lambda'(u + sT(u)) - \lambda'(u), T(u)\rangle (u + sT(u)) + (\lambda(u + sT(u)) - \lambda(u))T(u).$$

Because T is bounded on M, and both $\lambda(u)$ and $\lambda'(u)$ are uniformly continuous, we have

$$\lim_{s\to 0} R(u,s) = 0$$

uniformly on M. Therefore,

$$\begin{aligned}
B(\sigma(u,t)) - B(u) &= \int_0^t \langle b(\sigma(u,s)), R(u,s) + T(u)\rangle \mathrm{d}s\\
&:= \int_0^t \langle b(u), T(u)\rangle + r(u,s)\mathrm{d}s,
\end{aligned}$$

where

$$r(u,s) = \langle b(\sigma(u,s)) - b(u), R(u,s) + T(u)\rangle + \langle b(u), R(u,s)\rangle.$$

Since b is uniformly continuous as proved in Lemma 3.1, the properties that $\lim_{s\to 0} \sigma(u,s) = u$ and $\lim_{s\to 0} R(u,s) = 0$ leads to that

$$\lim_{s\to 0} r(u,s) = 0$$

uniformly on M. Moreover, a direct computation implies that

$$\begin{aligned}
&\langle b(u), T(u)\rangle \\
&= \left\langle b(u), \mathcal{J}(D(u)) - \frac{\langle -x_1^p \Delta_{p\mathbb{B}} u, \mathcal{J}(D(u))\rangle}{\langle -x_1^p \Delta_{p\mathbb{B}} u, u\rangle} u \right\rangle \\
&= \langle b(u), \mathcal{J}(D(u))\rangle - \frac{\langle b(u), u\rangle \langle -x_1^p \Delta_{p\mathbb{B}} u, \mathcal{J}(D(u))\rangle}{\langle -x_1^p \Delta_{p\mathbb{B}} u, u\rangle} \\
&= \langle D(u), \mathcal{J}(D(u))\rangle = \|D(u)\|^2_{\mathcal{H}_p^{-1,-\frac{N}{p}}},
\end{aligned}$$

which completes the proof.

Consider the level set, for $\beta > 0$,

$$\Phi_\beta = \{u \in M \mid B(u) \geq \beta\}.$$

Then we have the following deformation result

Lemma 3.5 *Let $\beta > 0$ be fixed. Assume that there exists an open set $U \subset M$ such that for some constants $\delta > 0$ and $0 < \rho < \beta$, it holds that*

$$\|D(u)\|_{\mathcal{H}_p^{-1,-\frac{N}{p}}} \geq \delta \quad \text{if } u \in V_\rho = \{u \in M \mid u \notin U, |B(u) - \beta| \leq \rho\}.$$

Then there exist $\varepsilon > 0$ and an operator η_ε such that
(i) *η_ε is odd and continuous,*
(ii) *$\eta_\varepsilon(\Phi_{\beta-\varepsilon} - U) \subset \Phi_{\beta+\varepsilon}$.*

Proof Take t_0 and $r(u,s)$ as in Lemma 3.4. Consider $t_1 \in [0, t_0]$, such that for $s \in [-t_1, t_1]$,

$$|r(u,s)| \leq \frac{1}{2}\delta^2$$

for all $u \in M$. Then for $u \in V_\rho$ and $t \in [0, t_1]$, we have

$$B(\sigma(u,t)) - B(u) = \int_0^t (\|D(u)\|^2_{\mathcal{H}_p^{-1-\frac{N}{p}}} + r(u,s))\mathrm{d}s \geq \int_0^t (\delta^2 - \frac{1}{2}\delta^2)\mathrm{d}s = \frac{1}{2}\delta^2 t. \tag{3.8}$$

Choosing $\varepsilon = \min\{\rho, \frac{1}{4}\delta^2 t_1\}$. If $u \in V_\rho \cap \Phi_{\beta-\varepsilon}$, then

$$|B(u) - \beta| \leq \rho,$$

and from (3.8) we have

$$B(\sigma(u, t_1)) \geq B(u) + \frac{1}{2}\delta^2 t_1 \geq \beta + \varepsilon. \tag{3.9}$$

By Lemma 3.4, fixing $u \in V_\rho$, the functional $B(\sigma(u, \cdot))$ is increasing in some interval $[0, s_0) \subset [0, t_1)$. Then for

$$u \in V_\varepsilon = \{u \in M \mid u \notin U, |B(u) - \beta| \leq \varepsilon\},$$

the functional

$$t_\varepsilon(u) = \min\{t \geq 0 \mid B(\sigma(u,t)) = \beta + \varepsilon\}$$

is well defined and verifies that
(i) $0 < t_\varepsilon(u) \leq t_1$,

(ii) $t_\varepsilon(u)$ is continuous in V_ε.

In fact, (3.9) implies (i). The continuity of $\sigma(\cdot, s)$ and the continuity of $B(\cdot)$ induce (ii).

Define

$$\eta_\varepsilon(u) = \begin{cases} \sigma(u, t_\varepsilon(u)) & \text{if } u \in V_\varepsilon, \\ u & \text{if } u \in \Phi_{\beta-\varepsilon} - (U \cup V_\varepsilon) \end{cases} \tag{3.10}$$

such that

$$\eta_\varepsilon : \Phi_{\beta-\varepsilon} - U \to \Phi_{\beta+\varepsilon}.$$

Indeed, since $\sigma(u, t)$ is odd and uniformly continuous with respect to u, then we have $\eta_\varepsilon(u)$ is odd and continuous.

We now prove the existence of a sequence of critical values and critical points by applying a min-max argument. For each $k \in \mathbb{N}$, consider the class

$$\mathcal{A}_k = \{A \subset M \mid A \text{ is closed}, A = -A, \gamma(A) \geq k\}, \tag{3.11}$$

where γ is the genus as in Definition 2.5.

Lemma 3.6 *Let $\mathcal{A}_k$ be defined in* (3.11)*, define β_k as follows*

$$\beta_k = \sup_{A \in \mathcal{A}_k} \min_{u \in A} B(u), \tag{3.12}$$

then for each k and $\beta_k > 0$, there exists a sequence $\{u_{k_j}\} \subset M$ such that as $j \to \infty$ it holds that

$$\begin{cases} \text{(i) } B(u_{k_j}) \to \beta_k, \\ \text{(ii) } D(u_{k_j}) \to 0. \end{cases} \tag{3.13}$$

Proof By Definition 2.5, for the manifold M given as in (3.4), we have $\gamma(M) = +\infty$. Hence it holds that $\mathcal{A}_k \neq \emptyset$ for all $k > 0$. For each k, given $A \in \mathcal{A}_k$, we have

$$\min_{u \in A} B(u) > 0,$$

which implies that $\beta_k > 0$ for all k. Assume that there is no sequence in M verifying the conditions (3.13), then there must exist constants $\delta > 0$ and $\rho > 0$ such that

$$\|D(u)\|_{\mathcal{H}_p^{-1,-\frac{N}{p}}} \geq \delta \quad \text{if } u \in \{u \in M \mid |B(u) - \beta_k| \leq \rho\}.$$

Without loss of generality, assume that $\delta < \beta_k$. Applying Lemma 3.5 with $U = \emptyset$, there exist $\varepsilon > 0$ and an odd continuous mapping η_ε such that

$$\eta_\varepsilon(\Phi_{\beta_k-\varepsilon}) \subset \Phi_{\beta_k+\varepsilon}.$$

By the definition of β_k in (3.12), there exists a set $A_\varepsilon \in \mathcal{A}_k$ such that

$$B(u) \geq \beta_k - \varepsilon \quad \text{in } A_\varepsilon,$$

namely, $A_\varepsilon \subset \Phi_{\beta_k-\varepsilon}$. Then $B(u) \geq \beta_{k+\varepsilon}$ in $\eta_\varepsilon(A_\varepsilon)$. Since $A_\varepsilon \in \mathcal{A}_k$, we have $\gamma(A_\varepsilon) \geq k$. By Proposition 2.2 and the fact that η_ε is odd and continuous, we get

$$\gamma(\eta_\varepsilon(A_\varepsilon)) \geq k,$$

which implies that

$$\eta_\varepsilon(A_\varepsilon) \in \mathcal{A}_k.$$

This is a contradiction with the definition of β_k in (3.12). In this way, for each k, we obtain the sequence $\{u_{k_j}\} \subset M$ verifying the conditions (3.13).

To the end, we need the following local (PS) condition.

Lemma 3.7 *Let $\{u_j\} \subset M$ and $\beta > 0$ such that as $j \to \infty$,*

$$\begin{cases} \text{(i) } B(u_j) \to \beta, \\ \text{(ii) } D(u_j) \to 0 \quad \text{in } \mathcal{H}_p^{-1,-\frac{N}{p}}(\mathbb{B}). \end{cases} \tag{3.14}$$

Then there exists a convergent subsequence of $\{u_j\}$ in M.

Proof Since $\{u_j\} \subset M$, then $\{u_j\}$ is bounded in $\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$. Hence, it holds that

$$u_j \rightharpoonup u \quad \text{in } \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B}) \text{ as } j \to \infty.$$

According to the compact embedding stated in Lemma 2.1, it follows that up to a subsequence,

$$u_j \to u \quad \text{in } L_p^{\frac{N}{p}-1}(\mathbb{B}) \text{ as } j \to \infty,$$

which implies that $B(u) = \beta$, in fact

$$B(u_j) = \|u_j\|_{L_p^{\frac{N}{p}-1}} \to \|u\|_{L_p^{\frac{N}{p}-1}} = B(u) \text{ as } j \to \infty.$$

The second condition in (3.14) indicates that for any $v \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$,

$$\langle D(u_j), v\rangle = \langle b(u_j), v\rangle - \frac{\langle b(u_j), u_j\rangle}{\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_j|^{p-2} \nabla_{\mathbb{B}} u_j \cdot \nabla_{\mathbb{B}} v \mathrm{d}\sigma \to 0 \quad \text{as } j \to \infty.$$

This implies that, by setting $\lambda = \frac{\alpha}{\beta}$, the functional $J(\cdot)$ in (1.5) satisfies that for any $v \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$,

$$\langle J'(u_j), v\rangle = \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^{p-2} \nabla_{\mathbb{B}} u \cdot \nabla_{\mathbb{B}} v \mathrm{d}\sigma - \lambda \int_{\mathbb{B}} x_1^p |u|^{p-2} u v \mathrm{d}\sigma \to 0 \quad \text{as } j \to \infty.$$

Since $u_j \rightharpoonup u$ in $\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, we obtain that

$$\langle J'(u_j) - J'(u), u_j - u\rangle \to 0 \quad \text{as } j \to \infty.$$

A direct computation gives that

$$\begin{aligned} o(1) &= \langle J'(u_j) - J'(u), u_j - u\rangle \\ &= \int_{\mathbb{B}} (|\nabla_{\mathbb{B}} u_j|^{p-2} \nabla_{\mathbb{B}} u_j - |\nabla_{\mathbb{B}} u|^{p-2} \nabla_{\mathbb{B}} u)(\nabla_{\mathbb{B}} u_j - \nabla_{\mathbb{B}} u) \mathrm{d}\sigma \\ &\quad - \lambda \int_{\mathbb{B}} x_1^p (|u_j|^{p-2} u_j - |u|^{p-2} u)(u_j - u) \mathrm{d}\sigma \\ &=: I_1 - I_2. \end{aligned}$$

Choose some γ_1 such that $\frac{N}{p}-p<\gamma_1<\frac{N}{p}$. Due to Hölder inequality, we can derive that

$$\begin{aligned}
I_2 &= \lambda\int_{\mathbb{B}} x_1^p(|u_j|^{p-2}u_j-|u|^{p-2}u)(u_j-u)\mathrm{d}\sigma\\
&\le \lambda\Big(\int_{\mathbb{B}}|x_1^{\frac{N}{p}-\gamma_1}(u_j-u)|^p\mathrm{d}\sigma\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{B}}|x_1^{p-(\frac{N}{p}-\gamma_1)}(|u_j|^{p-2}u_j-|u|^{p-2}u)|^{\frac{p}{p-1}}\mathrm{d}\sigma\Big)^{\frac{p-1}{p}}\\
&:= \lambda\, T_1\cdot T_2.
\end{aligned}$$

Combining Lemma 2.1 and the fact that $\{u_j\}$ is bounded in $\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, we derive that $T_1\to 0$ and T_2 is bounded, which implies

$$I_2\to 0\quad \text{ as } j\to\infty.$$

Set $P_j(x)=(|\nabla_{\mathbb{B}}u_j|^{p-2}\nabla_{\mathbb{B}}u_j-|\nabla_{\mathbb{B}}u|^{p-2}\nabla_{\mathbb{B}}u)(x)(\nabla_{\mathbb{B}}u_j-\nabla_{\mathbb{B}}u)(x)$, then we arrive that

$$I_1=\int_{\mathbb{B}}P_j(x)\mathrm{d}\sigma\to 0\quad\text{as } j\to\infty. \tag{3.15}$$

Here, denote the i^{th} component of $\nabla_{\mathbb{B}}u$ by $(\nabla_{\mathbb{B}}u)_i$. We have the following ellipticity conditions that

$$P_j(x)\ge 0,\quad P_j(x)>0,\quad \text{if } \nabla_{\mathbb{B}}u_j\ne\nabla_{\mathbb{B}}u. \tag{3.16}$$

In fact, for any $x_0\in\operatorname{int}\mathbb{B}$, without loss of generality, we assume $(\nabla_{\mathbb{B}}u_j)_i(x_0)>(\nabla_{\mathbb{B}}u)_i(x_0)$. In the case of $(\nabla_{\mathbb{B}}u_j)_i(x_0)>(\nabla_{\mathbb{B}}u)_i(x_0)\ge 0$, $(\nabla_{\mathbb{B}}u_j)_i(x_0)\ge 0>(\nabla_{\mathbb{B}}u)_i(x_0)$ and $0>(\nabla_{\mathbb{B}}u_j)_i(x_0)>(\nabla_{\mathbb{B}}u)_i(x_0)$, we have

$$(|\nabla_{\mathbb{B}}u_j|^{p-2}\nabla_{\mathbb{B}}u_j-|\nabla_{\mathbb{B}}u|^{p-2}\nabla_{\mathbb{B}}u)_i(x_0)(\nabla_{\mathbb{B}}u_j-\nabla_{\mathbb{B}}u)_i(x_0)>0.$$

This shows (3.16). In the following, we verify that

$$(\nabla_{\mathbb{B}}u_j)_i\to(\nabla_{\mathbb{B}}u)_i\quad\text{for } 1\le i\le N\quad\text{as } j\to\infty$$

a.e. in $\operatorname{int}\mathbb{B}$, which can be deduced by contradiction. Assume, there exist a point $x_1\in\operatorname{int}\mathbb{B}$ and its neighborhood U_{x_1}, such that for any $x_0\in U_{x_1}$,

$$\lim_{k\to\infty}\nabla_{\mathbb{B}}u_j(x_0)\ne\nabla_{\mathbb{B}}u(x_0).$$

The convergence of (3.15) implies that $(\nabla_{\mathbb{B}}u_j|^{p-2}\nabla_{\mathbb{B}}u_j-|\nabla_{\mathbb{B}}u|^{p-2}\nabla_{\mathbb{B}}u)(\nabla_{\mathbb{B}}u_j-\nabla_{\mathbb{B}}u)$ is bounded, then it holds

$$(|\nabla_{\mathbb{B}}u_j|^{p-2}\nabla_{\mathbb{B}}u_j-|\nabla_{\mathbb{B}}u|^{p-2}\nabla_{\mathbb{B}}u)_i(x_0)(\nabla_{\mathbb{B}}u_j-\nabla_{\mathbb{B}}u)_i(x_0)\le c.$$

It follows that

$$\begin{aligned}
&(|\nabla_{\mathbb{B}}u_j|^{p-2}\nabla_{\mathbb{B}}u_j)_i(x_0)(\nabla_{\mathbb{B}}u_j)_i(x_0)\\
\le{}& c+(|\nabla_{\mathbb{B}}u_j|^{p-2}\nabla_{\mathbb{B}}u_j)_i(x_0)(\nabla_{\mathbb{B}}u)_i(x_0)+(|\nabla_{\mathbb{B}}u|^{p-2}\nabla_{\mathbb{B}}u)_i(x_0)(\nabla_{\mathbb{B}}u_j-\nabla_{\mathbb{B}}u)_i(x_0)\\
\le{}& c+(|\nabla_{\mathbb{B}}u_j|^{p-2}+|\nabla_{\mathbb{B}}u|^{p-2})(x_0)(\nabla_{\mathbb{B}}u_j)_i(x_0)(\nabla_{\mathbb{B}}u)_i(x_0),
\end{aligned}$$

which indicates that $|\nabla_{\mathbb{B}}u_j(x_0)|^p=\sum_{i=1}^{N}(|\nabla_{\mathbb{B}}u_j|^{p-2}\nabla_{\mathbb{B}}u_j)_i(x_0)(\nabla_{\mathbb{B}}u_j)_i(x_0)$ is bounded. There exists a subsequence, here still denoted by $\{u_j\}$ such that

$$(\nabla_{\mathbb{B}}u_j)(x_0)\to\xi'\ne\xi=\nabla_{\mathbb{B}}u(x_0)\quad\text{as } j\to\infty.$$

This induces that

$$P_j(x_0) = (|\nabla_{\mathbb{B}} u_j|^{p-2}\nabla_{\mathbb{B}} u_j - |\nabla_{\mathbb{B}} u|^{p-2}\nabla_{\mathbb{B}} u)(x_0)(\nabla_{\mathbb{B}} u_j - \nabla_{\mathbb{B}} u)(x_0) \to c_0 > 0$$

for any $x_0 \in U_{x_1}$ as $j \to \infty$. It follows that

$$I_1 = \int_{\mathbb{B}} P_j(x)\mathrm{d}\sigma \to c \neq 0 \quad \text{as } j \to \infty,$$

which contradicts to (3.15).

Applying Lemma 2.2 to $(\nabla_{\mathbb{B}} u_j)_i$ for $1 \le i \le N$, we have

$$\lim_{j\to\infty} (\|\nabla_{\mathbb{B}} u_j\|^p_{L_p^{\frac{N}{p}-1}(\mathbb{B})} - \|\nabla_{\mathbb{B}} u_j - \nabla_{\mathbb{B}} u\|^p_{L_p^{\frac{N}{p}-1}(\mathbb{B})}) = \|\nabla_{\mathbb{B}} u\|^p_{L_p^{\frac{N}{p}-1}(\mathbb{B})}. \tag{3.17}$$

What left is to show that

$$\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma \to \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma \quad \text{as } j \to \infty.$$

Due to Egorov theorem, we obtain that for any $\delta > 0$, there exists a subset $E \subset \operatorname{int}\mathbb{B}$ with the measure $m(E) < \delta$, such that

$$(\nabla_{\mathbb{B}} u_j)_i \to (\nabla_{\mathbb{B}} u)_i \quad \text{for } 1 \le i \le N \text{ as } j \to \infty$$

uniformly on $\operatorname{int}\mathbb{B} \setminus E$. It follows that

$$\int_{\mathbb{B}\setminus E} |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma \to \int_{\mathbb{B}\setminus E} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma \quad \text{as } j \to \infty. \tag{3.18}$$

Now we claim that for any $\varepsilon > 0$, there are $\delta(\varepsilon) > 0$ and a subset $E \subset \mathbb{B}$ with the measure $m(E) < \delta(\varepsilon)$, such that

$$\int_E |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma < \varepsilon. \tag{3.19}$$

In fact,

$$o(1) = I_1 = \int_{\mathbb{B}} (|\nabla_{\mathbb{B}} u_j|^{p-2}\nabla_{\mathbb{B}} u_j - |\nabla_{\mathbb{B}} u|^{p-2}\nabla_{\mathbb{B}} u)(\nabla_{\mathbb{B}} u_j - \nabla_{\mathbb{B}} u)\mathrm{d}\sigma,$$

which implies that

$$\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma = \int_{\mathbb{B}} (|\nabla_{\mathbb{B}} u_j|^{p-2} + |\nabla_{\mathbb{B}} u|^{p-2})(\nabla_{\mathbb{B}} u_j \cdot \nabla_{\mathbb{B}} u)\mathrm{d}\sigma - \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma + o(1).$$

For any $E \subset \mathbb{B}$, we have

$$\int_E |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma \le \int_E |\nabla_{\mathbb{B}} u_j|^{p-1}|\nabla_{\mathbb{B}} u|\mathrm{d}\sigma + \int_E |\nabla_{\mathbb{B}} u|^{p-1}|\nabla_{\mathbb{B}} u_j|\mathrm{d}\sigma + \int_E |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma.$$

According to Hölder inequality, it follows

$$\begin{aligned}\int_E |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma \le & \Big(\int_E |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma\Big)^{\frac{p-1}{p}} \Big(\int_E |\nabla u|^p \mathrm{d}\sigma\Big)^{\frac{1}{p}} \\ & + \Big(\int_E |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma\Big)^{\frac{1}{p}} \Big(\int_E |\nabla u|^p \mathrm{d}\sigma\Big)^{\frac{p-1}{p}} \\ & + \int_E |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma,\end{aligned}$$

which verifies (3.19). Hence, for any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and a subset $E \subset \operatorname{int}\mathbb{B}$, such that both (3.18) and (3.19) hold, then we have

$$\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_j|^p \mathrm{d}\sigma \to \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^p \mathrm{d}\sigma \quad \text{as } j \to \infty.$$

This finishes the proof.

Combining Lemmas 3.6 and 3.7, then for each k, we have a sequence $u_{k_j} \subset M$ such that

$$u_{k_j} \to u_k \quad \text{in } M,$$

which gives that $u_k \in M$ with $B(u_k) = \beta_k$ and $D(u_k) = 0$. This induces that for any $\varphi \in \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$ and for each $k \in \mathbb{N}$,

$$\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_k|^{p-2} \nabla_{\mathbb{B}} u_k \cdot \nabla_{\mathbb{B}} \varphi \mathrm{d}\sigma = \lambda_k \int_{\mathbb{B}} x_1^p |u_k|^{p-2} u_k \varphi \mathrm{d}\sigma$$

by setting $\lambda_k = \frac{\alpha}{\beta_k}$. This completes the proof of Theorem 1.1.

Remark 3.1 For the case $p = 2$ in (1.2), the problem involving the linear Fuchs type operator $-x_1^{-2}\mathrm{div}_{\mathbb{B}}(\nabla_{\mathbb{B}} u)$ turns to be linear, which is a simpler and similar case of (1.2).

4 Proof of Corollary 1.1

Consider $\{E_k\}$ to be a sequence of linear subspaces of $\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, such that
(i) $E_k \subset E_{k+1}$,
(ii) $\overline{\mathcal{L}(\cup E_k)} = \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$,
(iii) $\dim E_k = k$.
Define

$$\widetilde{\beta}_k = \sup_{A \in \mathcal{A}_k} \inf_{u \in A \cap E_{k-1}^c} B(u),$$

where E_k^c is the linear and topological complementary of E_k. It is obvious that

$$\widetilde{\beta}_k \geq \beta_k > 0.$$

Hence, it is sufficient to show that $\lim\limits_{k\to\infty} \widetilde{\beta}_k = 0$, which will be verified by contradiction as follows. Assume that for some positive constant $\gamma > 0$, we have $\widetilde{\beta}_k > \gamma > 0$ for all $k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$, there exists $A_k \in \mathcal{A}_k$ such that

$$\widetilde{\beta}_k \geq \inf_{u \in A_k \cap E_{k-1}^c} B(u) > \gamma.$$

Then there exists $u \in A_k \cap E_{k-1}^c$ such that

$$\widetilde{\beta}_k \geq B(u_k) > \gamma.$$

In this way, we have formed a sequence $\{u_k\} \subset M$, such that $B(u_k) > \gamma$ for all $k \in \mathbb{N}$. Since $\{u_k\} \subset M$, as before we know that $\{u_k\}$ is bounded in $\mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B})$, which implies that

$$u_k \rightharpoonup v \quad \text{in } \mathcal{H}_{p,0}^{1,\frac{N}{p}}(\mathbb{B}) \text{ as } k \to \infty,$$
$$u_k \to v \quad \text{in } L_p^{\frac{N}{p}-1}(\mathbb{B}) \text{ as } k \to \infty.$$

Hence we have

$$B(v) = \frac{1}{p}\|v\|^p_{L_p^{\frac{N}{p}-1}} > \gamma. \tag{4.1}$$

But the fact that $u_k \in E^c_{k-1}$ implies $v = 0$, which induces the contradiction with (4.1), and then we finish the proof.

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