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On Some Model Equations of Euler and Navier-Stokes Equations*

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Abstract The author proposes a two-dimensional generalization of Constantin-Lax-Majda model. Some results about singular solutions are given. This model might be the first step toward the singular solutions of the Euler equations. Along the same line (vorticity formulation), the author presents some further model equations. He possibly models various aspects of difficulties related with the singular solutions of the Euler and Navier-Stokes equations. Some discussions on the possible connection between turbulence and the singular solutions of the Navier-Stokes equations are made.

Keywords Euler equations, Navier-Stokes equations, Singular solutions, Turbulence
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1 Introduction

The incompressible Navier-Stokes equations describe the motion of incompressible viscous flow. Whether singular solutions exist in 3D is one of the famous seven millennium prize problems. In terms of singular solutions, the study of incompressible Euler equations looks most probably to be the first major step. For the sake of convenience, we will omit incompressible from now on. The first nonlocal model was constructed by Constatin, Lax and Majda [2]. They constructed a one-dimensional model and got the singular solution explicitly. The motivation is the vorticity formulation. There are many developments after this model (see [1, 3, 4, 5, 8, 14] and others). Shortly before the publication of the current paper, the author found out that Kiselev [9, Problem 6] has proposed to study the two dimensional analog of [2] in 2016. Essentially Model 1 and 1' in section two are some special cases of his proposal.

In this paper, we give some high dimensional generalizations of the Constantin-Lax-Majda model. The study of them might help the understanding of singular solutions to the Euler and Navier-Stokes equations. The motivation is still vorticity formulation.

We first present a two-dimensional zero order scalar model. One may think of it as a nonlocal ODE. The good understanding of it is possibly the first step toward singular solutions of the Euler and Navier-Stokes equations.

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Then we give further models. In some sense, the vorticity formulation provides an explanation why the singular solutions to the Navier-Stokes equations are so hard. Roughly speaking, the zero order term is pro-singularity term. The first and second order terms are perturbations. To be able to construct original solution from vorticity, we need the initial vorticity to be divergence-free. Any of them is hard to handle. The combination of them composes one of the most difficult problems in contemporary mathematics.

One potential major application of singular solutions of the Navier-Stokes equations is about turbulence. There is a prevailling viewpoint that the turbulent theory is deeply connected with the singular solutions of the Navier-Stokes equations. Actually one can interpret the behaviour of turbulence by the guessed properties of singular solutions to the Navier-Stokes equations.

This paper is organized as follows. In Section 2, we discuss the two-dimensional zero order scalar models. In Section 3, further models are given. The possible connection between singular solutions of the Navier-Stokes equations and turbulence theory is presented in Section 4. The notations we use are standard ones.

2 Zero Order Scalar Models

The vorticity formulation of the three-dimensional Euler equations is the following:

$$w_t + u \cdot \nabla w - \nabla u w = 0. \tag{2.1}$$

In $\mathbb{R}^3$, the velocity u is given by the Biot-Savart law:

$$u = \mathrm{curl}\ \Delta^{-1} w. \tag{2.2}$$

Note that (2.1)–(2.2) are well-defined when the vorticity is not divergence-free. Define

$$Z_{ij} = \partial_{ij}\Delta^{-1}, \quad x \in \Omega, \tag{2.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ or $\mathbb{R}^n$ itself, $n \geq 2$. If the domain is bounded, then the boundary condition for $\triangle$ is the homogeneous Drichilet boundary condition:

$$\Delta^{-1} w \mid_{\partial\Omega} = 0. \tag{2.4}$$

In $\mathbb{R}^3$, the term ∇u can be rewritten as

$$\begin{aligned} \nabla u &= \nabla \mathrm{curl}\, \Delta^{-1} w \\ &= \begin{pmatrix} Z_{21}w_3 - Z_{31}w_2 & Z_{31}w_1 - Z_{11}w_3 & Z_{11}w_2 - Z_{21}w_1 \\ Z_{22}w_3 - Z_{32}w_2 & Z_{32}w_1 - Z_{12}w_3 & Z_{12}w_2 - Z_{22}w_1 \\ Z_{23}w_3 - Z_{33}w_3 & Z_{33}w_3 - Z_{13}w_3 & Z_{13}w_2 - Z_{23}w_1 \end{pmatrix}. \end{aligned} \tag{2.5}$$

Remark 2.1 If the domain has boundary, then in general, (2.2) is not valid. Neither is (2.5). But conceptually, (2.5) is still close to be true. For Biot-Savart law in bounded domain, we refer to [10] for more details.

The Constantin-Lax-Majda model has the following form:

$$\theta_t = H(\theta)\theta, \tag{2.6}$$

where H is the Hilbert transform.

One nature generalization of Constantin-Lax-Majda model is the following equation.

Model 1

$$w_t = Z_{11}w\ w, \quad x \in \Omega \subset \mathbb{R}^2. \tag{2.7}$$

Proposition 2.1 *The model equation is locally well-posed in $W^{1,p}$, $p > 2$, i.e., $\forall\, w_0 \in W^{1,p}$, $\exists\, w \in C((0,T), W^{1,p})$, s.t. $w(x,0) = w_0$, w satifies* (2.7).

The proof is pretty standard and we omitted it.

Next we present some elementary singular solutions of Model 1. One feature of the zero order model is that self-similar singular solutions could be considered in bounded domain.

Let

$$w = \frac{1}{T-t} Q\Big(\frac{x}{T-t}\Big). \tag{2.8}$$

Then the equation for Q is

$$Z_{11}Q\ Q = Q. \tag{2.9}$$

The interesting thing is that, when the domain is an ellipse, (2.9) has constant solution.

Theorem 2.1 *Assume $\Omega = \{ax_1^2 + bx_1x_2 + cx_2^2 < 1\}$, $a, c > 0$, $b^2 - 4ac < 0$. Then* (2.9) *has a constant solution $Q = 1 + \frac{c}{a}$.*

Proof By the definition of ellipse, we know

$$\Delta^{-1}\ 1 = \frac{1}{2(a+c)}(ax_1^2 + bx_1x_2 + cx_2^2 - 1).$$

So

$$Z_{11}\ 1 = \frac{a}{a+c}.$$

Therefore $Q = 1 + \frac{c}{a}$ solves (2.9). The theorem is proven.

Going back to the original equation, we see that $w = \frac{a+c}{a} \cdot \frac{1}{T-t}$ is a singular solution to Model 1.

Consider the following simpler version of Model 1:

$$(Z_{11} + aZ_{22})w\ w = w. \tag{2.10}$$

Proposition 2.2 *Assume that $a > 0$, the domain Ω is rectangle or whole space. Then for any measurable set $E \subset \Omega$, (2.10) has solution $w \in L^2(\Omega)$ such that*

$$(Z_{11} + aZ_{22})w\big|_{\Omega\setminus E} = 1.$$

Proof Without loss of generality, we may assume that the rectangle is $(0,\pi)\times(0,\pi)$. In this case, $\sin k\cdot x,\ k = (k_1,k_2)$, k_i positive integers, are complete orthogonal basis and $w = \lambda_k \sin k\cdot x$. So we have

$$Z_{11}w = \sum_{k_1,k_2=1}^{\infty} \frac{k_1^2}{k_1^2+k_2^2}\lambda_k \sin k\cdot x.$$

Therefore

$$\int_{(0,\pi)\times(0,\pi)} Z_{11}ww\mathrm{d}x \geq 0. \tag{2.11}$$

In the whole space case, similarly we have

$$\int_{R^2} Z_{11}ww\mathrm{d}x = \int_{R^2} \frac{k_1^2}{k_1^2+k_2^2}|\widetilde{w}|^2\ \mathrm{d}k \geq 0, \tag{2.12}$$

where $\widetilde{w}$ is the Fourier transform of w. Also note that Z_{11} is self-adjoint.

Define

$$L_a = Z_{11} + aZ_{22}.$$

So under the assumptions of the current proposition, L_a is coercive:

$$\langle L_a w, w\rangle_{L^2} \geq a\|w\|_{L^2}. \tag{2.13}$$

The proof mainly comes from the coerciveness of L_a. Below is the details.

Note

$$(2.10) \Leftrightarrow (L_a w - 1)w = 0.$$

So the solving of (2.10) reduces to finding w such that

$$\begin{cases} w = 0, & x \in E, \\ L_a w = 1, & x \notin E. \end{cases}$$

Define

$$\begin{aligned} &\widetilde{L}_a\widetilde{w} = L_a w\big|_{\Omega\setminus E}, \\ &w = \begin{cases} 0, & x \in E, \\ \widetilde{w}, & x \notin E, \end{cases} \quad \widetilde{w} \in L^2(\Omega\setminus E). \end{aligned} \tag{2.14}$$

In some sense $\widetilde{L}_a$ is the restriction of L_a on $L^2(\Omega\setminus E)$. Note that $\widetilde{L}_a$ is also self-adjoint. Below we show $\widetilde{L}_a$ is one-to-one and the inverse of it is bounded.

(1) one-to-one.

Assume $\widetilde{w} \in L^2(\Omega \setminus E)$ and $\widetilde{L}_a \widetilde{w} = 0$. Define

$$w = \begin{cases} 0, & x \in E, \\ \widetilde{w}, & x \notin E \end{cases} \qquad \text{as in (2.14).}$$

Then we have

$$\langle L_a w, w \rangle_{L^2(\Omega)} = \langle \widetilde{L}_a \widetilde{w}, \widetilde{w} \rangle_{L^2(\Omega \setminus E)} = 0. \tag{2.15}$$

Therefore

$$w \equiv 0,$$

which implies $\widetilde{w} \equiv 0$. So $\widetilde{L}_a$ is one-to-one.

(2) The inverse is bounded.

Using (2.14)–(2.15), we get

$$\begin{aligned} \langle \widetilde{L}_a \widetilde{w}, \widetilde{w} \rangle &= \langle L_a w, w \rangle \\ &\geq a \|w\|_{L^2}^2 \\ &\geq a \|\widetilde{w}\|_{L^2}. \end{aligned}$$

So

$$\|\widetilde{w}\|^2 \leq a^{-1} \|\widetilde{w}\|_{L^2} \|\widetilde{L}_a \widetilde{w}\|_{L^2},$$

i.e.,

$$\|\widetilde{w}\|_{L^2} \leq a^{-1} \|\widetilde{L}_a \widetilde{w}\|_{L^2}.$$

Hence the inverse of $\widetilde{L}_a$ is bounded.

Next we show $\widetilde{L}_a$ is onto.

Assume the contrary. Then $\exists\, \widetilde{w} \notin \widetilde{L}_a(L^2(\Omega \backslash E))$. Since the inverse of $\widetilde{L}_a$ is bounded, the space $\widetilde{L}_a(L^2(\Omega \setminus E))$ is closed. Denote $\widetilde{w}_1$ the projection of $\widetilde{w}$ on $\widetilde{L}_a(L^2(\Omega \setminus E))$. Then

$$\langle \widetilde{w} - \widetilde{w}_1, \widetilde{L}_a(L^2(\Omega \setminus E)) \rangle = 0.$$

So $\forall\, \widetilde{u} \in L^2(\Omega \setminus E)$,

$$\langle \widetilde{L}_a(\widetilde{w} - \widetilde{w}_1), \widetilde{u} \rangle = 0.$$

This means $\widetilde{L}_a(\widetilde{w} - \widetilde{w}_1) = 0$. So $\widetilde{w} = \widetilde{w}_1$. A contradiction. Therefore $\widetilde{L}_a$ is onto.

After proving $\widetilde{L}_a$ is onto, the proof is essential finished. Let $\widetilde{w} = \widetilde{L}_a^{-1} 1$ and

$$w = \begin{cases} 0, & x \in E, \\ \widetilde{w}, & x \notin E. \end{cases}$$

Then w is a solution to (2.10). The claim is proven.

Remark 2.2 The claim above might help a little bit in the study of singular solutions of Model 1.

There is a small generalization of Model 1.

Model 1′

$$w_t = Z_{12}w\ w, \quad x \in \Omega \subset \mathbb{R}^2. \tag{2.16}$$

Model 1′ seems a little bit harder than Model 1. There are some related evidences. For instance, if we assume $\varphi \in L^2(R^2)$, then

$$\int_{R^2} Z_{12}\varphi\varphi \mathrm{d}x = \int_{R^2} \frac{k_1 k_2}{k_1^2 + k_2^2} |\widetilde{\varphi}|^2 \mathrm{d}k \text{ will change sign,}$$

where $\widetilde{\varphi}$ is the Fourier transform of φ. Also for the simple singular solution in the ellipse, we need b to be non-zero. More precisely, we have the following theorem.

Theorem 2.2 *Assume* $\Omega = \{ax_1^2 + bx_1x_2 + cx_2^2 < 1\},\ a, c > 0,\ b^2 - 4ac < 0,\ b \neq 0.$ *Then*

$$w(t) = \frac{1}{T-t} \cdot \frac{2a+2c}{b}$$

is a self-similar singular solution to (2.16).

Proof The proof is essentially the same as that of Theorem 2.1.

For zero order models, bounded domain case might be simpler than the whole space case since the former is compact region.

3 Further Models

The vorticity formulation of the three-dimensional Navier-Stokes equations is the following:

$$w_t - \Delta w + u \cdot \nabla w - \nabla u w = 0. \tag{3.1}$$

By simplifying the zero order term, removing first order or second order term, we could get various model equations of the Navier-Stokes equations. The usual way of simplifying zero order term is to replace ∇u with simpler zero order operater.

Examples

$$w_t = \begin{pmatrix} w_1 + Z_{11}w_1 & \frac{1}{2}w_1 \\ \frac{1}{2}w_1 & w_1 + Z_{11}w_1 \end{pmatrix} w, \quad x \in \Omega \subset \mathbb{R}^2,\ w \in \mathbb{R}^2, \tag{3.2}$$

$$w_t = \nabla u\ w, \quad x \in \Omega \subset \mathbb{R}^3,\ w \in \mathbb{R}^3, \tag{3.3}$$

$$w_t + u \cdot \nabla w - Z_{11}w\ w = 0, \quad x \in \mathbb{R}^2, \tag{3.4}$$

$$w_t - \Delta w - Z_{11}w\ w = 0, \quad x \in \mathbb{R}^2, \tag{3.5}$$

$$w_t - \Delta w + u \cdot \nabla w - Z_{11}w\ w = 0, \quad x \in \mathbb{R}^2. \tag{3.6}$$

In 2D, $u = (-\partial_{x_2}\Delta^{-1}w, \partial_{x_1}\Delta^{-1}w)$.

Roughly speaking, (3.2) is a simple situation when the model is a system (The matrix in (3.2) is symmetric, and the equation also has simple singular solutions similar with Theorem 2.1.). (3.3) models the pro-ingularity effect of zero order term in the vorticity formulation. (3.4)–(3.6) model the effects of first order perturbation, second order perturbation, first and second order perturbation combined for scalar equations.

Remark 3.1 Different from zero order model, it seems that for the first and second order models, the whole space case is inclined to be first considered. One reason is that self-similar singular solutions for PDEs only occur in whole space.

Remark 3.2 It is known that in the whole space situation, the vorticity formulation with divergence initial vorticity is equivalent to the original Euler/Navier-Stokes equations (see [11, p.78, Proposition 2.21]). In general, the requirement of divergence-free initial data will make the situation harder. For instance, most probably the self-similar singular solutions would not exist if we add the divergence-free initial data requirement.

If the dimension is higher than three, it is convenient to think the velocity as 1-form and vorticity as 2-form. In this case, the divergence-free requirement becomes $\mathrm{d}w_0 = 0$, where d is the exterior differential and w_0 is initial vorticity. We refer to [15] for more details in the case of $\mathbb{R}^n,\ n > 3$.

Skew-symmetry of zero order term In the roughest sense, one may think of the zero order term $\nabla u w$ as $w^2,\ w \in \mathbb{R}$. Therefore one might expect that it has some pro-singularity effect.

Proposition 3.1 *Singular solutions generated from constant do not hold true for* (3.3).

Proof Define the generalized Kronecker sign:

$$\delta^i_{jl} = \begin{cases} 1, & (i,j,l) \text{ is an even arrangement of } (1,2,3), \\ -1, & (i,j,l) \text{ is an odd arrangement of } (1,2,3), \\ 0, & \text{otherwise.} \end{cases}$$

So

$$\begin{aligned} (\nabla u)_{mi} &= \partial_m u_i \\ &= \partial_m \delta^i_{jl} \partial_j \Delta^{-1} w_l \\ &= \delta^i_{jl} Z_{jm} w_l. \end{aligned}$$

And

$$(\nabla u\ w)_m = \delta^i_{jl} Z_{jm} w_l w_i. \tag{3.7}$$

Given any constant vector $c \in \mathbb{R}^3$, $Z_{jm}\ c = a_{jm}c$. Here the domain is $\{a_{ij}x_ix_j < 1\}$, $a_{ij} = a_{ji}$, $\sum\limits_{i=1}^{3} a_{ii} = 1$ and (a_{ij}) is positive definite. Therefore

$$\begin{aligned} &(\nabla uw)_m \mid_{w=c} \\ &= \delta^i_{jl} Z_{jm} c_l c_i \\ &= \delta^i_{jl} a_{jm} c_l c_i \\ &= 0. \end{aligned} \tag{3.8}$$

The proposition is proven.

The Proposition 3.1 above suggests that the zero order term has certain algebraic skew-symmetry, which may cause some more trouble in the study of singular solutions.

Possible steps toward Euler equations In the luckiest scenario, the study of model equations might lead to the existence of singular solutions of the Euler equations and even Navier-Stokes equations. The following are possible steps toward Euler equations:

(1) Model 1,

(2) (3.3),

(3) (3.4),

(4) the whole Euler equations.

Remark 3.3 It was suggested in [6, p.3] that the degree of difficulty for singular solutions to Navier-Stokes equations may decrease a lot in higher dimensions. Probably this scenario will also hold true for certain second order model.

Remark 3.4 For zero order models, if there is no divergence-free requirement on the initial data, the self-similar singular solutions probably exist. But for more complicated situations, one might have to work on singular solutions with general form. One evidence is that the Navier-Stokes equations do not have self-similar singular solutions at any dimensions (see [12, 16]). There were also no reliable numerical evidence that Euler equations have self-similar singular solutions.

4 Possible Connection with Turbulence

It is well accepted that the main features of turbulence is irregular, random and chaotic. Based on what is known on Navier-Stokes equations and the features of turbulence, it seems reasonable to make the following guess.

Conjecture 4.1 The singular solutions of three-dimensional Navier-Stokes equations generically are fluctuated.

Using the conjecture above, we could interpret the turbulence in the following way. Since the solutions are fluctuatedly singular, the average of them are irregular. The randomness comes from the infinite amplifying effect of fluctuatedly singular solutions over arbitrarily small experimental error. The chaotic behavior could be explained in the similar way.

Remark 4.1 The results on model equations (see [7, 13]) and numerical simulation for Euler equations suggest that, in dimension five and higher the usual singular solutions possibly are also typical for Navier-Stokes equations. There is no information in dimension four so far.

The model system constructed by Plecháč and Šverák in [13] has the form:

$$u_t - \triangle u + 2au\nabla u + (1-a)\nabla|u|^2 + (\text{div } u)u = 0, \tag{4.1}$$

where $a \in [0,1]$. The system (4.1) has the similar mathematical structures with the Navier-Stokes equations. It is possible that Conjecture 4.1 also holds true for (4.1). The phenomenon controlled by the system (4.1) might be called Plecháč-Šverák turbulence. The advantage of this turbulence is that it would be a lot easier since the governing system is local.

At this stage little is known regarding the singular solutions of the Navier-Stokes equations. Therefore the application in the turbulence theory is not much. With the development of the mathematical theory on the singular solutions, more and more applications could be expected. To some degree, the good understanding of turbulence may depend on the good understanding of singular solutions to the three-dimensional Navier-Stokes equations.

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