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Translating Surfaces of the Non-parametric Mean Curvature Flow in Lorentz Manifold $M^2 \times \mathbb{R}$ *

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Abstract In this paper, for the Lorentz manifold $M^2 \times \mathbb{R}$ with M^2 a 2-dimensional complete surface with nonnegative Gaussian curvature, the authors investigate its space-like graphs over compact, strictly convex domains in M^2, which are evolving by the non-parametric mean curvature flow with prescribed contact angle boundary condition, and show that solutions converge to ones moving only by translation.

Keywords Translating surfaces, Mean curvature flow, Lorentz manifolds.
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1 Introduction

In Riemannian (or pseudo-Riemannian) geometry, the mean curvature flow (MCF for short) evolves a family of immersed submanifolds along their mean curvature vectors $\vec{H}$ with a speed $|\vec{H}|$. More precisely, let $X_0 : M^n \to N^{m+n}$ be an isometric immersion from an n-dimensional oriented Riemannian manifold M to an $(n+m)$-dimensional Riemannian (or pseudo-Riemannian) manifold N^{n+m} (or with a pseudo-Riemannian metric whose signature is $(p, n+m-p)$, $n \le p \le n+m-1$). The MCF corresponds to a one-parameter family $X(\cdot,t) = X_t$ of immersions $X_t : M^n \to N^{n+m}$ whose images $M_t^n = X_t(M^n)$ satisfy

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(x,t) = \vec{H}, & \text{on } M^n \times [0,\mathbb{T}), \\ X(x,0) = X_0(x), & \text{on } M^n \end{cases} \tag{1.1}$$

for some $\mathbb{T} > 0$. The MCF attracts a lot of attention since Huisken's significant work (see [8]), where, by using the method of L^p estimates, he proved that if M^n is a compact, strictly convex hypersurface in the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$, the MCF (1.1) has a unique smooth solution on the finite time interval $[0, \mathbb{T}_{\max})$ with $\mathbb{T}_{\max} < \infty$, and the evolving hypersurfaces M_t^n contract to a single point as $t \to \mathbb{T}_{\max}$. Moreover, after an area-preserving rescaling, the rescaled hypersurfaces converge in C^∞-topology to a round sphere having the same area as M^n. For the MCF (1.1), if

$$\vec{H} = -X^{\perp},$$

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then the submanifold $X_t : M^n \to \mathbb{R}^{n+m}$ is called a self-shrinker, which is a self-similar solution to (1.1). Here $(\cdot)^\perp$ denotes the normal projection of a prescribed vector to the normal bundle of M_t^n in $\mathbb{R}^{m+n}$. Self-shrinking solutions are important in the study of type-I singularities of the MCF. For instance, by proving the monotonicity formula, at a given type-I singularity of the MCF, Huisken [10] proved that the flow is asymptotically self-similar, which implies that in this situation the flow can be modeled by self-shrinking solutions. If there exists a constant unit vector V such that

$$\vec{H} = V^\perp,$$

then the submanifold $X_t : M^n \to \mathbb{R}^{n+m}$ is called a translating soliton of the MCF (1.1). It is easy to see that the translating soliton gives an eternal solution $X_t = X_0 + tV$ to (1.1), which is called the translating solution. Translating solitons play an important role in the study of type-II singularities of the MCF. For instance, Angenent and Velázquez [2–3] gave some examples of convergence which implies that type-II singularities of the MCF are modeled by translating surfaces.

From the above brief introduction, we know that translating solutions of the MCF are special solutions to the flow equation and are worthy of being investigated for understanding type-II singularities of the MCF. There exist some interesting results which we prefer to mention here. For instance, Shahriyari [12] proved that for the MCF, there are only three types of complete translating graphs in $\mathbb{R}^3$, i.e., entire graphs, graphs between two parallel planes and graphs in one side of a plane. Moreover, in the last two types, graphs are asymptotic to planes next to their boundaries. Xin [14] proved that any complete translating soliton in $\mathbb{R}^{n+m}$ has infinite volume and has Euclidean volume growth at least. Moreover, he showed that graphic translating soliton hypersurfaces are weighted area-minimizing. Huisken [9] investigated graphs over bounded domains (with $C^{2,\alpha}$ boundary) in $\mathbb{R}^n$ ($n \geq 2$), which are evolving by the MCF with vertical contact angle boundary condition, and proved that the evolution exists for all the time and the evolving graphs converge to a constant function as time tends to infinity (i.e., $t \to \infty$). Altschuler and Wu [1] proved that graphs, defined over compact, strictly convex domains in $\mathbb{R}^2$, evolved by the non-parametric MCF with prescribed contact angle (not necessary to be vertical), converge to translating surfaces as $t \to \infty$. Guan [7] investigated graphs over bounded domains in $\mathbb{R}^n$, which are evolving by the non-parametric MCF with prescribed contact angle, and proved that the flow exists for all the time. But an extra assumption about the prescribed contact angle should be added in order to get the asymptotical behavior of limiting solutions. Zhou [15] improved Altschuler-Wu's and Guan's conclusions to the case of general product spaces $M^n \times \mathbb{R}$ with closed manifold M^n having nonnegative Ricci curvature.

The purpose of this paper is to investigate the case of space-like graphs evolved by the non-parametric MCF with the prescribed contact angle boundary condition, and try to get interesting convergence conclusions.

Throughout this paper, M^2 denotes a 2-dimensional complete Riemannian manifold with a metric σ and Ω is a compact, strictly convex domain of M^2 with smooth boundary $\partial\Omega$. Let $\kappa > 0$ be the curvature function of $\partial\Omega$. Assume that a point on Ω is described by local coordinates $\{\omega^1, \omega^2\}$. Let ∂_i, $i = 1, 2$, be the corresponding coordinate vector fields and $\sigma_{ij} = \sigma(\partial_i, \partial_j)$, $i, j = 1, 2$. Similar to the basic introduction of geometry of graphs shown in [4–5], we know that for the space-like graph $\mathscr{G} := \{(x, u(x, \cdot)) \mid x \in \Omega\}$ defined over $\Omega \subset M^2$, in the Lorentz manifold $M^2 \times \mathbb{R}$ with the metric $\overline{g} := \sigma_{ij} \mathrm{d}w^i \otimes \mathrm{d}w^j - \mathrm{d}s \otimes \mathrm{d}s$, the tangent vectors are given by

$$\vec{e}_i = \partial_i + D_i u \partial_s, \quad i = 1, 2,$$

and the corresponding upward unit normal vector is given by

$$\vec{\gamma} = \frac{1}{\sqrt{1-|Du|^2}}(\partial_s + D^j u \partial_j),$$

where $D^j u = \sigma^{ij} D_i u$ with D the covariant derivative operator on M^2. Denote by ∇ the gradient operator on $M^2 \times \mathbb{R}$, and then the second fundamental form $h_{ij} dw^i \otimes dw^j$ of $\mathscr{G}$ is given by

$$h_{ij} = -\langle \nabla_{\vec{e}_i} \vec{e}_j, \vec{\gamma} \rangle = \frac{1}{\sqrt{1-|Du|^2}} D_i D_j u.$$

Moreover, the scalar mean curvature of $\mathscr{G}$ is

$$H = \sum_{i=1}^{2} h_i^i = \frac{1}{\sqrt{1-|Du|^2}} \Big(\sum_{i,k=1}^{2} g^{ik} D_k D_i u \Big) = \frac{\sum_{i,k=1}^{2} \big(\sigma^{ik} + \frac{D^i u D^k u}{1-|Du|^2}\big) D_k D_i u}{\sqrt{1-|Du|^2}}, \tag{1.2}$$

where g^{ik} is the inverse of the induced Riemannian metric on the space-like graph $\mathscr{G}$.

Let T be the counterclockwise unit smooth tangent vector of $\partial\Omega$ and N be the inward unit normal vector of $\partial\Omega$. Then one can smoothly extend N, T to a thin neighborhood of the boundary $\partial\Omega$ (see Subsection 2.1 for details).

In order to bring convenience to calculations in the sequel and state our main conclusion clearly, we use the following notations:

$$\begin{aligned} v &= \sqrt{1-|Du|^2}, \\ g_{ij} &= \sigma_{ij} - D_i u D_j u, \\ g^{ij} &= \sigma^{ij} + \frac{D^i u D^j u}{1-|Du|^2}, \\ u_t &= \frac{\partial u}{\partial t}. \end{aligned}$$

For vectors V, W or matrices A, B, we will use the shorthand as follows

$$\langle V, W \rangle_g = g^{ij} V_i W_j, \quad \langle V, W \rangle_\sigma = \sigma^{ij} V_i W_j, \quad \langle A, B \rangle_{g,\sigma} = g^{ij} \sigma^{kl} A_{ik} B_{jl}.$$

Define $g^{TN} := g^{ij} T_i N_j$, $g^{TT} := g^{ij} T_i T_j$ and $g^{NN} := g^{ij} N_i N_j$ on $\partial\Omega$. For the second-order covariant derivatives of a prescribed function, we have the formula

$$D_V D_W u = V^i W^j D_{ij}^2 u + \langle D_V W, Du \rangle.$$

For the space-like graphs $\mathscr{G}$, we consider the following initial-boundary value problem (IBVP for short)

$$(\sharp) \quad \begin{cases} u_t = \Big(\sigma^{ij} + \dfrac{D^i u D^j u}{1-|Du|^2}\Big) D_i D_j u, & \text{on } \Omega \times [0, \mathbb{T}], \\ D_N u = \phi(x) v, & \text{on } \partial\Omega \times [0, \mathbb{T}], \\ u(\cdot, 0) = u_0(\cdot), & \text{on } \Omega_0, \end{cases}$$

where $\Omega_t = \Omega \times \{t\}$ is a slice in $\Omega \times [0, \mathbb{T}]$, $\phi \in C^\infty(\partial\Omega)$ and $u_0 \in C^\infty(\overline{\Omega})$. Of course, on $\partial\Omega$,

$$D_N u_0 = \phi(x) \sqrt{1-|Du_0|^2}$$

holds, which is actually called the compatibility condition of the IBVP ($\sharp$). Clearly, the IBVP ($\sharp$) describes the evolution of space-like graphs $\mathscr{G}$ by the mean curvature vector with the specified contact angle, since by (1.2), the right-hand side of the first evolution equation in ($\sharp$) equals Hv. For the IBVP ($\sharp$), we can prove the following theorem.

Theorem 1.1 *If Ω is a compact, strictly convex domain in M^2 with nonnegative Gaussian curvature, then, for solutions to the* IBVP ($\sharp$)*, we have the followings*:

1. *There exists a constant $c_1 := c_1(u_0, \kappa_0, \phi_0, \phi_1, \phi_2) > 0$ such that $|Du|^2 \leq c_1 < 1$ on $\overline{\Omega} \times [0, \infty)$, thus $u(x,t) \in C^\infty(\overline{\Omega} \times [0,\infty))$, where*

$$\kappa_0 := \min_{x\in\partial\Omega} \kappa(x), \quad \phi_0 := \min_{x\in\partial\Omega} \phi(x), \quad \phi_1 := \max_{x\in\partial\Omega} \phi(x), \quad \phi_2 := \max_{x\in\partial\Omega} |D_T\phi(x)|;$$

2. *$u(x,t)$ converges as $t \to \infty$ to a space-like surface u_∞ (unique up to translation), which moves at a constant speed c_3 given by* (2.15);

3. *if $\int_{\partial\Omega} \phi = 0$ then $c_3 = 0$, hence u_∞ is a maximal space-like surface in the Lorentz manifold $M^2 \times \mathbb{R}$.*

Remark 1.1 Clearly, if $M^2 \equiv \mathbb{R}^2$, Theorem 1.1 gives the existence of translating solutions to the space-like non-parametric MCF with the prescribed contact angle boundary condition in the Minkowski 3-space $\mathbb{R}^{2,1}$.

The paper is organized as follows. The uniform estimates for the time derivative and the gradient of the solution to the IBVP ($\sharp$) are in Section 2, which can be used to get the solvability of the BVP ($*$), the elliptic version of ($\sharp$), and the long-time existence of the IBVP ($\sharp$). The existence of translating solutions to ($\sharp$) is shown in Section 3.

2 Estimates

2.1 The boundary

Let $\{\theta, r\}$, with $r(x)$ the Riemannian distance function $d(x, \partial\Omega)$ from x to the boundary $\partial\Omega$, be the local coordinates for a thin neighborhood of $\partial\Omega$ such that

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\rangle = 0, \quad \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1$$

and

$$\left\{ \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\}_{\partial\Omega} = \{N, T\}.$$

Define a function φ such that $\left|\varphi^{-1}\frac{\partial}{\partial\theta}\right|^2 = 1$. Then one can get the extended normal and tangent vectors to be the orthonormal frame $\left\{\frac{\partial}{\partial r}, \varphi^{-1}\frac{\partial}{\partial\theta}\right\}$ of the thin neighborhood of Ω, which is also denoted by $\{N, T\}$, that is, $\left\{\frac{\partial}{\partial r}, \varphi^{-1}\frac{\partial}{\partial\theta}\right\} = \{N, T\}$. By [1, Lemma 2.1], we have the following lemma.

Lemma 2.1 *On $\partial\Omega$, one has*
(i) $\nabla_T T = kN,\ \nabla_T N = -kT,\ \nabla_N T = \nabla_N N = 0$;
(ii) *For any $f \in C^\infty(\overline{\Omega})$, $D_N D_T f = D_T D_N f + k D_T f$.*

From the boundary condition of ($\sharp$), it is not hard to verify the following facts

$$|D_N u|^2 = \phi^2 v^2, \tag{2.1}$$

$$|D_T u|^2 = 1 - (1 + \phi^2)v^2. \tag{2.2}$$

Differentiating conditions (2.1)–(2.2) in the time and tangential direction respectively, we know that all the derivatives of u on $\partial\Omega$, except $D_N D_N u$, can be given in terms of the first derivatives of u. More precisely, we have

$$D_N u_t = -\phi \frac{DuDu_t}{\sqrt{1 - |Du|^2}}, \tag{2.3}$$

$$D_T D_N u = \phi D_T v + v D_T \phi, \tag{2.4}$$

$$D_N D_T u = \phi D_T v + v D_T \phi + k D_T u, \tag{2.5}$$

$$D_T D_T u = -\frac{v(1 + \phi^2) D_T v + v^2 \phi D_T \phi}{D_T u}. \tag{2.6}$$

Besides, elements of the inverse of the metric matrix of the space-like graphs in $M^2 \times \mathbb{R}$ are given by

$$g^{TT} = \frac{1 - (D_N u)^2}{v^2}, \tag{2.7}$$

$$g^{NN} = 1 + \phi^2, \tag{2.8}$$

$$g^{NT} = g^{TN} = \frac{D_N u D_T u}{v^2}. \tag{2.9}$$

2.2 Existence of solutions

The key point of the existence for small time and the uniqueness of solutions to the IBVP ($\sharp$) is to show that the evolution equation in ($\sharp$) is uniformly parabolic at $t = 0$, which can be assured by the assumption that the initial graphic surface over Ω is space-like. In fact, by the linearization theory (see [11]) and the inverse function theorem (see [13]), together with the space-like graphic assumption, the short-time existence and the uniqueness of solutions to the IBVP ($\sharp$) can be obtained.

Assume that the IBVP ($\sharp$) has smooth solutions on the time interval $[0, \mathbb{T}]$, which means that all derivatives of u have bounds on $[0, \mathbb{T}]$. In the following, we first establish a time independent priori estimate for the gradient of the solution (see Theorem 2.1), which leads to the space-like preserving property for the evolving graphic surfaces in $M^2 \times \mathbb{R}$, and then turn the quasilinear evolution equation into a uniformly parabolic equation. Furthermore, by the standard theory of the second-order parabolic PDEs, the higher order regularity follows, which leads to the long-time existence of smooth solutions of the IBVP ($\sharp$)—see the end of Subsection 2.4 for a brief explanation.

2.3 The time derivative estimate

By the maximum principe of the second-order parabolic PDEs, we have the following lemma.

Lemma 2.2 $\sup\limits_{\overline{\Omega} \times [0, \mathbb{T}]} |u_t|^2 = \sup\limits_{\Omega_0} |u_t|^2$, *that is, there exists a positive constant* $c_0 = c_0(u_0) \in \mathbb{R}^+$ *such that for any* $(x, t) \in \overline{\Omega} \times [0, \mathbb{T}]$, *we have*

$$|u_t|^2(x, t) \leq c_0.$$

Proof We first show that the maximum of u_t must occur on $(\partial\Omega \times [0, \mathbb{T}]) \cup \Omega_0$. Let $(g^{ij})'$ be the differential of $g^{ij} = g^{ij}(x, u, Du) = g^{ij}(x, z, p)$ with respect to p. For simplicity, denote by $D_iu := u_i$, $D_ju := u_j$, $D_iD_ju := u_{ij}$ for $i, j = 1, 2$. By a direct computation, we have

$$\begin{aligned}
\frac{\partial}{\partial t}|u_t|^2 &= 2u_t\frac{\partial u_t}{\partial t} \\
&= 2u_t\Big(\frac{\partial g^{ij}}{\partial t}u_{ij} + g^{ij}\frac{\partial u_{ij}}{\partial t}\Big) \\
&= 2u_t\Big(\frac{\partial g^{ij}}{\partial u^k}\frac{\partial u^k}{\partial t}u_{ij} + g^{ij}D_iD_ju_t\Big) \\
&= 2u_t\frac{\partial g^{ij}}{\partial u^k}\frac{\partial u^k}{\partial t}u_{ij} + g^{ij}(D_iD_j|u_t|^2 - 2D_iu_tD_ju_t) \\
&= 2u_t\frac{\partial g^{ij}}{\partial u^k}\frac{\partial u^k}{\partial t}u_{ij} + g^{ij}D_iD_j|u_t|^2 - 2\langle Du_t, Du_t\rangle_g \\
&= 2u_t\frac{\partial g^{ij}}{\partial u^k}\frac{\partial(\sigma^{ki}u_i)}{\partial t}u_{ij} + g^{ij}D_iD_j|u_t|^2 - 2\langle Du_t, Du_t\rangle_g \\
&= \frac{\partial g^{ij}}{\partial u^k}\sigma^{ki}u_{ij}D_i|u_t|^2 + g^{ij}D_iD_j|u_t|^2 - 2\langle Du_t, Du_t\rangle_g \\
&= D_iD_ju\langle(g^{ij})', \nabla|u_t|^2\rangle_\sigma + g^{ij}D_iD_j|u_t|^2 - 2\langle Du_t, Du_t\rangle_g,
\end{aligned}$$

where $D_iD_ju\langle(g^{ij})', \nabla|u_t|^2\rangle_\sigma = \sum_{i,j,k=1}^{2} D_iD_ju\big(\frac{\partial g^{ij}}{\partial p^k}\nabla_k|u_t|^2\big)$. The boundedness of all the coefficients of the above evolution equation in the bounded domain $\overline{\Omega}\times[0,\mathbb{T}]$ follows by the continuity, which implies that

$$\sup_{\overline{\Omega}\times[0,\mathbb{T}]}|u_t|^2 = \sup_{(\partial\Omega\times[0,\mathbb{T}])\cup\Omega_0}|u_t|^2$$

by directly applying the weak maximum principle.

Next, we exclude the possibility that the maximum occurs at $(\xi,\tau)\in\partial\Omega\times[0,\mathbb{T}]$. Assume that $\max_{\Omega\times\{t\}}|u_t|^2 = |u_t|^2(\xi,\tau) > 0$ for some $(\xi,\tau)\in\partial\Omega\times[0,\mathbb{T}]$, we have $(D_Tu_t)(\xi,\tau) = 0$. By (2.3), it follows that

$$\begin{aligned}
(D_Nu_t)(\xi,\tau) &= -\phi\frac{D_NuD_Nu_t + D_TuD_Tu_t}{\sqrt{1-|Du|^2}}(\xi,\tau) \\
&= -\phi\frac{D_NuD_Nu_t}{\sqrt{1-|Du|^2}}(\xi,\tau) \\
&= -\phi^2(D_Nu_t)(\xi,\tau),
\end{aligned}$$

which implies $(1+\phi^2)(D_Nu_t)(\xi,\tau) = 0$, i.e., $(D_Nu_t)(\xi,\tau) = 0$. By Hopf lemma, it follows that $D_N|u_t|^2(\xi,\tau) < 0$. On the other hand, by the boundary condition, one has

$$\begin{aligned}
D_N|u_t|^2|_{(\xi,\tau)} = 2u_tD_Nu_t|_{(\xi,\tau)} &= 2u_t\frac{\partial}{\partial t}(\phi(x)\sqrt{1-|Du|^2})\Big|_{(\xi,\tau)} \\
&= -2u_t\frac{\phi(x)}{\sqrt{1-|Du|^2}}DuD_Nu_t\Big|_{(\xi,\tau)} \\
&= 0.
\end{aligned}$$

This is a contradiction. Hence, the maximum cannot be obtained at $(\xi,\tau)\in\partial\Omega\times[0,\mathbb{T}]$. The conclusion of Lemma 2.2 follows by summing up the above argument.

2.4 The gradient estimate

First, we need the evolution equation of $|Du|^2$.

Lemma 2.3 *We have the evolution equation of* $|Du|^2$ *as follows*

$$\frac{\partial}{\partial t}|Du|^2 = \frac{D_k|Du|^2 D_i|Du|^2}{v^2}g^{ik} + g^{ij}D_iD_j|Du|^2 - |D^2u|^2 - |D|Du|^2|^2 - K|Du|^2,$$

where K *denotes the Gaussian curvature of* M^2.

Proof First, by direct computations, we have

$$\begin{aligned}\frac{\partial}{\partial t}|Du|^2 &= \frac{\partial}{\partial t}(u^m u_m) = 2u^m(u_m)_t = 2u^m(u_t)_m,\\ (u_t)_m &= (g^{ij}u_{ij})_m = (g^{ij})_m u_{ij} + g^{ij}(u_{ij})_m,\\ (g^{ij})_m &= \Big(\sigma^{ij} + \frac{u^iu^j}{1-|Du|^2}\Big)_m\\ &= \frac{(u^i)_m u^j}{1-|Du|^2} + \frac{u^i(u^j)_m}{1-|Du|^2} + \frac{2u^k(u_k)_m u^iu^j}{(1-|Du|^2)^2}\end{aligned}$$

and

$$(|Du|^2)_{ij} = 2\sigma^{mk}u_{kj}u_{mi} + 2\sigma^{mk}u_ku_{mij}.$$

Therefore, the evolution equation for the gradient is given as follows

$$\begin{aligned}\frac{\partial}{\partial t}|Du|^2 &= 2u^m\Big[\frac{(u^i)_m u^j}{1-|Du|^2} + \frac{u^i(u^j)_m}{1-|Du|^2} + \frac{2u^k(u_k)_m u^iu^j}{(1-|Du|^2)^2}\Big]u_{ij} + 2g^{ij}u^m(u_{ij})_m\\ &= 2u^m\Big[\frac{(\sigma^{ik}u_k)_m u^j}{v^2} + \frac{u^i(\sigma^{jk}u_k)_m}{v^2} + \frac{2u^k(u_k)_m u^iu^j}{v^4}\Big]u_{ij} + 2g^{ij}u^m(u_{ij})_m\\ &= 2u^m\Big(\frac{\sigma^{ik}u_{km}u^j}{v^2} + \frac{u^i\sigma^{jk}u_{km}}{v^2} + \frac{2u^ku_{km}u^iu^j}{v^4}\Big)u_{ij} + 2g^{ij}u^m(u_{ij})_m\\ &= 2u^m\frac{u_{km}}{v^2}\Big(\sigma^{ik}u^j + \sigma^{jk}u^i + \frac{2u^ku^iu^j}{v^2}\Big)u_{ij}\\ &\quad + g^{ij}(D_iD_j|Du|^2 - 2\sigma^{mk}u_{kj}u_{mi} - 2u^mu_lR^l_{imj})\\ &= \frac{2u^mu_{km}\sigma^{ik}u^ju_{ij}}{v^2} + \frac{2u^mu_{km}\sigma^{ik}u^ju_{ij}}{v^2} + \frac{4u^mu^ku_{km}u^iu^ju_{ij}}{v^4}\\ &\quad + g^{ij}D_iD_j|Du|^2 - 2g^{ij}\sigma^{mk}u_{kj}u_{mi} - 2g^{ij}u^mu^lR_{limj}\\ &= \frac{4u^mu_{km}u^ju_{ij}}{v^2}\Big(\sigma^{ik} + \frac{u^ku^i}{v^2}\Big) + g^{ij}D_iD_j|Du|^2 - 2g^{ij}\sigma^{mk}u_{kj}u_{mi} - 2g^{ij}u^mu^lR_{limj}\\ &= \frac{2u^mu_{km}2u^ju_{ij}}{v^2}g^{ik} + g^{ij}D_iD_j|Du|^2 - 2g^{ij}\sigma^{mk}u_{kj}u_{mi} - 2g^{ij}u^mu^lR_{limj}\\ &= \frac{D_k|Du|^2D_i|Du|^2}{v^2}g^{ik} + g^{ij}D_iD_j|Du|^2 - 2g^{ij}\sigma^{mk}u_{kj}u_{mi} - 2g^{ij}u^mu^lR_{limj}\\ &= \frac{D_k|Du|^2D_i|Du|^2}{v^2}g^{ik} + g^{ij}D_iD_j|Du|^2 - 2\Big(\sigma^{ij} + \frac{u^iu^j}{v^2}\Big)\sigma^{mk}u_{kj}u_{mi}\\ &\quad - 2\Big(\sigma^{ij} + \frac{u^iu^j}{v^2}\Big)u^mu^lR_{limj}\\ &= \frac{D_k|Du|^2D_i|Du|^2}{v^2}g^{ik} + g^{ij}D_iD_j|Du|^2 - |D^2u|^2 - 2\frac{u^iu_{im}u^ju_{jk}\sigma^{mk}}{v^2}\end{aligned}$$

$$
\begin{aligned}
&- 2\sigma^{ij}u^m u^l R_{limj}\\
&= \frac{D_k|Du|^2 D_i|Du|^2}{v^2} g^{ik} + g^{ij}D_iD_j|Du|^2 - |D^2u|^2 - |D|Du|^2|^2 - 2\sigma^{ij}u^m u^l R_{limj}\\
&= \frac{D_k|Du|^2 D_i|Du|^2}{v^2} g^{ik} + g^{ij}D_iD_j|Du|^2 - |D^2u|^2 - |D|Du|^2|^2 - K|Du|^2,
\end{aligned}
$$

where R_{limj}, $1 \le l, i, m, j \le 2$ are the components of the curvature tensor on M^2.

Then, we begin to estimate the gradient of u as follows.

Theorem 2.1 *Under the assumptions of Theorem* 1.1, *there exists a positive constant* $c_1 = c_1(u_0, \kappa_0, \phi_0, \phi_1, \phi_2)$ *such that*

$$\sup_{\overline{\Omega}\times[0,\mathbb{T}]} |Du|^2 \le c_1 < 1.$$

Proof We first show that the maximum of $|Du|^2$ must occur on $(\partial\Omega \times [0, \mathbb{T}]) \cup \Omega_0$. Since M^2 has nonnegative Gaussian curvature, by Lemma 2.3, we can get the following estimate

$$\frac{\partial}{\partial t}|Du|^2 \le \frac{D_k|Du|^2 D_i|Du|^2}{v^2} g^{ik} + g^{ij}D_iD_j|Du|^2.$$

By applying the weak maximum principle to the above evolution inequality, we have

$$\sup_{\overline{\Omega}\times[0,\mathbb{T}]} |Du|^2 = \sup_{(\partial\Omega\times[0,\mathbb{T}])\cup\Omega_0} |Du|^2.$$

If the maximum of $|Du|^2$ occurs at Ω_0, then $\sup\limits_{\overline{\Omega}\times[0,\mathbb{T}]} |Du|^2 \le \sup\limits_{\Omega_0} |Du|^2 < 1$, where the last inequality holds since $\{(x, u(x, 0)) \,|\, x \in M^2\}$ is a space-like graph in $M^2 \times \mathbb{R}$. Now, we assume that the maximum of $|Du|^2$ occurs at $(\xi, \tau) \in \partial\Omega \times [0, \mathbb{T}]$ and divide the argument into two cases:

Case (1) If $|D_T u|(\xi, \tau) \le \frac{1}{2}$, then applying the fact that $(1 + \phi^2)v^2 = 1 - |D_T u|^2$, we have

$$|Du|^2(\xi, \tau) \le 1 - \frac{3}{4(1 + \phi^2)} < 1,$$

which establishes an upper bound for $|Du|^2$ on $\overline{\Omega} \times [0, \mathbb{T}]$ already.

Case (2) If $|D_T u|(\xi, \tau) > \frac{1}{2}$, then at (ξ, τ), one has

$$D_N|Du|^2(\xi, \tau) \le 0,$$

$$D_T|Du|^2(\xi, \tau) = 0 = D_T v(\xi, \tau).$$

Therefore, at (ξ, τ), (2.4)–(2.6) can be simplified as follows

$$D_T D_N u = v D_T \phi, \tag{2.10}$$

$$D_N D_T u = v D_T \phi + \kappa D_T u, \tag{2.11}$$

$$D_T D_T(u) = \frac{-v^2 \phi D_T \phi}{D_T u}. \tag{2.12}$$

Our target is to show that the following inequality

$$(D_N u)(D_N D_N u(\xi, \tau)) + (D_T u)(D_N D_T u)(\xi, \tau) \le 0 \tag{2.13}$$

can be rewritten as the one only involving the first derivatives of u. In order to get this, we need to consider the evolution equation of u. In fact, by using the assumption that $|Du|^2$ gets its maximum at (ξ, τ), (2.1)–(2.3) and (2.10)–(2.12), we get that at (ξ, τ), the following identity

$$
\begin{aligned}
u_t &= g^{TT} D_T D_T u + g^{TN} D_T D_N u + g^{NT} D_N D_T u + g^{NN} D_N D_N u \\
&\quad - g^{TT}\langle D_T T, Du\rangle - g^{TN}\langle D_T N, Du\rangle - g^{NT}\langle D_N T, Du\rangle - g^{NN}\langle D_N N, Du\rangle \\
&= \frac{1-|D_N u|^2}{v^2}\frac{(-vD_T v(1+\phi^2) - v^2\phi D_T\phi)}{D_T u} + \frac{D_N u D_T u}{v^2}(D_T\phi + \phi D_T v) \\
&\quad + \frac{D_N u D_T u}{v^2}(D_T\phi + \phi D_T v + \kappa D_T u) + (1+\phi^2) D_N D_N u - \frac{1-|D_N u|^2}{v^2}\langle \kappa N, Du\rangle \\
&\quad - \frac{D_N u D_T u}{v^2}\langle -\kappa T, Du\rangle \\
&= \frac{1-\phi^2 v^2}{v^2}\Big[\frac{-vD_T v(1+\phi^2) - v^2\phi D_T\phi}{D_T u} - \kappa\phi v\Big] + (1+\phi^2) D_N D_N u \\
&\quad + 2\frac{\phi v D_T u}{v^2}(D_T\phi + \phi D_T v + \kappa D_T u) \\
&= \frac{1-\phi^2 v^2}{v^2}\Big(\frac{-v^2\phi D_T\phi}{D_T u} - \kappa\phi v\Big) + (1+\phi^2) D_N D_N u + 2\frac{\phi v D_T u}{v^2}(D_T\phi + \kappa D_T u)
\end{aligned}
$$

holds. That is,

$$
\begin{aligned}
(1+\phi^2) D_N D_N u &= u_t - \frac{1-\phi^2 v^2}{v^2}\Big(\frac{-v^2\phi D_T\phi}{D_T u} - \kappa\phi v\Big) - 2\frac{\phi v D_T u}{v^2}(D_T\phi + \kappa D_T u) \\
&= u_t + \frac{\phi D_T\phi}{D_T u}(1 - \phi^2 v^2 - 2|D_T u|^2) + \frac{(1-\phi^2 v^2)\kappa\phi}{v} - \frac{2\kappa\phi|D_T u|^2}{v} \\
&= u_t + \frac{\phi D_T\phi}{D_T u}(v^2 - |D_T u|^2) - \frac{2\kappa\phi|D_T u|^2}{v} + \frac{\kappa\phi(1-\phi^2 v^2)}{v}.
\end{aligned}
$$

Substituting the above identity into (2.13), together with (2.11), yields

$$
\phi v\Big[u_t + \frac{\phi D_T\phi}{D_T}(v^2 - |D_T u|^2) - \frac{2\kappa\phi u_T^2}{v} + \frac{\kappa\phi(1-\phi^2 v^2)}{v}\Big] + (1+\phi^2) D_T u(v D_T\phi + \kappa u_T) \leq 0,
$$

which is equivalent to

$$
\begin{aligned}
&\phi v u_t + \frac{\phi^2 v D_T\phi}{D_T u}(v^2 - |D_T u|^2) - 2\kappa\phi^2|D_T u|^2 + \kappa\phi^2(1-\phi^2 v^2) + \\
&(1+\phi^2) D_T u(v D_T\phi + \kappa D_T u) \leq 0.
\end{aligned} \tag{2.14}
$$

It is easy to verify that

$$
-2\kappa\phi^2|D_T u|^2 + \kappa\phi^2(1-\phi^2 v^2) + (1+\phi^2)\kappa|D_T u|^2 = \kappa(1-v^2)
$$

and

$$
\frac{\phi^2 v D_T\phi}{D_T u}(v^2 - |D_T u|^2) + (1+\phi^2) D_T u D_T\phi v = \frac{v D_T\phi}{D_T u}(1-v^2).
$$

Using the above two identities, (2.14) can be simplified as follows

$$
\phi v u_t + \kappa(1-v^2) + \frac{v D_T\phi}{D_T u}(1-v^2) \leq 0.
$$

Note that

$$1 - v^2 = \frac{\phi^2}{1+\phi^2} + \frac{|D_T u|^2}{1+\phi^2}.$$

Hence,

$$\phi v u_t + \kappa(1-v^2) + \frac{vD_T\phi}{D_T u}\frac{\phi^2}{1+\phi^2} + \frac{vD_T\phi D_T u}{1+\phi^2} \leq 0.$$

In view of Lemma 2.2, $|D_T u|(\xi,\tau) > \frac{1}{2}$ and Ω is strictly convex, we can find a constant κ_0 such that

$$0 < \kappa_0(1-v^2) \leq \kappa(1-v^2) \leq c_2 v,$$

where c_2 is a positive constant depending on c_0, ϕ_0, ϕ_1, ϕ_2. Therefore,

$$|Du|^2 \leq c_1 := \frac{\sqrt{c_2^4 + 4c_2^2\kappa_0^2} - c_2^2}{2\kappa_0^2} < 1.$$

Our proof is finished.

Remark 2.1 The gradient estimate in Theorem 2.1 makes sure that the evolving graphs $\mathscr{G}_t := \{(x, u(x,t)) \,|\, x \in \Omega, 0 \leq t \leq \mathbb{T}\}$ are space-like under the non-parametric MCF, which is the core of the IBVP ($\sharp$).

By Lemma 2.2 and Theorem 2.1, together with the Schauder estimate for parabolic PDEs, we can get uniform estimates in any C^k-norm for the derivatives of u, and locally (in time) uniform bounds for the C^0-norm, which leads to the long-time existence, with uniform bounds on all higher derivatives of u, of the IBVP ($\sharp$). This finishes the proof of the first assertion of Theorem 1.1.

2.5 Boundary value problems

Applying the above gradient estimate in Theorem 2.1, one can solve the following boundary value problem (BVP for short)

$$(*)\quad \begin{cases} \Big(\sigma^{ij} + \dfrac{D^i u D^j u}{1-|Du|^2}\Big) D_i D_j u = c_3 & \text{on } \Omega, \\ D_N u = \phi(x) v & \text{on } \partial\Omega, \end{cases}$$

where c_3 is a constant determined uniquely by (2.15) below. Clearly, the BVP $(*)$ can be seen as the elliptic version of the IBVP ($\sharp$).

In fact, since the left-hand side of the first equation in the BVP $(*)$ can be written as

$$\sqrt{1-|Du|^2}\, D_i\Big(\frac{D_i u}{\sqrt{1-|Du|^2}}\Big),$$

integrating by parts, one can easily get

$$c_3 = -\frac{\int_{\partial\Omega}\phi}{\int_\Omega (1-|Du|^2)^{-\frac{1}{2}}}, \tag{2.15}$$

where, for convenience, we have dropped volume elements of the domain Ω and its boundary $\partial\Omega$ simultaneously.

One method for solving BVP $(*)$ is to consider the solvability of the following BVP:

$$(**) \qquad \begin{cases} \Big(\sigma^{ij} + \dfrac{D^i u_\varepsilon D^j u_\varepsilon}{1 - |Du_\varepsilon|^2}\Big) D_i D_j u_\varepsilon = \varepsilon u_\varepsilon & \text{on } \Omega, \\ D_N u_\varepsilon = \phi(x)\sqrt{1 - |Du_\varepsilon|^2} & \text{on } \partial\Omega. \end{cases}$$

Theorem 2.2 *The* BVP $(*)$ *has a unique, smooth solution.*

Proof We will use an argument similar to those in [1, 11]. It is known that the BVP $(**)$ has solutions for $\varepsilon > 0$. Therefore, one can replace u_t with $\varepsilon u_\varepsilon$ in the gradient estimate of Theorem 2.1 and get a conclusion that a limit solution to $(**)$ exists as $\varepsilon \to 0$, provided that there exists some c_0, independent of ε, such that $|\varepsilon u_\varepsilon|^2 \le c_0$.

Let ψ be a smooth function defined on Ω satisfying $D_N\psi < \phi\sqrt{1 - |D\psi|^2}$ on $\partial\Omega$. This kind of smooth functions can always be constructed. For instance, let d be the distance function to $\partial\Omega$ and A be a constant such that $\frac{A}{\sqrt{1-A^2}} < \phi$ on $\partial\Omega$. It is easy to check that a function ψ defined to be Ad near $\partial\Omega$ and extended to be a smooth function on all of Ω would satisfy the requirements that $\psi \in C^\infty(\partial\Omega)$, $D_N\psi < \phi\sqrt{1 - |D\psi|^2}$.

Assume that $\psi - u_\varepsilon$ attains its minimum at some point $\xi \in \Omega$.

If $\xi \in \partial\Omega$, then $D_T\psi(\xi) = D_T u_\varepsilon(\xi)$ and $D_N\psi(\xi) \ge D_N u_\varepsilon(\xi)$ hold. One can get

$$\phi(\xi) > \frac{D_N\psi}{\sqrt{1 - |D_T\psi|^2 - |D_N\psi|^2}}(\xi) \ge \frac{D_N u_\varepsilon}{\sqrt{1 - |D_T u_\varepsilon|^2 - |D_N u_\varepsilon|^2}}(\xi) = \phi(\xi),$$

since the function $\frac{q}{\sqrt{1-b^2-q^2}}$ with b a fixed constant is monotone nondecreasing in q. This is a contradiction.

Therefore, $\xi \in \Omega$, $D\psi(\xi) = Du_\varepsilon(\xi)$ and $D^2\psi(\xi) \ge D^2 u_\varepsilon(\xi)$. There exists a constant $c_4 = c_4(\psi)$ such that

$$c_4 \ge \Big(\sigma^{ij} + \frac{D^i\psi(\xi)D^j\psi(\xi)}{1 - |D\psi(\xi)|^2}\Big) D_i D_j \psi(\xi) \ge \Big(\sigma^{ij} + \frac{D^i u_\varepsilon(\xi) D^j u_\varepsilon(\xi)}{1 - |Du_\varepsilon(\xi)|^2}\Big) D_i D_j u_\varepsilon(\xi) = \varepsilon u_\varepsilon(\xi).$$

Together with the fact that $\varepsilon\psi(z) - \varepsilon u_\varepsilon(z) \ge \varepsilon\psi(\xi) - \varepsilon u_\varepsilon(\xi)$ for any $z \in \Omega$, we have

$$\varepsilon u_\varepsilon(z) \le \varepsilon\psi(z) - \varepsilon\psi(\xi) + \varepsilon u_\varepsilon(\xi) \le \varepsilon\psi(z) - \varepsilon\psi(\xi) + c_4$$

for any $z \in \Omega$. By a similar barrier argument, one can get a lower bound for $\varepsilon u_\varepsilon$. As in [11], $|Du_\varepsilon|^2 \le c_1$ implies $|D(\varepsilon u_\varepsilon)|^2 \to 0$ as $\varepsilon \to 0$, and then we have $\varepsilon u_\varepsilon \to c_3$. This gives the existence of solutions to the BVP $(*)$.

Now, in what follows, we would like to show the uniqueness of the solutions. Assume that the BVP $(*)$ has two solutions u_1, u_2 with constants c_5, c_6 on the right-hand side of $(*)$ and $c_5 < c_6$. Without loss of generality, we assume $u_1 \ge u_2$. By the linearization process, one easily knows that $\mathcal{U} := u_1 - u_2$ satisfies a linear elliptic differential inequality $\mathcal{L}(\mathcal{U}) < 0$. By the maximum principle, the minimum of $\mathcal{U}$ must be achieved at some point $\zeta \in \partial\Omega$, which implies that $|D_T u_1|^2(\zeta) = |D_T u_2|^2(\zeta) = a^2$ for some $a \in \mathbb{R}^+$. Since

$$\frac{D_N u_1}{\sqrt{1 - a^2 - |D_N u_1|^2}}(\zeta) = \frac{D_N u_2}{\sqrt{1 - a^2 - |D_N u_2|^2}}(\zeta),$$

it follows that $D_N u_1(\zeta) = D_N u_2(\zeta)$ at $\zeta \in \partial\Omega$ by using the fact that the function $\frac{q}{\sqrt{1-a^2-q^2}}$ is monotone nondecreasing in q. However, this is contradict with the Hopf boundary point lemma. So, $c_5 \geq c_6$. Reversing the roles of c_5 and c_6, one has $c_5 \leq c_6$. Therefore, one can get $c_5 = c_6$. By a similar argument, one can also obtain $u_1 = u_2$. This gives the uniqueness of solutions to the BVP $(*)$.

Our proof is finished.

Remark 2.2 Clearly, if $u = u(x)$ is a solution to the BVP $(*)$, then $\widetilde{u}(x,t) = u(x) + c_3 t$ is a solution to the IBVP $(\sharp)$. That is to say $\widetilde{u}$ is a translating solution with constant speed $|c_3|$.

3 Convergence

Now, we can show the following uniqueness conclusion of limit solutions to the IBVP $(\sharp)$ (up to translation) by applying the strong maximum principle of the second-order linear parabolic PDEs.

Lemma 3.1 *Let $\widetilde{u}_1$ and $\widetilde{u}_2$ be any two solutions to the* IBVP $(\sharp)$ *and let $\widetilde{\mathcal{U}} = \widetilde{u}_1 - \widetilde{u}_2$. Then $\widetilde{\mathcal{U}}$ becomes a constant function as $t \to \infty$. In particular, if u is a solution to the* BVP $(*)$, *then all limit solutions to the* IBVP $(\sharp)$ *are of the form $u + c_3 t$.*

Proof By the linearization process, one can easily get that $\widetilde{\mathcal{U}}$ satisfies the following linear parabolic equation

$$\frac{\partial}{\partial t}\widetilde{\mathcal{U}} = \widetilde{g}^{ij} D_i D_j \widetilde{\mathcal{U}} + \widetilde{b}^i D_i \widetilde{\mathcal{U}} \quad \text{on } \Omega \times [0, \mathcal{T}]$$

with the boundary condition

$$0 = \left\langle \frac{Du_1}{\sqrt{1-|Du_1|^2}} - \frac{Du_2}{\sqrt{1-|Du_2|^2}}, N \right\rangle := \widetilde{c}^{ij} N_j D_i \widetilde{\mathcal{U}} \quad \text{on } \partial\Omega \times [0, \mathcal{T}],$$

where

$$\widetilde{g}^{ij} = \int_0^1 g^{ij}(\theta Du_1 + (1-\theta) Du_2) \mathrm{d}\theta$$

and $\widetilde{b}^i$, $\widetilde{c}^{ij}$ are similarly determined (see [6]), N_j are components of the unit normal vector N. Note that $\widetilde{c}^{ij}$ is a positive definite matrix. By the strong maximum principle, we know that the oscillation function $\mathrm{osc}(t) := \max \widetilde{\mathcal{U}}(\cdot, t) - \min \widetilde{\mathcal{U}}(\cdot, t) \geq 0$ is strictly decreasing in t unless $\widetilde{\mathcal{U}}$ is constant.

The long-time existence of solutions to the IBVP $(\sharp)$ has been explained in Subsection 2.2 provided that the time-independent priori gradient estimate can be obtained. Therefore, together with estimates in Subsections 2.3–2.4, we have $\mathbb{T} = \infty$ here.

We claim that $|\widetilde{\mathcal{U}}|$ must be uniformly bounded on $\overline{\Omega} \times [0, \infty)$. By the maximum principle, we know that the minimum of $\widetilde{\mathcal{U}}$ should be achieved at some point $(\xi, t_0) \in (\partial\Omega \times [0, \infty)) \cup \Omega_0$. If $(\xi, t_0) \in \partial\Omega \times [0, \infty)$, then $D_T \widetilde{\mathcal{U}}(\xi, t_0) = 0$ and $D_N \widetilde{\mathcal{U}}(\xi, t_0) \geq 0$. That is, $D_T u_1(\xi, t_0) = D_T u_2(\xi, t_0)$ and $D_N u_1(\xi, t_0) \geq D_N u_2(\xi, t_0)$. Therefore, one has

$$\frac{D_N u_1}{\sqrt{1-|D_T u_1|^2 - |D_N u_1|^2}}(\xi, t_0) > \frac{D_N u_2}{\sqrt{1-|D_T u_2|^2 - |D_N u_2|^2}}(\xi, t_0),$$

since the function $\frac{q}{\sqrt{1-b^2-q^2}}$ with b a fixed constant is strictly increasing in q. However, this is contradict with the boundary condition

$$\frac{D_N u_1}{\sqrt{1-|D_T u_1|^2 - |D_N u_1|^2}}(\xi, t_0) = \frac{D_N u_2}{\sqrt{1-|D_T u_2|^2 - |D_N u_2|^2}}(\xi, t_0) = \phi(\xi).$$

Therefore, $(\xi, t_0) \in \Omega_0$, i.e., $\xi \in \Omega$ and $t_0 = 0$. This means that $\widetilde{\mathcal{U}}$ attains its minimum on Ω_0. The same situation happens to the maximum of $\widetilde{\mathcal{U}}$. Hence, we have $|\widetilde{\mathcal{U}}| = |u_1 - u_2| \leq c_7(u_0)$ for some nonnegative constant $c_7(u_0)$ depending only on u_0.

Since $|\widetilde{\mathcal{U}}|$ is uniformly bounded on $\overline{\Omega} \times [0, \infty)$, we can take a sequence $\{t_n\}$, $n \in \mathbb{Z}^+$ with $\mathbb{Z}^+$ the set of all positive integers, such that the limit $\lim\limits_{t_n \to \infty} \widetilde{\mathcal{U}}(\cdot, t_n)$ exists. If $\lim\limits_{t \to \infty} \widetilde{\mathcal{U}}(\cdot, t)$ is not a constant function, then a limit of $\widetilde{\mathcal{U}}_n(\cdot, t) := \widetilde{\mathcal{U}}(\cdot, t + t_n)$ as $t_n \to \infty$ would yield a solution on $\Omega \times [0, \infty)$ which would not be constant but $\mathrm{osc}(t)$ would be constant. However, this is contradict with the strict monotonicity of $\mathrm{osc}(t)$. Therefore, $\lim\limits_{t \to \infty} \widetilde{\mathcal{U}}(\cdot, t)$ should be a constant function, which implies the first assertion.

By Remark 2.2, we know that $u + c_3 t$ is a solution to the IBVP ($\sharp$) provided that u is a solution to the BVP ($*$). Hence, for any solution ω of ($\sharp$), by the first assertion, one has $\omega - (u + c_3 t)$ tends to a constant as $t \to \infty$, which implies that ω tends to $u + c_3 t$ for a different t. This completes the proof of the second assertion.

By applying Lemma 3.1 directly, we have the following corollary.

Corollary 3.1 *For a solution $u = u(x, t)$ of the* IBVP ($\sharp$)*, there exists a positive constant $c_8 \in \mathbb{R}^+$ such that $|u(x, t) - c_3 t| \leq c_8$.*

Now, we show that if $\int_{\partial\Omega} \phi = 0$, the limiting surface $u_\infty := \lim\limits_{t \to \infty} u(\cdot, t)$, with $u(\cdot, t)$ the solution to the IBVP ($\sharp$), should be maximal space-like.

Lemma 3.2 *If $\int_{\partial\Omega} \phi = 0$, then $c_3 = 0$ and $\lim\limits_{t \to \infty} u_t = 0$. That is, solutions $u(\cdot, t)$ to the* IBVP ($\sharp$) *converge to a maximal space-like surface u_∞ in the Lorentz manifold $M^2 \times \mathbb{R}$.*

Proof The first assertion follows directly from (2.15). By a direct calculation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega v = -\int_\Omega \frac{D_i u_t D_i u}{v} = \int_\Omega \frac{u_t^2}{v} + \int_{\partial\Omega} u_t \phi,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(\int_\Omega v - \int_{\partial\Omega} u\phi \Big) = \int_\Omega \frac{u_t^2}{v}.$$

Applying Theorem 2.1 and Corollary 3.1, we know that there exists a positive constant $c_9 \in \mathbb{R}^+$ depending on c_1 and c_8 such that

$$\int_0^\infty \int_\Omega \frac{u_t^2}{v} \leq c_9,$$

which implies the second assertion of Lemma 3.2.

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