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The Stochastic Control Model for Use Conversion of Land*

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Abstract In this paper, the authors investigate the optimal conversion rate at which land use is irreversibly converted from biodiversity conservation to agricultural production. This problem is formulated as a stochastic control model, then transformed into a HJB equation involving free boundary. Since the state equation has singularity, it is difficult to directly derive the boundary value condition for the HJB equation. They provide a new method to overcome the difficulty via constructing another auxiliary stochastic control problem, and impose a proper boundary value condition. Moreover, they establish the existence and uniqueness of the viscosity solution of the HJB equation. Finally, they propose a stable numerical method for the HJB equation involving free boundary, and show some numerical results.

Keywords Optimal stochastic control, HJB equation, Free boundary, Land use
2020 MR Subject Classification 93E20, 49L99, 49N90, 35Q99, 65N06

1 Introduction

Land is one of the most important nature resources, which has at least two common uses: Agricultural production and biodiversity conservation. It is a popular and important topic that how to allot the land to production and conservation appropriately, which is focused on in this paper.

There is a vast literature on this subject. In [1, 7], the authors discover the optimal conversion rules via two-period stochastic models. The authors extend the similar arguments by means of continuous time models in [2–3]. But there are no direct feedback expressions of the optimal conversion strategy in the above papers. In [10], the authors formulate this problem into a stochastic control model, and explore the properties of the optimal feedback between the conversion decision and the conservation benefit. They first transform the optimal stochastic control model into its associated Hamilton-Jacobi-Bellman (HJB for short) equation, and obtain some numerical results via the numerical method for partial differential equation (PDE for

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short). Unfortunately, there remains some uncertainty about the boundary value condition for the HJB equation and room for model improvement, which will be presented later.

In this paper, we suppose that land has only mutually exclusive uses: Agricultural production use and biodiversity conservation use. Land use can be converted from conservation to production, but the inverse transformation is inadmissible. Moreover, we assume that the transformation rate has an upper bound, and our aim is to obtain maximum benefit through adjusting the transformation rate. This problem can be formulated into a stochastic control model. In this model, the states are taken as the land area used as biodiversity reserve R (suppose that the total area is one unit) and the biodiversity conservation benefit per unit Y, the control variable is just the transformation rate v, and the objective functional $\mathfrak{F}$ is the expectation of the total benefit from agriculture and conservation minus the converting cost. The specific formulation of this model will be presented in Section 2.

In summary, the state equations in the optimal stochastic control problem are as follows:

$$R_t^{r;v} = r - \int_0^t v_s \, \mathrm{d}s, \tag{1.1}$$

$$Y_t^{y,r;v} = y + \int_0^t \Big(\alpha - \frac{(m-1)v_s}{R_s^{r;v}}\Big) Y_s^{y,r;v} \mathrm{d}s + \int_0^t \sqrt{\sigma_E^2 + \frac{\sigma_C^2 v_s}{R_s^{r;v}}}\; Y_s^{y,r;v} \mathrm{d}Z_s, \tag{1.2}$$

where α, m, σ_E, σ_C are positive constants, whose practical meanings will be explained in Section 2, and Z is a one-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ satisfying the usual condition. The vector (y, r) is the initial state with $y > 0$, $0 < r \leq 1$, and v is the admissible strategy, which belongs to the following admissible strategy set

$$\mathcal{A}(y,r) := \{v : v \text{ is } \mathbb{F}\text{-progressively measurable}, v_s \in [0, \overline{v}],\ R^{r;v} > 0,\ Y^{y,r;v} > 0\}, \tag{1.3}$$

where $\overline{v}$ is a positive constant, and represents the upper bound of the transformation rate. The objective functional takes the form of

$$\mathfrak{F}(y,r;v) := \mathbb{E}\Big\{ \int_0^\infty \mathrm{e}^{-\rho t} \Big[\frac{\phi}{1-\gamma}(1 - R_t^{r;v})^{1-\gamma} + R_t^{r;v} Y_t^{y,r;v} - k v_t\Big] \mathrm{d}t \Big\}, \tag{1.4}$$

where ρ, ϕ, γ, k are constants satisfying $\rho > \alpha$, $\phi > 0$, $\gamma \in (0,1)$ and $k \geq 0$.

We aim to find an optimal transformation rate $v^* \in \mathcal{A}(y,r)$ satisfying

$$F(y,r) := \mathfrak{F}(y,r;v^*) = \sup_{v \in \mathcal{A}(y,r)} \mathfrak{F}(y,r;v), \tag{1.5}$$

where the function F is called the value function of the stochastic control problem (1.5). From the upper estimate of F in Theorem 3.1, we know the existence of the value function F.

Applying the stochastic control theory in [8, 13], we derive that the value function F is the viscosity solution of the following HJB equation

$$\mathcal{L}F := \sup_{0 \leq v \leq \overline{v}} \Big\{ \frac{1}{2}\Big(\sigma_E^2 + \frac{\sigma_C^2 v}{r}\Big) y^2 F_{yy} + \Big(\alpha - \frac{(m-1)v}{r}\Big) y F_y - v F_r - \rho F$$

$$+\Big(\frac{\phi}{1-\gamma}(1-r)^{1-\gamma}+yr-kv\Big)\Big\}=0,\quad y>0,\ 0<r<1. \tag{1.6}$$

As we all know, the boundary value condition is important to a PDE problem. PDE with improper boundary value condition might have multiple solutions. So, the proper boundary value condition is important to ensure that the viscosity solution of PDE (1.6) is really the value function of Problem (1.5). In the PDE (1.6), there are four boundaries $y=0$, $r=0$, $y\to+\infty$ and $r=1$, on which the appropriate boundary values are necessary[1]. Particularly, on the boundary $r=0$, since the state equation (1.2) makes no sense, it is difficult to impose the valid boundary value condition.

In fact, Leroux, et al. consider a similar model in [10], where they use the biodiversity reserve R and the biodiversity conservation benefit $B=YR$ as the state processes. From Itô's formula, B is governed by the following stochastic differential equation (SDE for short):

$$B_t^{b,r;v}=b+\int_0^t\Big(\alpha-\frac{mv_s}{R_s^{r;v}}\Big)B_s^{b,r;v}\mathrm{d}s+\int_0^t\sqrt{\sigma_E^2+\frac{\sigma_C^2v_s}{R_s^{r;v}}}\,B_s^{b,r;v}\mathrm{d}Z_s,\qquad b=yr. \tag{1.7}$$

In their model, the admissible set and the objective functional are the same as those in (1.3) and (1.4), respectively. The authors deduce that the value function $\widehat{F}(b,r)$ of the corresponding optimization problem is the viscosity solution of the following HJB equation:

$$\begin{aligned}\widehat{\mathcal{L}}\widehat{F}:=\sup_{0\le v\le\overline{v}}\Big\{&\frac{1}{2}\Big(\sigma_E^2+\frac{\sigma_C^2v}{r}\Big)b^2\widehat{F}_{bb}+\Big(\alpha-\frac{mv}{r}\Big)b\widehat{F}_b-v\widehat{F}_r-\rho\widehat{F}\\&+\Big(\frac{\phi}{1-\gamma}(1-r)^{1-\gamma}+b-kv\Big)\Big\}=0.\end{aligned} \tag{1.8}$$

In their paper, they give the boundary value conditions as follows, via "practical meaning" rather than mathematical method,

$$\widehat{F}_{br}(b,0)=\widehat{F}_{bbr}(b,0)=0,\quad \widehat{F}(0,0)=\frac{\phi}{\rho(1-\gamma)},$$

$$\widehat{F}_b(0,r)=\mathrm{e}^{\left(\frac{\alpha-m\overline{v}}{r-\rho}\right)\frac{r}{\overline{v}}}-1,\qquad \lim_{b\to+\infty}\widehat{F}_b(b,r)=\mathrm{e}^{\left(\frac{\alpha-m\overline{v}}{r-\rho}\right)\frac{r}{\overline{v}}}-1.$$

They impose the values of $\widehat{F}_{br}$, $\widehat{F}_{bbr}$ or $\widehat{F}_b$ rather than $\widehat{F}$ on the boundaries, except for the value at the point $(0,0)$. As we all know, $\widehat{F}$ is a viscosity solution rather than a classical solution, and might have low regularity such that $\widehat{F}_{br}$ and $\widehat{F}_{bbr}$ make no sense. So, the above boundary value conditions are improper. In fact, we can show that the numerical result in [10] remains to be discussed via Theorem A.1 in our paper, where we prove that $\widehat{F}$ is linear growth with respect to b , rather than superlinear with respect to b in [10, Figure 3].

Our contributions can be summarized as follows. The first contribution is that we deduce a more feasible boundary value condition for the HJB equation (1.6), which is provided in Section 3 by means of some stochastic control methods. The main difficulty in imposing the proper boundary value conditions comes from the singularity at $r=0$, which means that the state

[1] In fact, we will show that it is sufficient to impose the boundary value condition when $r=0$ in Section 3.

equation (1.2) makes no sense if $r = 0$. So, it is impossible to directly achieve the value of F at $r = 0$. In order to overcome the difficulty, we first obtain the upper bound of the upper limit of $F(y, r)$ as $r \to 0^+$ via estimating the upper bound of F, and then obtain the lower bound of the lower limit of $F(y, r)$ as $r \to 0^+$ via another auxiliary stochastic control problem. Since we prove that the upper bound is equal to the lower bound, they are just the limit of $F(y, r)$ as $r \to 0^+$. Thus, the value function $F(y, r)$ is continuous with respect to r at $r = 0$ if we let its value at $r = 0$ be equal to the limit. Moreover, we expect that the approach proposed in this paper has wider applicability, and can be used to impose the boundary value condition for the kind of HJB equation, which is arisen from the stochastic control problem with the state equation involving singularity.

The second contribution is to establish the existence and uniqueness of the viscosity solution of the HJB equation (1.6). We can not directly apply the classical result for the existence and uniqueness of the viscosity solution of the HJB equation (1.6), due to $\frac{1}{r}$ in the coefficient functions in (1.6) blowing up when r tends to zero. In order to obtain the existence and uniqueness of the viscosity solution, we firstly define the viscosity solution of the HJB equation (1.6) according to Definition 4.1, where the solution must have linear-growth with respect to y and proper asymptotic property as r tends to zero, which is different from the standard definition in [5]. Then we construct a sequence of classical stochastic control problems (4.1) without blowing up characteristic to approximate to the original problem (1.5), and show the existence and uniqueness of the viscosity solution of the HJB equation (1.6) via establishing some proper estimates on the value functions of the stochastic control problems (4.1).

The third contribution is that we provide a stable numerical method for this kind of HJB equation associated with some free boundaries, which is first applied to compute the viscosity solution of HJB equation in [8–9], then extended to some financial PDE problems involving some free boundaries (e.g. in [4]). In fact, the authors apply another numerical method to the problem in [10], which looks like arising from "practical meaning" rather than from rigorous mathematical method. They assume that there exists a curve (i.e., the free boundary) splitting the overall solution domain into two parts: The conservation region, where the optimal transformation rate $v^* = 0$, and the conversion region with $v^* = \overline{v}$, and the free boundary is just the intersection between these two regions. Moreover, they assume that the value function $\widehat{F}$ is smooth in these two regions, and its first order derivative continuously crosses the free boundary. Though the assumptions may be true in this practical problem, no existing mathematical result ensures that these assumptions are right. In fact, there are several free boundaries rather than only one free boundary in some other practical problems, such as the example given in [11]. In order to avoid these assumptions, we apply the finite difference method for the viscosity solution of HJB equation in [4, 8–9] to the HJB equation (1.6), and provide some numerical results.

The forth contribution is to build a model which is more reasonable than the model introduced in [10] (Model L for short), where the authors use the biodiversity reserve R and the biodiversity conservation benefit $B = YR$ as the state processes. We will show that the value function $\widehat{F}$ in Model L always achieves its maximum value at $r = 0$ by means of Theorem A.1.

It means that the value of land is maximum when all land is used for agriculture production, which is puzzling. The key of the puzzle comes from the fact that the biodiversity conservation benefit per unit is infinite when $b > 0$ and $r \to 0^+$, which is impossible. So we change the state B by Y, which represents the biodiversity conservation benefit per unit. We will show that the benefit achieves its maximum at a proper biodiversity reserve by means of the numerical result in Table 2.

This paper is organized as follows: We formulate the model, deduce the HJB equation in Section 2, and derive the boundary value condition in Section 3. We prove that the value function is a viscosity solution of the HJB equation (1.6), and establish the existence and uniqueness of the viscosity solution of the HJB equation (1.6) in Section 4. In Section 5, we apply the finite difference method for HJB equations to Problem (1.6), and obtain some numerical results. Finally, we apply the methods in Section 3 and Section 5 to Model L in Appendix A, and show a more reasonable numerical result.

2 Optimal Conversion of Land Use Model

In this section, we first formulate the model as a stochastic control problem, then transform it into its associated HJB equation.

2.1 Assumptions and model

Consider an area of land, normalized to unity. The land use can be converted from biodiversity conservation to agricultural production, and the conversion is irreversible. Let A_t and R_t be the area of the land used for production and conservation at time t, respectively. Hence, $A_t + R_t = 1$.

Suppose that the area of the land for biodiversity reserve $R^{r;v}$ satisfies the ordinary differential equation (ODE for short) (1.1), where v is the conversion rate from reservation land to agricultural land, and the superscript $r; v$ means that R depends on its initial state r and the conversion strategy v. It is clear that A_t and R_t are nonnegative. Moreover, it is natural to assume that the initial conservation land $r \neq 0$ as the conversion is only from conservation land to agriculture land. So, we suppose that $0 < r, R^{r;v} \leq 1$.

The biodiversity conservation benefit B is governed by the SDE (1.7), where parameters α, m, σ_E and σ_C are positive constants, and represent the increasing rate of the species value in conservation land, the elasticity coefficient of conservation benefit, the volatility of the species value and the volatility of the ecosystem-specific species density, respectively. For more details, refer to [10]. Z is a one-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual condition.

Since the biodiversity conservation benefit B depends on the area of conservation land, we would rather adopt the biodiversity conservation benefit per unit Y as the state variable. Applying Itô's formula, we deduce that Y is governed by the SDE (1.2), which is clearly positive.

Since the conversion is irreversible, and its rate has a maximum value $\overline{v}$, we let $v_s \in [0, \overline{v}]$, and the admissible strategy set takes the form of (1.3).

The decision-maker aims to find an admissible strategy v^* to maximize the objective functional defined as (1.4), in which the constants ϕ, γ, k and ρ satisfying $\phi > 0$, $\gamma \in (0,1)$, $k \geq 0$ and $\rho > \alpha$, represent the return per unit area, the elasticity coefficient of the utility function, the marginal cost of conversion and the discount rate, respectively. For more details, we refer to [10]. In (1.4), the first term in the braces represents the flow benefit from the land for production, the second term represents the total value of biodiversity from the land for conservation, and the last term means the total cost of land conversion.

2.2 Associated HJB equation

In summary, the model can be formulated as a stochastic control problem. The decision-maker aims to find v^* from the admissible strategy set (1.3) to maximize the objective functional (1.4) subject to (1.1)–(1.2).

From the stochastic control theory in [8, 13], we know that the value function F is a viscosity solution of the HJB equation (1.6).

3 Boundary Value Condition

In this section, we derive the boundary value condition for the HJB equation (1.6) via the original stochastic control model and the theory on PDE.

Since the HJB equation (1.6) is degenerate at $y = 0$ and satisfies Fichera condition in [6], the boundary value of F at $y = 0$ should be determined by itself. Specifically, we should first deduce the ODE for $F(0, r)$ through taking $y = 0$ in the HJB equation (1.6), and obtain the value of $F(0, r)$ by means of solving the ODE. Since F has a linear growth with respect to y, which will be proven in Theorem 3.1, the standard theory for Cauchy problem (refer to [11–12]) implies that the boundary value of $F(y, r)$ at $y \to +\infty$ is also unnecessary. Note that the derivative of F with respect to r and the second-order derivative of F with respect to y in the HJB equation (1.6) are non-positive and non-negative, respectively. So, the HJB equation (1.6) is a forward parabolic PDE, and the boundary value at $r = 1$ is not needed. Hence, it is sufficient to impose the boundary value of F at $r = 0$.

However, it is not trivial to obtain the boundary value of F at $r = 0$ since the SDE of Y is meaningless when $r = 0$. The key idea to overcome the difficulty is to consider the limit of $F(y, r)$ as $r \to 0^+$. In fact, we should find a continuous solution of the HJB equation (1.6). So, $F(y, 0)$ is just the limit of $F(y, r)$ as $r \to 0^+$, and it is sufficient to find the limit.

Theorem 3.1 *The boundary value $F(y, 0) = \frac{\phi}{\rho(1-\gamma)}$ for any $y \geq 0$. Moreover, F has the following estimate*

$$0 \leq \frac{\phi(1-r)^{1-\gamma}}{\rho(1-\gamma)} \leq F(y, r) \leq \frac{\phi}{\rho(1-\gamma)} + \frac{yr}{\rho - \alpha} \quad \textit{for any } y \geq 0,\ 0 < r \leq 1. \tag{3.1}$$

Proof The proof is divided into two steps: Finding the lower bound of the lower limit of $F(y, r)$ as $r \to 0^+$, and finding the upper bound of the upper limit, which are equal.

Step 1 Take the special control $v \equiv 0$, then we deduce $R^{r;0} = r$, and

$$F(y,r) \geq \mathbb{E}\Big\{ \int_0^\infty \frac{\phi\,\mathrm{e}^{-\rho t}}{1-\gamma}(1-R_t^{r;0})^{1-\gamma}\,\mathrm{d}t \Big\} = \frac{\phi}{\rho(1-\gamma)}(1-r)^{1-\gamma} \tag{3.2}$$

for any $y > 0, 0 < r \leq 1$. Taking $r \to 0^+$ in (3.2), we obtain the lower bound of the lower limit of $F(y,r)$ as follows:

$$\liminf_{r\to 0^+} F(y,r) \geq \frac{\phi}{\rho(1-\gamma)} \quad \text{for any } y > 0. \tag{3.3}$$

Step 2 Construct another stochastic control problem: The state equation takes the form of

$$\widetilde{B}_t^{b;\widetilde{v}} = b + \int_0^t \Big(\alpha - m\widetilde{v}_s\Big)\widetilde{B}_s^{b;\widetilde{v}}\mathrm{d}s + \int_0^t \sqrt{\sigma_E^2 + \sigma_C^2\widetilde{v}_s}\,\widetilde{B}_s^{b;\widetilde{v}}\mathrm{d}Z_s,$$

where the new control $\widetilde{v}$ is similar to $\frac{v}{R^{r;v}}$ in the SDE (1.7).

The objective functional is described as

$$\widetilde{\mathfrak{F}}(b;\widetilde{v}) = \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\rho t}\Big(\frac{\phi}{1-\gamma} + \widetilde{B}_t^{b;\widetilde{v}}\Big)\,\mathrm{d}t\Big]$$

and the stochastic control problem is to find the optimal strategy $\widetilde{v}^* \in \widetilde{\mathcal{A}}(b)$ such that

$$\widetilde{F}(b) = \widetilde{\mathfrak{F}}(b;\widetilde{v}^*) = \sup_{\widetilde{v}\in\widetilde{\mathcal{A}}(b)} \widetilde{\mathfrak{F}}(b;\widetilde{v})$$

with the admissible strategy set

$$\widetilde{\mathcal{A}}(b) = \Big\{\widetilde{v} : \widetilde{v} \text{ is } \mathbb{F}\text{-progressively measurable, } \widetilde{v} \geq 0, \int_0^T \widetilde{v}_s\mathrm{d}s < +\infty \text{ for any } T \in (0,+\infty) \text{ a.s. in } \Omega,\ \widetilde{B}^{b;\widetilde{v}} \geq 0\Big\}.$$

It is not difficult to check that

$$\frac{v}{R^{r;v}} \in \widetilde{\mathcal{A}}(yr), \quad \widetilde{B}^{yr;\frac{v}{R^{r;v}}} = B^{yr,r;v} = Y^{y,r;v}R^{r;v}, \quad \widetilde{\mathfrak{F}}\Big(yr;\frac{v}{R^{r;v}}\Big) \geq \mathfrak{F}(y,r;v)$$

for any $v \in \mathcal{A}(y,r), y > 0, 0 < r \leq 1$. So, we deduce that $\widetilde{F}(yr) \geq F(y,r)$ for any $y > 0, 0 < r \leq 1$. Denote

$$\widehat{B}_t^{yr;\widetilde{v}} = \mathrm{e}^{-\alpha t}\widetilde{B}_t^{yr;\widetilde{v}} \geq 0,$$

then $\widehat{B}^{\,yr;\widetilde{v}}$ satisfies

$$\begin{aligned}\widehat{B}_t^{yr;\widetilde{v}} &= yr - \int_0^t m\widetilde{v}_s\widehat{B}_s^{yr;\widetilde{v}}\mathrm{d}s + \int_0^t \sqrt{\sigma_E^2 + \sigma_C^2\widetilde{v}_s}\,\widehat{B}_s^{yr;\widetilde{v}}\mathrm{d}Z_s \\ &\leq yr + \int_0^t \sqrt{\sigma_E^2 + \sigma_C^2\widetilde{v}_s}\,\widehat{B}_s^{yr;\widetilde{v}}\mathrm{d}Z_s.\end{aligned}$$

Since the term on the right hand side of this inequality is a non-negative local martingale, we have that

$$\mathbb{E}(\widehat{B}_t^{yr;\widetilde{v}}) \leq yr + \mathbb{E}\Big(\int_0^t \sqrt{\sigma_E^2 + \sigma_C^2\widetilde{v}_s}\,\widehat{B}_s^{yr;\widetilde{v}}\mathrm{d}Z_s\Big) \leq yr.$$

Hence, we deduce that

$$F(y,r) \le \widetilde{F}(yr) = \max_{\widetilde{v}\in\widetilde{\mathcal{A}}(yr)} \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\rho t}\Big(\frac{\phi}{1-\gamma} + \mathrm{e}^{\alpha t}\widehat{B}_t^{yr;\widetilde{v}}\Big)\,\mathrm{d}t\Big]$$

$$\le \frac{\phi}{\rho(1-\gamma)} + \frac{yr}{\rho-\alpha} \quad \text{for any } y>0, 0<r\le 1. \tag{3.4}$$

Taking $r \to 0^+$ in the above inequality, we obtain the upper bound of the upper limit of $F(y,r)$ as follows:

$$\limsup_{r\to 0^+} F(y,r) \le \frac{\phi}{\rho(1-\gamma)} \quad \text{for any } y>0.$$

Combining (3.3), we derive that

$$\lim_{r\to 0^+} F(y,r) = \frac{\phi}{\rho(1-\gamma)} \quad \text{for any } y>0. \tag{3.5}$$

Then (3.1) follows from (3.2) and (3.4).

From Theorem 3.1, we guess that the value function F of the stochastic control problem (1.5) is the unique viscosity solution of the HJB equation (1.6) with boundary value $F(y,0) = \frac{\phi}{\rho(1-\gamma)}$.

4 Existence and Uniqueness of the Viscosity Solution

In this section, we present the accurate definition of the viscosity solution of the HJB equation (1.6), and establish the existence and uniqueness of the viscosity solution.

Definition 4.1 *A function $F \in C([0,+\infty)\times[0,1])$ is called a viscosity solution of the* HJB *equation* (1.6) *with boundary value $F(y,0)=\frac{\phi}{\rho(1-\gamma)}$, if F satisfies* (3.1) *and the following properties*:

(1) *for any $f \in C^2((0,+\infty)\times(0,1))$, whenever $F-f$ attains a local maximum at $(y,r)\in(0,+\infty)\times(0,1)$, then $\mathcal{L}f(y,r)\ge 0$;*

(2) *for any $f \in C^2((0,+\infty)\times(0,1))$, whenever $F-f$ attains a local minimum at $(y,r)\in(0,+\infty)\times(0,1)$, then $\mathcal{L}f(y,r)\le 0$.*

Next, we prove that the value function F of the stochastic control problem (1.5) satisfies Definition 4.1, and then is a viscosity solution of the HJB equation (1.6) with the boundary value $F(y,0)=\frac{\phi}{\rho(1-\gamma)}$. This proof is not trivial. Since the coefficient functions in the state equation of Y and the HJB equation (1.6) blow up, the classical result can not be directly applied to this problem.

Theorem 4.1 *The value function F of the stochastic control problem* (1.5) *is a viscosity solution of the* HJB *equation* (1.6) *with the boundary value $F(y,0)=\frac{\phi}{\rho(1-\gamma)}$.*

Proof First of all, from Theorem 3.1, we see that the value function F of the stochastic control problem (1.5) satisfies (3.1).

In order to prove the F satisfies Properties (1)–(2), we construct the following approximation stochastic control problem

$$F_n(y,r) := \sup_{v\in\mathcal{A}(y,r)} \mathbb{E}\Big\{ \int_0^{\tau_n} \mathrm{e}^{-\rho t} \Big[\frac{\phi}{1-\gamma}(1-R_t^{r;v})^{1-\gamma} + R_t^{r;v} Y_t^{y,r;v} - kv_t \Big] \mathrm{d}t + \mathrm{e}^{-\rho\tau_n} F(Y_{\tau_n}^{y,r;v}, R_{\tau_n}^{r;v}) \Big\} \tag{4.1}$$

subject to (1.1)–(1.2), where $n \in \mathbb{N}_+$, and

$$\tau_n := \inf\Big\{ t \geq 0 : R_t^{r;v} \leq \frac{1}{n} \Big\}.$$

By the dynamic programming principle, we see that $F_n(y,r) = F(y,r)$ for any $(y,r) \in A_n := (0,+\infty) \times \big(\frac{1}{n}, 1\big]$. Noting that the coefficient functions in the state equations (1.1)–(1.2) are smooth in the set A_n, we know that the above approximation stochastic control problems are classic problems for any $n \in \mathbb{N}_+$. So, F_n is a viscosity solution of the HJB equation (1.6) in the domain A_n via the theory in [8]. Hence, taking $n \to +\infty$, we deduce that F satisfies Properties (1)–(2).

We will present the existence and uniqueness of the viscosity solution of the HJB equation (1.6) with the boundary value $F(y,0) = \frac{\phi}{\rho(1-\gamma)}$ by means of the following theorem.

Theorem 4.2 *There exists a unique viscosity solution of the* HJB *equation* (1.6) *with boundary value* $F(y,0) = \frac{\phi}{\rho(1-\gamma)}$.

Proof First, the upper estimate in (3.1) implies that the value function F of the stochastic control problem (1.5) exists. From Theorem 4.1, we know that the value function F is a viscosity solution of the HJB equation (1.6) with the boundary value $F(y,0) = \frac{\phi}{\rho(1-\gamma)}$. So, we obtain the existence of the viscosity solution of the HJB equation (1.6) with the boundary value $F(y,0) = \frac{\phi}{\rho(1-\gamma)}$.

In the following, we prove the uniqueness of the viscosity solution. Otherwise, there exist at least two viscosity solutions F^1 and F^2. Consider the approximation stochastic control problem (4.1), and denote F_n by F_n^i with the boundary value $F^i, i = 1,2$, respectively. It is clear that for any fixed $(y,r) \in A_n$,

$$\begin{aligned}
&|F^1(y,r) - F^2(y,r)| = |F_n^1(y,r) - F_n^2(y,r)| \\
\leq& \sup_{v\in\mathcal{A}(y,r)} \mathbb{E}(\mathrm{e}^{-\rho\tau_n} |F^1(Y_{\tau_n}^{y,r;v}, R_{\tau_n}^{r;v}) - F^2(Y_{\tau_n}^{y,r;v}, R_{\tau_n}^{r;v})|) \\
\leq& \sup_{v\in\mathcal{A}(y,r)} \mathbb{E}\Big[\mathrm{e}^{-\rho\tau_n} \Big(\frac{\phi}{\rho(1-\gamma)} + \frac{Y_{\tau_n}^{y,r;v} R_{\tau_n}^{r;v}}{\rho-\alpha} - \frac{\phi(1-R_{\tau_n}^{r;v})^{1-\gamma}}{\rho(1-\gamma)} \Big) \Big] \\
\leq& \sup_{v\in\mathcal{A}(y,r)} \mathbb{E}\Big[\mathrm{e}^{-\rho\tau_n} \Big(\frac{Y_{\tau_n}^{y,r;v} R_{\tau_n}^{r;v}}{\rho-\alpha} + \frac{\phi R_{\tau_n}^{r;v}}{\rho(1-\gamma)} \Big) \Big] \\
\leq& \sup_{v\in\mathcal{A}(y,r)} \frac{1}{\rho-\alpha} (\mathbb{E}(\mathrm{e}^{-\rho\tau_n} Y_{\tau_n}^{y,r;v} (R_{\tau_n}^{r;v})^{\beta})^k)^{\frac{1}{k}} (\mathbb{E}(R_{\tau_n}^{r;v})^{(1-\beta)k^*})^{\frac{1}{k^*}} + \frac{\phi}{\rho(1-\gamma)n},
\end{aligned} \tag{4.2}$$

where we use (3.1) in the second inequality, the fact that $(1-R^{r;v}_{\tau_n})^{1-\gamma} \geq 1 - R^{r;v}_{\tau_n}$ for any $0<\gamma<1$ in the third inequality, and the hölder inequality in the forth inequality. Moreover, we adopt the following notation

$$\beta := 1 - \frac{\min(1,m)}{2} \in (0,1), \quad k := 1 + 2\min\Big(\frac{\rho-\alpha}{\sigma_E^2}, \frac{m+\beta-1}{\sigma_C^2}\Big) > 1, \quad k^* = \frac{k}{k-1}.$$

Next, we establish an estimate of $X^{y,r;v} := \mathrm{e}^{-\rho\cdot}Y^{y,r;v}(R^{r;v})^{\beta}$. The direct computation shows that

$$\mathrm{d}X^{y,r;v}_t = \Big(\alpha - \rho - \frac{(m+\beta-1)v_s}{R^{r;v}_s}\Big) X^{y,r;v}_s \,\mathrm{d}s + \sqrt{\sigma_E^2 + \frac{\sigma_C^2 v_s}{R^{r;v}_s}}\, X^{y,r;v}_s \,\mathrm{d}Z_s,$$

$$\mathrm{d}(X^{y,r;v}_t)^k = k\, a_s (X^{y,r;v}_s)^k \,\mathrm{d}s + k\sqrt{\sigma_E^2 + \frac{\sigma_C^2 v_s}{R^{r;v}_s}}\, (X^{y,r;v}_s)^k \,\mathrm{d}Z_s,$$

where

$$\begin{aligned} a_s :&= \Big(\alpha - \rho - \frac{(m+\beta-1)v_s}{R^{r;v}_s}\Big) + \frac{k-1}{2}\Big(\sigma_E^2 + \frac{\sigma_C^2 v_s}{R^{r;v}_s}\Big) \\ &= \Big(\alpha - \rho + \frac{k-1}{2}\sigma_E^2\Big) - \Big((m+\beta-1) - \frac{k-1}{2}\sigma_C^2\Big)\frac{v_s}{R^{r;v}_s} \leq 0. \end{aligned}$$

So, we deduce that

$$(\mathrm{e}^{-\rho t}Y^{y,r;v}_t(R^{r;v}_t)^{\beta})^k = (X^{y,r;v}_t)^k \leq y^k r^{k\beta} + k\int_0^t \sqrt{\sigma_E^2 + \frac{\sigma_C^2 v_s}{R^{r;v}_s}}(X^{y,r;v}_s)^k \,\mathrm{d}Z_s.$$

Noting the term on the left hand side of this inequality is non-negative, and the Itô integral on the right hand side of this inequality is a local martingale, we derive that the Itô integral is a super-martingale, and

$$\mathbb{E}(\mathrm{e}^{-\rho\tau_n}Y^{y,r;v}_{\tau_n}(R^{r;v}_{\tau_n})^{\beta})^k \leq y^k r^{k\beta}.$$

Combining (4.2), we have the following estimate

$$|F^1(y,r) - F^2(y,r)| \leq \frac{yr^{\beta}}{(\rho-\alpha)n^{1-\beta}} + \frac{\phi}{\rho(1-\gamma)n} \to 0$$

as n tends to infinity. Hence, we prove that $F^1 = F^2$, and it follows that the viscosity solution of the HJB equation (1.6) with boundary value $F(y,0) = \frac{\phi}{\rho(1-\gamma)}$ is unique.

5 Numerical Method and Results

In this section, we apply the numerical method for HJB equations in [8–9] to PDE (1.6), and obtain some numerical illustrations of our model.

In order to finish the computation in finite steps, we restrict the HJB equation (1.6) in a bounded domain rather than the unbounded domain $y>0$, $0<r<1$. So, we use the solution

$F^{\overline{y}}$ of the following PDE in a bounded domain to approximate the solution F of the HJB equation (1.6) in the unbounded domain as $\overline{y} \to +\infty$,

$$\begin{cases} \sup\limits_{0\le v\le \overline{v}} \Big\{ \dfrac{1}{2}\Big(\sigma_E^2 + \dfrac{\sigma_C^2 v}{r}\Big) y^2 F^{\overline{y}}_{yy} + \Big[\alpha - \dfrac{(m-1)v}{r}\Big] y F^{\overline{y}}_y - vF^{\overline{y}}_r - \rho F^{\overline{y}} \\ \qquad + \Big[\dfrac{\phi}{1-\gamma}(1-r)^{1-\gamma} + yr - kv\Big]\Big\} = 0, \quad 0 < y < \overline{y},\ 0 < r < 1, \\ F^{\overline{y}}(y,0) = \dfrac{\phi}{\rho(1-\gamma)}, \quad F^{\overline{y}}(\overline{y},r) = F^{\overline{y}}(0,r) + \dfrac{\overline{y}r}{\rho-\alpha}, \quad 0 < y < \overline{y},\ 0 < r \le 1. \end{cases} \tag{5.1}$$

Remark 5.1 We add the Dirichlet boundary condition at $y = \overline{y}$. According to the standard method for Cauchy problem (refer to [11–12]), it is not difficult to deduce that $F^{\overline{y}}$ converges to F as $\overline{y} \to +\infty$. So, $F^{\overline{y}}(y,r)$ is very close to $F(y,r)$ if $\overline{y}$ is large enough and y is far from $\overline{y}$.

Apply the numerical method for the viscosity solutions of HJB equations in [8–9], and define the grid in the (y,r) coordinate system as (y_i, r_j), $y_i = (i-1)d_y, r_j = (j-1)d_r, i = 1,2,\cdots,M, j = 1,2,\cdots,N$, where $d_y = \frac{\overline{y}}{M-1}, d_r = \frac{1}{N-1}$. Thus, we have the following discrete implicit finite difference scheme of the PDE (5.1)

$$\begin{cases} \max\limits_{0\le v\le \overline{v}} \Big\{ \dfrac{1}{2}\Big(\sigma_E^2 + \dfrac{\sigma_C^2 v}{r_j}\Big) y_i^2 \Delta_y^2 F^{\overline{y}}_{i,j} - \Big(\alpha - \dfrac{(m-1)v}{r_j}\Big)^- y_i \Delta_y^- F^{\overline{y}}_{i,j} + \Big(\alpha - \dfrac{(m-1)v}{r_j}\Big)^+ \\ \quad y_i \Delta_y^+ F^{\overline{y}}_{i,j} - v\Delta_r F^{\overline{y}}_{i,j} - \rho F^{\overline{y}}_{i,j} + \Big(\dfrac{\phi}{1-\gamma}(1-r_j)^{1-\gamma} + y_i r_j - kv\Big)\Big\} = 0, \\ F^{\overline{y}}_{1,j} = E_j, \quad F^{\overline{y}}_{M,j} = E_j + \dfrac{\overline{y} r_j}{\rho-\alpha}, \quad F^{\overline{y}}_{i,1} = \dfrac{\phi}{\rho(1-\gamma)}, \end{cases} \tag{5.2}$$

where $x^+ = \max(x,0)$, $x^- = \max(-x,0)$,

$$F^{\overline{y}}_{i,j} = F^{\overline{y}}(y_i, r_j), \quad \Delta_r F^{\overline{y}}_{i,j} = \frac{F^{\overline{y}}_{i,j} - F^{\overline{y}}_{i,j-1}}{d_r}, \quad \Delta_y^+ F^{\overline{y}}_{i,j} = \frac{F^{\overline{y}}_{i+1,j} - F^{\overline{y}}_{i,j}}{d_y},$$

$$\Delta_y^- F^{\overline{y}}_{i,j} = \frac{F^{\overline{y}}_{i,j} - F^{\overline{y}}_{i-1,j}}{d_y}, \quad \Delta_y^2 F^{\overline{y}}_{i,j} = \frac{F^{\overline{y}}_{i+1,j} + F^{\overline{y}}_{i-1,j} - 2F^{\overline{y}}_{i,j}}{d_{yy}}, \quad d_{yy} = (d_y)^2,$$

and E_j satisfies

$$\begin{cases} \max\limits_{0\le v\le \overline{v}} \Big\{ - v\Delta_r E_j - \rho E_j + \Big(\dfrac{\phi}{1-\gamma}(1-r_j)^{1-\gamma} - kv\Big)\Big\} = 0, \\ E_1 = \dfrac{\phi}{\rho(1-\gamma)}. \end{cases}$$

Since the expression in the braces of (5.2) is a linear function with respect to v, we know that the maximum of the expression achieves at $v = 0$ or $v = \overline{v}$. So, we can transform the difference equation (5.2) into the following difference equation

$$\begin{cases} \dfrac{1}{2}\sigma_E^2 y_i^2 r_j \Delta_y^2 F^{\overline{y}}_{i,j} - \rho r_j F^{\overline{y}}_{i,j} + \Big[\dfrac{\phi r_j}{1-\gamma}(1-r_j)^{1-\gamma} + y_i r_j^2\Big] + h_{i,j}(v^*_{i,j}) = 0, \\ F^{\overline{y}}_{1,j} = E_j, \quad F^{\overline{y}}_{M,j} = E_j + \dfrac{\overline{y} r_j}{\rho-\alpha}, \quad F^{\overline{y}}_{i,1} = \dfrac{\phi}{\rho(1-\gamma)}, \end{cases} \tag{5.3}$$

where

$$v_{i,j}^* = \operatorname{argmax}\{h_{i,j}(v) : v = 0, \overline{v}\} \tag{5.4}$$

and

$$h_{i,j}(v) = \Big(\frac{\sigma_C^2}{2} y_i^2 \Delta_y^2 F_{i,j}^{\overline{y}} - r_j \Delta_r F_{i,j}^{\overline{y}} - k r_j\Big) v - [\alpha r_j - (m-1)v]^- \, y_i \Delta_y^- F_{i,j}^{\overline{y}}$$
$$+ [\alpha r_j - (m-1)v]^+ y_i \Delta_y^+ F_{i,j}^{\overline{y}}.$$

Through the above difference equations and the standard iteration method, we obtain some numerical illustrations. Unless otherwise stated, values of parameters used in this section are presented in Table 1, which are the same as in [10].

Table 1 The parameters.

the values of parameters				
$m = 0.25$	$\phi = 29.279$	$\lambda = 0.887$	$k = 0$	$\rho = 0.07$
$\overline{v} = 0.025$	$\sigma_c = 0.5$	$\sigma_E = 0.1$	$\alpha = 0.05$	

Figure 1 shows the optimal conversion boundary $r = R(y)$ (the blue curve), under which is the conservation zone, and above which is the conversion zone. If (y, r) lies in the conservation zone, then the optimal strategy is no conversion, i.e., $v^* = 0$, keeping the reserve land for biodiversity conservation. In the case of (y, r) belonging to the conversion zone, the optimal strategy is to convert the land for conservation into that for production at the rate of $\overline{v}$ until the land area for biodieversity conservation r first hits the optimal conversion boundary. Moreover, we find that the optimal conversion boundary is increasing with respect to biodiversity conservation benefit per unit y. It means that the optimal land area for conservation at high y should be more than that at lower y, whereas, the optimal land area for agricultural production at high y should be less than that at lower y. It is clear that the characteristics are consistent with reality.

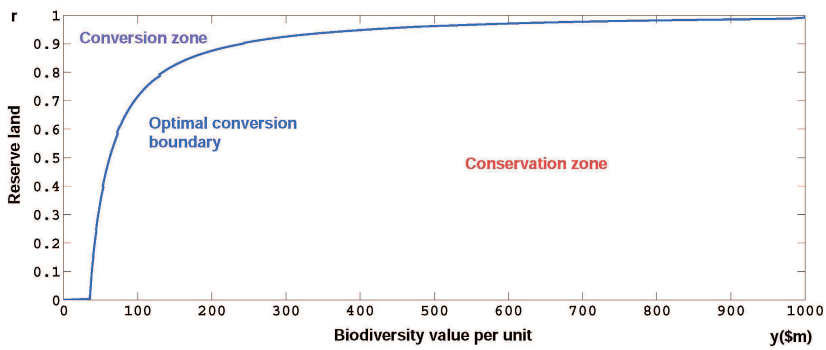

Figure 1 The optimal conversion boundary.

Figure 2 plots land value F as a function of biodiversity value per unit land y and reserve land r. From this figure, we find that land value F is increasing with respect to y, but it does not have the same monotonicity with respect to r, which is clearly illustrated in Table 2. In Table 2, the bold numbers represent the maximum value of F with respect to r for some y. Table 2 shows that F first increases and then decreases with respect to r, and the maximum value of F achieves at some $r^* \in (0, 1)$ rather than $r^* = 0$. The results are reasonable and coincide with our intuition.

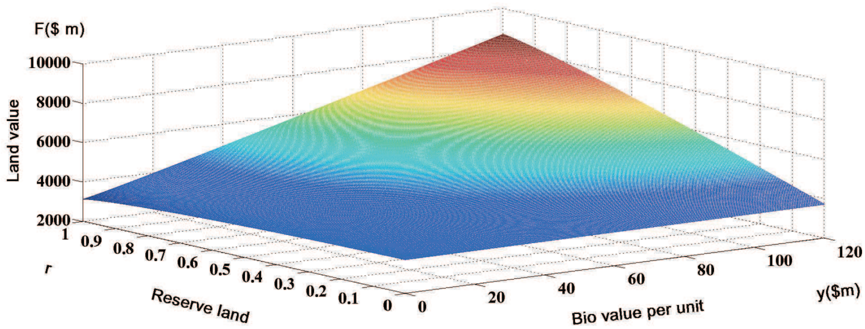

Figure 2 Land values F, as a function of reserve land r and biodiversity value per unit land y.

Table 2 Land values F (in billions of dollars)(in body of table), in terms of biodiversity value per unit land y (in millions of dollars) and reserve land r.

y	r										
	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
50	**5.32**	5.29	5.17	5.02	4.85	4.68	4.50	4.31	4.11	3.91	3.70
25	4.16	**4.24**	4.23	4.19	4.14	4.08	4.02	3.94	3.87	3.79	3.70
20	3.94	4.05	**4.06**	4.04	4.01	3.98	3.93	3.88	3.82	3.77	3.70
15	3.71	3.84	3.87	**3.88**	3.88	3.86	3.84	3.81	3.78	3.74	3.70
12	3.59	3.73	3.78	3.80	**3.81**	3.80	3.79	3.78	3.76	3.73	3.70
10	3.50	3.65	3.71	3.74	3.76	**3.76**	3.76	3.76	3.74	3.73	3.70
8	3.40	3.56	3.62	3.67	3.70	3.72	3.72	**3.73**	3.73	3.72	3.70
5	3.28	3.45	3.54	3.59	3.63	3.66	3.68	3.70	3.71	**3.71**	3.70

Appendix Some Results for Model L

In this section, we use the theoretical method in Section 3 and the numerical method in Section 5 to achieve some results of Model L introduced in [10], showing our improvement and perfection of this research. Specifically speaking, we will reveal that $\widehat{F}$ has a linear growth with respect to b via (5.5), rather than superlinear growth w.r.t b, which is shown via Figure 3 in [10]. And we will prove that $\widehat{F}$ always achieves its maximum value with respect to r at $r = 0$ for any $b > 0$.

Theorem 5.1 *The boundary value for the* HJB *equation* (1.8) *is* $\widehat{F}(b, 0) = \frac{\phi}{\rho(1-\gamma)} + \frac{b}{\rho-\alpha}$. *Moreover,* $\widehat{F}$ *has the following estimate*

$$0 \le \frac{\phi}{\rho(1-\gamma)}(1-r)^{1-\gamma} + \frac{b}{\rho-\alpha} \le \widehat{F}(b, r) \le \frac{\phi}{\rho(1-\gamma)} + \frac{b}{\rho-\alpha}. \tag{5.5}$$

Proof The proof is similar to that of Theorem 3.1 in Section 3.
Step 1 Taking the special control $v \equiv 0$, we compute that $R_t^{r;0} = r$ and

$$B_t^{b,r;0} = b + \int_0^t \alpha B_s^{b,r;0} \mathrm{d}s + \int_0^t \sigma_E B_s^{b,r;0} \mathrm{d}Z_s, \quad B_t^{b,r;0} = b \exp\Big\{\Big(\alpha - \frac{\sigma_E^2}{2}\Big)t + \sigma_E Z_t\Big\}.$$

So, we conclude that

$$\mathbb{E}(B_t^{b,r;0}) = \frac{b}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\Big\{\Big(\alpha - \frac{\sigma_E^2}{2}\Big)t + \sigma_E z - \frac{z^2}{2t}\Big\} \mathrm{d}z = b\mathrm{e}^{\alpha t}.$$

Combining the expression of the objective functional $\mathfrak{F}$ in (1.4), we deduce that

$$\begin{aligned}
\widehat{F}(b,r) &\geq \mathbb{E}\Big\{\int_0^\infty \mathrm{e}^{-\rho t}\Big[\frac{\phi}{1-\gamma}(1-R_t^{r;0})^{1-\gamma} + B_t^{b,r;0}\Big]\mathrm{d}t\Big\} \\
&= \int_0^\infty \mathrm{e}^{-\rho t}\Big[\frac{\phi}{1-\gamma}(1-r)^{1-\gamma} + \mathbb{E}(B_t^{b,r;0})\Big]\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-\rho t}\Big[\frac{\phi}{1-\gamma}(1-r)^{1-\gamma} + b\mathrm{e}^{\alpha t}\Big]\mathrm{d}t \\
&= \frac{\phi}{\rho(1-\gamma)}(1-r)^{1-\gamma} + \frac{b}{\rho-\alpha}. \qquad (5.6)
\end{aligned}$$

Taking $r \to 0^+$ in the above inequality, we know that

$$\liminf_{r\to 0^+} \widehat{F}(b,r) \geq \frac{\phi}{\rho(1-\gamma)} + \frac{b}{\rho-\alpha} \quad \text{for any } b > 0. \qquad (5.7)$$

Step 2 Construct the same stochastic control problem and repeat the argument as that in the proof of Theorem 3.1, we obtain that

$$\widehat{F}(b,r) \leq \widetilde{F}(b) \leq \frac{\phi}{\rho(1-\gamma)} + \frac{b}{\rho-\alpha} \quad \text{for any } b > 0,\ 0 < r \leq 1. \qquad (5.8)$$

We derive the following inequality through taking $r \to 0^+$ in (5.8),

$$\limsup_{r\to 0^+} \widehat{F}(b,r) \leq \frac{\phi}{\rho(1-\gamma)} + \frac{b}{\rho-\alpha} \quad \text{for any } b > 0.$$

Combining (5.7), we conclude the result via the continuity of $\widehat{F}$,

$$\widehat{F}(b,0) = \lim_{r\to 0^+} \widehat{F}(b,r) = \frac{\phi}{\rho(1-\gamma)} + \frac{b}{\rho-\alpha} \quad \text{for any } b > 0.$$

Then (5.5) follows from (5.6) and (5.8).

The results in Theorem A.1 show two results: $\widehat{F}$ has a linear growth with respect to b, and $\widehat{F}$ achieves its maximum value with respect to r when $r = 0$. In fact, these theoretical results are testified by the below numerical results as well.

Applying the numerical method in Section 4 to the HJB equation (1.8) with the boundary value condition in Theorem A.1, we provide some numerical results for Model L, which are shown in Figure 3 and Table 3. In Figure 3, we can find that $\widehat{F}$ really has a linear growth with respect to b, rather than super linear growth in [10]. In addition, we find that $\widehat{F}$ is decreasing with respect to r, which implies that for any fixed b, the maximum of $\widehat{F}$ arrives at $r = 0$. It means that the value of the land without biodiversity reserve achieves maximum (refer to Table 3), which is puzzling. As a matter of fact, we present that the maximum of the land value is achieved at a proper biodiversity reserve $r^* \in (0, 1)$ in our model (refer to Table 2), which is more reasonable.

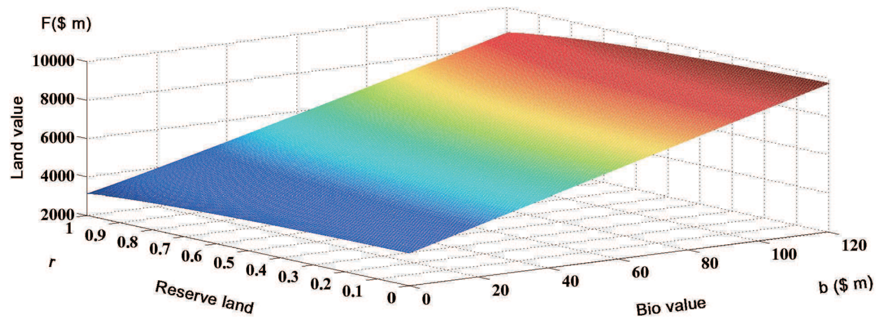

Figure 3 Land values F, as a function of reserve land r and biodiversity value b.

Table 3 Land values $\widehat{F}$ (in billions of dollars)(in body of table), in terms of biodiversity value b (in millions of dollars) and reserve land r.

b	r										
	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
25	4.19	4.38	4.48	4.56	4.63	4.69	4.76	4.82	4.87	4.91	**4.95**
13	3.64	3.83	3.93	4.00	4.06	4.11	4.16	4.21	4.26	4.31	**4.35**
12	3.60	3.79	3.89	3.96	4.02	4.07	4.12	4.16	4.21	4.26	**4.30**
10	3.52	3.70	3.80	3.87	3.93	3.98	4.03	4.07	4.12	4.16	**4.21**
9	3.46	3.65	3.75	3.82	3.88	3.93	3.97	4.01	4.06	4.10	**4.15**
7	3.38	3.57	3.67	3.74	3.80	3.85	3.89	3.92	3.96	4.01	**4.05**
6	3.34	3.53	3.63	3.70	3.76	3.81	3.85	3.88	3.92	3.96	**4.00**
5	3.30	3.49	3.59	3.66	3.72	3.77	3.81	3.84	3.87	3.91	**3.95**
4	3.26	3.45	3.55	3.62	3.68	3.73	3.77	3.80	3.83	3.86	**3.91**

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