Chin. Ann. Math. Ser. B
42(3), 2021, 327–332
DOI: 10.1007/s11401-021-0261-x

Chinese Annals of
Mathematics, Series B

An Explicit Example of a Reich Sequence for a Uniquely Extremal Quasiconformal Mapping

Xue MENG[1] Sihui ZHANG[2]

Abstract An explicit example of a Reich sequence for a uniquely extremal quasiconformal mapping in a borderline case between uniqueness and non-uniqueness is given.

Keywords Quasiconformal mapping, Uniquely extremal quasiconformal mapping, Reich sequence
2000 MR Subject Classification 30C62, 30C75

1 Introduction

Let Ω be a plane domain with at least two boundary points. The Teichmüller space $T(\Omega)$ is the space of equivalence classes of quasiconformal maps f from Ω to a variable domain $f(\Omega)$. Two quasiconformal maps f from Ω to $f(\Omega)$ and g from Ω to $g(\Omega)$ are equivalent if there is a conformal map c from $f(\Omega)$ onto $g(\Omega)$ and a homotopy through quasiconformal maps h_t mapping Ω onto $g(\Omega)$ such that $h_0 = c \circ f$, $h_1 = g$ and $h_t(p) = c \circ f(p)$ for every $t \in [0, 1]$ and every p in the boundary of Ω. Denote by $[f]$ the Teichmüller equivalence class of f; also sometimes denote the equivalence class by $[\mu]$ where μ is the Beltrami coefficient of f.

Denote by $M(\Omega)$ the open unit ball in $L^\infty(\Omega)$. For $\mu \in M(\Omega)$, define

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$

We say that μ is extremal in $[\mu]$ if $\|\mu\|_\infty = k_0([\mu])$, and uniquely extremal if $\|\nu\|_\infty > k_0(\mu)$ for any other $\nu \in [\mu]$. The corresponding f is also called extremal or uniquely extremal.

A quasiconformal mapping f will be said to be of Teichmüller type if its Beltrami cofficient μ is of Teichmüller type, i.e.,

$$\mu(z) = \frac{f_{\overline{z}}(z)}{f_z(z)} = k\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}, \quad z \in \Omega, \tag{1.1}$$

where $k \in (0, 1)$ is a constant and $\varphi_0(z) \neq 0$ a.e. is a measurable function in Ω. In particular, if φ_0 is holomorphic in Ω, we call f a Teichmüller mapping and the corresponding μ a Teichmüller differential.

Manuscript received August 13, 2020. Revised November 8, 2020.
[1]College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China.
E-mail: mengxuexq@163.com
[2]Corresponding author. College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China. E-mail: 071018005@fudan.edu.cn

Let $B(\Omega)$ be the Banach space consisting of holomorphic functions $\varphi(z)$ belonging to $L^1(\Omega)$, with norm

$$\|\varphi\| = \iint_\Omega |\varphi(z)|\mathrm{d}x\mathrm{d}y < \infty.$$

For $\mu \in M(\Omega)$ and $\varphi \in B(\Omega)$, set

$$\delta[\varphi] = k\|\varphi\| - \mathrm{Re} \iint_\Omega \mu(z)\varphi(z)\mathrm{d}x\mathrm{d}y.$$

In [4], Reich proved the following theorem.

Theorem 1.1 (see [4]) *Let $\mu \in M(\Omega)$ be given by (1.1). Suppose that there exists a sequence of functions $\varphi_n \in B(\Omega), n = 1, 2, \cdots$, such that*

$$\lim_{n\to\infty} \varphi_n(z) = \varphi_0(z) \tag{1.2}$$

pointwise a.e. in Ω,

$$\lim_{n\to\infty} \delta[\varphi_n] = 0. \tag{1.3}$$

Then f is uniquely extremal.

In [5], Reich showed that (1.3) can be replaced by the weaker assumption of boundedness of $\{\delta[\varphi_n]\}$, provided that (1.2) is strengthened appropriately. This was done in his theorem which is stated below.

Theorem 1.2 (see [5]) *Let $\mu \in M(\Omega)$ be given by (1.1). Suppose that there exists a sequence of functions $\varphi_n \in B(\Omega)$, $n = 1, 2, \cdots$, such that*

$$\lim_{n\to\infty} \varphi_n(z) = \varphi_0(z) \tag{1.4}$$

pointwise a.e. in Ω, $\varphi_0 \in L^1_{\mathrm{loc}}(\Omega)$,

$$\delta[\varphi_n] \le M, \quad n = 1, 2, \cdots, \tag{1.5}$$

$$\varliminf_{A\to\infty} \iint_{\Omega(n,A)} |\varphi_n(z)|\mathrm{d}x\mathrm{d}y = 0 \tag{1.6}$$

uniformly with respect to n, where $\Omega(n, A) = \{z \in \Omega : A|\varphi_0(z)| < |\varphi_n(z)|\}$. Then f is uniquely extremal.

Generally, if μ satisfies Reich's condition above, following [3] we call $\{\varphi_n\}$ a Reich sequence for μ on Ω. Note that a Reich sequence $\{\varphi_n\}$ for μ is not necessarily convergent pointwise on Ω (see [1–2]). Therefore, if in addition φ_n converges to some φ_0 pointwise a.e. on Ω, we call $\{\varphi_n\}$ a normal Reich sequence for μ on Ω.

Let $\mu \in M(\Omega)$ be given by (1.1). We say that μ satisfies weak Reich's condition on Ω if there exists a sequence $\{\varphi_n\}$ in $B(\Omega)$ such that (1.4)–(1.6) hold. If μ satisfies weak Reich's condition, we call $\{\varphi_n\}$ a weak Reich sequence for μ on Ω.

In order to show the unique extremality of f, when we cannot find the normal Reich sequence by Theorem 1.1, we can find the weak Reich sequence by Theorem 1.2 instead. For example, define

$$\Omega_a = \{z = x + \mathrm{i}y : x > |y|^a\}, \quad 1 < a < \infty, \quad \Omega_\infty = \{z = x + \mathrm{i}y : x > 0,\ |y| < 1\}. \tag{1.7}$$

Ever since [8], the family (1.7) has been known to possess a precise transition point $a = 3$. It is known that the horizontal stretch of Ω_a is uniquely extremal if and only if $3 \le a \le \infty$. It is easy to check that

$$\varphi_n(z) = \mathrm{e}^{\frac{-z}{n}}$$

provides a normal Reich sequence when $3 < a \le \infty$, but it fails to do so for the critical case $a = 3$. For a long time, in [7], Reich gave an explicit example of a normal Reich sequence for the critical case $a = 3$.

In this paper, we consider another uniquely extremal quasiconformal mapping, in a borderline case between uniqueness and non-uniqueness, a normal Reich sequence is given explicitly in this case for the first time.

Define $\omega_a = \{z = x + \mathrm{i}y : y > |x|^a,\ z \ne \mathrm{i}b,\ b > 0\}$, $T(z)$ the Teichmüller mapping generated by the quadratic differential $\varphi_0(z) = \frac{\mathrm{i}}{(z-\mathrm{i}b)}$. The complex dilatation is $\mu(z) = k\frac{z-\mathrm{i}b}{\mathrm{i}|z-\mathrm{i}b|}$, $0 < k < 1$.

In [6], it was proved that if $a \ge \frac{3}{2}$, then $T(z)$ is uniquely extremal, and in the proof, the normal Reich sequence is given when $a > \frac{3}{2}$, but when $a = \frac{3}{2}$, the normal Reich sequence is failed to work. In this paper, we construct the normal Reich sequence for this case.

Theorem 1.3 *The functions*

$$\varphi_n(z) = \frac{\mathrm{i} \cdot \exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]}{z - \mathrm{i}b}, \quad z \in \omega_{\frac{3}{2}},\ n = 1, 2, \cdots,$$

where $z^{\frac{1}{n}}$ *denotes the branch in the right half-plane that is real on the positive y-axis, provide a normal Reich sequence for* $\omega_{\frac{3}{2}}$.

2 Proof of Theorem 1.3

We have

$$\lim_{n\to\infty} \varphi_n(z) = \varphi_0(z),$$

$$|\varphi_n(z)| = \frac{|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]|}{|z - \mathrm{i}b|}$$

and

$$\delta[\varphi_n] = k \iint\limits_{\omega_{\frac{3}{2}}} \frac{|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]| - \mathrm{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\}}{|z - \mathrm{i}b|} \mathrm{d}x\mathrm{d}y.$$

We mainly focus on the left half part of $\omega_{\frac{3}{2}}$ to the upper of the line $\operatorname{Im} z = N$, $N > \max\{2b, 1\}$ first, and denote it by Ω, i.e., $\Omega = \{z : z \in \omega_{\frac{3}{2}},\ \operatorname{Re} z < 0,\ \operatorname{Im} z > N\}$. We also let $\widetilde{\Omega} = \{z : z \in \omega_{\frac{3}{2}},\ \operatorname{Re} z > 0,\ \operatorname{Im} z > N\}$. Obviously,

$$k \iint\limits_{\Omega\cup\widetilde{\Omega}} \frac{|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]| - \operatorname{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\}}{|z-\mathrm{i}b|}\mathrm{d}x\mathrm{d}y$$

equals

$$2k \iint\limits_{\Omega} \frac{|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]| - \operatorname{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\}}{|z-\mathrm{i}b|}\mathrm{d}x\mathrm{d}y.$$

Consider

$$k \iint\limits_{\Omega} \frac{|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]| - \operatorname{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\}}{|z-\mathrm{i}b|}\mathrm{d}x\mathrm{d}y.$$

Let $E(z) = |\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]| - \operatorname{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\}$, and let $\theta = \arg(-\mathrm{i}z)$, $z = x+\mathrm{i}y$, $-\mathrm{i}z = -\mathrm{i}x + y$. Hence $|z|\cos\theta = y$, $|z|\sin\theta = -x$. It is easy to have $\theta < \frac{\pi}{4}, 1 > \frac{\sin\theta}{\theta} > \frac{2\sqrt{2}}{\pi}$. From $\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}} < 1 < \frac{\pi\sin\theta}{2\sqrt{2}\theta}$, we have

$$\sin\frac{\theta}{n} < \frac{\pi\sin\theta}{2\sqrt{2}n}.$$

From

$$\operatorname{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\} = \exp\Big[-\frac{|z|^{\frac{1}{n}}\cos\frac{\theta}{n}}{2^n}\Big]\cdot\cos\Big(\frac{|z|^{\frac{1}{n}}\sin\frac{\theta}{n}}{2^n}\Big)$$

and

$$\begin{aligned}\cos\Big(\frac{|z|^{\frac{1}{n}}\sin\frac{\theta}{n}}{2^n}\Big) &> \cos\Big(\frac{|z|^{\frac{1}{n}}\pi\sin\theta}{2^n 2\sqrt{2}n}\Big) = \cos\Big(\frac{|z|^{\frac{1-n}{n}}\cdot\pi|x|}{2^n 2\sqrt{2}n}\Big)\\ &> \cos\Big(\frac{y^{\frac{1-n}{n}}\cdot\pi y^{\frac{2}{3}}}{2^n 2\sqrt{2}n}\Big) = \cos\Big(\frac{\pi\cdot y^{\frac{3-n}{3n}}}{2^n 2\sqrt{2}n}\Big)\\ &> \cos\Big(\frac{\pi}{2^n 2\sqrt{2}n}\Big) > 0,\end{aligned}$$

when n is sufficiently large, we obtain that

$$\operatorname{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\} > 0. \tag{2.1}$$

Since

$$|w| - \operatorname{Re} w = \frac{(\operatorname{Im} w)^2}{|w| + \operatorname{Re} w} < \frac{(\operatorname{Im} w)^2}{|w|}$$

when $\operatorname{Re} w > 0$, from (2.1) we have

$$E(z) \le \exp\Big[-\frac{|z|^{\frac{1}{n}}\cos\frac{\theta}{n}}{2^n}\Big]\cdot\sin^2\Big(\frac{|z|^{\frac{1}{n}}\sin\frac{\theta}{n}}{2^n}\Big). \tag{2.2}$$

Now we estimate the terms on the right of (2.2) when n is sufficiently large. From

$$\cos\frac{\theta}{n} \ge \cos\Big(\frac{2\cdot\pi}{3\cdot 4}\Big) = \frac{\sqrt{3}}{2} \triangleq c,$$

we get

$$\exp\Big[-\frac{|z|^{\frac{1}{n}}\cos\frac{\theta}{n}}{2^n}\Big]\le\exp\Big[-\frac{y^{\frac{1}{n}}c}{2^n}\Big].$$

From

$$\Big||z|^{\frac{1}{n}}\sin\frac{\theta}{n}\Big|\le|z|^{\frac{1-n}{n}}\cdot|z|\cdot\frac{\pi|\sin\theta|}{2\sqrt{2}n}\le y^{\frac{1-n}{n}}\cdot|x|\cdot\frac{\pi}{2\sqrt{2}n},$$

we obtain that

$$\sin^2\Big(\frac{|z|^{\frac{1}{n}}\sin\frac{\theta}{n}}{2^n}\Big)\le\frac{1}{2^{2n}}\cdot\frac{y^{\frac{2-2n}{n}}\cdot x^2\cdot\pi^2}{8n^2}.$$

Since $N>2b$, it is easy to see $|z-\mathrm{i}b|>\frac{y}{2}$ when $z\in\Omega$, hence

$$\begin{aligned}
&\iint\limits_{\Omega}\frac{|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]|-\mathrm{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\}}{|z-\mathrm{i}b|}\mathrm{d}x\mathrm{d}y\\
&\le\iint\limits_{\Omega}\exp\Big[\frac{-y^{\frac{1}{n}}c}{2^n}\Big]\frac{\frac{y^{\frac{2-2n}{n}}\cdot x^2\cdot\pi^2}{2^{2n}\cdot 8n^2}}{\frac{y}{2}}\mathrm{d}x\mathrm{d}y\\
&=\frac{2\pi^2}{2^{2n}8n^2}\cdot\int_N^{\infty}\exp\Big(\frac{-y^{\frac{1}{n}}c}{2^n}\Big)\cdot y^{\frac{2-3n}{n}}\mathrm{d}y\int_{-y^{\frac{2}{3}}}^{0}x^2\mathrm{d}x\\
&<\frac{2\pi^2}{2^{2n}24n^2}\int_0^{+\infty}\exp\Big(\frac{-y^{\frac{1}{n}}c}{2^n}\Big)y^{\frac{2-n}{n}}\mathrm{d}y.
\end{aligned}$$

Let $u=\frac{y^{\frac{1}{n}}}{2^n}$. Then $y=2^{n^2}u^n$, $\mathrm{d}y=2^{n^2}\cdot nu^{n-1}\mathrm{d}u$, hence

$$\begin{aligned}
\frac{2\pi^2}{2^{2n}24n^2}\int_0^{+\infty}\exp\Big(\frac{-y^{\frac{1}{n}}c}{2^n}\Big)y^{\frac{2-n}{n}}\mathrm{d}y&=\frac{2\pi^2}{2^{2n}24n^2}\int_0^{+\infty}\exp^{-cu}2^{2-n^n}u^{2-n}nu^{n-1}\mathrm{d}u\\
&=\frac{2\pi^2}{24n}\int_0^{+\infty}\exp^{-cu}u\mathrm{d}u=\frac{\pi^2}{12c^3n}.
\end{aligned}$$

From above we obtain

$$\iint\limits_{\Omega}\frac{|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]|-\mathrm{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\}}{|z-\mathrm{i}b|}\mathrm{d}x\mathrm{d}y<\frac{\pi^2}{12c^3n}. \tag{2.3}$$

Now we estimate another part of

$$\delta[\varphi_n]=k\iint\limits_{\omega_{\frac{3}{2}}}\frac{|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]|-\mathrm{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\}}{|z-\mathrm{i}b|}\mathrm{d}x\mathrm{d}y.$$

Let $w=\mathrm{e}^{\mathrm{i}\theta}=r\cos\theta+\mathrm{i}r\sin\theta$. Then

$$\begin{aligned}
|\mathrm{e}^w|-\mathrm{Re}(\mathrm{e}^w)&=\mathrm{e}^{r\cos\theta}-\mathrm{e}^{r\cos\theta}\cos(r\sin\theta)\\
&=\mathrm{e}^{r\cos\theta}\cdot2\sin^2\frac{r\sin\theta}{2}<\frac{1}{2}\mathrm{e}^{r\cos\theta}r^2\sin^2\theta\le\frac{1}{2}\mathrm{e}^r r^2.
\end{aligned}$$

Let

$$\Omega_N=\omega_{\frac{3}{2}}-(\Omega\cup\widetilde{\Omega}).$$

Then $0 \le y \le N$, $0 \le |x| \le N^{\frac{2}{3}}$, we obtain that when $z \in \Omega_N$, $|-\mathrm{i}z| = \sqrt{x^2+y^2} < 2N$. Let

$$w = -\frac{(-\mathrm{i}z)^{\frac{1}{n}}}{2^n}, \quad r = |w| = \Big| -\frac{(-\mathrm{i}z)^{\frac{1}{n}}}{2^n}\Big| < \frac{(2N)^{\frac{1}{n}}}{2^n}.$$

From above we get

$$\begin{aligned} &|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]| - \mathrm{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\} \\ &< \frac{1}{2}\cdot \mathrm{e}^{|-\frac{(-\mathrm{i}z)^{\frac{1}{n}}}{2^n}|}\cdot \Big| -\frac{(-\mathrm{i}z)^{\frac{1}{n}}}{2^n}\Big|^2 < \frac{1}{2}\cdot \mathrm{e}^{\frac{(2N)^{\frac{1}{n}}}{2^n}}\cdot \frac{(2N)^{\frac{2}{n}}}{2^{2n}} < \frac{3}{2}\cdot \frac{(2N)^{\frac{2}{n}}}{2^{2n}}. \end{aligned}$$

From $-\frac{N}{2} < -b < y-b < N-b < N$ and $|z-\mathrm{i}b| = \sqrt{(y-b)^2+x^2} < \sqrt{N^2+N^2} < 2N$, we obtain

$$\begin{aligned} &\iint\limits_{\Omega_N} \frac{|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]| - \mathrm{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\}}{|z-\mathrm{i}b|}\mathrm{d}x\mathrm{d}y \\ &< \frac{3}{2}\cdot\frac{(2N)^{\frac{2}{n}}}{2^{2n}}\cdot \int\limits_{\Omega_N} \frac{\mathrm{d}x\mathrm{d}y}{|z-\mathrm{i}b|} \le \frac{3(2N)^{\frac{2}{n}}}{2^{2n+1}}\cdot \int_0^{2N}\int_0^{2\pi} \frac{r\mathrm{d}r\mathrm{d}\theta}{r} = \frac{3\pi(2N)^{\frac{2}{n}}N}{2^{2n-1}}. \end{aligned}$$

From above we get

$$\iint\limits_{\Omega_N} \frac{|\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]| - \mathrm{Re}\{\exp[-(2^{-n}(-\mathrm{i}z)^{\frac{1}{n}})]\}}{|z-\mathrm{i}b|}\mathrm{d}x\mathrm{d}y < \frac{3\pi(2N)^{\frac{2}{n}}N}{2^{2n-1}}. \tag{2.4}$$

Hence from (2.3)–(2.4), we obtain

$$\delta[\varphi_n] < k\Big(\frac{\pi^2}{6c^3n} + \frac{3\pi(2N)^{\frac{2}{n}}N}{2^{2n-1}}\Big),$$

which approaches 0 when $n \to \infty$. The proof of Theorem 1.3 is complete.

Acknowledgement The authors sincerely thank the referees for their valueable suggestions.

References

[1] Bozin, V., Lakic, N., Marković, V. and Mateljević, M., Unique extremality, *J. Anal. Math.*, **75**, 1998, 299–338.

[2] Bozin, V., Marković, V. and Mateljević, M., The unique extremality in the tangent space of the universal Teichmüller space, *Integral Transform. Spec. Funct.*, **6**, 1998, 145–149.

[3] Mateljević, M., Unique extremality of quasiconformal mappings, *J. Anal. Math.*, **107**, 2009, 39–63.

[4] Reich, E., A criteria for unique extremality of Teichmüller mapppings, *Indiana Univ. Math. J.*, **3**, 1981, 441–447.

[5] Reich, E., On criteria for unique extremality of Teichmüller mappings, *Ann. Acad. Sci. Fenn. Ser. A Math.*, **6**, 1981, 289–301.

[6] Reich, E., On the structure of the family of extremal quasiconformal mappings of parabolic regions, *Complex Variables*, **5**, 1986, 289–300.

[7] Reich, E., A critical exmple for the necessary and sufficient condition for unique quasiconformal extremality, *Contemporary Mathematics*, **382**, 2005, 343–348.

[8] Reich, E. and Kurt, S., On the extremality of certain Teichmüller mappings, *Comment. Math. Helv.*, **45**, 1970, 353–362.