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On the Minimal Solutions of Variational Inequalities in Orlicz-Sobolev Spaces

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Abstract In this paper, the author studies the existence of the minimal nonnegative solutions of some elliptic variational inequalities in Orlicz-Sobolev spaces on bounded or unbounded domains. She gets some comparison results between different solutions as tools to pass to the limit in the problems and to show the existence of the minimal solutions of the variational inequalities on bounded domains or unbounded domains. In both cases, coercive and noncoercive operators are handled. The sufficient and necessary conditions for the existence of the minimal nonnegative solution of the noncoercive variational inequality on bounded domains are established.

Keywords Orlicz-Sobolev spaces, Elliptic variational inequalities, Minimal nonnegative solutions, Bounded domains, Unbounded domains
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1 Introduction

In [10–12, 23], the sub-supersolution method was used to prove the existence of solutions and extremal solutions, confined between their subsolutions and supersolutions, for elliptic variational inequalities and nonlinear elliptic equations in Sobolev spaces or Orlicz-Sobolev spaces defined on bounded domains. There are many existence results for elliptic equations in Orlicz-Sobolev spaces on bounded domains such as [2–4, 7–9, 13–14, 16–19]. Landes [22] proved the existence of a weak solution of a quasilinear elliptic equation without any lower order term on an unbounded domain. In [6], a different method was used to establish the existence of the minimal solution to some elliptic variational inequalities in Sobolev spaces defined on bounded or unbounded domains. When trying to weaken the restriction on the operators in [6], one is led to replace Sobolev spaces by Orlicz-Sobolev spaces. In this paper, we will take into account the minimal solutions of elliptic variational inequalities on bounded and unbounded domains by using a method in [6] different from the above.

In the present work, we will show the existence of the minimal nonnegative solutions for some variational inequalities with coercive operators or noncoercive operators on bounded domains or unbounded domains. Some comparison principles are investigated for the minimal solutions of variational inequalities on bounded domains or unbounded domains. In particular noncoercive one, the sufficient and necessary conditions for the existence of the minimal nonnegative solution are established on bounded domains. We will use the idea introduced in [6].

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The paper is organized as follows. Section 2 contains some preliminaries which will be needed in the next two sections. In Section 3, we start with the coercive case and prove the existence theorems of nonnegative solutions for some elliptic variational inequalities on bounded domain. Some comparison results between different solutions are established as tools to pass to the limit in the problems and we show the existence of the minimal solutions for the variational inequalities with coercive operators or noncoercive operators defined on bounded domains. The sufficient and necessary conditions for the existence of the minimal nonnegative solution for the noncoercive one are established. Section 4 is devoted to showing the existence of the minimal solutions for variational inequalities on unbounded domains. In both cases, coercive and noncoercive operators are handled.

2 Preliminaries

For quick reference, we recall some basic results of Orlicz spaces. Good references are [1, 7, 18, 21, 25].

2.1 *N*-function

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e., M is continuous, convex, with $M(u) > 0$ for $u > 0$, $\frac{M(u)}{u} \to 0$ as $u \to 0$, and $\frac{M(u)}{u} \to +\infty$ as $u \to +\infty$. Equivalently, M admits the representation $M(u) = \int_0^u \varphi(t)\mathrm{d}t$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing, right continuous function, with $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$, and $\varphi(t) \to +\infty$ as $t \to +\infty$.

Clearly, $M(u) \le u\varphi(u) \le M(2u)$ for all $u \ge 0$.

The N-function $\overline{M}$ conjugated to M is defined by $\overline{M}(v) = \int_0^v \phi(s)\mathrm{d}s$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\phi(s) = \sup\{t : \varphi(t) \le s\}$.

φ, ϕ are called the right-hand derivatives of $M, \overline{M}$, respectively.

The N-function M is said to satisfy the Δ_2 condition near infinity ($M \in \Delta_2$ for short), if for some $k > 1$ and $u_0 > 0$, $M(2u) \le kM(u)$, $\forall\, u \ge u_0$. It is readily seen that this will be the case if and only if for every $l > 1$ there exists a positive constant $k = k(l)$ and $\widetilde{u} > 0$, such that

$$M(lu) \le kM(u), \quad \forall\, u \ge \widetilde{u}. \tag{2.1}$$

Moreover, one has the following Young inequality: $uv \le M(u) + \overline{M}(v)$, $\forall\, u, v \ge 0$, and the equality holds if and only if $v = \varphi(u)$ or $u = \phi(v)$.

We will extend these N-functions into even functions on all $\mathbb{R}$.

Let P, Q be two N-functions. We say that P grows essentially less rapidly than Q near infinity, denoted as $P \ll Q$, if for every $\varepsilon > 0$, $\frac{P(t)}{Q(\varepsilon t)} \to 0$ as $t \to +\infty$. This is the case if and only if $\lim\limits_{t\to+\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

For a measurable function u on Ω, its modular is defined by $\rho_M(u) = \int_\Omega M(|u(x)|)\mathrm{d}x$.

The Sobolev conjugate N-function M_* of M is defined by

$$M_*^{-1}(t) = \int_0^t \frac{M^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\mathrm{d}\tau, \quad t \ge 0.$$

2.2 Orlicz spaces

Let Ω be an open and bounded subset of $\mathbb{R}^N$ and M be an N-function. The Orlicz class $\mathcal{K}_M(\Omega)$ (respectively, the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\rho_M(u) < +\infty \Big(\text{respectively, } \rho_M\Big(\frac{u}{\lambda}\Big) < +\infty \text{ for some } \lambda > 0\Big).$$

$L_M(\Omega)$ is a Banach space under the (Luxemburg) norm

$$\|u\|_{(M)} = \inf\Big\{\lambda > 0 : \rho_M\Big(\frac{u}{\lambda}\Big) \le 1\Big\},$$

and $\mathcal{K}_M(\Omega)$ is a convex subset of $L_M(\Omega)$ but not necessarily a linear space.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if $M \in \Delta_2$, moreover, $L_M(\Omega)$ is separable.

$L_M(\Omega)$ is reflexive if and only if $M \in \Delta_2$ and $\overline{M} \in \Delta_2$.

Convergences in norm and in modular are equivalent if and only if $M \in \Delta_2$.

The dual space of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)\mathrm{d}x$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{(\overline{M})}$.

2.3 Orlicz-Sobolev spaces

We now turn to the Orlicz-Sobolev space: $W^1L_M(\Omega)$ (respectively, $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional partial derivatives lie in $L_M(\Omega)$ (respectively, $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{\Omega,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{(M)}.$$

Denote $\|Du\|_{(M)} = \||Du|\|_{(M)}$ and $\|u\|_{1,M,\Omega} = \|u\|_{(M)} + \|Du\|_{(M)}$. Clearly, $\|u\|_{1,M,\Omega}$ is equivalent to $\|u\|_{\Omega,M}$.

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

If $M \in \Delta_2$, then $W^1L_M(\Omega) = W^1E_M(\Omega)$. If $M \in \Delta_2$ and $\overline{M} \in \Delta_2$, then $W^1L_M(\Omega) = W^1E_M(\Omega)$ are reflexive and the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ are equivalent.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathscr{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_M)$ closure of $\mathscr{D}(\Omega)$ in $W^1L_M(\Omega)$. If $M \in \Delta_2$, then $W_0^1L_M(\Omega) = W_0^1E_M(\Omega)$ and $W_0^1L_M(\Omega)$ is separable. If $M \in \Delta_2$ and $\overline{M} \in \Delta_2$, then $W_0^1L_M(\Omega)$ is reflexive.

3 Variational Inequalities in Bounded Domains

This section is devoted to studying the existence of nonnegative solutions and their minimal solutions for some quasilinear variational inequalities in bounded domains. We investigate variational inequalities with coercive operators in Subsection 3.1 and with noncoercive operators in Subsection 3.2, respectively.

3.1 Variational inequalities with coercive operator

Let Ω be a bounded domain in $\mathbb{R}^N (N \ge 1)$ with Lipschitz boundary, M be an N-function and $\overline{M}$ be the complementary function of M, and φ, ϕ are the right-hand derivatives of $M, \overline{M}$, respectively. Assume that $M, \overline{M} \in \Delta_2$.

In what follows we denote by $L^0(\Omega)$ the set of all (equivalence classes of) Lebesgue measurable functions from Ω to $\mathbb{R}$. For $u, v \in L^0(\Omega)$, $U, V \subset L^0(\Omega)$, we use the standard notation: $u \le v \Leftrightarrow u(x) \le v(x)$ for a.e. $x \in \Omega$, $u \wedge v = \min\{u, v\}, u \vee v = \max\{u, v\}$,

$U \wedge V = \{u \wedge v : u \in U, v \in V\}$, $U \vee V = \{u \vee v : u \in U, v \in V\}$, $u^+ := u \vee 0, u^- := -u \wedge 0$. We consider the usual ordering relation $(W_0^1 L_M(\Omega), \leq)$.

We denote by $\mathcal{K}$ a closed convex subset of $W_0^1 L_M(\Omega)$ containing 0 such that the lattice condition

$$\mathcal{K} \vee \mathcal{K} \subset \mathcal{K}, \quad \mathcal{K} \wedge \mathcal{K} \subset \mathcal{K}$$

is satisfied. This type of lattice convex sets usually occurs in applications. For example, $\mathcal{K} = W_0^1 L_M(\Omega)$ for equations, and $\mathcal{K} = \{u \in W_0^1 L_M(\Omega) : u(x) \geq \psi(x) \text{ for a.e. } x \in \Omega\}$ for obstacle problems, where $\psi : \Omega \to \mathbb{R}$ is an obstacle function (see [8–9, 13, 19]).

Let $a(x,\xi) = (a_i(x,\xi))_{1\leq i\leq N}$ and $a_0(x,\xi)$ be a family of Carathéodory's functions defined on $\Omega \times \mathbb{R}^{N+1}$ such that

$$(x,\xi) \mapsto a_i(x,\xi) \text{ is measurable on } \Omega \times \mathbb{R}^{N+1}, \quad i = 0,1,\cdots,N, \tag{3.1}$$

$$\xi \mapsto a_i(x,\xi) \text{ is continuous on } \mathbb{R}^{N+1}, \quad i = 0,1,\cdots,N, \tag{3.2}$$

$$\sum_{i=0}^{N} a_i(x,\xi)\xi_i \geq \alpha \sum_{i=0}^{N} M(|\xi_i|), \tag{3.3}$$

$$\sum_{i=0}^{N} [a_i(x,\xi) - a_i(x,\xi')](\xi_i - \xi_i') \geq 0, \tag{3.4}$$

$$|a_i(x,\xi)| \leq \vartheta(x) + \beta \sum_{i=0}^{N} \overline{M}^{-1}(M(|\xi_i|)), \quad i = 0,1,\cdots,N \tag{3.5}$$

for a.e. $x \in \Omega$ and for all $\xi = (\xi_i)_{0\leq i\leq N}$, $\xi' = (\xi_i')_{0\leq i\leq N} \in \mathbb{R}^{N+1}$, where $\alpha, \beta > 0$ and $\vartheta \in L_{\overline{M}}(\Omega)$.

Let $f \in L_{\overline{M}}(\Omega)$. We consider the following variational inequality:

$$\begin{cases} u \in \mathcal{K}, \\ \langle Au, v-u\rangle \geq \displaystyle\int_\Omega f(v-u)\mathrm{d}x, \quad \forall\, v \in \mathcal{K}, \end{cases} \tag{3.6}$$

where A is a nonlinear operator defined from $W_0^1 L_M(\Omega)$ into its dual by

$$Au(x) := -\operatorname{div} a(x,u,\nabla u) + a_0(x,u,\nabla u).$$

From (3.5), the operator A is well defined. For simplicity we set the dual pairing $\langle \cdot,\cdot \rangle = \langle \cdot,\cdot \rangle_{(W_0^1 L_M(\Omega))^*, W_0^1 L_M(\Omega)}$. The above duality is equivalent to

$$\langle Au, v\rangle = \int_\Omega a(x,u,\nabla u)\nabla v \mathrm{d}x + \int_\Omega a_0(x,u,\nabla u) v \mathrm{d}x \quad \text{for all } v \in W_0^1 L_M(\Omega). \tag{3.7}$$

Let $\ell > 0$ be a real number and Ω_l be the cylinder $(-\ell,\ell) \times \Omega$. The points in $\mathbb{R}^{N+1}$ are denoted by (y,x) with $x = (x_1,\cdots,x_N) \in \mathbb{R}^N$ and the gradient operator defined over $\mathbb{R}^{N+1}$ is also denoted by $\nabla' = (\partial_y, \nabla)$ with $\nabla = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_N})$. We set

$$\mathcal{K}_\ell := \{v \in W_0^1 L_M(\Omega_\ell) \mid v(y,\cdot) \in \mathcal{K} \quad \text{a.e. in } (-\ell,\ell)\}.$$

This is a closed convex subset of $W_0^1 L_M(\Omega_\ell)$.

Let $\psi : [0,+\infty) \to [0,+\infty)$ be an increasing function. By [20, Lemma 8.2], $\psi(|\xi|)\frac{\xi}{|\xi|}$ is monotone on $\mathbb{R}^n$ $(n \geq 1)$. However, we will show the strict monotonicity of $\psi(|\xi|)\frac{\xi}{|\xi|}$ on $\mathbb{R}^n$ and give another method to proof the monotonicity of $\psi(|\xi|)\frac{\xi}{|\xi|}$.

Lemma 3.1 *If $\psi : [0, +\infty) \to [0, +\infty)$ is increasing, then $\psi(|\xi|)\frac{\xi}{|\xi|}$ is monotone on $\mathbb{R}^n$ ($n \geq 1$). Moreover, if ψ is strictly increasing, then $\psi(|\xi|)\frac{\xi}{|\xi|}$ is strictly monotone.*

Proof Let ξ, η be two nonzero vectors in $\mathbb{R}^n$, with angle $\theta \in [0, \pi]$ between them at the origin. Then

$$\begin{aligned}&\Big[\frac{\psi(|\xi|)}{|\xi|}\xi - \frac{\psi(|\eta|)}{|\eta|}\eta\Big](\xi - \eta)\\ &= [\psi(|\xi|) - \psi(|\eta|)](|\xi| - |\eta|) + \Big[\frac{\psi(|\xi|)}{|\xi|} + \frac{\psi(|\eta|)}{|\eta|}\Big](|\xi||\eta| - \xi \cdot \eta).\end{aligned} \tag{3.8}$$

The first term of the right-hand side of (3.8) is nonnegative since ψ is increasing. Since

$$\xi \cdot \eta = |\xi||\eta| \cos\theta, \tag{3.9}$$

the second term of the right-hand side of (3.8) is nonnegative. Consequently,

$$\Big[\frac{\psi(|\xi|)}{|\xi|}\xi - \frac{\psi(|\eta|)}{|\eta|}\eta\Big](\xi - \eta) \geq 0.$$

Moreover, suppose that there exist $\xi_1, \xi_2 \in \mathbb{R}^N$ with $\xi_1 \neq \xi_2$ such that $\frac{\psi(|\xi_1|)}{|\xi_1|}\xi_1 = \frac{\psi(|\xi_2|)}{|\xi_2|}\xi_2$. It follows from (3.8)–(3.9) that

$$[\psi(|\xi_1|) - \psi(|\xi_2|)](|\xi_1| - |\xi_2|) = 0 \tag{3.10}$$

and $\cos\theta_1 = 1$, where $\theta_1 \in [0, \pi]$ is the angle between ξ_1 and ξ_2 at the origin. This yields that $\theta_1 = 0$. Therefore, there exists $\lambda > 0$ with $\lambda \neq 1$ such that $\xi_1 = \lambda\xi_2$. It implies that $|\xi_1| \neq |\xi_2|$. Immediately, $\psi(|\xi_1|) \neq \psi(|\xi_2|)$ since ψ is strictly increasing. It contradicts (3.10). This completes the proof of the lemma.

We recall the following notation which will be used later. It can be referred to [24, p. 25] or [5, Definition 1].

Definition 3.1 (see [24]) *Let X be a reflexive Banach space. The operator $T : X \to X^*$ is called pseudomonotone if*

(i) *T is bounded; and*

(ii) *for any sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u_0$ weakly in X and $\limsup\limits_{n\to\infty}\langle T(u_n), u_n - u_0\rangle \leq 0$, the inequality*

$$\liminf_{n\to\infty}\langle T(u_n), u_n - v\rangle \geq \langle T(u_0), u - v\rangle, \quad \forall\, v \in X$$

holds.

Theorem 3.1 *Assume that $f \in L_{\overline{M}}(\Omega)$ is nonnegative and the assumptions (3.1)–(3.5) are satisfied. Then for every $\ell > 0$, the following variational inequality*

$$\begin{cases} u_\ell \in \mathcal{K}_\ell, \\ \displaystyle\int_{\Omega_\ell} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell \partial_y (v - u_\ell)\mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell}\langle Au_\ell, v - u_\ell\rangle \mathrm{d}y \geq \int_{\Omega_\ell} f(v - u_\ell)\mathrm{d}x\mathrm{d}y, \quad \forall\, v \in \mathcal{K}_\ell \end{cases} \tag{3.11}$$

has at least one solution u_ℓ with $u_\ell \geq 0$. Moreover, if φ is strictly increasing, or a and a_0 are strictly monotone, i.e., for a.e. $x \in \Omega$ and all $\xi = (\xi_i)_{0\leq i\leq N}$, $\xi' = (\xi'_i)_{0\leq i\leq N} \in \mathbb{R}^{N+1}$,

$$\sum_{i=0}^{N}[a_i(x, \xi) - a_i(x, \xi')](\xi_i - \xi'_i) > 0, \quad \textit{with } \xi \neq \xi', \tag{3.12}$$

then there exists a unique solution u_ℓ of (3.11) and $u_\ell \geq 0$.

Proof For $\ell > 0$, we define the operator $\mathscr{T}_\ell : W_0^1 L_M(\Omega_\ell) \to (W_0^1 L_M(\Omega_\ell))^*$ by

$$\langle \mathscr{T}_\ell u, v\rangle = \int_{\Omega_\ell} \frac{\varphi(|\partial_y u|)}{|\partial_y u|}\partial_y u \partial_y v \mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell} \langle Au, v\rangle \mathrm{d}y, \quad \forall\, u, v \in W_0^1 L_M(\Omega_\ell).$$

Denote $z = (y, x)$. Thanks to (2.1), (3.5) and the Young inequality, we can deduce that there exist positive constants $C(\beta, N, K)$ and $C(N, \ell, \widetilde{u})$ such that

$$\begin{aligned} |\langle \mathscr{T}_\ell u, v\rangle| \le C(\beta, N, K) \int_{\Omega_\ell} [&M(|u(z)|) + M(|\nabla' u(z)|) \\ &+ M(|v(z)|) + M(|\nabla' v(z)|)]\mathrm{d}z + C(N, \ell, \widetilde{u}) \end{aligned} \tag{3.13}$$

for all $u, v \in W_0^1 L_M(\Omega_\ell)$. Therefore, $\mathscr{T}_\ell$ is well defined. From (3.13), it is easy to see that $\mathscr{T}_\ell$ is bounded. In view of (3.4) and Lemma 3.1, we can show that $\mathscr{T}_\ell$ is monotone. In a similar way as [10, Lemma 8], we can check that $\mathscr{T}_\ell$ is continuous. Thanks to [26, Propositon 27.6(a)], $\mathscr{T}_\ell$ is pseudomonotone. For $u_0 \in \mathcal{K}_\ell$, in a way similar to the proof of [10, Theorem 11], we have

$$\langle \mathscr{T}_\ell(u) - f, u - u_0\rangle \ge C_1 \int_{\Omega_\ell} \Big[M(|\partial_y u|) + \sum_{i=0}^{N} M(|\partial_{x_i} u|)\Big]\mathrm{d}x\mathrm{d}y - C_2 \tag{3.14}$$

for some positive constants C_1 and C_2. For $u \in W_0^1 L_M(\Omega_\ell)$, define

$$\|u\| = \inf\Big\{\lambda > 0 : \int_{\Omega_\ell} \Big[M\Big(\frac{|\partial_y u|}{\lambda}\Big) + \sum_{i=0}^{N} M\Big(\frac{|\partial_{x_i} u|}{\lambda}\Big)\Big]\mathrm{d}x\mathrm{d}y \le 1\Big\}.$$

Then $\frac{1}{N+2}\|u\| \le \|u\|_{1,M,\Omega_\ell} \le (N+2)\|u\|$. In view of (3.14), for any $\varepsilon > 0$, we have

$$\begin{aligned} \langle \mathscr{T}_\ell(u) - f, u - u_0\rangle &\ge C_1(\|u\| - \varepsilon) \int_{\Omega_\ell} \Big[M\Big(\frac{|\partial_y u|}{\|u\| - \varepsilon}\Big) + \sum_{i=0}^{N} M\Big(\frac{|\partial_{x_i} u|}{\|u\| - \varepsilon}\Big)\Big]\mathrm{d}x\mathrm{d}y - C_2 \\ &\ge C_1(\|u\| - \varepsilon) - C_2. \end{aligned}$$

Therefore, $\langle \mathscr{T}_\ell(u) - f, u - u_0\rangle > 0$ when $\|u\|_{1,M,\Omega_\ell}$ is sufficiently large. From the above results, we can deduce that the conditions (i)–(iv) in [19] hold. According to [19, Proposition 1], the variational inequality (3.11) has at least one solution u_ℓ.

Now, we prove that u_ℓ is nonnegative. Taking $v = u_\ell^+ \in \mathcal{K}_\ell$ in (3.11) we have

$$\int_{\Omega_\ell} \varphi(|\partial_y u_\ell^-|)|\partial_y u_\ell^-|\mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell} \langle A(-u_\ell^-), (-u_\ell^-)\rangle \mathrm{d}y \le -\int_{\Omega_\ell} f u_\ell^- \mathrm{d}x\mathrm{d}y \le 0.$$

In view of (3.4), we have

$$\int_{\Omega_\ell} M(|\partial_y u_\ell^-|)\mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell}\int_{\Omega} \alpha \sum_{i=0}^{N} M(|\partial_{x_i} u_\ell^-|)\mathrm{d}x\mathrm{d}y \le 0.$$

Consequently, $u_\ell \ge 0$.

Suppose that there exists another solution u_ℓ' of (3.11) with $u_\ell \ne u_\ell'$. When we take $v = u_\ell'$ in (3.11) and $v = u_\ell$ in (3.11) written for u_ℓ' and add the two inequalities, it comes

$$\int_{\Omega_\ell} \Big[\frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell - \frac{\varphi(|\partial_y u_\ell'|)}{|\partial_y u_\ell'|}\partial_y u_\ell'\Big]\partial_y(u_\ell - u_\ell')\mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell} \langle Au_\ell - Au_\ell', u_\ell - u_\ell'\rangle \mathrm{d}y \le 0.$$

If φ is strictly increasing or a and a_0 are strictly monotone, we can deduce, by Lemma 3.1 and (3.4), or by (3.12), $u_\ell = u_\ell'$. It is a contradiction.

Similar to the proof of Theorem 3.1, it is easy to show the following theorem.

Theorem 3.2 *Assume that $f \in L_{\overline{M}}(\Omega)$ is nonnegative and the assumptions* (3.1)–(3.5) *are satisfied. Then* (3.6) *has at least one solution u with $u \geq 0$. Moreover, if a and a_0 satisfy* (3.12), *then there exists a unique solution of* (3.6) *and $u \geq 0$.*

Does there exist the minimal nonnegative solution of problem (3.6) when it has more than one solution? The following theorem will give the answer.

Theorem 3.3 *Assume that $f \in L_{\overline{M}}(\Omega)$ is nonnegative, φ is strictly increasing and the assumptions* (3.1)–(3.5) *are satisfied. Then the pointwise limit of $\{u_\ell\}_\ell$ is the minimal nonnegative solution of* (3.6), *where u_ℓ is the solution of* (3.11), *for any $\ell > 0$.*

Moreover, if u_1 and u_2 are the minimal nonnegative solutions of (3.6) *obtained by replacing f with f_1 and f_2 respectively, then $f_1 \leq f_2$ implies $u_1 \leq u_2$.*

Proof Step 1 The sequence $\{u_\ell\}_\ell$ is nondecreasing and bounded above by any nonnegative solution of (3.6).

Let $0 < \ell < \ell'$. Extending u_ℓ by 0 on $\Omega_{\ell'}$ and since $u_{\ell'}$ is nonnegative, when we take $v = u_\ell - (u_\ell - u_{\ell'})^+ \in \mathcal{K}_\ell$ in (3.11) and $v = u_{\ell'} + (u_\ell - u_{\ell'})^+ \in \mathcal{K}_{\ell'}$ in (3.11) written for $u_{\ell'}$ and $\Omega_{\ell'}$ and add the two inequalities, it comes

$$\begin{aligned}
&\int_{\Omega_\ell} \Big[\frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell - \frac{\varphi(|\partial_y u_{\ell'}|)}{|\partial_y u_{\ell'}|}\partial_y u_{\ell'}\Big]\partial_y (u_\ell - u_{\ell'})^+ \mathrm{d}x\mathrm{d}y \\
&\quad + \int_{-\ell}^{\ell} \langle A u_\ell - A u_{\ell'}, (u_\ell - u_{\ell'})^+\rangle \mathrm{d}y \\
&\leq - \int_{\Omega_{\ell'}\setminus\Omega_\ell} \frac{\varphi(|\partial_y u_{\ell'}^-|)}{|\partial_y u_{\ell'}^-|}\partial_y u_{\ell'}^- \mathrm{d}x\mathrm{d}y - \int_{(-\ell',\ell')\setminus(-\ell,\ell)} \langle A(-u_{\ell'}^-), -u_{\ell'}^-\rangle \mathrm{d}y - \int_{\Omega_{\ell'}\setminus\Omega_\ell} f u_{\ell'}^- \mathrm{d}x\mathrm{d}y \\
&\leq 0.
\end{aligned}$$

Thanks to the condition (3.4) we deduce

$$\int_{\Omega_\ell} \Big[\frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell - \frac{\varphi(|\partial_y u_{\ell'}|)}{|\partial_y u_{\ell'}|}\partial_y u_{\ell'}\Big]\partial_y (u_\ell - u_{\ell'})^+ \mathrm{d}x\mathrm{d}y \leq 0.$$

Using Lemma 3.1, we have $u_\ell(y,x) \leq u_{\ell'}(y,x)$ for a.e. $(y,x) \in \Omega_\ell$, which shows that the sequence $\{u_\ell\}_\ell$ is nondecreasing.

Let $\ell > 0$ and u be a nonnegative solution of (3.6). Taking $v = u + (u_\ell(y,\cdot) - u)^+ \in \mathcal{K}$ as a test function in (3.6), for a.e. $y \in (-\ell,\ell)$, and integrating in y we derive

$$\int_{-\ell}^{\ell} \langle A u, (u_\ell - u)^+\rangle \mathrm{d}y \geq \int_{\Omega_\ell} f (u_\ell - u)^+ \mathrm{d}x\mathrm{d}y. \tag{3.15}$$

Taking $v = u_\ell - (u_\ell - u)^+ \in \mathcal{K}_\ell$ as a test function in (3.11) we can deduce that

$$\begin{aligned}
&- \int_{\Omega_\ell} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell \partial_y (u_\ell - u)^+ \mathrm{d}x\mathrm{d}y - \int_{-\ell}^{\ell} \langle A u_\ell, (u_\ell - u)^+\rangle \mathrm{d}y \\
&\geq - \int_{\Omega_\ell} f (u_\ell - u)^+ \mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.16}$$

Adding the two inequalities (3.15) and (3.16) and using the fact that u is independent of y and the monotone condition (3.4) we obtain that

$$\int_{\Omega_\ell} \Big[\frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell - \frac{\varphi(|\partial_y u|)}{|\partial_y u|}\partial_y u\Big]\partial_y (u_\ell - u)^+ \mathrm{d}x\mathrm{d}y \leq 0.$$

By Lemma 3.1,

$$u_\ell(y,x) \le u(x) \quad \text{for a.e. } x \in \Omega_\ell. \tag{3.17}$$

Immediately, $\{u_\ell\}_\ell$ is bounded above by any nonnegative solution of (3.6).

Step 2 The pointwise limit of $\{u_\ell\}_\ell$ is independent of y.

It follows from Step 1 that u_ℓ has a pointwise limit we denote by $\widehat{u}$ such that

$$u_\ell \to \widehat{u} \quad \text{a.e. in } \Omega_\ell. \tag{3.18}$$

Let $h > 0$. Denote $\mathcal{T}_h u_\ell(y,x) = u_\ell(y+h,x)$. Then the functions $\mathcal{T}_h u_\ell(y,x)$ and $(\mathcal{T}_h u_\ell(y,x) - u_{\ell+h}(y,x))^+$ are supported in the closure of $\Omega_\ell^h := (-l-h, l-h) \times \Omega$. Thanks to (3.11), we have

$$\begin{aligned}&\int_{\Omega_\ell^h} \frac{\varphi(|\partial_y \mathcal{T}_h u_\ell|)}{|\partial_y \mathcal{T}_h u_\ell|} \partial_y \mathcal{T}_h u_\ell \partial_y (v - \mathcal{T}_h u_\ell) \mathrm{d}x\mathrm{d}y + \int_{-\ell-h}^{\ell-h} \langle A\mathcal{T}_h u_\ell, v - \mathcal{T}_h u_\ell \rangle \mathrm{d}y \\ &\ge \int_{\Omega_\ell^h} f(v - \mathcal{T}_h u_\ell)\mathrm{d}x\mathrm{d}y, \quad \forall\, v \in \mathcal{K}_{\ell,h},\end{aligned} \tag{3.19}$$

where $\mathcal{K}_{\ell,h} := \{\mathcal{T}_h v \mid v \in \mathcal{K}_\ell\} = \{v \in W_0^1 L_M(\Omega_\ell^h) \mid v(y,\cdot) \in \mathcal{K} \text{ a.e. in } (-\ell-h, \ell-h)\}$. Choosing $v = \mathcal{T}_h u_\ell - (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \in \mathcal{K}_{\ell,h}$ in (3.19) and $v = u_{\ell+h} + (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \in \mathcal{K}_{\ell+h}$ in (3.11) written for $u_{\ell+h}$ and adding the two inequalities, we have

$$\begin{aligned}&\int_{\Omega_\ell^h} \Big[\frac{\varphi(|\partial_y \mathcal{T}_h u_\ell|)}{|\partial_y \mathcal{T}_h u_\ell|}\partial_y \mathcal{T}_h u_\ell - \frac{\varphi(|\partial_y u_{\ell+h}|)}{|\partial_y u_{\ell+h}|}\partial_y u_{\ell+h}\Big] \partial_y (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \mathrm{d}x\mathrm{d}y \\ &+ \int_{-\ell-h}^{\ell-h} \langle A\mathcal{T}_h u_\ell - Au_{\ell+h}, (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \rangle \mathrm{d}y \le 0.\end{aligned}$$

Using (3.4) we can obtain that

$$\int_{\Omega_\ell^h} \Big[\frac{\varphi(|\partial_y \mathcal{T}_h u_\ell|)}{|\partial_y \mathcal{T}_h u_\ell|}\partial_y \mathcal{T}_h u_\ell - \frac{\varphi(|\partial_y u_{\ell+h}|)}{|\partial_y u_{\ell+h}|}\partial_y u_{\ell+h}\Big] \partial_y (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \mathrm{d}x\mathrm{d}y \le 0.$$

By Lemma 3.1,

$$u_\ell(y+h,x) \le u_{\ell+h}(y,x) \quad \text{for a.e. } (y,x) \in \Omega_\ell. \tag{3.20}$$

Letting $\ell \to +\infty$ in (3.20), we have

$$\widehat{u}(y+h,x) \le \widehat{u}(y,x) \quad \text{for a.e. } (y,x) \in \Omega_\ell. \tag{3.21}$$

Similarly, we can show that (3.21) holds whenever $h < 0$ for a.e. $(y,x) \in \Omega_\ell$. Since h is arbitrary, $\widehat{u}(y,x) = \widehat{u}(x)$ for a.e. $(y,x) \in \Omega_\ell$, that is, $\widehat{u}$ is independent of y.

Step 3 For all $\ell_0 > 0$, there exists a constant $C(\ell_0)$ independent of ℓ such that

$$\|u_\ell\|_{1,M,\Omega_{\ell_0}} \le C(\ell_0). \tag{3.22}$$

Let now $\ell_0 > 0$. Clearly, it is need to consider the case $\ell > \ell_0$. Let $\varrho \in \mathscr{D}(-2\ell_0, 2\ell_0)$ such that $0 \le \varrho \le 1, \varrho = 1$ on $(-\ell_0, \ell_0)$. Let u be a nonnegative solution of (3.6). Taking $v = u_\ell - \varrho^2(u_\ell - u) \in \mathcal{K}_\ell$ as a test function in (3.11), we derive that

$$\int_{\Omega_\ell} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|} \partial_y u_\ell \partial_y [\varrho^2(u_\ell - u)]\mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell} \langle Au_\ell, \varrho^2(u_\ell - u)\rangle \mathrm{d}y$$

$$\leq \int_{\Omega_\ell} f\varrho^2(u_\ell - u)\mathrm{d}x\mathrm{d}y \leq 0.$$

By the fact $\varrho = 1$ on $(-\ell_0, \ell_0)$ and u is independent of y, it yields that

$$\begin{aligned}
&\int_{\Omega_{\ell_0}} M(|\partial_y u_\ell|)\mathrm{d}x\mathrm{d}y + \alpha\int_{-\ell_0}^{\ell_0}\int_\Omega \sum_{i=0}^N M(|\partial_{x_i} u_\ell|)\mathrm{d}x\mathrm{d}y\\
\leq &\int_{\Omega_\ell} \varrho^2\varphi(|\partial_y u_\ell|)|\partial_y u_\ell|\mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell}\varrho^2\langle Au_\ell, u_\ell\rangle\mathrm{d}y\\
\leq &\int_{\Omega_\ell} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell 2\varrho\partial_y\varrho(u - u_\ell)\mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell}\varrho^2\langle Au_\ell, u\rangle\mathrm{d}y.
\end{aligned}$$

By (2.1), (3.3), (3.5) and the Young inequality, we can obtain that

$$\begin{aligned}
&\int_{\Omega_{\ell_0}} M(|\partial_y u_\ell|)\mathrm{d}x\mathrm{d}y + \alpha\int_{-\ell_0}^{\ell_0}\int_\Omega \sum_{i=1}^N M(|\partial_{x_i} u_\ell|)\mathrm{d}x\mathrm{d}y\\
\leq\ &C\varepsilon\int_{\Omega_\ell}\Big[M(|\partial_y u_\ell|) + \sum_{i=0}^N M(|\partial_{x_i} u_\ell|)\Big]\mathrm{d}x\mathrm{d}y + C\int_{\Omega_{2\ell_0}} M\Big(\frac{1}{\varepsilon}|u_\ell|\Big)\mathrm{d}x\mathrm{d}y\\
&+ C\int_{\Omega_{2\ell_0}}\Big[M\Big(\frac{1}{\varepsilon}|u|\Big) + M(|u|) + \overline{M}(|\vartheta(x)|) + \varepsilon M(\widetilde{u})\Big]\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where $\varepsilon \in (0, 1)$ and the constant $C > 0$ is independent of ℓ. Since $M \in \Delta_2$, there exists a constant $K_\varepsilon > 0$ and some $t_\varepsilon > 0$ such that $M\big(\frac{1}{\varepsilon}t\big) \leq K_\varepsilon M(t)$ for all $t > t_\varepsilon$. In view of (3.17), we get

$$\begin{aligned}
&\int_{\Omega_{\ell_0}} M(|\partial_y u_\ell|)\mathrm{d}x\mathrm{d}y + \alpha\int_{-\ell_0}^{\ell_0}\int_\Omega \sum_{i=1}^N M(|\partial_{x_i} u_\ell|)\mathrm{d}x\mathrm{d}y\\
\leq\ &C\varepsilon\int_{\Omega_\ell}\Big[M(|\partial_y u_\ell|) + \sum_{i=0}^N M(|\partial_{x_i} u_\ell|)\Big]\mathrm{d}x\mathrm{d}y\\
&+ C(\varepsilon)\int_{\Omega_{2\ell_0}}[M(|u|) + \overline{M}(|\vartheta(x)|) + M(\widetilde{u}) + M(t_\varepsilon)]\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where $\varepsilon \in (0, 1)$ and the positive constants C and $C(\varepsilon)$ are independent of ℓ. Choosing ε small enough, we get

$$\int_{\Omega_{\ell_0}}[M(|u_\ell|) + M(|\nabla' u_\ell|)]\mathrm{d}x\mathrm{d}y \leq C(\ell_0)$$

for some positive constant $C(\ell_0)$ independent of ℓ. For $u \in W^1L_M(\Omega_{\ell_0})$, define

$$\rho(u) := \int_{\Omega_{\ell_0}}[M(|u_\ell|) + M(|\nabla' u_\ell|)]\mathrm{d}x\mathrm{d}y$$

and

$$\|u\|_{\rho,\Omega_{\ell_0}} := \inf\Big\{\lambda < 0 : \rho\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}.$$

Then $\|u\|_{\rho,\Omega_{\ell_0}}$ is a norm of $W^1L_M(\Omega_{\ell_0})$ equivalent to $\|u\|_{1,M,\Omega_{\ell_0}}$ (see [15]). It implies (3.22).

Step 4 $\widehat{u}$ is a solution of (3.6).

Let $\ell_0 > 0$. In view of (3.22), $\{u_\ell\}_\ell$ is bounded in $W_0^1L_M(\Omega_{\ell_0})$ for all $\ell > 0$. Consequently, $\{\frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell\}_\ell$ and $\{a_i(x, u_\ell, \nabla u_\ell)\}_\ell$ $(i = 0, 1, \cdots, N)$ are bounded in $L_{\overline{M}}(\Omega_{\ell_0})$. Hence, there exist d and d_i in $L_{\overline{M}}(\Omega_{\ell_0})$ such that

$$u_\ell \to \widehat{u} \text{ strongly in } L_M(\Omega_{\ell_0}), \tag{3.23}$$

$$\nabla u_\ell \rightharpoonup \nabla\widehat{u} \text{ weakly in } (L_M(\Omega_{\ell_0}))^N \text{ for } \sigma\Big(\prod_{i=1}^N L_M(\Omega_{\ell_0}), \prod_{i=1}^N E_{\overline{M}}(\Omega_{\ell_0})\Big), \tag{3.24}$$

$$\frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell \rightharpoonup d \text{ weakly in } L_{\overline{M}}(\Omega_{\ell_0}) \text{ for } \sigma(L_{\overline{M}}(\Omega_{\ell_0}), E_M(\Omega_{\ell_0})), \tag{3.25}$$

$$a_i(x, u_\ell, \nabla u_\ell) \rightharpoonup d_i \text{ weakly in } L_{\overline{M}}(\Omega_{\ell_0}) \text{ for } \sigma(L_{\overline{M}}(\Omega_{\ell_0}), E_M(\Omega_{\ell_0})), \tag{3.26}$$

as $\ell \to +\infty$, $i = 0, 1, \cdots, N$. The two first convergences hold for the whole sequence since $\{u_\ell\}_\ell$ is nondecreasing, which guarantees the uniqueness of the limit and the last two convergences hold up to a subsequence.

Let ω be a nonnegative function in $\mathscr{D}(-\ell_0, \ell_0)$. By (3.4), (3.23) and (3.26), it follows that

$$\begin{aligned}\liminf_{\ell\to+\infty}\int_{-\ell_0}^{\ell_0}\omega\langle Au_\ell, u_\ell\rangle \mathrm{d}y &\geq \liminf_{\ell\to+\infty}\int_{-\ell_0}^{\ell_0}\omega\langle Au_\ell, \widehat{u}\rangle \mathrm{d}y + \liminf_{\ell\to+\infty}\int_{-\ell_0}^{\ell_0}\omega\langle A\widehat{u}, u_\ell - \widehat{u}\rangle \mathrm{d}y\\ &= \int_{\Omega_{\ell_0}}\omega\sum_{i=0}^N d_i\partial_{x_i}\widehat{u}\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{3.27}$$

where $\partial_{x_0}\widehat{u} = \widehat{u}$.

On the other hand, taking $v = u_\ell - \frac{\omega}{|\omega|_\infty}(u_\ell - \widehat{u}) \in \mathcal{K}_\ell$ as a test function in (3.11), one has

$$\int_{\Omega_{\ell_0}}\frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell\partial_y[\omega(u_\ell - \widehat{u})]\mathrm{d}x\mathrm{d}y + \int_{-\ell_0}^{\ell_0}\omega\langle Au_\ell, u_\ell - \widehat{u}\rangle\mathrm{d}y \leq \int_{\Omega_{\ell_0}}\omega f(u_\ell - \widehat{u})\mathrm{d}x\mathrm{d}y \leq 0$$

and thus

$$\begin{aligned}&\int_{\Omega_{\ell_0}}\omega M(|\partial_y u_\ell|)\mathrm{d}x\mathrm{d}y + \int_{-\ell_0}^{\ell_0}\omega\langle Au_\ell, u_\ell\rangle\mathrm{d}y\\ &\leq \int_{\Omega_{\ell_0}}\frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell\partial_y\omega(\widehat{u} - u_\ell)\mathrm{d}x\mathrm{d}y + \int_{-\ell_0}^{\ell_0}\omega\langle Au_\ell, \widehat{u}\rangle\mathrm{d}y,\end{aligned}$$

since $\widehat{u}$ is independent of y. Consequently,

$$\limsup_{\ell\to+\infty}\Big[\int_{\Omega_{\ell_0}}\omega M(|\partial_y u_\ell|)\mathrm{d}x\mathrm{d}y + \int_{-\ell_0}^{\ell_0}\omega\langle Au_\ell, u_\ell\rangle\mathrm{d}y\Big] \leq \int_{\Omega_{\ell_0}}\omega\sum_{i=0}^N d_i\partial_{x_i}\widehat{u}\mathrm{d}x\mathrm{d}y. \tag{3.28}$$

Combining (3.27) and (3.28), we have

$$\lim_{\ell\to+\infty}\int_{-\ell_0}^{\ell_0}\omega\langle Au_\ell, u_\ell\rangle\mathrm{d}y = \int_{\Omega_{\ell_0}}\omega\sum_{i=0}^N d_i\partial_{x_i}\widehat{u}\mathrm{d}x\mathrm{d}y \tag{3.29}$$

and

$$\lim_{\ell\to+\infty}\int_{\Omega_{\ell_0}}\omega M(|\partial_y u_\ell|)\mathrm{d}x\mathrm{d}y = 0. \tag{3.30}$$

Let $w \in \mathcal{K}$ and $\omega \not\equiv 0$ be a nonnegative function in $\mathscr{D}\big(-\frac{\ell_0}{2}, \frac{\ell_0}{2}\big)$. Taking $v = u_\ell + \frac{\omega}{|\omega|_\infty}(w - u_\ell) \in \mathcal{K}_\ell$ as a test function in (3.11), we have

$$\int_{\Omega_{\frac{\ell_0}{2}}} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell \partial_y[\omega(w-u_\ell)]\mathrm{d}x\mathrm{d}y + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \omega\langle Au_\ell, w-u_\ell\rangle \mathrm{d}y \geq \int_{\Omega_{\frac{\ell_0}{2}}} \omega f(w-u_\ell)\mathrm{d}x\mathrm{d}y.$$

Thanks to (3.4), one has

$$\int_{\Omega_{\frac{\ell_0}{2}}} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell \partial_y[\omega(w-u_\ell)]\mathrm{d}x\mathrm{d}y + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \omega\langle Aw, w-u_\ell\rangle \mathrm{d}y \geq \int_{\Omega_{\frac{\ell_0}{2}}} \omega f(w-u_\ell)\mathrm{d}x\mathrm{d}y.$$

From (3.30), it is clear that $\partial_y u_\ell \to 0$ strongly in $L_M(\Omega_{\frac{\ell_0}{2}})$. This can imply that $\varphi(|\partial_y u_\ell|) \to 0$ strongly in $L_{\overline{M}}(\Omega_{\frac{\ell_0}{2}})$. Passing to the limit in the above inequality as $\ell \to +\infty$ yields

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \omega\langle Aw, w-\widehat{u}\rangle \mathrm{d}y \geq \int_{\Omega_{\frac{\ell_0}{2}}} \omega f(w-\widehat{u})\mathrm{d}x\mathrm{d}y.$$

This implies

$$\langle Aw, w-\widehat{u}\rangle \geq \int_\Omega f(w-\widehat{u})\mathrm{d}x, \quad \forall\, w \in \mathcal{K}.$$

Choosing $w = \widehat{u} + t(v - \widehat{u})$, where $0 < t < 1$ and $v \in \mathcal{K}$, and passing to the limit as $t \to 0$ we get

$$\langle A\widehat{u}, v-\widehat{u}\rangle \geq \int_\Omega f(v-\widehat{u})\mathrm{d}x, \quad \forall\, v \in \mathcal{K},$$

that is, $\widehat{u}$ is a solution of (3.6).

Step 5 $\widehat{u}$ is the minimal nonnegative solution of (3.6).

Let u be an arbitrary solution of the problem (3.6). Letting $\ell \to +\infty$ in (3.17), we deduce $\widehat{u} \leq u$. This means that $\widehat{u}$ is the minimal solution of the problem (3.6).

Step 6 $u_1 \leq u_2$.

Let $u_{\ell,1}$ and $u_{\ell,2}$ be the solutions of (3.11) obtained if we replace f by f_1 and f_2, respectively. Then $u_{\ell,1}$ and $u_{\ell,2}$ converge to u_1 and u_2, respectively, as $\ell \to +\infty$. Taking $v = u_{\ell,1} - (u_{\ell,1} - u_{\ell,2})^+$ and $v = u_{\ell,1} + (u_{\ell,1} - u_{\ell,2})^+$ in (3.11) for f_1 and f_2, respectively, we get

$$\begin{aligned}&\int_{\Omega_\ell} \Big(\frac{\varphi(|\partial_y u_{\ell,1}|)}{|\partial_y u_{\ell,1}|}\partial_y u_{\ell,1} - \frac{\varphi(|\partial_y u_{\ell,2}|)}{|\partial_y u_{\ell,2}|}\partial_y u_{\ell,2}\Big)\partial_y(u_{\ell,1}-u_{\ell,2})^+\mathrm{d}x\mathrm{d}y \\ &+ \int_{-\ell}^{\ell} \langle Au_{\ell,1} - Au_{\ell,2}, (u_{\ell,1}-u_{\ell,2})^+\rangle \mathrm{d}y \leq \int_{\Omega_\ell} (f_1 - f_2)(u_{\ell,1}-u_{\ell,2})^+\mathrm{d}x\mathrm{d}y \leq 0.\end{aligned}$$

Using the condition (3.4) one has

$$\int_{\Omega_\ell} \Big(\frac{\varphi(|\partial_y u_{\ell,1}|)}{|\partial_y u_{\ell,1}|}\partial_y u_{\ell,1} - \frac{\varphi(|\partial_y u_{\ell,2}|)}{|\partial_y u_{\ell,2}|}\partial_y u_{\ell,2}\Big)\partial_y(u_{\ell,1}-u_{\ell,2})^+\mathrm{d}x\mathrm{d}y \leq 0.$$

This implies $u_{\ell,1} \leq u_{\ell,2}$ a.e. in Ω_ℓ. Letting $\ell \to +\infty$, it follows that $u_1 \leq u_2$.

3.2 Noncoercive variational inequalities

We keep the notation and the assumptions of Subsection 3.1. Then we consider the following problem

$$\begin{cases} u \in \mathcal{K}, \\ \langle Au, v-u \rangle \geq \displaystyle\int_\Omega F(x,u)(v-u)\mathrm{d}x, \quad \forall\, v \in \mathcal{K}, \end{cases} \tag{3.31}$$

where $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonnegative function satisfying

$$F(\cdot, t) : \Omega \to \mathbb{R} \text{ is measurable for all } t \in \mathbb{R}, \tag{3.32}$$

$$F(x, \cdot) : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing for a.e. } x \in \Omega, \tag{3.33}$$

$$F(x, u) \in L_{\overline{M}}(\Omega) \text{ for all } u \in L_{M_*}(\Omega). \tag{3.34}$$

Clearly, (3.31) is the extension of (3.6).

Remark 3.1 If $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonnegative Carathéodory function such that for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, $|F(x,t)| \leq q(x)$, where $q(x) \in L_{\overline{M}}(\Omega)$, then F satisfies the conditions (3.32)–(3.34).

Lemma 3.2 *Let F be a nonnegative function satisfying the hypotheses* (3.32)–(3.34)*, φ be strictly increasing, and suppose that the assumptions* (3.1)–(3.5) *are fulfilled. Define that u_n is the minimal nonnegative solution of the variational inequality in the last line of the following problem*:

$$\begin{cases} u_0 = 0, \\ u_n \in \mathcal{K}, \\ \langle Au_n, v-u_n \rangle \geq \displaystyle\int_\Omega F(x,u_{n-1})(v-u_n)\mathrm{d}x, \quad \forall\, v \in \mathcal{K}, \end{cases} \tag{3.35}$$

$\forall\, n \geq 1$. *Then the sequence $\{u_n\}_{n\in\mathbb{N}}$ is well defined and nondecreasing.*

Proof The existence of u_n is insured by Theorem 3.3 since $F(x, u_{n-1}) \in L_{\overline{M}}(\Omega)$. In a way similar to the proof in [6], the sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ is nonnegative and nondecreasing. This completes the proof of Lemma 3.2.

Denote by u_∞ the pointwise nonnegative limit of $\{u_n\}_n$ which is not necessarily in $L_M(\Omega)$ and may equal ∞. We also denote $F_\infty := \lim\limits_{n\to\infty} F(\cdot, u_n)$, which may also be infinite on some subset. Assume that

$$F_\infty \in L_{\overline{M}}(\Omega). \tag{3.36}$$

Note that the above assumption is satisfied. For example, $\sup\limits_{t\geq 0} F(\cdot, t) \in L_{\overline{M}}(\Omega)$. Then we have the following result.

Theorem 3.4 *Let F be a nonnegative function satisfying the hypotheses* (3.32)–(3.34)*, φ be strictly increasing, and suppose that the assumptions* (3.1)–(3.5) *are fulfilled. Then we have the equivalence between the following assertions*:

(1) (3.31) *has at least one nonnegative solution,*

(2) (3.31) *has a minimal nonnegative solution,*

(3) *the hypothesis* (3.36) *holds.*
Moreover if the hypothesis (3.36) *holds, then* u_∞*, the limit of* u_n*, belongs to* $\mathcal{K}$ *and is the minimal solution of* (3.31)*.*

Proof Clearly, (2) $\Rightarrow$ (1). Suppose that (3.31) has a nonnegative solution $w \in \mathcal{K}$. According to Theorem 3.3, there exits a minimal nonnegative solution $\underline{w} \in \mathcal{K}$ of the following problem:

$$\begin{cases} u \in \mathcal{K}, \\ \langle Au, v-u\rangle \geq \displaystyle\int_\Omega F(x,w)(v-u)\mathrm{d}x, \quad \forall\, v \in \mathcal{K}. \end{cases} \tag{3.37}$$

It is clearly that w is also a solution of (3.37). Then $w \geq \underline{w}$. Since w is nonnegative and F is nondecreasing in the second variable, $F(x,w) \geq F(x,0)$ for a.e. $x \in \Omega$. By Theorem 3.3, we have $\underline{w} \geq u_1$, where u_1 is the minimal nonnegative solution of the variational inequality in the last line of (3.35) for $n=1$. Consequently, $F(x,w) \geq F(x,\underline{w}) \geq F(x,u_1)$ for a.e. $x \in \Omega$. This implies that $\underline{w} \geq u_2$ by Theorem 3.3, where u_2 is the minimal nonnegative solution of the variational inequality in the last line of (3.35) for $n=2$. By induction, we can obtain that

$$w \geq \underline{w} \geq u_n, \quad \forall\, n \in \mathbb{N}, \tag{3.38}$$

and

$$F(x,w) \geq F(x,\underline{w}) \geq F(x,u_n) \quad \text{for a.e. } x \in \Omega, \quad \forall\, n \in \mathbb{N},$$

which yields (3.36). Hence, (1) $\Rightarrow$ (3).

Let the hypothesis (3.36) hold. By Theorem 3.3, there exists the minimal solution $\underline{u}_\infty$ of the following problem:

$$\begin{cases} u \in \mathcal{K}, \\ \langle Au, v-u\rangle \geq \displaystyle\int_\Omega F_\infty(v-u)\mathrm{d}x, \quad \forall\, v \in \mathcal{K}. \end{cases}$$

Since $F(\cdot,u_{n-1}) \leq F_\infty$, $\forall\, n \geq 1$, and thanks to Theorem 3.3, we deduce that

$$u_n \leq \underline{u}_\infty, \quad \forall\, n \in \mathbb{N}. \tag{3.39}$$

It follows that there exists u_∞ such that

$$u_n \to u_\infty \quad \text{a.e. in } \Omega, \tag{3.40}$$

as $n \to \infty$. Therefore, $u_\infty \leq \underline{u}_\infty$, and $F_\infty = F(\cdot,u_\infty)$ a.e. in Ω. Consequently, $u_\infty \in L_{\overline{M}}(\Omega)$.

Taking $v = \underline{u}_\infty$ as a test function in (3.35), using (3.3), (3.5), (3.33), (3.36), (3.39), Young inequality, and the fact $M \in \Delta_2$, we obtain that

$$\begin{aligned} \alpha \int_\Omega \sum_{i=0}^N M(|\partial_{x_i} u_n|)\mathrm{d}x &\leq \int_\Omega F(x,u_{n-1})(u_n - \underline{u}_\infty)\mathrm{d}x + \langle Au_n, \underline{u}_\infty\rangle \\ &\leq \int_\Omega F_\infty(u_n - \underline{u}_\infty)\mathrm{d}x + \langle Au_n, \underline{u}_\infty\rangle \\ &\leq C\varepsilon \int_\Omega \sum_{i=0}^N M(|\partial_{x_i} u_n|)\mathrm{d}x + C(\varepsilon) \end{aligned}$$

for some positive constants C and $C(\varepsilon)$ independent of n, where $\varepsilon \in (0,1)$ and $\partial_{x_0} u_n = u_n$. Choosing ε small enough, we have

$$\int_\Omega [M(|u_n|) + M(|\nabla u_n|)]\mathrm{d}x \leq C$$

for some constant $C > 0$ independent of n. This yields that $\|u_n\|_{1,M,\Omega} \leq C$ for some positive constant C independent of n. Combining the above results, it follows, as $n \to \infty$, that

$$u_n \to u_\infty \text{ strongly in } L_M(\Omega), \tag{3.41}$$

$$u_n \rightharpoonup u_\infty \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}). \tag{3.42}$$

Since $\mathcal{K}$ is a closed convex subset of $W_0^1 L_M(\Omega)$, it is also weakly closed, which yields that $u_\infty \in \mathcal{K}$.

From (3.4) and (3.35), it follows that

$$\langle Aw, w - u_n \rangle - \int_\Omega F(x, u_{n-1})(w - u_n)\mathrm{d}x \geq \langle Aw - Au_n, w - u_n \rangle \geq 0, \quad \forall\, w \in \mathcal{K}. \tag{3.43}$$

Since $\overline{M} \in \Delta_2$, from (3.40), (3.33) and (3.36), it is easy to see that $F(x, u_n)$ is strongly convergent to $F(x, u_\infty)$ in $L_{\overline{M}}(\Omega)$ as $n \to \infty$. Then passing to the limit as $n \to \infty$ in (3.43), we have that

$$\langle Aw, w - u_\infty \rangle \geq \int_\Omega F(x, u_\infty)(w - u_\infty)\mathrm{d}x, \quad \forall\, w \in \mathcal{K}.$$

Taking $w = u_\infty + t(v - u_\infty)$ with $0 < t < 1$ and $v \in \mathcal{K}$, and letting $t \to 0$, one has

$$\langle Au_\infty, v - u_\infty \rangle \geq \int_\Omega F(x, u_\infty)(v - u_\infty)\mathrm{d}x, \quad \forall\, v \in \mathcal{K},$$

that is, u_∞ is a solution of (3.31). Therefore, (3) $\Rightarrow$ (1).

Suppose that the hypothesis (3.36) holds. Let $w \in \mathcal{K}$ be a nonnegative solution of (3.31). By the above arguments, letting $\ell \to +\infty$ in (3.38), we get $w \geq u_\infty$. Since w is an arbitrary solution of (3.31), we have that u_∞ is the minimal solution of (3.31). Hence (3) $\Rightarrow$ (2) and the proof is achieved.

4 Variational Inequalities in Unbounded Domains

This section is devoted to studying the existence of nonnegative solutions and their minimal solutions for some quasilinear variational inequalities in unbounded domains. We investigate variational inequalities with coercive operators in Subsection 4.1 and with noncoercive operators in Subsection 4.2, respectively.

4.1 Variational inequalities with coercive operator

Let G be a bounded domain in $\mathbb{R}^{N-1}$ ($N \geq 2$) with Lipschitz boundary, and $\mathcal{K}_G$ be a closed convex subset of $W_0^1 L_M(G)$ containing 0 such that the lattice condition

$$\mathcal{K}_G \wedge \mathcal{K}_G \subset \mathcal{K}_G, \quad \mathcal{K}_G \vee \mathcal{K}_G \subset \mathcal{K}_G$$

is satisfied. Let M be an N-function, $\overline{M}$ be the complementary function of M, and φ, ϕ are the right-hand derivatives of $M, \overline{M}$, respectively. Assume that $M, \overline{M} \in \Delta_2$.

For $x \in \mathbb{R} \times G$, denote $x = (x_1, X_2)$ with $X_2 = (x_2, \cdots, x_N)$, and

$$W^1_{\mathrm{loc}} L_M(\mathbb{R} \times \overline{G}) = \{u \mid u \in W^1 L_M((-a,a) \times G), \ \forall\, a > 0\}.$$

We set

$$\begin{aligned}\widetilde{\mathcal{K}} &:= W^1_{\mathrm{loc}} L_M(\mathbb{R}; \mathcal{K}_G) \\ &:= \{v \in W^1_{\mathrm{loc}} L_M(\mathbb{R} \times \overline{G}) \mid v = 0 \text{ on } \mathbb{R} \times \partial G \text{ and } v(x_1, \cdot) \in \mathcal{K}_G \text{ for a.e. } x_1 \in \mathbb{R}\}.\end{aligned}$$

Then $\widetilde{\mathcal{K}}$ is a closed convex subset of $W^1_{\mathrm{loc}} L_M(\mathbb{R} \times \overline{G})$. We also denote

$$L^{\mathrm{loc}}_{\overline{M}}(\mathbb{R} \times \overline{G}) = \{f \mid f \in L_{\overline{M}}((-a,a) \times G), \ \forall\, a > 0\}$$

and

$$L^{\mathrm{loc}}_{\overline{M}}(\mathbb{R}, L_{\overline{M}}(G)) := \{f \in L^{\mathrm{loc}}_{\overline{M}}(\mathbb{R} \times \overline{G}) \mid f(x_1, \cdot) \in L_{\overline{M}}(G) \text{ for a.e. } x_1 \text{ in } \mathbb{R}\}.$$

For a nonnegative f in $L^{\mathrm{loc}}_{\overline{M}}(\mathbb{R}, L_{\overline{M}}(G))$, we consider the following nonlinear variational inequality defined on the infinite cylinder $\mathbb{R} \times G$,

$$\begin{cases} u \in \widetilde{\mathcal{K}}, \\ \displaystyle\int_{\mathbb{R} \times G} a(x, u, \nabla u) \cdot \nabla(\varrho(v-u))\mathrm{d}x + \int_{\mathbb{R} \times G} a_0(x, u, \nabla u)\varrho(v-u)\mathrm{d}x \\ \displaystyle\geq \int_{\mathbb{R} \times G} f\varrho(v-u)\mathrm{d}x \end{cases} \tag{4.1}$$

for all $v \in \widetilde{\mathcal{K}}$, and all $\varrho \in \mathscr{D}(\mathbb{R})$ with $\varrho \geq 0$.

Note that if $\partial_{x_1} a_1(x, u, \nabla u) \in L^{\mathrm{loc}}_{\overline{M}}(\mathbb{R}, L_{\overline{M}}(G))$, the above variational inequality can be written as

$$\begin{aligned} &\int_G \sum_{2 \leq i \leq N} a_i(x, u, \nabla u)\partial_{x_i}(v-u)(x_1, \cdot)\mathrm{d}X_2 \\ &+ \int_G [a_0(x, u, \nabla u) - \partial_{x_1} a_1(x, u, \nabla u)](v-u)(x_1, \cdot)\mathrm{d}X_2 \\ \geq &\int_G f(v-u)(x_1, \cdot)\mathrm{d}X_2, \quad \forall\, v \in \widetilde{\mathcal{K}} \text{ a.e. } x_1 \text{ in } \mathbb{R}. \end{aligned}$$

Since the domain is unbounded and f is not necessarily in the dual of $W^1_0 L_M(\mathbb{R} \times G)$, the existence of nonnegative solutions to problem (4.1) is not an ordinary issue. Once this is ensured, we can then look for the minimal nonnegative solution. Here, we will use the same approach as in Subsection 3.1 to prove these existence results. To this end, in addition to the hypotheses (3.1)–(3.5), assume that

$$\begin{aligned} a_i(x_1, X_2, \xi_0, 0, \xi_2, \cdots, \xi_N) &= a_i(X_2, \xi_0, 0, \xi_2, \cdots, \xi_N) \\ &:= a_i(X_2, \xi_0, \xi_2, \cdots, \xi_N), \quad \forall\, \xi_j \in \mathbb{R}, \ j = 0, 2, \cdots, N, \end{aligned} \tag{4.2}$$

$i = 0, 1, \cdots, N$. That is to say if $\xi_1 = 0$ then the coefficients a_i for $i = 0, 1, \cdots, N$ are independent of x_1. We also assume that there exists $h \in L_{\overline{M}}(G)$ such that

$$f(x_1, X_2) \leq h(X_2) \quad \text{for a.e. } (x_1, X_2) \in \mathbb{R} \times G. \tag{4.3}$$

For $\ell > 0$, let $\Omega_{\ell,G} = (-\ell,\ell)^2 \times G$. For simplicity we denote $\langle\cdot,\cdot\rangle_G = \langle\cdot,\cdot\rangle_{(W_0^1L_M(G))^*,W_0^1L_M(G)}$ and $\langle\cdot,\cdot\rangle_{\ell,G} = \int_{-\ell}^{\ell}\langle\cdot,\cdot\rangle_G \mathrm{d}x_1$. We set

$$\widetilde{\mathcal{K}}_\ell := \{v \in W_0^1L_M(\Omega_{\ell,G}) \mid v(y,x_1,\cdot) \in \mathcal{K}_G \quad \text{for a.e. } (y,x_1) \in (-\ell,\ell)^2\}.$$

Then $\widetilde{\mathcal{K}}_\ell$ is a closed convex subset of $W_0^1L_M(\Omega_{\ell,G})$.

Let $\ell > 0$. Consider the following variational inequalities:

$$\begin{cases} u \in \mathcal{K}_G, \\ \langle A_G u, v-u\rangle_G \geq \displaystyle\int_G h(v-u)\mathrm{d}X_2, \quad \forall\, v \in \mathcal{K}_G, \end{cases} \tag{4.4}$$

and

$$\begin{cases} u_\ell \in \widetilde{\mathcal{K}}_\ell, \\ \displaystyle\int_{\Omega_{\ell,G}} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell \partial_y(v-u_\ell)\mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell}\langle Au_\ell, v-u_\ell\rangle_{\ell,G}\mathrm{d}y \\ \geq \displaystyle\int_{\Omega_{\ell,G}} f(v-u_\ell)\mathrm{d}x\mathrm{d}y, \quad \forall\, v \in \widetilde{\mathcal{K}}_\ell, \end{cases} \tag{4.5}$$

where $A_G u = -\sum\limits_{2\leq i\leq N}\partial_{x_i}a_i(X_2,u,\nabla_{X_2}u) + a_0(X_2,u,\nabla_{X_2}u)$ with $\nabla_{X_2} = (\partial_{x_2},\cdots,\partial_{x_N})$ and A is the nonlinear operator given by (3.7). Under the above assumptions, the problems (4.4) has the minimal nonnegative solution by Theorem 3.3 and (4.5) has a unique solution $u_\ell \in \widetilde{\mathcal{K}}_\ell$ with $u_\ell \geq 0$ for every $\ell > 0$ by Theorem 3.1, when φ is strictly increasing. Then, we have the following theorem.

Theorem 4.1 *Suppose that the assumptions* (3.1)–(3.5) *and* (4.2)–(4.3) *are satisfied, where* Ω *is replaced by* $\mathbb{R}\times G$ *in* (3.1)–(3.5). *Assume that* φ *is strictly increasing. Then the pointwise limit of* $\{u_\ell\}_\ell$ *is the minimal nonnegative solution of* (4.1), *where* u_ℓ *is the solution of* (4.5), *for every* $\ell > 0$.

Moreover, the following assertions hold:

(i) *Let* u_1 *and* u_2 *be the minimum nonnegative solutions of* (4.1) *obtained by replacing* f *with* f_1 *and* f_2, *respectively. If* $f_1 \leq f_2$, *then* $u_1 \leq u_2$.

(ii) *Let* $\widehat{u}_1$ *be the minimum nonnegative solution of* (4.1) *obtained by replacing* f *with* f_1, *and* $\widehat{u}_2$ *be a nonnegative solution of* (4.1) *obtained by replacing* f *with* f_2, *where* f_2 *does not necessarily satisfy* (4.3). *If* $f_1 \leq f_2$, *then* $\widehat{u}_1 \leq \widehat{u}_2$.

Proof Step 1 The sequence $\{u_\ell\}_\ell$ is nondecreasing and bounded by any solution u of (4.4).

As the same arguments in the proof of Theorem 3.3, we can get $\{u_\ell\}_\ell$ is a nondecreasing sequence.

Taking $v = u+(u_\ell(y,x_1,\cdot)-u)^+ \in \mathcal{K}_G$ as a test function in (4.4) and integrating on $(-\ell,\ell)^2$, we obtain by (4.2) that

$$\int_{-\ell}^{\ell}\langle Au,(u_\ell-u)^+\rangle_{\ell,G}\mathrm{d}y \geq \int_{\Omega_{\ell,G}} h(u_\ell-u)^+\mathrm{d}x\mathrm{d}y. \tag{4.6}$$

Taking $v = u_\ell-(u_\ell-u)^+ \in \widetilde{\mathcal{K}}_\ell$ as a test function in (4.5) and adding the resulting inequality with (4.6) one yields

$$\int_{\Omega_{\ell,G}} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell \partial_y(u_\ell-u)^+\mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell}\langle Au_\ell - Au,(u_\ell-u)^+\rangle_{\ell,G}\mathrm{d}y$$

$$\leq \int_{\Omega_{\ell,G}} (f-h)(u_\ell - u)^+ \mathrm{d}x\mathrm{d}y.$$

In view of (3.4) and (4.3), we derive

$$\int_{\Omega_{\ell,G}} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|} \partial_y u_\ell \partial_y (u_\ell - u)^+ \mathrm{d}x\mathrm{d}y \leq 0.$$

Since u is independent of y, this implies that

$$\int_{\Omega_{\ell,G}} \Big[\frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|} \partial_y u_\ell - \frac{\varphi(|\partial_y u|)}{|\partial_y u|} \partial_y u\Big] \partial_y (u_\ell - u)^+ \leq 0.$$

It follows that $u_\ell \leq u$.

Step 2 The pointwise limit of u_ℓ is independent of y.

By Step 1, there exists $\widehat{u} \geq 0$, such that

$$u_\ell \to \widehat{u} \quad \text{a.e. in } \mathbb{R}^2 \times G \text{ as } \ell \to +\infty. \tag{4.7}$$

Following the arguments as in the proof of Theorem 3.3, we can show that $\widehat{u}$ is independent of y.

Step 3 For all $\ell_0 > 0$, there exists a constant C_{ℓ_0} independent of ℓ such that

$$\|u_\ell\|_{1,M,\Omega_{\ell_0,G}} \leq C_{\ell_0}. \tag{4.8}$$

Let $\varrho \in \mathscr{D}((-2\ell_0, 2\ell_0)^2)$ such that

$$0 \leq \varrho \leq 1 \text{ and } \varrho = 1 \quad \text{on } (-\ell_0, \ell_0)^2.$$

Let $\ell_0 > 0$ and u be a nonnegative solution of (4.4). Taking $v = u_\ell - \varrho^2(u_\ell - u) \in \widetilde{\mathcal{K}}_\ell$ in (4.5), then following the same arguments as in the proof of Theorem 3.3 we can deduce (4.8).

Step 4 $\widehat{u}$ is a solution of (4.1).

For $\ell_0 > 0$, according to (4.7)–(4.8), we can deduce that

$$u_\ell \to \widehat{u} \text{ strongly in } L_M(\Omega_{\ell_0,G}) \tag{4.9}$$

and

$$u_\ell \rightharpoonup \widehat{u} \text{ weakly in } W^1 L_M(\Omega_{\ell_0,G}) \text{ for } \sigma\Big(\prod_{i=0}^{N+1} L_M(\Omega_{\ell_0,G}), \prod_{i=0}^{N+1} E_{\overline{M}}(\Omega_{\ell_0,G})\Big), \tag{4.10}$$

as $\ell \to +\infty$.

Since $\widetilde{\mathcal{K}}_{\ell_0}$ is closed and convex, it is also weakly closed and by consequence $\widehat{u} \in \widetilde{\mathcal{K}}_{\ell_0}$, i.e., $\widehat{u}(x_1, \cdot) \in \mathcal{K}_G$ for a.e. $x_1 \in (-\ell_0, \ell_0)$. Then by using the above convergence results, we can prove as in (3.29)–(3.30) that

$$\lim_{\ell \to +\infty} \int_{\Omega_{\ell_0,G}} \omega \langle A u_\ell, u_\ell \rangle_{\ell_0,G} \mathrm{d}x\mathrm{d}y = \int_{\Omega_{\ell_0,G}} \omega \sum_{i=0}^{N} d_i \partial_{x_i} \widehat{u} \mathrm{d}x\mathrm{d}y, \tag{4.11}$$

$$\lim_{\ell \to +\infty} \int_{\Omega_{\ell_0,G}} \omega M(|\partial_y u_\ell|) \mathrm{d}x\mathrm{d}y = 0, \quad \forall\, \omega \in \mathscr{D}((-\ell_0, \ell_0)^2),\ \omega \geq 0, \tag{4.12}$$

where d_i is the weak limit of $a_i(x, u_\ell, \nabla u_\ell)$ in $L_{\overline{M}}(\Omega_{\ell_0,G})$. (4.11)–(4.12) hold for a subsequence of $\{u_\ell\}_\ell$, still denoted by $\{u_\ell\}_\ell$.

Let $w \in \widetilde{\mathcal{K}}$ and $\omega \not\equiv 0$ be a nonnegative function in $\mathscr{D}\big(\big(-\frac{\ell_0}{2}, \frac{\ell_0}{2}\big)^2\big)$. Taking $v = u_\ell + \frac{\omega}{|\omega|_\infty}(w - u_\ell) \in \widetilde{\mathcal{K}}_\ell$ as a test function in (4.5), we get

$$\begin{aligned}&\int_{\Omega_{\frac{\ell_0}{2},G}} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell \partial_y(\omega(w-u_\ell))\mathrm{d}x\mathrm{d}y + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \langle Au_\ell, \omega(w-u_\ell)\rangle_{\frac{\ell_0}{2},G}\mathrm{d}y \\ &\geq \int_{\Omega_{\frac{\ell_0}{2},G}} \omega f(w-u_\ell)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Passing to the limit as $\ell \to +\infty$ and taking into account (4.9)–(4.12) we obtain

$$\int_{\Omega_{\frac{\ell_0}{2},G}} \sum_{i=0}^{N} d_i \partial_{x_i}(\omega(w-\widehat{u}))\mathrm{d}x\mathrm{d}y \geq \int_{\Omega_{\frac{\ell_0}{2},G}} \omega f(w-\widehat{u})\mathrm{d}x\mathrm{d}y. \tag{4.13}$$

Let $t > 0$ and ψ be a nonnegative function in $\mathscr{D}((-\frac{\ell_0}{2}, \frac{\ell_0}{2})^2)$. Then it follows from the condition (3.4) that

$$\begin{aligned}&\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \psi\langle A(\widehat{u}+t\omega(w-\widehat{u})) - Au_\ell, \widehat{u}-u_\ell\rangle_{\frac{\ell_0}{2},G}\mathrm{d}y \\ &+ \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \psi t\langle A(\widehat{u}+t\omega(w-\widehat{u})) - Au_\ell, \omega(w-\widehat{u})\rangle_{\frac{\ell_0}{2},G}\mathrm{d}y \geq 0.\end{aligned}$$

Passing to the limit as $\ell \to +\infty$, it follows from (3.5), (4.9)–(4.11) that

$$\int_{\Omega_{\frac{\ell_0}{2},G}} \psi \sum_{i=0}^{N}[a_i(x, \widehat{u}+t\omega(w-\widehat{u}), \nabla(\widehat{u}+t\omega(w-\widehat{u}))) - d_i]\partial_{x_i}(\omega(w-\widehat{u}))\mathrm{d}x\mathrm{d}y \geq 0,$$

$\forall\, \psi \in \mathscr{D}\big(\big(-\frac{\ell_0}{2}, \frac{\ell_0}{2}\big)^2\big)$, $\psi \geq 0$. Consequently, letting $t \to 0$, one has

$$\begin{aligned}&\int_{\Omega_{\frac{\ell_0}{2},G}} \psi \sum_{i=0}^{N}[a_i(x, \widehat{u}, \nabla\widehat{u}) - d_i]\partial_{x_i}(\omega(w-\widehat{u}))\mathrm{d}x\mathrm{d}y \geq 0, \\ &\qquad\qquad \forall\, \psi \in \mathscr{D}\Big(\Big(-\frac{\ell_0}{2}, \frac{\ell_0}{2}\Big)^2\Big),\ \psi \geq 0,\end{aligned}$$

which implies that

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \langle A\widehat{u}, \omega(w-\widehat{u})\rangle_{\frac{\ell_0}{2},G}\mathrm{d}y \geq \int_{\Omega_{\frac{\ell_0}{2},G}} \sum_{i=0}^{N} d_i\partial_{x_i}(\omega(w-\widehat{u}))\mathrm{d}x\mathrm{d}y. \tag{4.14}$$

Combining (4.13) and (4.14), we have

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \langle A\widehat{u}, \omega(w-\widehat{u})\rangle_{\frac{\ell_0}{2},G}\mathrm{d}y \geq \int_{\Omega_{\frac{\ell_0}{2},G}} \omega f(w-\widehat{u})\mathrm{d}x\mathrm{d}y \tag{4.15}$$

for all $w \in \widetilde{\mathcal{K}}$, $\omega \in \mathscr{D}\big(\big(-\frac{\ell_0}{2}, \frac{\ell_0}{2}\big)^2\big)$, $\omega \geq 0$. Taking $\omega(y, x_1) = \widetilde{\varrho}(y)\varrho(x_1)$ in (4.15) where $\widetilde{\varrho} \not\equiv 0$ and ϱ are nonnegative functions in $\mathscr{D}\big(-\frac{\ell_0}{2}, \frac{\ell_0}{2}\big)$, we obtain

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \widetilde{\varrho}(y)\langle A\widehat{u}, \varrho(w - \widehat{u})\rangle_{\frac{\ell_0}{2},G}\mathrm{d}y \geq \int_{\Omega_{\frac{\ell_0}{2},G}} \widetilde{\varrho}(y)\varrho f(w - \widehat{u})\mathrm{d}x\mathrm{d}y.$$

The fact that $\widehat{u}$ is independent of y implies that

$$\langle A\widehat{u}, \varrho(w - \widehat{u})\rangle_{\frac{\ell_0}{2},G} \geq \int_{(-\frac{\ell_0}{2},\frac{\ell_0}{2})\times G} \varrho f(w - \widehat{u})\mathrm{d}x, \quad \forall\, w \in \widetilde{\mathcal{K}},\ \varrho \in \mathscr{D}\Big(-\frac{\ell_0}{2}, \frac{\ell_0}{2}\Big),\ \varrho \geq 0.$$

Since ℓ_0 is arbitrary, we get

$$\langle A\widehat{u}, \varrho(w - \widehat{u})\rangle_{\mathbb{R}\times G} \geq \int_{\mathbb{R}\times G} \varrho f(w - \widehat{u})\mathrm{d}x, \quad \forall\, w \in \widetilde{\mathcal{K}},\ \varrho \in \mathscr{D}(\mathbb{R}),\ \varrho \geq 0.$$

Therefore, $\widehat{u}$ is a solution of (4.1).

Step 5 $\widehat{u}$ is the minimal nonnegative solution of (4.1).

Let u be an arbitrary nonnegative solution of (4.1). Then $(u_\ell - u)^+$ are supported in the closure of $\Omega_{\ell,G}$. Choosing $\varrho \in \mathscr{D}(\mathbb{R})$ such that $\varrho = 1$ on $(-\ell, \ell)$, taking $v = u + (u_\ell(y, \cdot) - u)^+ \in \widetilde{K}$ as a test function in (4.1), and integrating on $(-\ell, \ell)$, we have

$$\int_{-\ell}^{\ell} \langle Au, (u_\ell - u)^+\rangle_{\ell,G}\mathrm{d}y \geq \int_{\Omega_{\ell,G}} f(u_\ell - u)^+\mathrm{d}x\mathrm{d}y. \tag{4.16}$$

Taking $v = u_\ell - (u_\ell - u)^+ \in \widetilde{K}_\ell$ as a test function in (4.5) and summing the produced inequality with (4.16), we obtain

$$-\int_{\Omega_{\ell,G}} \frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell \partial_y(u_\ell - u)^+\mathrm{d}x\mathrm{d}y + \int_{-\ell}^{\ell} \langle Au - Au_\ell, (u_\ell - u)^+\rangle_{\ell,G}\mathrm{d}y \geq 0.$$

According to (3.4), we get

$$\int_{\Omega_{\ell,G}} \Big[\frac{\varphi(|\partial_y u_\ell|)}{|\partial_y u_\ell|}\partial_y u_\ell - \frac{\varphi(|\partial_y u|)}{|\partial_y u|}\partial_y u\Big]\partial_y(u_\ell - u)^+\mathrm{d}x\mathrm{d}y \leq 0,$$

since u is independent of y. It follows from Lemma 3.1 that $u_\ell \leq u$ a.e. in Ω_ℓ. Letting $\ell \to +\infty$ in (4.7), we have $\widehat{u} \leq u$ a.e. in $\mathbb{R} \times G$. This means that $\widehat{u}$ is the minimal nonnegative solution of (4.1).

Let u_ℓ^1 and u_ℓ^2 be the converging sequences defined above as solutions of (4.5) for $f = f_1$ and $f = f_2$, respectively. By the same arguments as in the proof of Theorem 3.3, $u_\ell^1 \leq u_\ell^2$, since $f_1 \leq f_2$. Letting $\ell \to +\infty$, it follows that $\widehat{u}_1 \leq \widehat{u}_2$ a.e. in $\mathbb{R} \times G$, where $\widehat{u}_1$ and $\widehat{u}_2$ are the minimum nonnegative solutions of (4.1) obtained by replacing f with f_1 and f_2, respectively. Hence, the assertion (i) holds.

Let u_ℓ^1 be the converging sequence defined above as the solution of (4.5) for $f = f_1$, $\widehat{u}_1$ be the minimum nonnegative solution of (4.1) obtained by replacing f with f_1, and $\widehat{u}_2$ be a nonnegative solution of (4.1) obtained by replacing f with f_2, where f_2 does not necessarily satisfy (4.3). Note that $(u_\ell^1 - \widehat{u}_2)^+$ are supported in the closure of $\Omega_{\ell,G}$. Taking $v = u_\ell^1 - (u_\ell^1 - \widehat{u}_2)^+ \in \widetilde{\mathcal{K}}_\ell$ as a test function in (4.5) for $f = f_1$ and $v = \widehat{u}_2 + (u_\ell^1(y, \cdot) - \widehat{u}_2)^+ \in \widetilde{\mathcal{K}}$ as a test function in

(4.1) for $f=f_2$, choosing $\varrho\in\mathscr{D}(\mathbb{R})$ such that $\varrho=1$ on $(-\ell,\ell)$ and integrating on $(-\ell,\ell)$, then summing the produced inequalities we have

$$\begin{aligned}&-\int_{\Omega_{\ell,G}}\frac{\varphi(|\partial_y u_\ell^1|)}{|\partial_y u_\ell^1|}\partial_y u_\ell^1\partial_y(u_\ell^1-\widehat{u}_2)^+\mathrm{d}x\mathrm{d}y+\int_{-\ell}^{\ell}\langle A\widehat{u}_2-Au_\ell^1,(u_\ell^1-\widehat{u}_2)^+\rangle_{\ell,G}\mathrm{d}y\\ &\ge\int_{\Omega_{\ell,G}}(f_2-f_1)(u_\ell^1-\widehat{u}_2)^+\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Using the fact that $\widehat{u}_2$ is independent of y and the condition (3.4) we derive

$$\int_{\Omega_{\ell,G}}\Big[\frac{\varphi(|\partial_y u_\ell^1|)}{|\partial_y u_\ell^1|}\partial_y u_\ell^1-\frac{\varphi(|\partial_y \widehat{u}_2|)}{|\partial_y \widehat{u}_2|}\partial_y\widehat{u}_2\Big]\partial_y(u_\ell^1-\widehat{u}_2)^+\mathrm{d}x\mathrm{d}y\le 0.$$

Therefore, $u_\ell^1\le\widehat{u}_2$. Letting $\ell\to+\infty$, it yields that $\widehat{u}_1\le\widehat{u}_2$ a.e. in $\mathbb{R}\times G$. Consequently, the assertion (ii) holds.

Consider the following nonlinear elliptic problem defined on the infinite cylinder $\mathbb{R}\times G$,

$$\begin{cases}u\in W^1_{\mathrm{loc}}L_M(\mathbb{R}\times\overline{G}), \quad u=0 \quad \text{on } \mathbb{R}\times\partial G,\\ Au=f \quad \text{in } \mathbb{R}\times G.\end{cases}\tag{4.17}$$

A function u is called a (weak) solution of (4.17) if $u\in W^1_{\mathrm{loc}}L_M(\mathbb{R}\times\overline{G})$ and

$$\int_{\mathbb{R}\times G}a(x,u,\nabla u)\nabla v\mathrm{d}x+\int_{\mathbb{R}\times G}a_0(x,u,\nabla u)v\mathrm{d}x=\int_{\mathbb{R}\times G}fv\mathrm{d}x,\quad \forall\, v\in\mathscr{D}(\mathbb{R}\times G).\tag{4.18}$$

Then any solution of problem (4.1) for $K_G=W_0^1L_M(G)$ is a solution of (4.17) and vice versa. Thus the existence of nonnegative solutions of problem (4.17) is proved in Theorem 4.1. Indeed, let $u\in\widetilde{\mathcal{K}}$ be a solution of (4.1). Choosing $v=u\pm v'$ with $v'\in\mathscr{D}(\mathbb{R}\times G)$ in (4.1) and $\varrho=1$ on the support of v', we can obtain (4.18). The converse is an immediate consequence of (4.18).

Therefore, we have the following result as an immediate consequence of Theorem 4.1.

Corollary 4.1 *Under the assumptions of Theorem* 4.1, *there exists a minimal nonnegative solution of* (4.17). *Moreover, let $\widehat{u}_1$ and $\widehat{u}_2$ be the minimal nonnegative solutions of* (4.17) *obtained by replacing f with f_1 and f_2 respectively. Then if $f_1\le f_2$ we have $\widehat{u}_1\le\widehat{u}_2$.*

4.2 Noncoercive variational inequalities

We consider the following nonlinear variational inequality defined on the infinite cylinder $\mathbb{R}\times G$,

$$\begin{cases}u\in\widetilde{\mathcal{K}},\\ \displaystyle\int_{\mathbb{R}\times G}a(x,u,\nabla u)\nabla(\varrho(v-u))\mathrm{d}x+\int_{\mathbb{R}\times G}a_0(x,u,\nabla u)\varrho(v-u)\mathrm{d}x\\ \displaystyle\ge\int_{\mathbb{R}\times G}F(x,u)\varrho(v-u)\mathrm{d}x,\quad\forall\, v\in\widetilde{\mathcal{K}},\ \varrho\in\mathscr{D}(\mathbb{R}),\ \varrho\ge 0,\end{cases}\tag{4.19}$$

where F is defined as in Subsection 3.2, replacing Ω by $\mathbb{R}\times G$. In addition, we assume that

$$h(X_2,r):=\sup_{x_1\in\mathbb{R}}F(x_1,X_2,r)\tag{4.20}$$

satisfies

$$h(X_2,u)\in L_{\overline{M}}(G),\quad\forall\, u\in L_{M_*}(G).\tag{4.21}$$

Lemma 4.1 *Let F and h be nonnegative functions satisfying the hypotheses above. Suppose that the assumptions* (3.1)–(3.5) *and* (4.2) *hold, where Ω is replaced by $\mathbb{R}\times G$ in* (3.1)–(3.5)*. Assume that φ is strictly increasing. Define $\underline{u}_n$ and u_n are respectively the minimal nonnegative solutions of variational inequalities in the last line of the following problems:*

$$\begin{cases} \underline{u}_0 = 0, \\ \underline{u}_n \in \mathcal{K}_G, \\ \langle A_G \underline{u}_n, v - \underline{u}_n \rangle_G \geq \displaystyle\int_G h(X_2, \underline{u}_{n-1})(v - \underline{u}_n)\mathrm{d}X_2, \quad \forall\, v \in \mathcal{K}_G, \end{cases} \tag{4.22}$$

and

$$\begin{cases} u_0 = 0, \\ u_n \in \widetilde{\mathcal{K}}, \\ \displaystyle\int_{\mathbb{R}\times G} a(x, u_n, \nabla u_n)\nabla(\varrho(v - u_n))\mathrm{d}x + \int_{\mathbb{R}\times G} a_0(x, u_n, \nabla u_n)\varrho(v - u_n)\mathrm{d}x \\ \geq \displaystyle\int_{\mathbb{R}\times G} F(x, u_{n-1})\varrho(v - u_n)\mathrm{d}x, \quad \forall\, v \in \widetilde{\mathcal{K}},\ \forall\, \varrho \in \mathscr{D}(\mathbb{R}),\ \varrho \geq 0, \end{cases} \tag{4.23}$$

respectively, for every $n \geq 1$. Then the sequences $\{\underline{u}_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ are well defined and nondecreasing satisfying

$$u_n \leq \underline{u}_n, \quad F(x, u_n) \leq h(X_2, \underline{u}_n), \quad \forall\, n \in \mathbb{N} \textit{ for a.e. } x \in \mathbb{R}\times G. \tag{4.24}$$

Proof It is clear that u_0, $\underline{u}_0$ satisfy (4.24). Suppose that $\{\underline{u}_{n-1}\}$ and $\{u_{n-1}\}$ are defined and satisfy (4.24), i.e.,

$$u_{n-1} \leq \underline{u}_{n-1}, \quad F(x, u_{n-1}) \leq h(X_2, \underline{u}_{n-1}) \quad \text{for a.e. } x \in \mathbb{R}\times G. \tag{4.25}$$

Thanks to (4.21), one has $h(X_2, \underline{u}_{n-1}) \in L_{\overline{M}}(G)$. Consequently, $\underline{u}_n$ exists by Theorem 3.3. In view of (4.25), $F(x, u_{n-1}) \in L_{\overline{M}}(G)$. Therefore, u_n exists by Theorem 4.1. Arguing as Step 1 in the proof of Theorem 4.1 one can deduce that $u_n \leq \underline{u}_n$. According to (3.33) and (4.20), we have

$$F(x, u_n) \leq F(x, \underline{u}_n) \leq h(X_2, \underline{u}_n), \quad \forall\, n \in \mathbb{N} \text{ for a.e. } x \in \mathbb{R}\times G,$$

i.e., (4.24) holds.

We denote $h_\infty = \lim\limits_{n\to\infty} h(\cdot, \underline{u}_n)$, $F_\infty = \lim\limits_{n\to\infty} F(\cdot, \underline{u}_n)$ and assume that

$$h_\infty \in L_{\overline{M}}(G). \tag{4.26}$$

Theorem 4.2 *Let F and h be nonnegative functions satisfying the hypotheses above and suppose that the assumptions* (3.1)–(3.5) *and* (4.2) *hold, where Ω is replaced by $\mathbb{R}\times G$ in* (3.1)–(3.5)*. Assume that φ is strictly increasing. Then there exists a minimal nonnegative solution of* (4.19)*.*

Proof By (3.33), $F(\cdot, u_{n-1}) \leq F_\infty$, $\forall\, n \geq 1$ a.e. in $\mathbb{R}\times G$. It follows, by using Theorem 4.1, that $\{u_n\}_{n\in\mathbb{N}}$ is nondecreasing and

$$u_n \leq \underline{u}, \tag{4.27}$$

where $\underline{u}$ is the minimal solution of

$$\begin{cases} u \in \widetilde{\mathcal{K}}, \\ \displaystyle\int_{\mathbb{R}\times G} a(x,u,\nabla u)\cdot\nabla(\varrho(v-u))\mathrm{d}x + \int_{\mathbb{R}\times G} a_0(x,u,\nabla u)\varrho(v-u)\mathrm{d}x \\ \displaystyle\geq \int_{\mathbb{R}\times G} F_\infty \varrho(v-u)\mathrm{d}x \end{cases} \tag{4.28}$$

for all $v \in \widetilde{\mathcal{K}}$, and all $\varrho \in \mathscr{D}(\mathbb{R})$ with $\varrho \geq 0$.

Note that $F_\infty \leq h_\infty$ a.e. in $\mathbb{R}\times G$, and h_∞ is independent of x_1. By Lemma 4.1, there exists $\widetilde{u}_\infty$ such that $u_n \to \widetilde{u}_\infty$ a.e. on $\mathbb{R}\times G$.

Let $\ell_0 > 0$. Taking $v = u_n - \varrho(u_n - \underline{u})$ as a test function in (4.23) and choosing ϱ such that $\varrho = 1$ on $(-\ell_0, \ell_0)$, we have

$$\begin{aligned} &\int_{\mathbb{R}\times G} a(x,u_n,\nabla u_n)2\varrho\partial_{x_1}\varrho(u_n-\underline{u})\mathrm{d}x + \int_{\mathbb{R}\times G} a(x,u_n,\nabla u_n)\varrho^2\nabla(u_n-\underline{u})\mathrm{d}x \\ &+ \int_{\mathbb{R}\times G} a_0(x,u_n,\nabla u_n)\varrho^2(u_n-\underline{u})\mathrm{d}x \leq \int_{\mathbb{R}\times G} F(x,u_{n-1})\varrho^2(u_n-\underline{u})\mathrm{d}x, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_{\mathbb{R}\times G} a(x,u_n,\nabla u_n)\varrho^2\nabla u_n\mathrm{d}x + \int_{\mathbb{R}\times G} a_0(x,u_n,\nabla u_n)\varrho^2 u_n\mathrm{d}x \\ &\leq -\int_{\mathbb{R}\times G} a(x,u_n,\nabla u_n)2\varrho\partial_{x_1}\varrho u_n\mathrm{d}x + \int_{\mathbb{R}\times G} a(x,u_n,\nabla u_n)2\varrho\partial_{x_1}\varrho\underline{u}\mathrm{d}x \\ &\quad + \int_{\mathbb{R}\times G} a(x,u_n,\nabla u_n)\varrho^2\nabla\underline{u}\mathrm{d}x + \int_{\mathbb{R}\times G} a_0(x,u_n,\nabla u_n)\varrho^2\underline{u}\mathrm{d}x \\ &\quad + \int_{\mathbb{R}\times G} F(x,u_{n-1})\varrho^2(u_n-\underline{u})\mathrm{d}x. \end{aligned}$$

Since $M \in \Delta_2$, using the conditions (3.3), (3.5), (4.27) and the Young inequality, we can deduce

$$\int_{(-\ell_0,\ell_0)\times G} [M(|u_n|) + M(|\nabla u_n|)]\mathrm{d}x \leq C(\ell_0)$$

as Theorem 3.3, for some constant $C = C(\ell_0)$ independent of n, and consequently,

$$\|u_n\|_{1,M,(-\ell_0,\ell_0)\times G} \leq C(\ell_0).$$

Therefore, one has

$$u_n \to \widetilde{u}_\infty \text{ strongly in } L_M((-\ell_0,\ell_0)\times G)$$

and

$$\begin{aligned} &u_n \rightharpoonup \widetilde{u}_\infty \text{ weakly in } W_0^1 L_M((-\ell_0,\ell_0)\times G) \\ &\text{for } \sigma\Big(\prod_{i=0}^N L_M((-\ell_0,\ell_0)\times G), \prod_{i=0}^N E_{\overline{M}}((-\ell_0,\ell_0)\times G)\Big), \end{aligned}$$

as $n \to \infty$. Since ℓ_0 is arbitrary, $\widetilde{u}_\infty \in \widetilde{\mathcal{K}}$.

By the same arguments as in the proof of Theorem 4.1, it is easy to see that $\widetilde{u}_\infty$ is a nonnegative solution of (4.19).

Let w be a nonnegative solution of (4.19). Then w is a solution of the following problem:

$$\begin{cases} u \in \widetilde{\mathcal{K}}, \\ \displaystyle\int_{\mathbb{R}\times G} a(x,u,\nabla u)\nabla(\varrho(v-u))\mathrm{d}x + \int_{\mathbb{R}\times G} a_0(x,u,\nabla u)\varrho(v-u)\mathrm{d}x \\ \geq \displaystyle\int_{\mathbb{R}\times G} F(x,w)\varrho(v-u)\mathrm{d}x, \quad \forall\, v \in \widetilde{\mathcal{K}},\ \varrho \in \mathscr{D}(\mathbb{R}),\ \varrho \geq 0. \end{cases}$$

Since $w \geq 0$, $F(x,w) \geq F(x,0) = F(x,u_0)$ for a.e. $x \in \mathbb{R}\times G$. By Lemma 4.1 and Theorem 4.1(ii), $w \geq u_1$. Therefore, $F(x,w) \geq F(x,u_1)$ for a.e. $x \in \mathbb{R}\times G$. By Lemma 4.1 and Theorem 4.1(ii), $w \geq u_2$. By induction, we can obtain that $w \geq u_n$, $\forall\, n \in \mathbb{N}$. Letting $n \to \infty$, $w \geq \widetilde{u}_\infty$, that is, $\widetilde{u}_\infty$ is the minimal solution of (4.19).

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