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A Metric Space with Transfinite Asymptotic Dimension $2\omega + 1$*

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Abstract The authors construct a metric space whose transfinite asymptotic dimension and complementary-finite asymptotic dimension are both $2\omega + 1$, where ω is the smallest infinite ordinal number. Therefore, an example of a metric space with asymptotic property C is obtained.

Keywords Transfinite asymptotic dimension, Complementary-finite asymptotic dimension, Asymptotic property C
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1 Introduction

M. Gromov introduced the notion of asymptotic dimension to study finitely generated groups in [1]. In 1998, Guoliang Yu discovered a successful application of asymptotic dimension. He proved that a group with finite asymptotic dimension satisfies the higher Novikov signature conjecture (see [2]). In 2000, N. Higson and J. Roe proved that metric space with bounded geometry and finite asymptotic dimension has property A (see [3]). There is a large class of groups with finite asymptotic dimension, such as finite generated commutative groups, finite rank free groups, Gromov hyperbolic groups and so on. In [4], A. Dranishnikov introduced asymptotic property C which is a natural extension of asymptotic dimension. To classify the metric spaces with infinite asymptotic dimension, T. Radul defined the transfinite asymptotic dimension (trasdim) and found that asymptotic property C can be characterized by transfinite asymptotic dimension. i.e., a metric space X has asymptotic property C if and only if $\mathrm{trasdim}(X) < \infty$ (see [5]). There are examples of metric spaces with trasdim$= \infty$, and with trasdim$= \omega$ as well, where ω is the smallest infinite ordinal number (see [5]). In [6], we constructed a metric space X with $\mathrm{trasdim}(X) = \omega + 1$ which is the first example we found out with transfinite asymptotic dimension greater than ω. By the technique developed in [6], we constructed metric space $X_{\omega+k}$ with $\mathrm{trasdim}(X_{\omega+k}) = \omega + k$ in [7], which generalized the results in [6]. In this paper, we construct a metric space $X_{2\omega+1}$ with $\mathrm{coasdim}(X_{2\omega+1}) =\mathrm{trasdim}(X_{2\omega+1}) = 2\omega + 1$.

This paper is organized as follows: In Section 2, we recall some definitions and properties of transfinite asymptotic dimension and complementary-finite asymptotic dimension. In Section 3, we construct a concrete metric space $X_{2\omega+1}$, whose transfinite asymptotic dimension and

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complementary-finite asymptotic dimension are both $2\omega+1$, where ω is the smallest infinite ordinal number.

2 Preliminaries

Let (X,d) be a metric space and $U,V\subseteq X$,

$$\operatorname{diam} U=\sup\{d(x,y)\mid x,y\in U\}\quad\text{and}\quad d(U,V)=\inf\{d(x,y)\mid x\in U,y\in V\}.$$

Let $R>0$ and $\mathcal{U}$ be a family of subsets of X. $\mathcal{U}$ is said to be R-bounded if

$$\operatorname{diam}\mathcal{U}\doteq\sup\{\operatorname{diam} U\mid U\in\mathcal{U}\}\le R.$$

In this case, $\mathcal{U}$ is said to be uniformly bounded. Let $r>0$, a family $\mathcal{U}$ is said to be r-disjoint if

$$d(U,V)\ge r\quad\text{for every } U,V\in\mathcal{U} \text{ with } U\ne V.$$

In this paper, we denote $\bigcup\{U\mid U\in\mathcal{U}\}$ by $\bigcup\mathcal{U}$, denote $\{U\mid U\in\mathcal{U}_1 \text{ or } U\in\mathcal{U}_2\}$ by $\mathcal{U}_1\cup\mathcal{U}_2$ and denote $\{N_\delta(U)\mid U\in\mathcal{U}\}$ by $N_\delta(\mathcal{U})$ for some $\delta>0$. Letting A be a subset of X, we denote $\{x\in X\mid d(x,A)<\epsilon\}$ by $N_\epsilon(A)$ for some $\epsilon>0$.

Definition 2.1 (see [8]) *The asymptotic dimension of a metric space X does not exceed n (denoted by $\operatorname{asdim}(X)\le n$) which means if there exists $n\in\mathbb{N}$, such that for every $r>0$, there exists a sequence of uniformly bounded families $\{\mathcal{U}_i\}_{i=0}^n$ of subsets of X such that $\bigcup_{i=0}^{n}\mathcal{U}_i$ covers X and each $\mathcal{U}_i$ is r-disjoint for $i=0,1,\cdots,n$. In this case, we say that X has finite asymptotic dimension.*

In [5], T. Radul generalized asymptotic dimension of a metric space X to transfinite asymptotic dimension denoted by $\operatorname{trasdim}(X)$.

Definition 2.2 (see [5]) *Let $\mathrm{Fin}\mathbb{N}$ denote the collection of all finite, nonempty subsets of $\mathbb{N}$, and let $M\subseteq\mathrm{Fin}\mathbb{N}$. For $\sigma\in\{\emptyset\}\bigcup\mathrm{Fin}\mathbb{N}$, let*

$$M^\sigma=\{\tau\in\mathrm{Fin}\mathbb{N}\mid\tau\cup\sigma\in M \text{ and } \tau\cap\sigma=\emptyset\}.$$

Let M^a be the abbreviation for $M^{\{a\}}$ for $a\in\mathbb{N}$. Define the ordinal number $\mathrm{Ord}M$ inductively as follows:

$$\begin{aligned}
&\mathrm{Ord}M=0\Leftrightarrow M=\emptyset.\\
&\mathrm{Ord}M\le\alpha\Leftrightarrow\forall\ a\in\mathbb{N},\ \textit{there exists } \beta<\alpha,\quad \textit{such that } \mathit{Ord}M^a\le\beta.\\
&\mathrm{Ord}M=\alpha\Leftrightarrow\mathrm{Ord}M\le\alpha \textit{ and } \mathrm{Ord}M\le\beta \textit{ is not true for any } \beta<\alpha.\\
&\mathrm{Ord}M=\infty\Leftrightarrow\mathrm{Ord}M\le\alpha \textit{ is not true for any ordinal number } \alpha.
\end{aligned}$$

Lemma 2.1 (see [9]) *Let $M\subseteq\mathrm{Fin}\mathbb{N}$ and $k\in\mathbb{N}$, $\mathrm{Ord}M\le\omega+k$ if and only if $\mathrm{Ord}M^\tau<\omega$ for every $\tau\in\mathrm{Fin}\mathbb{N}$ with $|\tau|=k+1$.*

Definition 2.3 (see [5]) *Given a metric space X, define the following collection:*

$$A(X)=\{\sigma\in\mathrm{Fin}\mathbb{N}\mid\ \textit{there are no uniformly bounded families } \mathcal{U}_i \textit{ for } i\in\sigma,$$

$$\text{such that each } \mathcal{U}_i \text{ is } i\text{-disjoint and } \bigcup_{i\in\sigma}\mathcal{U}_i \text{ covers } X\}.$$

The transfinite asymptotic dimension of X is defined as $\mathrm{trasdim}(X)=\mathrm{Ord}A(X)$.

Definition 2.4 *Let $\{Z_i\}_{i=1}^{\infty}$ be a sequence of subspaces of a metric space (Z, d_Z). Let*

$$X=\bigsqcup_{i=1}^{\infty}(0,\cdots,0,Z_i,0,\cdots).$$

For every $x, y \in X$, there exist unique $l, k \in \mathbb{N}$, $x_l \in Z_l$ and $y_k \in Z_k$, such that $x = (0,\cdots,0,x_l,0,\cdots)$ and $y=(0,\cdots,0,y_k,0,\cdots)$. Assume that $l \le k$. Let $c=0$ if $l=k$ and $c = l+(l+1)+\cdots+(k-1)$ if $l<k$. Define a metric on X by

$$d(x,y)=d_Z(x_l,y_k)+c.$$

The metric space (X,d) is said to be an asymptotic union of $\{Z_i\}_{i=1}^{\infty}$, which is denoted by $\mathrm{as}\bigsqcup_{i=1}^{\infty}Z_i$. *And we denote* $\mathrm{as}\bigsqcup_{i=n}^{\infty}Z_i$ *as a subspace of* $\mathrm{as}\bigsqcup_{i=1}^{\infty}Z_i$ *for every $n\in\mathbb{N}$.*

For every $k, n \in \mathbb{N}$, let

$$X_{\omega+k}^{(n)}=\{(x_1,\cdots,x_n)\in\mathbb{R}^n \mid |\{j \mid x_j\notin 2^n\mathbb{Z}\}|\le k\} \quad\text{and}\quad X_{\omega+k}=\mathrm{as}\bigsqcup_{n=1}^{\infty}X_{\omega+k}^{(n)},$$

where $X_{\omega+k}^{(n)}$ is considered as a subspace of the metric space $(\bigoplus\mathbb{R}, d_{\max})$.

Lemma 2.2 (see [7]) $\mathrm{coasdim}(X_{\omega+k})=\omega+k$ *for every $k\in\mathbb{N}$.*

Let $Y_{\omega+k}^{(n)}=\{(x_1,\cdots,x_n)\in(2^k\mathbb{Z})^n \mid |\{j \mid x_j\notin 2^n\mathbb{Z}\}|\le k\}$, and let

$$Y_{\omega+k}=\mathrm{as}\bigsqcup_{n=1}^{\infty}Y_{\omega+k}^{(n)} \quad\text{and}\quad Y_{2\omega}=\mathrm{as}\bigsqcup_{k=1}^{\infty}Y_{\omega+k},$$

where $Y_{\omega+k}$ is a subspace of the metric space $\mathrm{as}\bigsqcup_{j=1}^{\infty}\mathbb{R}^j$ for each $k\in\mathbb{N}$.

Lemma 2.3 (see [7]) *For every $k\in\mathbb{N}$,* $\mathrm{trasdim}(Y_{\omega+k})=\omega+k$ *and* $\mathrm{trasdim}(Y_{2\omega})=2\omega$.

Definition 2.5 (see [9]) *Every ordinal number γ can be represented as $\gamma=\lambda(\gamma)+n(\gamma)$, where $\lambda(\gamma)$ is the limit ordinal or 0 and $n(\gamma)\in\mathbb{N}$. Letting X be a metric space, we define the complementary-finite asymptotic dimension of X* ($\mathrm{coasdim}(X)$) *inductively as follows:*

- $\mathrm{coasdim}(X)=-1 \Leftrightarrow X=\emptyset$.
- $\mathrm{coasdim}(X)\le\lambda(\gamma)+n(\gamma) \Leftrightarrow$ *for every $r>0$, there exist r-disjoint uniformly bounded families $\mathcal{U}_0,\cdots,\mathcal{U}_{n(\gamma)}$ of subsets of X such that* $\mathrm{coasdim}\big(X\setminus\bigcup\bigcup_{i=0}^{n(\gamma)}\mathcal{U}_i\big)<\lambda(\gamma)$.
- $\mathrm{coasdim}(X)=\gamma \Leftrightarrow \mathrm{coasdim}(X)\le\gamma$ *and* $\mathrm{coasdim}(X)\nleq\beta$ *for every $\beta<\gamma$.*
- $\mathrm{coasdim}(X)=\infty \Leftrightarrow \mathrm{coasdim}(X)\nleq\gamma$ *for every ordinal γ.*

X is said to have complementary-finite asymptotic dimension if $\mathrm{coasdim}(X)\le\gamma$ *for some ordinal number γ.*

Lemma 2.4 (see [9]) *Let X be a metric space with $X_1, X_2\subseteq X$. Then*

$$\mathrm{coasdim}(X_1\cup X_2)\le\max\{\mathrm{coasdim}(X_1),\mathrm{coasdim}(X_2)\}.$$

Lemma 2.5 (see [10]) *Letting X be a metric space, if X has complementary-finite asymptotic dimension, then* $\mathrm{trasdim}(X) \le \mathrm{coasdim}(X)$.

3 Main Result

Let

$$\begin{aligned} &X((p_1,\cdots,p_n),(q_1,\cdots,q_n)) \\ &= \Big\{(x_i)_{i=1}^{\sum_{j=1}^{n} q_j} \in (2^{p_1}\mathbb{Z})^{\sum_{j=1}^{n} q_j} \;\Big|\; |\{i : x_i \notin 2^{p_k}\mathbb{Z}\}| \le \sum_{j=1}^{k-1} q_j \quad \text{for every } k = 2,3,\cdots,n\Big\}, \end{aligned}$$

where $(p_1,\cdots,p_n),(q_1,\cdots,q_n) \in \mathbb{N}^n$ and $p_1 \le \cdots \le p_n$. Then $Y_{\omega+k}^{(n)} = X((k,n),(k,n-k))$ when $n \ge k$. Since for every $k \in \mathbb{N}$,

$$\mathrm{trasdim}\Big(\mathrm{as}\bigsqcup_{n=k}^{\infty} Y_{\omega+k}^{(n)}\Big) = \mathrm{trasdim}\Big(\mathrm{as}\bigsqcup_{n=1}^{\infty} Y_{\omega+k}^{(n)}\Big) = \mathrm{trasdim}(Y_{\omega+k}) = \omega + k,$$

We have

$$\mathrm{trasdim}\Big(\mathrm{as}\bigsqcup_{k=1}^{\infty}\Big(\mathrm{as}\bigsqcup_{n=k}^{\infty} X((k,n),(k,n-k))\Big)\Big) = \mathrm{trasdim}\Big(\mathrm{as}\bigsqcup_{k=1}^{\infty}\Big(as\bigsqcup_{n=k}^{\infty} Y_{\omega+k}^{(n)}\Big)\Big) < 2\omega$$

is not true.

Since $\mathrm{as}\bigsqcup_{k=1}^{\infty}\big(\mathrm{as}\bigsqcup_{n=k}^{\infty} X((k,n),(k,n-k))\big) \subseteq Y_{2\omega}$ and $\mathrm{trasdim}(Y_{2\omega}) = 2\omega$, we get

$$\mathrm{trasdim}\Big(\mathrm{as}\bigsqcup_{k=1}^{\infty}\Big(\mathrm{as}\bigsqcup_{n=k}^{\infty} X((k,n),(k,n-k))\Big)\Big) = 2\omega.$$

Now we prove that $\mathrm{trasdim}\big(\mathrm{as}\bigsqcup_{k=1}^{\infty}\big(\mathrm{as}\bigsqcup_{n=k}^{\infty} X((0,k,n),(1,k,n-k))\big)\big) = 2\omega + 1$.

Lemma 3.1 *For every $r \in \mathbb{N}$, there are r-disjoint uniformly bounded families $\mathcal{U}_0$ and $\mathcal{U}_1$, such that $\mathcal{U}_0 \cup \mathcal{U}_1$ covers* $\mathrm{as}\bigsqcup_{k=2r}^{\infty}\big(\mathrm{as}\bigsqcup_{n=k}^{\infty} X((0,k),(1,n))\big)$.

Proof For every $r \in \mathbb{N}$, $k,n \in \mathbb{N}$ and $k \ge 2r$, $n \ge k$. Let $\mathcal{V}_0^{(k)} = \{[i2^k - r, i2^k + r] \mid i \in \mathbb{Z}\}$.

- If $\big[\frac{2^k-2r}{r}\big] = 2m$, then let

$$\begin{aligned} \mathcal{V}_1^{(k)} &= \{[i2^k + 2jr, i2^k + (2j+1)r] \mid i \in \mathbb{Z}, j = 1,2,\cdots,m-1\} \\ &\quad \cup \{[i2^k + 2mr, (i+1)2^k - r] \mid i \in \mathbb{Z}\}, \\ \mathcal{V}_2^{(k)} &= \{[i2^k + (2j-1)r, i2^k + 2jr] \mid i \in \mathbb{Z}, j = 1,2,\cdots,m\}. \end{aligned}$$

- If $\big[\frac{2^k-2r}{r}\big] = 2m+1$, then let

$$\begin{aligned} \mathcal{V}_1^{(k)} &= \{[i2^k + 2jr, i2^k + (2j+1)r] \mid i \in \mathbb{Z}, j = 1,2,\cdots,m\}, \\ \mathcal{V}_2^{(k)} &= \{[i2^k + (2j-1)r, i2^k + 2jr] \mid i \in \mathbb{Z}, j = 1,2,\cdots,m\} \\ &\quad \cup \{[i2^k + 2mr + r, (i+1)2^k - r] \mid i \in \mathbb{Z}\}. \end{aligned}$$

Note that $\mathcal{V}_0^{(k)} \cup \mathcal{V}_1^{(k)}, \mathcal{V}_2^{(k)}$ are r-disjoint, $2r$-bounded and $\mathcal{V}_0^{(k)} \cup \mathcal{V}_1^{(k)} \cup \mathcal{V}_2^{(k)}$ covers $\mathbb{R}$. Let

$$\mathcal{W}_{0,k}^{(n)} = \Big\{ \Big(\prod_{i=1}^{n+1} V_i \Big) \cap X((0,k),(1,n)) \big| V_i \in \mathcal{V}_0^{(k)} \Big\},$$

$$\mathcal{W}_{1,k}^{(n)} = \Big\{ \prod_{i=1}^{j-1} \{2^k n_i\} \times V_j \times \prod_{i=j+1}^{n+1} \{2^k n_i\} \big| n_i \in \mathbb{Z}, V_j \in \mathcal{V}_1^{(k)}, j = 1,2,3,\cdots,n+1 \Big\},$$

$$\mathcal{W}_{2,k}^{(n)} = \Big\{ \prod_{i=1}^{j-1} \{2^k n_i\} \times V_j \times \prod_{i=j+1}^{n+1} \{2^k n_i\} \big| n_i \in \mathbb{Z}, V_j \in \mathcal{V}_2^{(k)}, j = 1,2,3,\cdots,n+1 \Big\}.$$

It is easy to see that $\mathcal{W}_{i,k}^{(n)}$ are r-disjoint and $2r$-bounded for $i = 0,1,2$. Let

$$\mathcal{U}_{0,k}^{(n)} = \mathcal{W}_{0,k}^{(n)} \cup \mathcal{W}_{1,k}^{(n)}, \quad \mathcal{U}_{1,k}^{(n)} = \mathcal{W}_{2,k}^{(n)}.$$

Then $\mathcal{U}_{0,k}^{(n)}$ and $\mathcal{U}_{1,k}^{(n)}$ are r-disjoint and $2r$-bounded families.

For every $x \in X((0,k),(1,n))$, without loss of generality, we assume that $x = (x_1, \cdots, x_n) \in \mathbb{Z} \times 2^k\mathbb{Z} \times \cdots \times 2^k\mathbb{Z}$. Then $x_i \in 2^k\mathbb{Z}$ for $i = 2,3,\cdots,n$ and x_1 is in one of the following cases.

- $x_1 \in [2^k i - r, 2^k i + r]$ for some $i \in \mathbb{Z}$, it is easy to see that $x \in \bigcup \mathcal{W}_{0,k}^{(n)}$.
- $x_1 \in V$ for some $V \in \mathcal{V}_1^{(k)}$, it is easy to see that $x \in \bigcup \mathcal{W}_{1,k}^{(n)}$.
- $x_1 \in V$ for some $V \in \mathcal{V}_2^{(k)}$, it is easy to see that $x \in \bigcup \mathcal{W}_{2,k}^{(n)}$.

So $\mathcal{U}_{0,k}^{(n)} \cup \mathcal{U}_{1,k}^{(n)}$ covers $X((0,k),(1,n))$. Let

$$\mathcal{U}_{0,k} = \bigcup_{n \geq k} \mathcal{U}_{0,k}^{(n)}, \quad \mathcal{U}_{1,k} = \bigcup_{n \geq k} \mathcal{U}_{1,k}^{(n)}.$$

Since for every $n, m \geq k$ and $n \neq m$,

$$d(X((0,k),(1,n)), X((0,k),(1,m))) \geq k > r,$$

$\mathcal{U}_{0,k}, \mathcal{U}_{1,k}$ are r-disjoint and $2r$-bounded families such that $\mathcal{U}_{0,k} \cup \mathcal{U}_{1,k}$ covers as $\bigsqcup_{n=k}^{\infty} X((0,k),(1,n))$.

Similarly, Let $\mathcal{U}_0 = \bigcup_{k \geq 2r} \mathcal{U}_{0,k}$ and $\mathcal{U}_1 = \bigcup_{k \geq 2r} \mathcal{U}_{1,k}$. Since for every $k, k' \geq 2r$ and $k \neq k'$,

$$d\Big(\text{as} \bigsqcup_{n=k}^{\infty} X((0,k),(1,n)), \text{as} \bigsqcup_{n=k'}^{\infty} X((0,k),(1,n))\Big) \geq 2r > r,$$

$\mathcal{U}_0, \mathcal{U}_1$ are r-disjoint and $2r$-bounded families such that $\mathcal{U}_0 \cup \mathcal{U}_1$ covers

$$\text{as} \bigsqcup_{k=2r}^{\infty} \Big(\text{as} \bigsqcup_{n=k}^{\infty} X((0,k),(1,n))\Big).$$

Proposition 3.1 *Let*

$$X = \text{as} \bigsqcup_{k=1}^{\infty} \Big(\text{as} \bigsqcup_{n=k}^{\infty} X((0,k,n),(1,k,n-k))\Big).$$

Then $\text{coasdim}(X) \leq 2\omega + 1$.

Proof Since for $k \le n$, $X((0,k,n),(1,k,n-k)) \subseteq X((0,k),(1,n))$ and by Lemma 3.1, for any $r \in \mathbb{N}$, there are r-disjoint uniformly bounded families $\mathcal{U}_0$ and $\mathcal{U}_1$ such that

$$\bigcup(\mathcal{U}_0 \cup \mathcal{U}_1) = \text{ as } \bigsqcup_{k=2r}^{\infty} \Big(\text{as} \bigsqcup_{n=k}^{\infty} X((0,k,n),(1,k,n-k))\Big).$$

Note that

$$\text{as} \bigsqcup_{n=k}^{\infty} X((0,k,n),(1,k,n-k)) \subseteq X_{\omega+k+1} = \text{as} \bigsqcup_{n=1}^{\infty} X_{\omega+k+1}^{(n)}.$$

Then by Lemma 2.2 and Lemma 2.4,

$$\begin{aligned} &\text{coasdim}\Big(\text{as} \bigsqcup_{k=1}^{2r-1} \Big(\text{as} \bigsqcup_{n=k}^{\infty} X((0,k,n),(1,k,n-k))\Big)\Big) \\ \le\ &\text{coasdim}\Big(\text{as} \bigsqcup_{k=1}^{2r-1} X_{\omega+k+1}\Big) \\ \le\ &\text{coasdim}(X_{\omega+2r}) = \omega + 2r < 2\omega. \end{aligned}$$

i.e., $\text{coasdim}(X \backslash (\bigcup \bigcup_{i=0}^{1} \mathcal{U}_i)) < 2\omega$. Therefore,

$$\text{coasdim}\Big(\text{as} \bigsqcup_{k=1}^{\infty} \Big(\text{as} \bigsqcup_{n=k}^{\infty} X((0,k,n),(1,k,n-k))\Big)\Big) \le 2\omega + 1.$$

Proposition 3.2 $\text{trasdim}\big(\text{as} \bigsqcup_{k=1}^{\infty} \big(\text{as} \bigsqcup_{n=k}^{\infty} X((0,k,n),(1,k,n-k))\big)\big) \le 2\omega + 1.$

Proof It can be obtained easily by Lemma 2.5 and Proposition 3.1.

Definition 3.1 (see [11]) *Let X be a metric space and let A, B be a pair of disjoint subsets of X. We say that a subset $L \subset X$ is a partition of X between A and B, if there exist open sets $U, W \subset X$ satisfying the following conditions*

$$A \subset U, B \subset W \quad \textit{and} \quad X = U \sqcup L \sqcup W.$$

Definition 3.2 (see [7]) *Let X be a metric space and let A, B be a pair of disjoint subsets of X. For any $\epsilon > 0$, we say that a subset $L \subset X$ is an ϵ-partition of X between A and B, if there exist open sets $U, W \subset X$ satisfying the following conditions*

$$A \subset U, \quad B \subset W, \quad X = U \sqcup L \sqcup W, \quad d(L, A) > \epsilon, \quad d(L, B) > \epsilon.$$

Clearly, an ϵ-partition L of X between A and B is a partition of X between A and B.

Lemma 3.2 (see [7]) *Let $L_0 \doteq [0, B]^n$ for some $B > 0$, F_i^+, F_i^- be the pairs of opposite faces of L_0, where $i = 1, 2, \cdots, n$ and let $0 < \epsilon < \frac{1}{6}B$. For $k = 1, 2, \cdots, n$, let $\mathcal{U}_k$ be an ϵ-disjoint and $\frac{1}{3}B$-bounded family of subsets of $[0, B]^n$. Then there exists an ϵ-partition L_{k+1} of L_k between $F_{k+1}^+ \cap L_k$ and $F_{k+1}^- \cap L_k$, such that $L_{k+1} \subseteq L_k \cap (\bigcup \mathcal{U}_{k+1})^c$ and $L_{k+1} \subset L_k$ for $k = 0, 1, 2, \cdots, n-1$.*

Proof For every $k \in \{1, \cdots, n\}$, let $\mathcal{A}_k \doteq \{U \in \mathcal{U}_k \mid d(U, F_k^+) \le 2\epsilon\}$ and $\mathcal{B}_k \doteq \{U \in \mathcal{U}_k \mid d(U, F_k^+) > 2\epsilon\}$. Clearly, $\mathcal{A}_k \cup \mathcal{B}_k = \mathcal{U}_k$. Let

$$A_k = \bigcup\{N_{\frac{\epsilon}{3}}(U) : U \in \mathcal{A}_k\},$$
$$B_k = \bigcup\{N_{\frac{\epsilon}{3}}(U) : U \in \mathcal{B}_k\}.$$

Then $d(A_k, F_k^-) > B - \frac{1}{3}B - 2\epsilon - \frac{\epsilon}{3} > \frac{4}{3}\epsilon$ and $d(B_k, F_k^+) > 2\epsilon - \frac{\epsilon}{3} > \frac{4}{3}\epsilon$. It follows that

$$(A_k \cup N_{\frac{4}{3}\epsilon}(F_k^+)) \cap (B_k \cup N_{\frac{4}{3}\epsilon}(F_k^-)) = \emptyset.$$

Let

$$L_{k+1} \doteq L_k \setminus ((A_{k+1} \cup N_{\frac{4}{3}\epsilon}(F_{k+1}^+)) \cup (B_{k+1} \cup N_{\frac{4}{3}\epsilon}(F_{k+1}^-)))$$

for $k = 0, 1, 2, \cdots, n-1$. Therefore,

$$L_{k+1} = L_k \setminus ((A_{k+1} \cup N_{\frac{4}{3}\epsilon}(F_{k+1}^+)) \cup (B_{k+1} \cup N_{\frac{4}{3}\epsilon}(F_{k+1}^-))) \subset L_k.$$

And L_{k+1} is an ϵ-partition of L_k between $F_{k+1}^+ \cap L_k$ and $F_{k+1}^- \cap L_k$ such that $L_{k+1} \subset L_k \cap (\bigcup \mathcal{U}_{k+1})^c$.

Lemma 3.3 (see [11, Lemma 1.8.19]) *Let F_i^+ and F_i^- be the pairs of opposite faces of $I^n \doteq [0,1]^n$, where $i \in \{1, \cdots, n\}$. If $I^n = L_0 \supset L_1 \supset \cdots \supset L_n$ is a decreasing sequence of closed sets such that L_i is a partition of L_{i-1} between $L_{i-1} \cap F_i^+$ and $L_{i-1} \cap F_i^-$ for $i \in \{1, 2, \cdots, n\}$, then $L_n \ne \emptyset$.*

Proposition 3.3 *Let*

$$X = \mathrm{as}\bigsqcup_{k=1}^{\infty}\Big(\mathrm{as}\bigsqcup_{n=k}^{\infty} X((0,k,n),(1,k,n-k))\Big).$$

Then $\mathrm{trasdim}(X) \le 2\omega$ *is not true.*

Proof Suppose that $\mathrm{trasdim}\big(\mathrm{as}\bigsqcup_{k=1}^{\infty}\big(\mathrm{as}\bigsqcup_{n=k}^{\infty} X((0,k,n),(1,k,n-k))\big)\big) \le 2\omega$. Then for every $a \in \mathbb{N}$, $\mathrm{Ord}A(X)^a \le \omega + m$ for some $m = m(a) \in \mathbb{N}$. By Lemma 2.1, for every $\tau \in \mathrm{Fin}\mathbb{N}$ satisfying $a \notin \tau$ and $|\tau| = m+1$, $\mathrm{Ord}A(X)^{\{a\}\sqcup\tau} \le n$ for some $n = n(a,\tau) > 1$. Then for any $\sigma \in \mathrm{Fin}\mathbb{N}$ with $|\sigma| = n+1$ and $(\{a\}\sqcup\tau)\cap\sigma = \emptyset$, $\{a\}\sqcup\tau\sqcup\sigma \notin A(X)$. Let

$$\tau = \{a + 2^{m+3}, a + 2^{m+3} + 1, \cdots, a + 2^{m+3} + m\}$$

and

$$\sigma = \{a + 2^{m+n+4} + m, a + 2^{m+n+4} + m + 1, \cdots, a + 2^{m+n+4} + m + n\}.$$

Then there are a-disjoint B-bounded family $\mathcal{U}$, $(a+2^{m+3})$-disjoint B-bounded families $\mathcal{V}_1, \cdots, \mathcal{V}_{m+1}$ and $(a + 2^{m+n+4} + m)$-disjoint B-bounded families $\mathcal{W}_1, \cdots, \mathcal{W}_{n+1}$, such that $\mathcal{U} \cup (\bigcup_{i=1}^{m+1} \mathcal{V}_i) \cup (\bigcup_{j=1}^{n+1} \mathcal{W}_j)$ covers X for some $B > 2^{m+n+4} + a + m$. It follows that $\mathcal{U} \cup (\bigcup_{i=1}^{m+1} \mathcal{V}_i) \cup (\bigcup_{j=1}^{n+1} \mathcal{W}_j)$ covers

$$X((0, m+1, m+n+2)(1, m+1, n+1)) \cap [0, 6B]^{m+n+3}.$$

We assume that $p = \frac{6B}{2^{m+n+2}} \in \mathbb{N}$. Take a bijection

$$\psi : \{1, 2, \cdots, p^{m+n+3}\} \to \{0, 1, 2, \cdots, p-1\}^{m+n+3},$$

and let

$$Q(t) = \prod_{j=1}^{m+n+3} [2^{m+n+2}\psi(t)_j, 2^{m+n+2}(\psi(t)_j + 1)],$$

where $\psi(t)_j$ is the jth coordinate of $\psi(t)$.

Let $\mathcal{Q} = \{Q(t) \mid t \in \{1, 2, \cdots, p^{m+n+3}\}\}$, then $[0, 6B]^{m+n+3} = \bigcup\limits_{Q\in\mathcal{Q}} Q$.

Let $L_0 = [0, 6B]^{m+n+3}$. By Lemma 3.2, since $N_{2^{m+n+2}}(\mathcal{W}_1)$ is $(a+m+2^{m+n+3})$-disjoint and $(2^{m+n+3} + B)$-bounded, there exists a $(a + m + 2^{m+n+3})$-partition L_1 of $[0, 6B]^{m+n+3}$ between F_1^+ and F_1^- such that

$$L_1 \subset \Big(\bigcup N_{2^{m+n+2}}(\mathcal{W}_1)\Big)^c \cap [0, 6B]^{m+n+3},$$
$$d(L_1, F_1^{+/-}) > (a + m + 2^{m+n+3}).$$

Let $\mathcal{M}_1 = \{Q \in \mathcal{Q} \mid Q \cap L_1 \neq \emptyset\}$ and $M_1 = \bigcup \mathcal{M}_1$. Since L_1 is a $(a+m+2^{m+n+3})$-partition of $[0, 6B]^{m+n+3}$ between F_1^+ and F_1^-, M_1 is a partition of $[0, 6B]^{m+n+3}$ between F_1^+ and F_1^-. i.e., $[0, 6B]^{m+n+3} = M_1 \sqcup A_1 \sqcup B_1$ such that A_1, B_1 are open in $[0, 6B]^{m+n+3}$ and A_1, B_1 contain two opposite facets F_1^-, F_1^+ respectively. Let

$$L_1' = \partial_{m+n+2} M_1 \doteq \bigcup \{\partial_{m+n+2} Q \mid Q \in \mathcal{M}_1\},$$

where $\partial_{m+n+2}Q$ is the set of $(m+n+2)$-skeleton of Q. Then $[0, 6B]^{m+n+3} \setminus (L_1' \sqcup A_1 \sqcup B_1)$ is the union of some disjoint open $(m + n + 3)$-dimensional cubes with length of edge being 2^{m+n+2}. So L_1' is a partition of $[0, 6B]^{m+n+3}$ between F_1^+ and F_1^-, and $L_1' \subset \big(\bigcup \mathcal{W}_1\big)^c \cap [0, 6B]^{m+n+3}$.

Similarly, by Lemma 3.2, there exists a $(a+m+2^{m+n+3})$-partition L_2 of L_1' between $L_1' \cap F_2^+$ and $L_1' \cap F_2^-$ such that

$$L_2 \subset \Big(\bigcup N_{2^{m+n+2}}(\mathcal{W}_2)\Big)^c \cap L_1'.$$

Let $\mathcal{M}_2 = \{Q \in \mathcal{M}_1 \mid Q \cap L_2 \neq \emptyset\}$ and $M_2 = \bigcup \mathcal{M}_2$. Since L_2 is a $(a+m+2^{m+n+3})$-partition of L_1' between $L_1' \cap F_2^+$ and $L_1' \cap F_2^-$, $M_2 \cap L_1'$ is a partition of L_1' between $L_1' \cap F_2^+$ and $L_1' \cap F_2^-$, i.e., $L_1' = (M_2 \cap L_1') \sqcup A_2 \sqcup B_2$ such that A_2, B_2 are open in L_1' and A_2, B_2 contain two opposite facets $L_1' \cap F_2^-$, $L_1' \cap F_2^+$ respectively. Let $L_2' = L_1' \cap (\partial_{m+n+1} M_2) = L_1' \cap \big(\bigcup \{\partial_{m+n+1} Q \mid Q \in \mathcal{M}_2\}\big)$, then $L_1' \setminus (L_2' \sqcup A_2 \sqcup B_2)$ is the union of some disjoint open $(m+n+2)$-dimensional cubes with length of edge $= 2^{m+n+2}$. So L_2' is also a partition of L_1' between $L_1' \cap F_2^+$ and $L_1' \cap F_2^-$ and $L_2' \subset \big(\bigcup (\mathcal{W}_1 \cup \mathcal{W}_2)\big)^c \cap [0, 6B]^{m+n+3}$.

After $n + 1$ steps above, we obtain a partition L_{n+1}' of L_n' between $L_n' \cap F_{n+1}^+$ and $L_n' \cap F_{n+1}^-$such that

$$L_{n+1}' \subset \Big(\bigcup (\mathcal{W}_1 \cup \cdots \cup \mathcal{W}_{n+1})\Big)^c \cap [0, 6B]^{m+n+3}$$

and L_{n+1}' is $m + 2$-skeleton. Note that $L_{n+1}' \subset \{(x_i)_{i=1}^{n+m+3} \in [0, 6B]^{m+n+3} \mid |\{j \mid x_j \notin 2^{m+n+2}\mathbb{Z}\}| \leq m + 2\}$.

By Lemma 3.2 and since $N_{2^{m+1}}(\mathcal{V}_1)$ is $(a + 2^{m+2})$-disjoint and $(2^{m+2} + B)$-bounded, there exists a $(a + 2^{m+2})$-partition L_{n+2} of L_{n+1}' between $L_{n+1}' \cap F_{n+2}^+$ and $L_{n+1}' \cap F_{n+2}^-$, such that $L_{n+2} \subset \big(\bigcup N_{2^{m+1}}(\mathcal{V}_1)\big)^c \cap L_{n+1}'$.

Similarly to L_0, L'_{n+1} can be represented as the union of $(m+2)$-dimensional cubes with length of edges being 2^{m+1}. Let $\mathcal{Q}'$ be a family of $(m+2)$-dimensional cubes above with length of edges being 2^{m+1}. Let

$$\mathcal{M}_{n+2} = \{Q' \in \mathcal{Q}' \mid Q' \cap L_{n+2} \neq \emptyset\} \quad \text{and} \quad M_{n+2} = \bigcup \mathcal{M}_{n+2}.$$

Since L_{n+2} is a $(a + 2^{m+2})$-partition of L'_{n+1} between $L'_{n+1} \cap F^+_{n+2}$ and $L'_{n+1} \cap F^-_{n+2}$, $M_{n+2} \cap L'_{n+1}$ is a partition of L'_{n+1} between $L'_{n+1} \cap F^+_{n+2}$ and $L'_{n+1} \cap F^-_{n+2}$, i.e., $L'_{n+1} = (M_{n+2} \cap L'_{n+1}) \sqcup A_{n+2} \sqcup B_{n+2}$ such that A_{n+2}, B_{n+2} are open in L'_{n+1} and A_{n+2}, B_{n+2} contain two opposite facets $L'_{n+1} \cap F^+_{n+2}$, $L'_{n+1} \cap F^-_{n+2}$, respectively. Let $L'_{n+2} = L'_{n+1} \cap (\partial_{m+1} M_{n+2})$. Then $L'_{n+1} \setminus (L'_{n+2} \sqcup A_{n+2} \sqcup B_{n+2})$ is the union of some disjoint open $(m+2)$-dimensional cubes with length of edge being 2^{m+1}. So L'_{n+2} is a partition of L'_{n+1} between $L'_{n+1} \cap F^+_{n+2}$ and $L'_{n+1} \cap F^-_{n+2}$, and $L'_{n+2} \subset (\mathcal{V}_1)^c \cap L'_{n+1}$.

After $m+1$ steps above, we have L'_{m+n+2} to be a partition of L'_{m+n+1} between $L'_{m+n+1} \cap F^+_{m+n+2}$ and $L'_{m+n+1} \cap F^-_{m+n+2}$.

Since

$$L'_{n+m+2} \subset \{(x_i)_{i=1}^{n+m+3} \in \mathbb{R}^{n+m+3} \mid |\{j \mid x_j \notin 2^{m+n+2}\mathbb{Z}\}| \le m+2 \text{ and } |\{j \mid x_j \notin 2^{m+1}\mathbb{Z}\}| \le 1\},$$

we have

$$\begin{aligned} L'_{n+m+2} \subset {} & \Big(\bigcup \bigcup_{i=1}^{m+1} \mathcal{V}_i\Big)^c \cap \Big(\bigcup \bigcup_{j=1}^{n+1} \mathcal{W}_j\Big)^c \\ & \cap X((0, m+1, m+n+2)(1, m+1, n+1)) \cap [0, 6B]^{m+n+3}. \end{aligned}$$

By Lemma 3.2 and $\mathcal{U}$ is a-disjoint and B-bounded, there exists a partition L_{n+m+3} of L'_{n+m+2} such that $L_{n+m+3} \subset \big(\bigcup \mathcal{U}\big)^c \cap L'_{n+m+2}$. Then

$$\begin{aligned} L_{n+m+3} \subseteq {} & \Big(\bigcup \mathcal{U}\Big)^c \cap \Big(\bigcup \bigcup_{i=1}^{m+1} \mathcal{V}_i\Big)^c \cap \Big(\bigcup \bigcup_{j=1}^{n+1} \mathcal{W}_j\Big)^c \\ & \cap X((0, m+1, m+n+2), (1, m+1, n+1)) \cap [0, 6B]^{m+n+3} = \emptyset, \end{aligned}$$

which is a contradiction to Lemma 3.3.

Proposition 3.4 *Let*

$$X = \text{as} \bigsqcup_{k=1}^{\infty} \Big(\text{as} \bigsqcup_{n=k}^{\infty} X((0, k, n), (1, k, n-k))\Big).$$

Then $\operatorname{coasdim}(X) \le 2\omega$ *is not true.*

Proof By Lemma 2.5 and Proposition 3.3, $\operatorname{coasdim}(X) \le 2\omega$ is not true.

Proposition 3.5 *Let*

$$X = \text{as} \bigsqcup_{k=1}^{\infty} \Big(\text{as} \bigsqcup_{n=k}^{\infty} X((0, k, n), (1, k, n-k))\Big).$$

Then $\operatorname{coasdim}(X) = \operatorname{trasdim}(X) = 2\omega + 1$.

Proof By Proposition 3.1 and Proposition 3.4, $\operatorname{coasdim}(X) = 2\omega + 1$. Moreover, by Proposition 3.2 and Proposition 3.3, $\operatorname{trasdim}(X) = 2\omega + 1$.

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