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On a Class of Generalized Curve Flows for Planar Convex Curves*

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Abstract In this paper, the authors consider a class of generalized curve flow for convex curves in the plane. They show that either the maximal existence time of the flow is finite and the evolving curve collapses to a round point with the enclosed area of the evolving curve tending to zero, i.e., $\lim_{t\to T} A(t) = 0$, or the maximal time is infinite, that is, the flow is a global one. In the case that the maximal existence time of the flow is finite, they also obtain a convergence theorem for rescaled curves at the maximal time.

Keywords Curve flow, Convex curve, Longtime existence, Convergence
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1 Introduction

In this paper, we introduce a new curve flow in the plane and along the flow the isoperimetric defect is a monotone quantity. So the interesting question is to study the behavior of this flow. This is the main goal of this paper and the precise results will be stated as theorems below. With no doubt, in the last decades, there are many interesting progress about curve flows in the plane such as curve shortening flows, expanding flows, and nonlocal flows. Motivated by problems from fluid mechanics (see [2]), many people have considered different kinds of curve shortening flow problems. The most widely studied curve shortening flow in the plane is the family of evolving curves $\gamma(t)$ such that

$$\frac{\partial}{\partial t}\gamma(t) = kN, \tag{1.1}$$

where k and N are the curvature of the curve γ and the (inward pointing) unit normal vector to the curve respectively. It has been known that the embedding property is preserved along the flow (1.1), and any simple closed curve can be evolved by (1.1) into a convex one in finite time (see [12]) and at the finite maximal existing time the flow shrinks to a round point in the sense that it becomes asymptotically circular (see [7–9], for a summary of this problem see also [3]). Expanding evolution flows of planar curves also attract a lot of attention. Chow and Tsai [4] have studied the expanding flow such as

$$\frac{\partial}{\partial t}\gamma(t) = -G\Big(\frac{1}{k}\Big)N,$$

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where G is a positive smooth function with $G' > 0$ everywhere. Andrew [1] has studied more general expanding flows, especially flows with isotropic speeds. Nonlocal curve flows have also been considered in last decades. The interesting parts of such flows are that they preserve some geometric quantities. Gage [10] has introduced an area-preserving flow

$$\frac{\partial}{\partial t}\gamma(t) = \Big(k - \frac{2\pi}{L}\Big)N,$$

where L is the length of the evolving curve γ. Then he has proved that the length of the evolving curve is non-increasing and the flow finally converges to a circle. Later on, many people try to find various curve flows which preserve the length of the evolving curve or the area enclosed. One may refer to the interesting papers of Pan, Ma and their coauthors (see [11, 13–14, 16–17] for such results). In particular, in [13], Ma and Cheng have considered an area-preserving flow

$$\frac{\partial}{\partial t}\gamma(t) = \Big(\alpha(t) - \frac{1}{k}\Big)N,$$

where $\alpha(t) = \frac{1}{L}\int_0^L \frac{1}{k}\mathrm{d}s$. Then, they have shown that if the initial curve is any convex curve, the evolving curve converges to a circle in classical sense. Apart from those area-preserving and length-preserving flows, Dallaston and McCue [5–6], Tsai and Wang [19] have considered the flow

$$\frac{\partial}{\partial t}r(t) = [k - q(t)]N, \tag{1.2}$$

where $\gamma(t) \subset \mathbb{R}^2$ is a parametrization of any initial smooth embedded closed curve γ_0, $q(t) = \frac{2\pi-\beta}{L(t)}$ and β is a real constant. For such a flow, the enclosed area $A(t)$ of the evolving curve satisfies

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = -\beta,$$

i.e., the changing rate of the enclosed area $A(t)$ is a fixed constant. In the papers [5] and [6] for $\beta \geq 0$, the authors have derived possible extinction shapes as the curve contracts to a point. In [19], the authors have considered the case $-\infty < \beta < \infty$ and they have concluded the following conclusions. (1) When $\beta > 0$, the flow converges to a point p as t tends to the finite maximal time. Especially when $\beta \in (0, 2\pi]$, the rescaled evolving curve $\widetilde{\gamma}(t) = \sqrt{\frac{\pi}{A}}(\gamma - p)$ converges to the unit circle in the sense that its curvature $\widetilde{k} \to 1$ in the C^∞ norm. (2) When $\beta < 0$ the rescaled evolving curve $\widetilde{\gamma}(t) = \sqrt{\frac{\pi}{A}}\gamma$ converges to the unit circle S^1 centered at the origin $O = (0, 0)$ in the C^∞ norm.

By considering the first order expansion of the function $q(t)$ at $t = 0$ in the generalized curve flow (1.2), we may simply have $q(t) = a + bt$ with a, b being two real constants. The corresponding flow is still a geometric flow in the sense of [18], since the quantity $q(t)$ is geometric one in the sense that it is independent from the parametrization of the curve $\gamma(t)$. We refer to [18] for the generalized curve flow

$$\frac{\partial}{\partial t}C(t) = \beta(t)N(C(t)), \tag{1.3}$$

where $\beta(t)$ is a geometric quantity in the sense that it is independent from the parametrization of the curve $C(t)$. Note that up to a change of a time scale, $C(t) = \psi(t)\gamma(t)$ for a smooth function $\psi(t)$, we may change the evolution (1.1) into the geometric form as (1.3),

$$\frac{\partial}{\partial t}C(t) = \Big(k - p(t)\frac{\psi_t(t)}{\psi(t)^3}\Big)N(C(t)),$$

where $p(t) = \langle C(t), N(C(t))\rangle$ is the support function of the curve $C(t)$. The first order approximation of the quantity $p(t)\frac{\psi_t(t)}{\psi(t)^3}$ also leads to the flow (1.4) below. We shall let $\beta(t) = k - q(t) = k - \beta t$ with β being a real number. Then we are led to a new curve flow

$$\begin{cases} \dfrac{\partial X}{\partial t}(\varphi, t) = K(\varphi, t)N_{\text{in}}(\varphi, t), \\ X(\varphi, 0) = X_0(\varphi), \quad \varphi \in S^1, \end{cases} \tag{1.4}$$

where $X_0(\varphi) : S^1 \to \gamma_0 \subset \mathbb{R}^2$ is a parametrization of any given initial smoothly embedded closed curve γ_0, $k(\varphi, t)$ is the curvature of the evolving curve $\gamma(\cdot, t)$ (parametrized by $X(\varphi, t)$), $N_{\text{in}}(\varphi, t)$ is the inward unit normal vector of $\gamma(\cdot, t)$ and $K(\varphi, t) = k(\varphi, t) - \beta t$ with β being a real constant. Note that our new flow does not preserve the area or the length of the initial data, even the changing rate of the enclosed area or length.

Remark 1.1 When $\beta = 0$, the flow (1.4) is the flow (1.1).

As we shall see soon that, though the length and the area of the evolving curve may not be decreasing along the flow (1.4), the isoperimetric defect is a monotone quantity. This interesting property of the flow motivates us to consider the question if the flow has a nice global behavior. Using the standard arguments (see [9–10]), we may obtain the short time existence result about the flow (1.4) for any immersed closed curve. To understand the global behavior of the flow, we need to calculate some evolution equations for the curvature of the flow (1.4). We notice that the convexity of the evolving curve is preserved. Under the assumption that there are positive lower bound and upper bound of the enclosed area $A(t)$, we shall show that there is a lower bound of the curvature. Meanwhile, we can obtain an integral estimate and a gradient estimate of the curvature along the flow. Then we can show that the curvature will not blow up along the flow, that is to say, there is an upper bound of the curvature. Then the standard parabolic regularity guarantees that all space-time derivatives of the curvature are bounded. Thus we may conclude the long time existence of the flow in below.

Theorem 1.1 *Let $\gamma_0 \subset \mathbb{R}^2$ be a smooth initial convex closed curve and let β be a constant. Then the flow* (1.4) *has a smooth solution for short time* $[0, T)$ *and each evolving curve $\gamma(\cdot, t)$ is a smooth convex curve on* $[0, T)$. *Moreover, the flow* (1.4) *exists as long as its enclosed area $A(t)$ remains positive and finite.*

Remark 1.2 As mentioned above, without convexity assumption, we always have the short time solution to (1.4). That is to say, if the initial data $\gamma_0 \subset \mathbb{R}^2$ is a smooth embedded closed curve in the plane, there is a positive constant $T > 0$ such that the solution to (1.4) exists in $[0, T)$ and each evolving curve $\gamma(\cdot, t)$ is still a smooth embedded closed curve in the time interval $[0, T)$.

The main goal is to consider the behavior of the maximal time existing flow and we show that there is a convergence result in the case that $\beta \geq 0$, which is stated in the following theorem.

Theorem 1.2 *Let $\gamma_0 \subset \mathbb{R}^2$ be a C^2 initial convex closed curve and let β be a nonnegative constant. Then we have the family of C^2 convex curves $\gamma(t)$, which satisfies the evolution equation* (1.4) *for $0 < t < T$, where $T > 0$ is the maximal existing time of the flow, such that* *either* (1) $T < \infty$, $\lim_{t\to T} A(t) = 0$, *the flow converges to a round point as* $t \to T$ *in the sense that*

$$\lim_{t\to T} \frac{L^2(t)}{A(t)} = 4\pi$$

and the normalized curves $\eta(t) = \sqrt{\frac{\pi}{A}}\gamma(t)$ *converge in the Hausdorff metric to the unit circle*; *or* (2) $T = \infty$, *i.e., the flow is a global one.*

Here we recall that for two closed convex sets A and B, the Hausdorff distance between them is $d_H(A,B) = \inf\{\epsilon \mid A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon\}$, where $A_\epsilon = \{x \in \mathbb{R}^2 \mid \mathrm{dist}(x, A) \le \epsilon\}$.

The higher order convergence about the normalized curves $\eta(t)$ is possible following the argument in [9], which is by now well-known, and we may omit the details. We point out that there may occur the case that $T = \infty$ for some initial data, which may be treated in latter chance. At this moment, we have no understanding about the omega limit of the flow at $t = \infty$. In the case that $\beta < 0$, we may know that the area and the length of evolving curves are both decreasing, however, we can not obtain a good estimate of the isoperimetric ratio. Thus we are unable to show any general asymptotically convergence result. We leave this problem open.

The paper is organized as follows. We shall give some evolution equations related to the curve flow (1.4) in Section 2. Then we prove the long time existence Theorem 1.1 and the convergence Theorem 1.2 in Section 3.

2 Convex Curves in the Plane

In this section, we assume that $q(t)$ is a continuous function on $[0,\infty)$ with $q(0) = 0$ and we consider the evolution of curvature and the evolution of isoperimetric defect for the curve flow (1.4) with $K = k - q(t)$. We also assume that each $X = X(\varphi, t) := \gamma$ is a C^2 planar curve.

We first consider the evolution of the length parameter $\mathrm{d}s = \left|\frac{\partial X}{\partial \varphi}\right|\mathrm{d}\varphi$. Recall that

$$\begin{aligned}\frac{\partial}{\partial t}\Big|\frac{\partial X}{\partial \varphi}\Big|^2 &= \frac{\partial}{\partial t}\Big\langle\frac{\partial X}{\partial \varphi}, \frac{\partial X}{\partial \varphi}\Big\rangle \\ &= 2\Big\langle\frac{\partial}{\partial t}\frac{\partial X}{\partial \varphi}, \frac{\partial X}{\partial \varphi}\Big\rangle \\ &= 2\Big\langle\frac{\partial}{\partial \varphi}\frac{\partial X}{\partial t}, \frac{\partial X}{\partial \varphi}\Big\rangle \\ &= 2\Big\langle\frac{\partial}{\partial \varphi}(KN), \frac{\partial X}{\partial \varphi}\Big\rangle \\ &= 2\Big\langle KN_\varphi, \frac{\partial X}{\partial \varphi}\Big\rangle \\ &= 2\Big\langle K(-k)\frac{\partial X}{\partial \varphi}, \frac{\partial X}{\partial \varphi}\Big\rangle \\ &= -2Kk\Big|\frac{\partial X}{\partial \varphi}\Big|^2.\end{aligned}$$

Then,

$$\frac{\partial}{\partial t}\Big|\frac{\partial X}{\partial \varphi}\Big| = -Kk\Big|\frac{\partial X}{\partial \varphi}\Big|. \tag{2.1}$$

Recall that

$$L = \int_\gamma \Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi$$

and

$$2A = -\int_\gamma \langle X, N\rangle \Big|\frac{\partial X}{\partial \varphi}\Big| \mathrm{d}\varphi.$$

Then we have

$$L_t = \int_\gamma \Big|\frac{\partial X}{\partial \varphi}\Big|_t \mathrm{d}\varphi = -\int_\gamma Kk\Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi = -\int_\gamma Kk\mathrm{d}s,$$

i.e.,

$$L_t = -\int_\gamma (k - q(t))k\mathrm{d}s = -\int_\gamma k^2\mathrm{d}s + 2\pi q(t). \tag{2.2}$$

Recall Gage's inequality (see [7]) that for convex closed curves,

$$\int_\gamma k^2\mathrm{d}s \geq \frac{\pi L(t)}{A(t)}.$$

Since $q(0) = 0$, we know that the length $L(t)$ of evolving curve $\gamma(\cdot, t)$ is decreasing for short time interval of 0. By the assumption that $q(t) \geq 0$ for $t \in (0, T)$, we have

$$L_t \leq 2\pi q(t) \leq 2\pi \max_{[0,T]} q(t), \quad L(t) \leq C_0(T)$$

for some uniform constant $C_0(T) > 0$.

For the area $A = A(t)$, we have

$$\begin{aligned}
2A_t &= -\int_\gamma \langle X, N\rangle_t \Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi - \int_\gamma \langle X, N\rangle \Big|\frac{\partial X}{\partial \varphi}\Big|_t \mathrm{d}\varphi \\
&= -\int_\gamma \langle X_t, N\rangle \Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi - \int_\gamma \langle X, N_t\rangle \Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi - \int_\gamma \langle X, N\rangle \Big|\frac{\partial X}{\partial \varphi}\Big|_t \mathrm{d}\varphi \\
&= -\int_\gamma K\Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi - \int_\gamma \langle X, N_t\rangle \Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi + \int_\gamma Kk\langle X, N\rangle \Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi.
\end{aligned}$$

Recall that

$$N_t = -\frac{k_\varphi}{\big|\frac{\partial X}{\partial \varphi}\big|^2} \cdot \frac{\partial X}{\partial \varphi} = -\frac{k_\varphi}{\big|\frac{\partial X}{\partial \varphi}\big|} T. \tag{2.3}$$

Then,

$$\begin{aligned}
2A_t &= -\int_\gamma K\Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi + \int_\gamma K_\varphi \Big\langle X, \frac{\partial X}{\partial \varphi}\Big\rangle \Big|\frac{\partial X}{\partial \varphi}\Big|^{-1}\mathrm{d}\varphi + \int_\gamma Kk\langle X, N\rangle \Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi \\
&= -\int_\gamma K\Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi + \int_\gamma \langle X, T\rangle K_\varphi \mathrm{d}\varphi + \int_\gamma Kk\langle X, N\rangle \Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi \\
&= -2\int_\gamma K\Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi.
\end{aligned}$$

It implies that

$$A_t = -\int_\gamma K\Big|\frac{\partial X}{\partial \varphi}\Big|\mathrm{d}\varphi = -\int_\gamma K\mathrm{d}s,$$

and then

$$A_t = -\int_\gamma (k - q(t))\mathrm{d}s = -2\pi + q(t)L(t). \tag{2.4}$$

Using $q(0) = 0$, we know again that the area $A(t)$ of evolving curve $\gamma(\cdot, t)$ is decreasing for short time interval of 0. Using $q(t) \geq 0$ for $t \in (0, T)$, we have

$$A_t \leq q(t)L \leq C_1(T), \quad A(t) \leq C(T)$$

for some uniform constant $C(T) > 0$.

Recall the isoperimetric inequality in the plane that

$$L^2 - 4\pi A \geq 0.$$

We now consider the evolution of the isoperimetric defect defined by

$$L^2 - 4\pi A.$$

By direct calculations, we obtain

$$\begin{aligned}(L^2 - 4\pi A)_t &= 2LL_t - 4\pi A_t \\ &= 2L\Big(-\int_\gamma Kk\mathrm{d}s\Big) + 4\pi\int_\gamma K\mathrm{d}s \\ &= -2L\int_\gamma k^2\mathrm{d}s + 2Lq\int_\gamma k\mathrm{d}s + 4\pi\int_\gamma k\mathrm{d}s - 4\pi qL \\ &= -2L\int_\gamma k^2\mathrm{d}s + 8\pi^2.\end{aligned}$$

Using Gage's inequality and the isoperimetric inequality for convex closed curves, we have

$$\int_\gamma k^2\mathrm{d}s \geq \frac{\pi L(t)}{A(t)} \geq \frac{4\pi^2}{L}. \tag{2.5}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(L^2 - 4\pi A) \leq -\frac{2\pi L^2}{A} + 8\pi^2 = -\frac{2\pi}{A}(L^2 - 4\pi A) \leq 0, \tag{2.6}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{L^2}{4\pi A} - 1\Big) = \frac{(L^2 - 4\pi A)_t}{4\pi A} - \frac{4\pi(L^2 - 4\pi A)(-2\pi + qL)}{(4\pi A)^2}.$$

Using $q(t) \geq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{L^2}{4\pi A} - 1\Big) \leq -\frac{qL}{A}\Big(\frac{L^2}{4\pi A} - 1\Big) \leq 0. \tag{2.7}$$

In particular, the last two inequalities illustrate that along the curve flow (1.4) with $q(t) = \beta t$, $\beta \geq 0$, the isoperimetric defect and isoperimetric ratio are both decreasing.

By (2.7), we have that for $t > 0$,

$$\frac{L^2}{4\pi A}(t) \leq \frac{L^2}{4\pi A}(0).$$

If we assume $A(t) \to 0$ as $t \to T$, we have

$$L^2(t) \leq A(t)\frac{L^2}{A}(0) \to 0. \tag{2.8}$$

Thus we have proved the below.

Lemma 2.1 *For the flow* (1.4) *with the finite maximal time $T > 0$ and with $A(t) \to 0$ as $t \to T$, the length of the evolving curve $\gamma(\cdot, t)$ tends to zero as $t \to T$.*

Recall the Bonnesen inequality (see [15]) for the planar convex curve γ that

$$\frac{L^2}{A} - 4\pi \geq \frac{\pi^2}{A}(r_{\text{out}} - r_{\text{in}})^2,$$

where r_{out} is the radius of the smallest possible circle that encloses γ, while r_{in} is the radius of the largest possible circle contained within the curve γ. By this inequality, we know that for the curve flow $\gamma(\cdot, t)$ with finite maximal time $T > 0$ and with $A(t) \to 0$ as $t \to T$, the flow $\gamma(\cdot, t)$ shrinks to a round point, i.e., its extinction shape is circular in the C^0 sense (see [6] also). Generally speaking, it is possible that the parabolic curve flow may develop singularities before it shrinks to a point, which is a subtle point in the study of planar curve flows.

We now consider the evolution of curvature along the curve flow (1.4) and obtain the following result.

Lemma 2.2 *The evolution of curvature along the curve flow* (1.4) *is*

$$k_t = \frac{\partial}{\partial \varphi}\Big(\frac{K_\varphi}{|\frac{\partial X}{\partial \varphi}|}\Big)\Big|\frac{\partial X}{\partial \varphi}\Big|^{-1} + Kk^2. \tag{2.9}$$

Proof Differentiating (2.3) with respect to φ, we obtain

$$\begin{aligned}
N_{t\varphi} &= -\frac{\partial}{\partial \varphi}\Big(\frac{K_\varphi}{|\frac{\partial X}{\partial \varphi}|}\Big) \cdot T - \frac{K_\varphi}{|\frac{\partial X}{\partial \varphi}|} \cdot kN\Big|\frac{\partial X}{\partial \varphi}\Big| \\
&= -\frac{\partial}{\partial \varphi}\Big(\frac{k_\varphi}{|\frac{\partial X}{\partial \varphi}|}\Big)T - K_\varphi kN \\
&= N_{\varphi t} \\
&= \frac{\partial}{\partial t}\frac{\partial}{\partial \varphi}(N) \\
&= \frac{\partial}{\partial t}\Big(-kT\Big|\frac{\partial X}{\partial \varphi}\Big|\Big) \\
&= -k_t\Big|\frac{\partial X}{\partial \varphi}\Big|T - k\Big|\frac{\partial X}{\partial \varphi}\Big|_t T - k\Big|\frac{\partial X}{\partial \varphi}\Big|T_t \\
&= -k_t\Big|\frac{\partial X}{\partial \varphi}\Big|T + Kk^2\Big|\frac{\partial X}{\partial \varphi}\Big|T - k\Big|\frac{\partial X}{\partial \varphi}\Big|T_t.
\end{aligned}$$

Then we have

$$k_t\Big|\frac{\partial X}{\partial \varphi}\Big| - Kk^2\Big|\frac{\partial X}{\partial \varphi}\Big| = \frac{\partial}{\partial \varphi}\Big(\frac{k_\varphi}{|\frac{\partial X}{\partial \varphi}|}\Big),$$

which is equivalent to the desired result.

Denote by $v = \big|\frac{\partial X}{\partial \varphi}\big|$ for the curve $\gamma(\varphi, t)$. Recall that the arc-length parameter ds equals vdφ. Then the operator $\frac{\partial}{\partial s}$ is given in terms of φ by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial \varphi}.$$

Meanwhile we can obtain the below.

Lemma 2.3 *Along the curve flow* (1.4),

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = Kk\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}. \tag{2.10}$$

Proof

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial s} &= \frac{\partial}{\partial t}\Big(\frac{1}{v}\frac{\partial}{\partial \varphi}\Big)\\ &= \frac{\partial}{\partial t}\Big(\frac{1}{v}\Big)\frac{\partial}{\partial \varphi} + \frac{1}{v}\frac{\partial^2}{\partial t\partial\varphi}\\ &= -\frac{\frac{\partial v}{\partial t}}{v^2}\frac{\partial}{\partial \varphi} + \frac{1}{v}\frac{\partial^2}{\partial t\partial\varphi}\\ &= \frac{Kkv}{v^2}\frac{\partial}{\partial \varphi} + \frac{1}{v}\frac{\partial}{\partial \varphi}\frac{\partial}{\partial t}\\ &= Kk\frac{1}{v}\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}\\ &= Kk\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}.\end{aligned}$$

Furthermore, we obtain the following lemma.

Lemma 2.4

$$\frac{\partial T}{\partial t} = \frac{\partial K}{\partial s}N = K_sN, \quad \frac{\partial N}{\partial t} = -\frac{\partial K}{\partial s}T = -K_sT. \tag{2.11}$$

Proof

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial^2 X}{\partial t\partial s}\\ &= \frac{\partial^2 X}{\partial s\partial t} + Kk\frac{\partial X}{\partial s}\\ &= \frac{\partial}{\partial s}(KN) + KkT\\ &= K_sN.\end{aligned}$$

By (2.3), we can easily get the second equality.

Let θ be the angle between the tangent vector and the x axis. Then we have the following lemma.

Lemma 2.5

$$\frac{\partial \theta}{\partial t} = K_s, \quad \frac{\partial \theta}{\partial s} = k. \tag{2.12}$$

Proof Since the unit tangent vector is $T = (\cos\theta, \sin\theta)$,

$$\frac{\partial T}{\partial t} = K_sN = K_s(-\sin\theta, \cos\theta) = (-\sin\theta, \cos\theta)\frac{\partial\theta}{\partial t}.$$

Then

$$\frac{\partial \theta}{\partial t} = K_s.$$

Hence, it follows that

$$\frac{\partial T}{\partial s} = kN = (-\sin\theta, \cos\theta)\frac{\partial \theta}{\partial s}.$$

By now, the curvature evolution can be given below.

Lemma 2.6

$$\frac{\partial k}{\partial t} = Kk^2 + K_{ss}. \tag{2.13}$$

Proof

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \theta}{\partial s} = Kk\frac{\partial \theta}{\partial s} + \frac{\partial}{\partial s}\frac{\partial \theta}{\partial t} = Kk^2 + K_{ss}$$

3 Proof of Main Result

In this section, we let $K = k - \beta t$ in the curve flow (1.4), where β is a real number. We mainly consider the case when $\beta \geq 0$. We shall show below that the convexity of the evolving curves of the flow is preserved provided the initial curve is a convex one. We shall study the behavior of the convex curve flow $\gamma(\cdot, t)$, $0 \leq t \leq T < \infty$ with T being the finite maximal existing time, and we can show that the curvature of the curve flow remains bounded before T and $A(t) \to 0$ as $t \to T$ (see Lemma 3.7 below).

We can use the angle θ of the tangent line as a parameter, so the curvature of the curve may be expressed by $k = k(\theta)$. To determine the evolution equation for curvature, we take $\tau = t$ as the time parameter and use θ as the other coordinate. Thus we change variables from (φ, t) to (θ, τ). Then we obtain the following equation for k in terms of θ and τ.

Lemma 3.1

$$\frac{\partial k}{\partial \tau} = k^2\Big(K + \frac{\partial^2 k}{\partial \theta^2}\Big). \tag{3.1}$$

Proof

$$\begin{aligned}
\frac{\partial k}{\partial \tau} &= \frac{\partial k}{\partial t} - \frac{\partial k}{\partial \theta}\frac{\partial \theta}{\partial t} \\
&= Kk^2 + \frac{\partial Ks}{\partial s} - \frac{\partial k}{\partial \theta}K_s \\
&= Kk^2 + \frac{\partial}{\partial s}\Big(\frac{\partial K}{\partial \theta}\cdot k\Big) - \frac{\partial k}{\partial \theta}\frac{\partial K}{\partial \theta}k \\
&= Kk^2 + \frac{\partial}{\partial \theta}(K_\theta)\frac{\partial \theta}{\partial s}\cdot k + \frac{\partial K}{\partial \theta}\frac{\partial k}{\partial \theta}\cdot k - \frac{\partial k}{\partial \theta}\frac{\partial K}{\partial \theta}k \\
&= Kk^2 + K_{\theta\theta}k^2 + K_\theta k_\theta k - \frac{\partial k}{\partial \theta}\frac{\partial K}{\partial \theta}k \\
&= Kk^2 + K_{\theta\theta}k^2 \\
&= k^2(K + k_{\theta\theta}).
\end{aligned}$$

In the rest of the paper, we shall only deal with this equation and for simplicity we replace τ by t. This also means that the formula above can be rewritten as

$$\frac{\partial k}{\partial t} = k^2(K + k_{\theta\theta}).$$

If $\beta < 0$, then

$$\frac{\partial k}{\partial t} = k^2(K + k_{\theta\theta}) = k^2(k - \beta t + k_{\theta\theta}) \geq k^2\Big(k + \frac{\partial^2 k}{\partial \theta^2}\Big).$$

By maximum principle, we can obtain

$$k(\theta, t) \geq k_{\min}(0)$$

for all $(\theta, t) \in S^1 \times [0, T)$, where $k_{\min}(t) = \inf_{\theta}\{k(\theta, t)\}$.

If $\beta \geq 0, t \in [0, T)$, we have $0 \leq \beta t < \beta T = C_1$. It follows that

$$\frac{\partial k}{\partial t} \geq k^2\Big(\frac{\partial^2 k}{\partial \theta^2} + k - C_1\Big), \quad (\theta, t) \in S^1 \times [0, T).$$

Then by [6, Lemma 2.1], we obtain

$$k(\theta, t) \geq \frac{k_{\min}(0)}{2 + C_1 k_{\min}(0)}, \quad \forall\, (\theta, t) \in S^1 \times [0, T).$$

Thus we obtain a lower bound of the curvature for evolving curves below.

Lemma 3.2

$$\begin{cases} k(\theta, t) \geq \dfrac{k_{\min}(0)}{2 + C_1 k_{\min}(0)}, & \beta \geq 0, \\ k(\theta, t) \geq k_{\min}(0), & \beta < 0, \end{cases} \quad \forall\, (\theta, t) \in S^1 \times [0, T), \tag{3.2}$$

where $k_{\min}(t) = \inf_{\theta} k(\theta, t)$.

Remark 3.1 The convexity of the evolving curves of the flow is preserved provided that the initial curve is a convex one.

We now suppose that the flow has a smooth convex solution on a finite time interval $[0, T)$ and $A(t)$ has positive upper bound and lower bound on $[0, T)$, i.e., there exist positive constants c and C such that

$$0 < c \leq A(t) \leq C, \quad \forall\, t \in [0, T). \tag{3.3}$$

Since $L^2 \geq 4\pi A$ and (2.6) hold, we have that for some constant $C(0)$ depending only on the initial curve, $L(t)$ satisfies

$$\sqrt{4\pi c} \leq L(t) \leq \sqrt{C(0) + 4\pi C}, \quad t \in [0, T). \tag{3.4}$$

As in [9], we define the median curvature by

$$k^*(t) = \sup\{b \mid k(\theta, t) > b \text{ on some interval of length } \pi\}.$$

We consider estimate of the median curvature $k^*(t)$ for the evolving curve $\gamma(\cdot, t)$. By the geometric estimate in [9] and our assumption (3.3), we can obtain

$$0 < k^* < \frac{L(t)}{A(t)} \leq \rho, \quad \rho = \frac{\sqrt{C(0) + 4\pi C}}{c}, \quad \forall\, t \in [0, T). \tag{3.5}$$

Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\log k\mathrm{d}\theta &= \int_0^{2\pi}\frac{k_t}{k}\mathrm{d}\theta\\ &= \int_0^{2\pi}k(k_{\theta\theta}+k-q)\mathrm{d}\theta\\ &= \int_0^{2\pi}(k^2-k_\theta^2-qk)\mathrm{d}\theta. \end{aligned}\tag{3.6}$$

For each time $t\in[0,T)$, we consider the open set $U=\{\theta\mid k(\theta,t)>k^*(t)\}$. By the definition of $k^*(t)$, We can write U as a countable union of disjoint open intervals I_i, each of which must have length less than or equal to π. At the endpoints of the closure of these intervals, $k(\theta,t)=k^*(t)$, and Wirtinger's inequality can be applied to the function $k(\theta,t)-k^*(t)$ over $\overline{I}_i$ to obtain

$$\int_{\overline{I}_i}(k-k^*)^2\mathrm{d}\theta\le\int_{\overline{I}_i}\Big(\frac{\partial}{\partial\theta}(k-k^*)\Big)^2\mathrm{d}\theta=\int_{\overline{I}_i}k_\theta^2\mathrm{d}\theta.$$

Then we have

$$\begin{aligned}\int_{\overline{I}_i}(k^2-k_\theta^2-qk)\mathrm{d}\theta &\le\int_{\overline{I}_i}[(2k^*-q)k-{k^*}^2]\mathrm{d}\theta\\ &\le(2k^*-q)\int_{\overline{I}_i}k\mathrm{d}\theta.\end{aligned}\tag{3.7}$$

On the compliment of U, we have the estimate $k\le k^*$. Then

$$\begin{aligned}\int_{U^c}(k^2-k_\theta^2-qk)\mathrm{d}\theta &\le\int_{U^c}{k^*}^2\mathrm{d}\theta-\int_{U^c}qk\mathrm{d}\theta\\ &={k^*}^2\int_{U^c}\mathrm{d}\theta-q\int_{U^c}k\mathrm{d}\theta.\end{aligned}\tag{3.8}$$

Combining (3.7)–(3.8) with (3.6), we can obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\log k\mathrm{d}\theta &\le(2k^*-q)\int_U k\mathrm{d}\theta+{k^*}^2\int_{U^c}\mathrm{d}\theta-q\int_{U^c}k\mathrm{d}\theta\\ &=2k^*\int_\gamma k\mathrm{d}\theta+{k^*}^2\int_{U^c}\mathrm{d}\theta-q\int_\gamma k\mathrm{d}\theta\\ &\le(2k^*-q)\int_\gamma k\mathrm{d}\theta+2\pi{k^*}^2.\end{aligned}$$

Recall that

$$L_t=-\int_0^{2\pi}K\mathrm{d}\theta=-\int_0^{2\pi}k\mathrm{d}\theta+2\pi q.$$

Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\log k\mathrm{d}\theta &\le(2k^*-q)\int_0^{2\pi}k\mathrm{d}\theta+2\pi{k^*}^2\\ &=(2k^*-q)(2\pi q-L_t)+2\pi{k^*}^2.\end{aligned}\tag{3.9}$$

Assume that $\beta\ge0$. If $2k^*-q\le0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\log k\mathrm{d}\theta\le2\pi{k^*}^2\le2\pi\rho^2,$$

where ρ is the constant defined in (3.5).

If $2k^* - q \geq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} \log k \mathrm{d}\theta \leq 2\rho(C_2 - L_t) + 2\pi\rho^2,$$

where $C_2 = 2\pi C_1$. Integrating over $[0, T]$, we can obtain

$$\int_0^{2\pi} \log k(\theta, t)\mathrm{d}\theta - \int_0^{2\pi} \log k(\theta, 0)\mathrm{d}\theta \leq (2\rho C_2 + 2\pi\rho^2)T + 2\rho L(0).$$

Assume that $\beta < 0$. Then $q \leq 0$ and $(2k^* - q) > 0$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} \log k \mathrm{d}\theta &\leq (2k^* - q)(2\pi q - L_t) + 2\pi {k^*}^2 \\ &\leq (2\rho - \beta T)(2\pi q - L_t) + 2\pi {k^*}^2 \\ &\leq (2\rho - \beta T)(-L_t) + 2\pi\rho^2.\end{aligned}$$

Again integrating over $[0, T]$, gives

$$\int_0^{2\pi} \log k(\theta, t)\mathrm{d}\theta \leq \int_0^{2\pi} \log k(\theta, 0)\mathrm{d}\theta + (2\rho - \beta T)L(0) + 2\pi\rho^2 T.$$

Thus, we obtain the following integral estimate of the curvature of the evolving curve.

Lemma 3.3 *Let $\rho > 0$ be the constant in* (3.5). *Then there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ and depending only on β, c, ρ such that*

$$\int_0^{2\pi} \log k(\theta, t)\mathrm{d}\theta \leq \int_0^{2\pi} \log k(\theta, 0)\mathrm{d}\theta + \lambda_1 L(0) + \lambda_2 T, \quad \forall\, t \in [0, T). \tag{3.10}$$

We can also obtain a gradient estimate of the curvature.

Lemma 3.4 *There exists a constant $C(0) \geq 0$ depending only on the initial curve such that*

$$\int_0^{2\pi} k_\theta^2(\theta, t)\mathrm{d}\theta \leq \int_0^{2\pi} k^2(\theta, t)\mathrm{d}\theta + 2\beta \int_0^t \int_0^{2\pi} k(\theta, s)\mathrm{d}\theta\mathrm{d}s + C(0). \tag{3.11}$$

Proof

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} (k^2 - k_\theta^2 - 2qk)\mathrm{d}\theta &= \int_0^{2\pi} (2kk_t - 2k_\theta k_{\theta t} - 2q'(t)k - 2qk_t)\mathrm{d}\theta \\ &= 2\int_0^{2\pi} (kk_t - k_\theta k_{\theta t} - q'k - qk_t)\mathrm{d}\theta \\ &= 2\int_0^{2\pi} (k - q)k_t\mathrm{d}\theta - 2\int_0^{2\pi} k_\theta k_{\theta t}\mathrm{d}\theta - 2q'\int_0^{2\pi} k\mathrm{d}\theta \\ &= 2\int_0^{2\pi} (k - q + k_{\theta\theta})k_t\mathrm{d}\theta - 2q'\int_0^{2\pi} k\mathrm{d}\theta \\ &= 2\int_0^{2\pi} \frac{k_t}{k^2}k_t\mathrm{d}\theta - 2\beta\int_0^{2\pi} k\mathrm{d}\theta \\ &\geq -2\beta\int_0^{2\pi} k\mathrm{d}\theta.\end{aligned}$$

So,

$$\int_0^{2\pi}(k^2(\theta,t)-k_\theta^2(\theta,t))\mathrm{d}\theta+2\beta\int_0^t\int_0^{2\pi}k\mathrm{d}\theta\mathrm{d}s\geq -C(0).$$

This completes the proof.

For every $t\in[0,T)$, we let $\overline{k}(t)=\max\limits_{\theta}k(\theta,t)$. Then there is a $\theta(t)\in S^1$ such that $\overline{k}(t)=k(\theta(t),t)$. By (3.11), we have

$$\begin{aligned}\overline{k}(t)&=k(\theta,t)+\int_\theta^{\theta(t)}k_\theta(s,t)\mathrm{d}s\\&\leq k(\theta,t)+|\theta(t)-\theta|^{\frac12}\Big(\int_0^{2\pi}k_\theta^2(\theta,t)\mathrm{d}\theta\Big)^{\frac12}\\&\leq k(\theta,t)+|\theta(t)-\theta|^{\frac12}\Big(\int_0^{2\pi}k^2\mathrm{d}\theta+2|\beta|\int_0^t\int_0^{2\pi}k\mathrm{d}\theta\mathrm{d}s+C(0)\Big)^{\frac12}\\&\leq k(\theta,t)+|\theta(t)-\theta|^{\frac12}[2\pi\overline{k}^2+4\pi|\beta|\overline{k}T+C(0)]^{\frac12}\\&\leq k(\theta,t)+|\theta(t)-\theta|^{\frac12}(\sqrt{2\pi}\overline{k}+\sqrt{2\pi}|\beta|T+\sqrt{C(0)})\\&\leq k(\theta,t)+|\theta(t)-\theta|^{\frac12}\sqrt{2\pi}\overline{k}+2\pi|\beta|T+\sqrt{2\pi C(0)}.\end{aligned}\tag{3.12}$$

Then, we can get the following estimate.

Lemma 3.5 *For sufficiently small $\epsilon>0$, there exist two constants $\delta>0$ and $D>0$ depending only on ϵ,β,T and the initial curve, such that*

$$(1-\epsilon)\overline{k}\leq k(\theta,t)+D\tag{3.13}$$

for all $\theta\in(\theta(t)-\delta^2,\theta(t)+\delta^2)$ and for all t sufficiently close to T.

By Lemma 3.5 above, we can obtain an upper bound of the curvature.

Lemma 3.6 *Suppose that the flow* (1.4) *with $q(t)=\beta t$ has a smooth convex solution on a finite time interval $[0,T)$ such that $A(t)$ has a positive upper bound C and a positive lower bound c on $[0,T)$. Then the curvature $k(\theta,t)$ will not blow up as $t\to T$.*

Proof We argue by contradiction. If the curvature blows up as $t\to T$, we would have $\overline{k}\to\infty$ as $t\to T$. However, (3.13) shows that for t sufficiently close to T, the curvature is uniformly large on some interval of fixed length $2\delta^2$. This leads to $\lim\limits_{t\to T}\log k(\theta,t)=\infty$. However, this is a contradiction to (3.10). Thus, the curvature $k(\theta,t)$ will not blow up as $t\to T$.

We now give the proof of Theorem 1.1.

Proof Note that as long as $A(t)$ remains positive and finite, by (3.4), the length $L(t)$ will remain positive and finite. Moreover, by Lemmas 3.2 and 3.6, the curvature $k(\theta,t)$ has positive upper bound and positive lower bound. Following the arguments in [9, p 84–86], via the standard parabolic regularity applied to (3.1), we can obtain that all space-time derivatives of $k(\theta,t)$ remain bounded. As a consequence of this, the flow can continue to evolve smoothly. This completes the proof.

To prove Theorem 1.2, we need some reparations. Roughly speaking, the idea of the proof of Theorem 1.2 is similar to the main theorem of [8].

Firstly note that we have the area decay at T.

Lemma 3.7 *For the curve flow* (1.4) *with finite maximal time* $T > 0$*, we have*

$$A(t) \to 0 \quad \text{as } t \to T.$$

Proof In fact, since T is the maximal existing time, we know that there is a sequence t_j such that $A(t_j) \to 0$ as $t_j \to T$. By (2.4) and (2.8), we know that for some uniform constant $C > 0$ depending on T,

$$A_t = -2\pi + q(t)L(t) \le -2\pi + C\sqrt{A(t)}.$$

In a small neighborhood of $t = t_j$, $A(t)$ is small such that $C\sqrt{A(t)} \le 2\pi$. Hence $A_t < 0$ in the neighborhood of t_j. This implies that $C\sqrt{A(t)} \le 2\pi$ for any $t > t_j$ and $A(t) \to 0$ as $t \to T$. This completes the proof of Lemma 3.7.

Furthermore, we have the following lemma.

Lemma 3.8 *If* $\lim\limits_{t\to T} A(t) = 0$*, then*

$$\lim_{t\to T} \inf L(t)\Big(\int_{\gamma(t)} k^2 \mathrm{d}s - \pi\frac{L(t)}{A(t)}\Big) \le 0. \tag{3.14}$$

Proof We now consider the isoperimetric ratio. By (2.2) and (2.4), we can obtain

$$\begin{aligned}\Big(\frac{L^2}{A}\Big)_t &= \frac{2LL_tA - A_tL^2}{A^2}\\ &= \frac{2L}{A}L_t - A_t\frac{L^2}{A^2}\\ &= -\frac{2L}{A}\int_{\gamma(t)} Kk\mathrm{d}s + \frac{L^2}{A^2}\int_{\gamma(t)} K\mathrm{d}s\\ &= -\frac{2L}{A}\int_{\gamma(t)} k^2\mathrm{d}s + \frac{4\pi L}{A}q + \frac{2\pi L^2}{A^2} - \frac{qL^3}{A^2}.\end{aligned}$$

For

$$\frac{4\pi L}{A}q - \frac{qL^3}{A^2} = \frac{Lq}{A}\Big(4\pi - \frac{L^2}{A}\Big) \le 0,$$

we have

$$\Big(\frac{L^2}{A}\Big)_t \le -\frac{2L}{A}\int_{\gamma(t)} k^2\mathrm{d}s + \frac{2\pi L^2}{A^2} = -\frac{2L}{A}\Big(\int_{\gamma(t)} k^2\mathrm{d}s - \frac{\pi L}{A}\Big).$$

If $L\big(\int_{\gamma(t)} k^2\mathrm{d}s - \frac{\pi L}{A}\big) > \epsilon$ in the time interval $[t_1, T)$, then

$$\Big(\frac{L^2}{A}\Big)_t \le -\frac{2\epsilon}{A}.$$

However,

$$(\log A)_t = \frac{A_t}{A} = -\frac{\int_{\gamma(t)} K\mathrm{d}s}{A} = -\frac{2\pi - qL}{A} \ge -\frac{2\pi}{A},$$

i.e., $-\frac{2}{A} \le \frac{1}{\pi}(\log A)_t$. This leads to

$$\Big(\frac{L^2}{A}\Big)_t \le \frac{\epsilon}{\pi}(\log A)_t, \quad t_1 \le t < T.$$

From which it follows by integration that

$$\frac{L^2}{A}(t) \leq \frac{L^2}{A}(t_1) - \frac{\epsilon}{\pi}\log(A(t_1)) + \frac{\epsilon}{\pi}\log(A(t)), \quad t_1 \leq t < T.$$

The left-hand side is at least 4π, but the right-hand side tends to negative infinity as $A(t)$ goes to zero. This gives us a contradiction. So we complete the proof.

We also need two lemmas which have been proved by Gage [8]. For convenience of readers, we present them here without proof.

Lemma 3.9 (see [8, Lemma 2]) *There is a non-negative function $F(\gamma)$ which is defined for all C^1 convex curves γ and which satisfies*

$$LA(1 - F(\gamma)) \geq \pi \int_\gamma p^2 \mathrm{d}s. \tag{3.15}$$

Here, $p = -\langle X, N\rangle$. Given a sequence of regular convex curves γ_i such that $\lim\limits_{i\to\infty} F(\gamma_i) = 0$, we consider the normalized curves $\eta_i = \sqrt{\frac{\pi}{A}}\gamma_i$. If these normalized curves lie in a fixed bounded region of the plane, then the laminae H_i enclosed converges to the unit disk in Hausdorff metric. Finally, $F(\gamma) = 0$ if and only if γ is a circle.

Lemma 3.10 (see [8, Lemma 3]) *For the same function $F(\gamma)$ as above, we have*

$$(1 - F(\gamma)) \int_\gamma k^2 \mathrm{d}s - \pi \frac{L}{A} \geq 0, \tag{3.16}$$

whenever γ is a C^2 convex curve in the plane.

We now prove Theorem 1.2.

Proof Lemma 3.7 shows $A(t) \to 0$ as $t \to T$, and by Lemma 2.1 we have $L(t) \to 0$. Hence, the flow (1.4) must converge to a point $p \in \mathbb{R}^2$ as $t \to T$.

By (3.16) we obtain

$$\int k^2 \mathrm{d}s - \pi\frac{L}{A} \geq \Big(\int k^2 \mathrm{d}s\Big) F(\gamma).$$

By the Cauchy-Schwartz inequality and the fact that the total curvature of a simple closed curve is 2π, we see that

$$L \int k^2 \mathrm{d}s \geq \Big(\int k \mathrm{d}s\Big)^2 = 4\pi^2.$$

Then we can obtain

$$L\Big(\int k^2 \mathrm{d}s - \pi\frac{L}{A}\Big) \geq 4\pi^2 F(\gamma). \tag{3.17}$$

By Lemma 3.8, we conclude that there is a subsequence of curves $\gamma(t_i)$ such that the left-hand side of (3.17) tends to zero. Then it follows that $F(\gamma(t_i))$ tends to zero.

Next, we want to show that the normalized curves lie in a bounded region. From the inequality (2.7), we observe that $\frac{L^2}{A}$ decreases under the curve flow. Using the Bonnesen inequality that

$$\frac{L^2}{A} - 4\pi \geq \frac{\pi^2}{A}(r_{\mathrm{out}} - r_{\mathrm{in}})^2, \tag{3.18}$$

we know that the outer radii of the normalized curves $\eta(t)$ are bounded for all $t \in [0, T)$ by a constant C. As the evolving convex curve shrinking as time increasing, we can choose one point as the origin in the homothetic expansion of $\mathbb{R}^2$. Then all of the normalized curves $\eta(t)$ will lie in a ball of radius $2C$ around this point.

Applying Lemma 3.9, we see that the sequence of normalized laminae $H(t_i)$ converges to the unit disk in the Hausdorff metric. Since L and A are continuous functions of convex laminae, $\frac{L^2}{A}$ converges to 4π for this sequence. Then, $\frac{L^2}{A}$ is decreasing under this curve evolution and therefore $\frac{L^2}{A}$ converges to 4π for the entire one parameter family of curves. For the normalized curves, (3.18) shows that both r_{out} and r_{in} converge to 1, forcing the normalized curves to converge to the unit circle. Thus we complete the proof.

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