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Willmore Surfaces in Spheres via Loop Groups IV: On Totally Isotropic Willmore Two-Spheres in S^6*

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Abstract In this paper the author derives a geometric characterization of totally isotropic Willmore two-spheres in S^6, which also yields to a description of such surfaces in terms of the loop group language. Moreover, applying the loop group method, he also obtains an algorithm to construct totally isotropic Willmore two-spheres in S^6. This allows him to derive new examples of geometric interests. He first obtains a new, totally isotropic Willmore two-sphere which is not S-Willmore (i.e., has no dual surface) in S^6. This gives a negative answer to an open problem of Ejiri in 1988. In this way he also derives many new totally isotropic, branched Willmore two-spheres which are not S-Willmore in S^6.

Keywords Totally isotropic Willmore two-spheres, Normalized potential, Iwasawa decompositions
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1 Introduction

Totally isotropic surfaces first appeared in the study of the global geometry of surfaces in the famous work of Calabi [11], where twistor bundle theory was applied to describe the geometry of minimal two-spheres in S^n. This led later to much progress in geometry and the theory of integrable systems (see for example [4, 8, 10]). In the study of Willmore two-spheres, totally isotropic surfaces play an important role as well. First we note that isotropic properties are conformally invariant. This indicates that they are of interest in the conformal geometry of surfaces. Moreover, the classical work of Ejiri [20] shows that isotropic surfaces in S^4 are automatically Willmore surfaces and furthermore they are Willmore surfaces with dual surfaces. He also showed that Willmore two-spheres in S^4 are either Möbius equivalent to minimal surfaces with planer ends in $\mathbb{R}^4$, or isotropic two-spheres (see [20]) (see also [7, 28–29]).

In [20], Ejiri also introduced the notion of S-Willmore surfaces. Roughly speaking, these surfaces can be viewed as Willmore surfaces admitting dual surfaces. Note that by Bryant's classical work, every Willmore surface in S^3 has a dual Willmore surface (see [5–6]). But when the codimension is bigger than 1, a Willmore surface may not have a dual surface (see [7, 20, 27]). Using the duality properties of S-Willmore surfaces, Ejiri provided furthermore a classification of S-Willmore two-spheres in S^{n+2} by constructing the holomorphic forms for these surfaces (see [20]). Especially, for Willmore two-spheres in S^4, a construction of a holomorphic 8-form indicates that these surfaces are automatically S-Willmore (see [7, 20, 26, 28–29]). In the end of Ejiri's paper, he asked whether all Willmore two-spheres in S^{n+2} are S-Willmore or not. If the

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answer is 'no', i.e., if some Willmore, but not S-Willmore two-spheres would exist, how would one construct and characterize them?

In this paper, we will answer Ejiri's open problem by a concrete construction of a totally isotropic Willmore two-sphere in S^6 which is not S-Willmore. Moreover, beyond the explicit construction of some new examples, the main goal of this paper is to characterize all totally isotropic Willmore two-spheres in S^6 via their geometric properties and their normalized potentials. This geometric description also supplies the basis for the work of [34], where we provide a coarse classification of Willmore two-spheres in spheres by using the loop group method for the construction of harmonic maps (see [8, 15, 18–19]).

Different from the case in S^4, where totally isotropic surfaces are automatically S-Willmore surfaces and of finite uniton type, totally isotropic surfaces in S^6 are not even Willmore in general. We refer to [8, 18, 33–34] for the definition of uniton. Moreover, even a totally isotropic Willmore surface in S^6 will, in general, not be of finite uniton type (see Remark 2.11). To this end, we first derive a geometric characterization of totally isotropic Willmore two-spheres in S^6, which is similar to the description of minimal two-spheres in S^n (see [4, 11]). Roughly speaking, the normal connection of a totally isotropic Willmore two-sphere has a special form and conversely, totally isotropic surfaces with such special normal connection are always Willmore and of finite uniton type (see Theorems 2.1–2.2). Application of this description yields a second description of such Willmore surfaces in terms of loop group language (see Theorems 2.8 and 3.3). The second description of totally isotropic Willmore two-spheres in S^6 contains also a concrete algorithm of constructions of explicit totally isotropic Willmore two-spheres. By this method, we derive many new examples of Willmore surfaces as follows, most of which have two branched points.

Example 1.1 Let $\lambda \in S^1$ and let (we refer to Section 2 for the definition of η)

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \widehat{B}_1 \\ -\widehat{B}_1^t I_{1,3} & 0 \end{pmatrix} \mathrm{d}z$$

with

$$\widehat{B}_1 = \frac{1}{2} \begin{pmatrix} \mathrm{i}pz^{p-1} & -pz^{p-1} & -\mathrm{i} & 1 \\ -\mathrm{i}pz^{p-1} & pz^{p-1} & -\mathrm{i} & 1 \\ -p & -\mathrm{i}p & -z^{p-1} & -\mathrm{i}z^{p-1} \\ \mathrm{i}p & -p & -\mathrm{i}z^{p-1} & z^{p-1} \end{pmatrix}. \tag{1.1}$$

Here $p \in \mathbb{Z}^+, p \geq 2$. The associated family of Willmore two-spheres x_λ, corresponding to η, is

$$x_\lambda = \frac{1}{\nu} \begin{pmatrix} 1 - r^{2p-2} - \dfrac{(p-1)^3(p+1)r^{2p}}{p^2} + \dfrac{p^2(p-1)^2 r^{2p+2}}{(p+1)^2} - \dfrac{(p-1)^2 r^{4p}}{p^2(p+1)^2} \\ -\mathrm{i}(z^{p-1} - \overline{z}^{p-1})\Big(1 + \dfrac{(p-1)^2 r^{2p+2}}{(p+1)^2}\Big) \\ (z^{p-1} + \overline{z}^{p-1})\Big(1 + \dfrac{(p-1)^2 r^{2p+2}}{(p+1)^2}\Big) \\ -\mathrm{i}(p-1)(\lambda^{-1} z^p - \lambda \overline{z}^p)\Big(1 - \dfrac{(p-1)r^{2p}}{p^2(p+1)}\Big) \\ (p-1)(\lambda^{-1} z^p + \lambda \overline{z}^p)\Big(1 - \dfrac{(p-1)r^{2p}}{p^2(p+1)}\Big) \\ -\mathrm{i}\frac{(p-1)r^{2p-2}}{p}(\lambda^{-1} z - \lambda \overline{z})\Big(1 + \dfrac{p^2(p-1)r^2}{p+1}\Big) \\ \frac{(p-1)r^{2p-2}}{p}(\lambda^{-1} z + \lambda \overline{z})\Big(1 + \dfrac{p^2(p-1)r^2}{p+1}\Big) \end{pmatrix}, \tag{1.2}$$

with

$$\nu = 1 + r^{2p-2} + \frac{(p-1)^2(p^2+1)r^{2p}}{p^2} + \frac{p^2(p-1)^2r^{2p+2}}{(p+1)^2} + \frac{(p-1)^2r^{4p}}{p^2(p+1)^2} \quad \text{and} \quad r = |z|.$$

Moreover $x_\lambda : S^2\backslash\{0,\infty\} \to S^6$ is a Willmore immersion in S^6, which is full, not S-Willmore, and totally isotropic. It is obvious that x_λ is S^1-equivariant. Note that x_λ is also immersed at 0 and ∞ when $p = 2$. When $p > 2$, x_λ has two branched points 0 and ∞, whose metrics tend to 0 with the same speed r^{2p-4}. To be concrete, near the point $z = 0$, $|\partial_z(x_\lambda)|^2 = 2(p-1)^2r^{2p-4} + o(r^{2p-4})$. Near the point $z = \infty$, setting $\widetilde{z} = \frac{1}{z}$ and $\widetilde{r} = |\widetilde{z}|$, we have $|\partial_{\widetilde{z}}(x_\lambda)|^2 = 2p^4(p-1)^2\widetilde{r}^{2p-4} + o(\widetilde{r}^{2p-4})$.

Recently there are several progresses on the discussions of branched points of Willmore surfaces (see for example [1, 12, 24–25]). We hope that these examples will help the understanding of branched points of Willmore surfaces. We only show the explicit computations in Appendix B for the case $p = 2$, since the construction of x_λ is the same for the other ones.

This paper is organized as follow: In Section 2, we first recall basic results of Willmore surfaces and derive a new geometric description of isotropic Willmore two-spheres in S^6. Moreover, we obtain a description of the normalized potentials of isotropic Willmore two-spheres in S^6. The converse part, that generically such normalized potentials will produce special totally isotropic Willmore surfaces in S^6, as well as new examples, makes up the main content of Section 3. The main idea is to perform a concrete Iwasawa decomposition for these normalized potentials to derive geometric properties of the corresponding Willmore surfaces, which also yields an algorithm to construct Willmore surfaces. We put the technical computations of Iwasawa decompositions and examples into two Appendixes for interested readers.

2 Isotropic Willmore Two-spheres in S^6

In Subsection 2.1, we first recall the basic theory of Willmore surfaces and then focus on isotropic Willmore surfaces in S^6. In Subsection 2.2 we will collect the basic DPW methods as well as Wu's formula for harmonic maps and then derive the normalized potentials for isotropic Willmore two-spheres in S^6.

2.1 Isotropic Willmore surfaces in S^6 and related holomorphic differentials

2.1.1 Willmore surfaces in spheres

For completeness we first recall briefly the basic surface theory. For more details, we refer to [16, Section 2], [17] and [34, Section 2] (see also [9, 26]).

Let $\mathbb{R}^{n+4}_1$ be the Lorentz-Minkowski space equipped with the Lorentzian metric

$$\langle x, y\rangle = -x_0y_0 + \sum_{j=1}^{n+3} x_jy_j = x^tI_{1,n+3}y, \quad I_{1,n+3} = \operatorname{diag}(-1, 1, \cdots, 1), \ \forall x, y \in \mathbb{R}^{n+4}.$$

We denote by $\mathcal{C}^{n+3}_+ = \{x \in \mathbb{R}^{n+4}_1 \mid \langle x, x\rangle = 0, x_0 > 0\}$ the forward light cone and by $Q^{n+2} = \mathcal{C}^{n+3}_+/\mathbb{R}^+$ the projective light cone. It is well-known that Riemannian space forms can be conformally embedded into Q^{n+2} (see [5, 7, 9, 20, 28–29]). For a conformal immersion $y : M \to S^{n+2}$, one has a canonical lift $Y = \mathrm{e}^{-\omega}(1, y)$ into $\mathcal{C}^{n+3}$ with respect to a local complex coordinate z of the Riemann surface M, where $\mathrm{e}^{2\omega} = 2\langle y_z, y_{\overline{z}}\rangle$. There exists a global bundle decomposition $M \times \mathbb{R}^{n+4}_1 = V \oplus V^\perp$, with $V = \operatorname{Span}\{Y, \operatorname{Re} Y_z, \operatorname{Im} Y_z, Y_{z\overline{z}}\}$, where $V^\perp$ denotes

the orthogonal complement of V. Let $V_{\mathbb{C}}$ and $V_{\mathbb{C}}^{\perp}$ be the complexifications of V and $V^{\perp}$. Let $\{Y, Y_z, Y_{\overline{z}}, N\}$ be a frame of $V_{\mathbb{C}}$ such that $\langle N, Y_z\rangle = \langle N, Y_{\overline{z}}\rangle = \langle N, N\rangle = 0$ and$\langle N, Y\rangle = -1$. Let D denote the normal connection on $V_{\mathbb{C}}^{\perp}$. For any section $\psi \in \Gamma(V_{\mathbb{C}}^{\perp})$ of the normal bundle and a canonical lift Y with respect to z, we obtain the structure equations:

$$\begin{cases} Y_{zz} = -\dfrac{s}{2}Y + \kappa, \\ Y_{z\overline{z}} = -\langle \kappa, \overline{\kappa}\rangle Y + \dfrac{1}{2}N, \\ N_z = -2\langle \kappa, \overline{\kappa}\rangle Y_z - sY_{\overline{z}} + 2D_{\overline{z}}\kappa, \\ \psi_z = D_z\psi + 2\langle \psi, D_{\overline{z}}\kappa\rangle Y - 2\langle \psi, \kappa\rangle Y_{\overline{z}}. \end{cases} \tag{2.1}$$

Here $\kappa\frac{\mathrm{d}z^2}{|\mathrm{d}z|}$ is named the conformal Hopf differential of y, and s is named the Schwarzian of y (see [9]). The conformal Gauss, Codazzi and Ricci equations as integrability conditions are as follows:

$$\begin{cases} \dfrac{1}{2}s_{\overline{z}} = 3\langle \kappa, D_z\overline{\kappa}\rangle + \langle D_z\kappa, \overline{\kappa}\rangle, \\ \mathrm{Im}\Big(D_{\overline{z}}D_{\overline{z}}\kappa + \dfrac{\overline{s}}{2}\kappa\Big) = 0, \\ R^D_{\overline{z}z}\psi := D_{\overline{z}}D_z\psi - D_zD_{\overline{z}}\psi = 2\langle \psi, \kappa\rangle\overline{\kappa} - 2\langle \psi, \overline{\kappa}\rangle\kappa. \end{cases} \tag{2.2}$$

Recall that y is a Willmore surface if and only if the Willmore equation holds (see [9])

$$D_{\overline{z}}D_{\overline{z}}\kappa + \frac{\overline{s}}{2}\kappa = 0. \tag{2.3}$$

Another equivalent condition of y being Willmore is the harmonicity of the conformal Gauss map $Gr : M \to Gr_{1,3}(\mathbb{R}^{n+4}_1) = SO^+(1, n+3)/SO^+(1,3) \times SO(n)$ of y (see [5, 20, 26]) with $Gr := Y \wedge Y_u \wedge Y_v \wedge N = -2\mathrm{i}\cdot Y \wedge Y_z \wedge Y_{\overline{z}} \wedge N$. A local lift of Gr is chosen as

$$F := \Big(\frac{1}{\sqrt{2}}(Y+N), \frac{1}{\sqrt{2}}(-Y+N), e_1, e_2, \psi_1, \cdots, \psi_n\Big) : U \to SO^+(1, n+3) \tag{2.4}$$

with its Maurer-Cartan form $\alpha = F^{-1}\mathrm{d}F = \begin{pmatrix} A_1 & B_1 \\ -B_1^t I_{1,3} & A_2 \end{pmatrix}\mathrm{d}z + \begin{pmatrix} \overline{A}_1 & \overline{B}_1 \\ -\overline{B}_1^t I_{1,3} & \overline{A}_2 \end{pmatrix}\mathrm{d}\overline{z}$, where

$$B_1 = \begin{pmatrix} \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n \\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n \\ -k_1 & \cdots & -k_n \\ -\mathrm{i}k_1 & \cdots & -\mathrm{i}k_n \end{pmatrix}, \tag{2.5}$$

$\{\psi_j\}$ is an orthonormal basis of $V^{\perp}$ on U and $\kappa = \sum\limits_j k_j\psi_j$, $D_{\overline{z}}\kappa = \sum\limits_j \beta_j\psi_j$, $k^2 = \sum\limits_j |k_j|^2$.

2.1.2 Isotropic Willmore surfaces in S^6

Recall that y is totally isotropic if and only if all the derivatives of y with respect to z are isotropic, that is,

$$\langle Y_z^{(m)}, Y_z^{(n)}\rangle = 0 \quad \text{for all} \ \ m, \ n \in \mathbb{Z}^+.$$

Here $Y_z^{(m)}$ means taking m times derivatives of Y by z. We refer to [4, 11, 16–17, 20, 26] for more discussions on isotropic surfaces. A well-known result states that y is totally isotropic

if and only if y is the projection of a holomorphic or anti-holomorphic curve into the twistor bundle $\mathfrak{T}S^{2n}$ of S^{2n} (see [11, 20]). For the basic theory about twistor bundles, we refer to [10].

Let y be a Willmore surface with an isotropic Hopf differential, i.e., $\langle\kappa,\kappa\rangle\equiv 0$. Note that one derives straightforwardly that $\langle\kappa, D_z\kappa\rangle = \langle\kappa, D_{\overline{z}}\kappa\rangle = 0$ by differentiating $\langle\kappa,\kappa\rangle = 0$. Applying the Willmore equation (2.3), we also have $\langle D_{\overline{z}}\kappa, D_{\overline{z}}\kappa\rangle\equiv 0$.

For isotropic Willmore surfaces, Ma introduced several holomorphic differentials, see [26, Theorem 5.4]. For our case, we only need that

$$\Omega \mathrm{d}z^4 := \langle D_{\overline{z}}\kappa, D_z\kappa\rangle \mathrm{d}z^4 \tag{2.6}$$

is a globally defined holomorphic differential on M. The fact that $\Omega\mathrm{d}z^4$ is holomorphic can be derived from a direct computation using $\langle\kappa,\kappa\rangle = 0$, Willmore equations and Ricci equations (see also [26]). Then, if $M = S^2$, we will have $\langle D_{\overline{z}}\kappa, D_z\kappa\rangle\equiv 0$.

Now we assume that y is not S-Willmore, then $D_{\overline{z}}\kappa$ is not parallel to κ (recall that y is called S-Willmore if y is Willmore with $D_{\overline{z}}\kappa\|\kappa$, see [16–17, 20]). So $D_{\overline{z}}\kappa$ and κ span a two-dimensional isotropic subspace $\mathrm{Span}_{\mathbb{C}}\{\kappa, D_{\overline{z}}\kappa\}$. Since $D_z\kappa$ is perpendicular to κ and $D_{\overline{z}}\kappa$, $D_z\kappa$ is contained in $\mathrm{Span}_{\mathbb{C}}\{\kappa, D_{\overline{z}}\kappa\}$. As a consequence, we also have $\langle D_z\kappa, D_z\kappa\rangle = 0$. Summing up, we obtain the following theorem. (This theorem can also be derived by the loop group theory. See the end of Subsection 2.2.)

Theorem 2.1 *Let y be a Willmore two-spheres in S^6 with isotropic Hopf differential, i.e., $\langle\kappa,\kappa\rangle = 0$. If y is not S-Willmore, then y is totally isotropic (and hence full) in S^6. Moreover, locally there exists an isotropic frame $\{E_1, E_2\}$ of the normal bundle $V_{\mathbb{C}}^{\perp}$ of y such that*

$$\begin{cases} \kappa,\ D_z\kappa,\ D_{\overline{z}}\kappa \in \mathrm{Span}_{\mathbb{C}}\{E_1, E_2\}, \\ \langle E_i, E_j\rangle = 0, \quad \langle E_i, \overline{E}_j\rangle = 2\delta_{ij}, \quad i,j = 1,2, \\ D_z E_i \in \mathrm{Span}_{\mathbb{C}}\{E_1, E_2\}, \quad D_{\overline{z}}E_i \in \mathrm{Span}_{\mathbb{C}}\{E_1, E_2\}, \quad i = 1,2. \end{cases} \tag{2.7}$$

That is, the normal connection is block diagonal under the frame $\{E_1, E_2, \overline{E}_1, \overline{E}_2\}$.

Note that (2.7) provides also sufficient conditions for y to be a Willmore surface.

Theorem 2.2 *Let y be a totally isotropic surface from U into S^6, with complex coordinate z. If there exists an isotropic frame $\{E_1, E_2\}$ of the normal bundle $V_{\mathbb{C}}^{\perp}$ of y such that (2.7) holds, then y is a Willmore surface.*

Proof By (2.7), we see that $D_{\overline{z}}D_{\overline{z}}\kappa + \frac{\overline{s}}{2}\kappa$ is an isotropic vector. Since $\mathrm{Im}(D_{\overline{z}}D_{\overline{z}}\kappa + \frac{\overline{s}}{2}\kappa) = 0$ by (2.2), we have $D_{\overline{z}}D_{\overline{z}}\kappa + \frac{\overline{s}}{2}\kappa = 0$. So y is Willmore.

2.2 Normalized potentials of totally isotropic Willmore two-spheres in S^6

This subsection aims to derive the description of totally isotropic Willmore two-spheres in S^6 in terms of the loop group methods. To this end, we will first collect the basic theory concerning the DPW construction of harmonic maps and the applications to Willmore surfaces. Then, we will derive the construction of normalized potentials of totally isotropic Willmore two-spheres via Wu's formula. For more details of the loop group method we refer to [17–19, 37].

2.2.1 Harmonic maps into a symmetric space

Let G/K be a symmetric space defined by the involution $\sigma : G \to G$, with $G^\sigma \supset K \supset (G^\sigma)_0$, and Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{k} = \mathrm{Lie}(K)$. The Cartan decomposition induced by σ on $\mathfrak{g}$ states that $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$.

Let f be a conformal harmonic map from a Riemann surface M into G/K. Let U be an open connected subset of M with complex coordinate z. Then there exists a frame $F : U \to G$ of f with a Maurer-Cartan form $F^{-1}\mathrm{d}F = \alpha$. The Maurer-Cartan equation reads $\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$. Decomposing with respect to the Cartan decomposition, we obtain $\alpha = \alpha_0 + \alpha_1$ with $\alpha_0 \in \Gamma(\mathfrak{k} \otimes T^*M)$, $\alpha_1 \in \Gamma(\mathfrak{p} \otimes T^*M)$. And the Maurer-Cartan equation becomes

$$\begin{cases} \mathrm{d}\alpha_0 + \dfrac{1}{2}[\alpha_0 \wedge \alpha_0] + \dfrac{1}{2}[\alpha_1 \wedge \alpha_1] = 0, \\ \mathrm{d}\alpha_1 + [\alpha_0 \wedge \alpha_1] = 0. \end{cases}$$

Decomposing α_1 further into the $(1,0)$-part α_1' and the $(0,1)$-part α_1'' and introducing $\lambda \in S^1$, we set

$$\alpha_\lambda = \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1'', \quad \lambda \in S^1. \tag{2.8}$$

It is well known (see [15]) that $f : M \to G/K$ is harmonic if and only if $\mathrm{d}\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ for all $\lambda \in S^1$.

Definition 2.1 *Let $F(z,\lambda)$ be a solution to the equation $\mathrm{d}F(z,\lambda) = F(z,\lambda)\alpha_\lambda$, $F(0,\lambda) = F(0)$. Then $F(z,\lambda)$ is called the extended frame of the harmonic map f. Note that $F(z,1) = F(z)$.*

2.2.2 Two decomposition theorems

To state the DPW constructions for harmonic maps, we need the Iwasawa and Birkhoff decompositions for loop groups. For simplicity, from now on we consider the concrete case for Willmore surfaces (see [17]). In this case, $G = SO^+(1,n+3)$, $K = SO^+(1,3) \times SO(n)$ and $\mathfrak{g} = \mathfrak{so}(1,n+3) = \{X \in \mathfrak{gl}(n+4,\mathbb{R}) \mid X^t I_{1,n+3} + I_{1,n+3}X = 0\}$. The involution is given by

$$\begin{array}{rccc} \sigma : & SO^+(1,n+3) & \to & SO^+(1,n+3), \\ & A & \mapsto & DAD^{-1}, \end{array} \quad \text{with } D = \begin{pmatrix} -I_4 & 0 \\ 0 & I_n \end{pmatrix}.$$

Note that $SO^+(1,n+3)^\sigma \supset SO^+(1,3) \times SO(n+2) = (SO^+(1,n+3)^\sigma)_0$. We also have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with

$$\mathfrak{k} = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \mid A_1^t I_{1,3} + I_{1,3}A_1 = 0, A_2 + A_2^t = 0 \right\}, \quad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,3} & 0 \end{pmatrix} \right\}.$$

Let $G^{\mathbb{C}} = SO^+(1,n+3,\mathbb{C}) := \{X \in SL(n+4,\mathbb{C}) \mid X^t I_{1,n+3} X = I_{1,n+3}\}$ with $\mathfrak{so}(1,n+3,\mathbb{C})$ its Lie algebra. We extend σ to an inner involution of $SO^+(1,n+3,\mathbb{C})$ with $K^{\mathbb{C}} = S(O^+(1,3,\mathbb{C}) \times O(n,\mathbb{C}))$ its fixed point group. Let $\Lambda G_\sigma^{\mathbb{C}}$ be the group of loops in $G^C = SO^+(1,n+3,\mathbb{C})$ twisted by σ.

Theorem 2.3 (see [16, Theorem 4.5], also see [15, 17] Iwasawa decomposition) *There exists a closed, connected solvable subgroup $S \subseteq K^{\mathbb{C}}$, such that the multiplication $\Lambda G_\sigma^0 \times \Lambda_S^+ G_\sigma^{\mathbb{C}} \to \Lambda G_\sigma^{\mathbb{C}}$ is a real analytic diffeomorphism onto the open subset $\Lambda G_\sigma^0 \cdot \Lambda_S^+ G_\sigma^{\mathbb{C}} = \mathcal{I}_e^{\mathcal{U}} \subset (\Lambda G_\sigma^{\mathbb{C}})^0$. Here $\Lambda_S^+ G_\sigma^{\mathbb{C}} := \{\gamma \in \Lambda^+ G_\sigma^S \mid \gamma|_{\lambda=0} \in S\}$.*

Let $\Lambda_*^- G_\sigma^{\mathbb{C}}$ be the group of loops that extend holomorphically into ∞ and take values I at infinity. Set also $\Lambda_{\mathcal{C}}^+ G_\sigma^{\mathbb{C}} := \{\gamma \in \Lambda^+ G_\sigma^{\mathbb{C}} \mid \gamma|_{\lambda=0} \in (K^{\mathbb{C}})^0\}$.

Theorem 2.4 (see [15–17] Birkhoff decomposition) *The multiplication $\Lambda_*^- G_\sigma^{\mathbb{C}} \times \Lambda_{\mathcal{C}}^+ G_\sigma^{\mathbb{C}} \to \Lambda G_\sigma^{\mathbb{C}}$ is an analytic diffeomorphism onto the open, dense subset $\Lambda_*^- G_\sigma^{\mathbb{C}} \cdot \Lambda_{\mathcal{C}}^+ G_\sigma^{\mathbb{C}}$ (big Birkhoff cell).*

2.2.3 The DPW construction

Let $\mathbb{D} \subset \mathbb{C}$ be a disk or $\mathbb{C}$ with complex coordinate z.

Theorem 2.5 (see [15]) (1) *Let $f : \mathbb{D} \to G/K$ denote a harmonic map with an extended frame $F(z, \overline{z}, \lambda) \in \Lambda G_\sigma$ and $F(0, 0, \lambda) = I$. Then there exists a Birkhoff decomposition of $F(z, \overline{z}, \lambda)$,*

$$F_-(z, \lambda) = F(z, \overline{z}, \lambda)F_+(z, \overline{z}, \lambda), \quad \text{with } F_+ \in \Lambda_S^+ G_\sigma^{\mathbb{C}},$$

such that $F_-(z, \lambda) : \mathbb{D} \to \Lambda_*^- G_\sigma^{\mathbb{C}}$ is meromorphic and the Maurer-Cartan form η of F_- is

$$\eta = F_-^{-1} \mathrm{d}F_- = \lambda^{-1}\eta_{-1}(z)\mathrm{d}z,$$

with η_{-1} independent of λ. The meromorphic 1-form η is called the normalized potential of f.

(2) *Let η be a $\lambda^{-1} \cdot \mathfrak{p} \otimes \mathbb{C}$-valued meromorphic 1-form on $\mathbb{D}$. Let $F_-(z, \lambda)$ be a solution to $F_-^{-1}\mathrm{d}F_- = \eta$, $F_-(0, \lambda) = I$. Then there exists an Iwasawa decomposition*

$$F_-(0, \lambda) = \widetilde{F}(z, \overline{z}, \lambda)\widetilde{F}^+(z, \overline{z}, \lambda), \quad \text{with } \widetilde{F} \in \Lambda G_\sigma, \quad \widetilde{F} \in \Lambda_S^+ G_\sigma^{\mathbb{C}}$$

on an open subset $\mathbb{D}_\mathfrak{I}$ of $\mathbb{D}$. Moreover, $\widetilde{F}(z, \overline{z}, \lambda)$ is an extended frame of some harmonic map from $\mathbb{D}_\mathfrak{I}$ to G/K with $\widetilde{F}(0, \lambda) = I$. All harmonic maps can be obtained in this way, since the above two procedures are inverse to each other if the normalization at some based point is fixed.

Note that in this paper since we consider the case with Identity, initial condition the Birkhoff decomposition (see Theorem 2.4) holds for our case (see [15, 30]). Moreover, Theorem 2.6 holds only if the Iwasawa decomposition and Birkhoff decomposition are satisfied, since the proof of the similar results in [15] replies only on these two decompositions. In this sense, [15] is sufficient for this paper, except the Iwasawa case, which is provided essentially in [16–17]. We also refer to Hélein's paper (see [22]) for another Iwasawa decomposition for some non-compact symmetric space (i.e, $SO^+(1, 4)/(SO^+(1, 1) \times SO(3))$) slightly differenting from the present one. We refer to [17] for more discussions on these two kinds of different harmonic maps related with Willmore surfaces.

The normalized potential can be determined from the Maurer-Cartan form of f (see [36]). Let f, $F(z, \lambda)$ and α_λ denote the stuff as above. Let δ_1 and δ_0 denote the sum of the holomorphic terms of z about $z = 0$ in the Taylor expansion of $\alpha_1'\big(\frac{\partial}{\partial z}\big)$ and $\alpha_0'\big(\frac{\partial}{\partial z}\big)$ respectively.

Theorem 2.6 (see [36] Wu's formula) *We retain the notions in Theorem* 2.5. *Then the normalized potential of f with respect to the based point* 0 *is given by $\eta = \lambda^{-1}F_0(z)\delta_1 F_0(z)^{-1}\mathrm{d}z$, where $F_0(z) : \mathbb{D} \to G^{\mathbb{C}}$ is the solution to $F_0(z)^{-1}\mathrm{d}F_0(z) = \delta_0 \mathrm{d}z$, $F_0(0) = I$.*

2.2.4 Normalized potentials of totally isotropic Willmore two-spheres in S^6

Let F be a frame of a Willmore surface y with $\alpha = F^{-1}\mathrm{d}F = \alpha_1' + \alpha_0 + \alpha_1''$ as above. Here

$$\alpha_0' = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \mathrm{d}z, \quad \alpha_1' = \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,3} & 0 \end{pmatrix} \mathrm{d}z.$$

Let δ_1' be the holomorphic part of α_1' and δ_0' be the holomorphic part of α_0'. Let $\widetilde{B}_1$ be the holomorphic part of B_1. Let $F_0 = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ be the solution to $F_0^{-1}\mathrm{d}F_0 = \delta_0'$, $F_0(z_0) = I_8$.

By Theorem 2.6, we have

$$\eta = F_0\delta_1' F_0^{-1} = \lambda^{-1}\begin{pmatrix} 0 & \widehat{B}_1 \\ -\widehat{B}_1^t I_{1,3} & 0 \end{pmatrix}\mathrm{d}z \quad \text{with} \quad \widehat{B}_1 = K_1\widetilde{B}_1 K_2^{-1}. \tag{2.9}$$

Applying Wu's formula, we obtain the following theorem.

Theorem 2.7 *Let y be a totally isotropic Willmore two-spheres in S^6. Then the normal bundle of y satisfies the properties* (2.7) *of Theorem* 2.1. *The normalized potential of y is of the form*

$$\eta = \lambda^{-1}\eta_{-1}\mathrm{d}z = \lambda^{-1}\begin{pmatrix} 0 & \widehat{B}_1 \\ -\widehat{B}_1^t I_{1,3} & 0 \end{pmatrix}\mathrm{d}z, \tag{2.10}$$

with (h_{ij} are meromorphic functions)

$$\widehat{B}_1 = \begin{pmatrix} h_{11} & \mathrm{i}h_{11} & h_{12} & \mathrm{i}h_{12} \\ h_{21} & \mathrm{i}h_{21} & h_{22} & \mathrm{i}h_{22} \\ h_{31} & \mathrm{i}h_{31} & h_{32} & \mathrm{i}h_{32} \\ h_{41} & \mathrm{i}h_{41} & h_{42} & \mathrm{i}h_{42} \end{pmatrix}, \quad \widehat{B}_1^t I_{1,3}\widehat{B}_1 = 0. \tag{2.11}$$

Lemma 2.1 *Set*

$$\mathfrak{k}_2^{\mathbb{C}} := \left\{ A_2 \mid A_2 = \begin{pmatrix} 0 & -\mathrm{b}_{12} & -\mathrm{b}_{13} & -\mathrm{b}_{14} \\ \mathrm{b}_{12} & 0 & \mathrm{b}_{14} & -\mathrm{b}_{13} \\ \mathrm{b}_{13} & -\mathrm{b}_{14} & 0 & -\mathrm{b}_{34} \\ \mathrm{b}_{14} & \mathrm{b}_{13} & \mathrm{b}_{34} & 0 \end{pmatrix} \in \mathfrak{so}(4,\mathbb{C}) \right\}. \tag{2.12}$$

Then, $\mathfrak{k}_2^{\mathbb{C}}$ is a Lie sub-algebra of $\mathfrak{so}(4,\mathbb{C})$. Moreover, let $\mathfrak{K}_2^{\mathbb{C}}$ be the subgroup of $SO(4,\mathbb{C})$ with Lie algebra $\mathfrak{k}_2^{\mathbb{C}}$. Then

$$\mathfrak{K}_2^{\mathbb{C}} = \left\{ K_2 | K_2 = \begin{pmatrix} \mathrm{t}_{11} & -\mathrm{t}_{12} & -\mathrm{t}_{13} & -\mathrm{t}_{14} \\ \mathrm{t}_{12} & \mathrm{t}_{11} & \mathrm{t}_{14} & -\mathrm{t}_{13} \\ \mathrm{t}_{13} & -\mathrm{t}_{14} & \mathrm{t}_{11} & \mathrm{t}_{12} \\ \mathrm{t}_{14} & \mathrm{t}_{13} & -\mathrm{t}_{12} & \mathrm{t}_{11} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\varphi & \sin\varphi \\ & & -\sin\varphi & \cos\varphi \end{pmatrix} \in SO(4,\mathbb{C}) \right\}. \tag{2.13}$$

Proof It is direct to show that

$$[\mathfrak{k}_2^{\mathbb{C}}, \mathfrak{k}_2^{\mathbb{C}}] = \widetilde{\mathfrak{k}_2^{\mathbb{C}}} = \left\{ A_2 \mid A_2 = \begin{pmatrix} 0 & -\mathrm{b}_{12} & -\mathrm{b}_{13} & -\mathrm{b}_{14} \\ \mathrm{b}_{12} & 0 & \mathrm{b}_{14} & -\mathrm{b}_{13} \\ \mathrm{b}_{13} & -\mathrm{b}_{14} & 0 & \mathrm{b}_{12} \\ \mathrm{b}_{14} & \mathrm{b}_{13} & -\mathrm{b}_{12} & 0 \end{pmatrix} \in \mathfrak{so}(4,\mathbb{C}) \right\}$$

and $\widetilde{\mathfrak{k}_2^{\mathbb{C}}}$ is the Lie algebra of

$$\left\{ K_2 \Big| K_2 = \begin{pmatrix} \mathrm{t}_{11} & -\mathrm{t}_{12} & -\mathrm{t}_{13} & -\mathrm{t}_{14} \\ \mathrm{t}_{12} & \mathrm{t}_{11} & \mathrm{t}_{14} & -\mathrm{t}_{13} \\ \mathrm{t}_{13} & -\mathrm{t}_{14} & \mathrm{t}_{11} & \mathrm{t}_{12} \\ \mathrm{t}_{14} & \mathrm{t}_{13} & -\mathrm{t}_{12} & \mathrm{t}_{11} \end{pmatrix} \in SO(4,\mathbb{C}) \right\}.$$

Since $K_\varphi^{-1}\widetilde{\mathfrak{k}_2^{\mathbb{C}}}K_\varphi \subset \mathfrak{k}_2^{\mathbb{C}}$, we see that $\mathfrak{K}_2^{\mathbb{C}}$ is the subgroup of $SO(4,\mathbb{C})$ with Lie algebra $\mathfrak{k}_2^{\mathbb{C}}$. Here

$$K_\varphi = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\varphi & \sin\varphi \\ & & -\sin\varphi & \cos\varphi \end{pmatrix}.$$

Remark 2.1 (2.13) shows that the subgroup $\widehat{\mathfrak{K}_2} = \{K_2 \in \mathfrak{K}_2 \mid K_2 = \overline{K}_2\}$ is diffeomorphic to $S^3 \times S^1$.

Proof of Theorem 2.7 If y is not S-Willmore, (2.7) comes from Theorem 2.1. If y is S-Willmore, first let E_1 be a basis of the bundle spanned by κ (this bundle is globally defined, since $D_{\overline{z}}\kappa \in \mathrm{Span}_{\mathbb{C}}\{\kappa\}$, see the proof of [17, Lemma 1.3] for a detailed proof). Next, we consider the sub-bundle $\mathbb{V}_2$ of the normal bundle perpendicular to $\{E_1, \overline{E}_1\}$. Since $\langle D_z\kappa, E_1\rangle = 0$, we can chose an isotropic basis $\{E_2, \overline{E}_2\}$ of $\mathbb{V}_2$, such that $\langle E_2, E_2\rangle = 0$, $\langle E_2, \overline{E}_2\rangle = 2$ and $\langle D_z\kappa, E_2\rangle = 0$. Then it is straightforward to verify that (2.7) holds.

Now we apply (2.7). Set $E_1 = \psi_1 + \mathrm{i}\psi_2$, $E_2 = \psi_3 + \mathrm{i}\psi_4$. Then we have a frame F of the form (2.4). Under this frame, we have

$$B_1 = \begin{pmatrix} \sqrt{2}\beta_1 & \sqrt{2}\mathrm{i}\beta_1 & \sqrt{2}\beta_3 & \sqrt{2}\mathrm{i}\beta_3 \\ -\sqrt{2}\beta_1 & -\sqrt{2}\mathrm{i}\beta_1 & -\sqrt{2}\beta_3 & -\sqrt{2}\mathrm{i}\beta_3 \\ -k_1 & -\mathrm{i}k_1 & -k_3 & -\mathrm{i}k_3 \\ -\mathrm{i}k_1 & k_1 & -\mathrm{i}k_3 & k_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -\mathrm{b}_{12} & -\mathrm{b}_{13} & -\mathrm{b}_{14} \\ \mathrm{b}_{12} & 0 & \mathrm{b}_{14} & -\mathrm{b}_{13} \\ \mathrm{b}_{13} & -\mathrm{b}_{14} & 0 & -\mathrm{b}_{34} \\ \mathrm{b}_{14} & \mathrm{b}_{13} & \mathrm{b}_{34} & 0 \end{pmatrix}.$$

Then the normalized potential of y is expressed by (2.9). The holomorphic part $\widetilde{B}_1$ of B_1 has the same form as B_1 and since K_1 does not change the relations between the columns of $\widetilde{B}_1$, we need only to consider the influence of K_2 on $\widetilde{B}_1$. Note that A_2 takes value in $\mathfrak{k}_2^{\mathbb{C}}$. So the holomorphic part $\widetilde{A}_2$ of A_2 also takes value in $\mathfrak{k}_2^{\mathbb{C}}$. Therefore, the integration $\widehat{A}_2 = \int_{z_0}^{z} \widetilde{A}_2 \mathrm{d}z$ of $\widetilde{A}_2$ also takes value in $\mathfrak{k}_2^{\mathbb{C}}$. By Lemma 2.1, K_2 takes value in $\mathfrak{K}_2^{\mathbb{C}}$. Summing up, we can assume that the following two equations hold:

$$K_2 = \begin{pmatrix} \mathrm{t}_{11} & -\mathrm{t}_{12} & -\mathrm{t}_{13} & -\mathrm{t}_{14} \\ \mathrm{t}_{12} & \mathrm{t}_{11} & \mathrm{t}_{14} & -\mathrm{t}_{13} \\ \mathrm{t}_{13} & -\mathrm{t}_{14} & \mathrm{t}_{11} & \mathrm{t}_{12} \\ \mathrm{t}_{14} & \mathrm{t}_{13} & -\mathrm{t}_{12} & \mathrm{t}_{11} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\varphi & \sin\varphi \\ & & -\sin\varphi & \cos\varphi \end{pmatrix},$$

$$K_1\widetilde{B}_1 = \begin{pmatrix} \widehat{\mathrm{h}}_{11} & \mathrm{i}\widehat{\mathrm{h}}_{11} & \widehat{\mathrm{h}}_{12} & \mathrm{i}\widehat{\mathrm{h}}_{12} \\ \widehat{\mathrm{h}}_{21} & \mathrm{i}\widehat{\mathrm{h}}_{21} & \widehat{\mathrm{h}}_{22} & \mathrm{i}\widehat{\mathrm{h}}_{22} \\ \widehat{\mathrm{h}}_{31} & \mathrm{i}\widehat{\mathrm{h}}_{31} & \widehat{\mathrm{h}}_{32} & \mathrm{i}\widehat{\mathrm{h}}_{32} \\ \widehat{\mathrm{h}}_{41} & \mathrm{i}\widehat{\mathrm{h}}_{41} & \widehat{\mathrm{h}}_{42} & \mathrm{i}\widehat{\mathrm{h}}_{42} \end{pmatrix}.$$

Then $K_1\widetilde{B}_1K_2^{-1}$ has the form

$$\begin{pmatrix} h_{11} & \mathrm{i}h_{11} & h_{12} & \mathrm{i}h_{12} \\ h_{21} & \mathrm{i}h_{21} & h_{22} & \mathrm{i}h_{22} \\ h_{31} & \mathrm{i}h_{31} & h_{32} & \mathrm{i}h_{32} \\ h_{41} & \mathrm{i}h_{41} & h_{42} & \mathrm{i}h_{42} \end{pmatrix}, \quad \text{with} \begin{cases} h_{j1} = \widehat{\mathrm{h}}_{j1}(\mathrm{t}_{11} - \mathrm{i}\mathrm{t}_{12}) - \widehat{\mathrm{h}}_{j2}(\mathrm{t}_{13} + \mathrm{i}\mathrm{t}_{14}), \\ h_{j2} = (\widehat{\mathrm{h}}_{j1}(\mathrm{t}_{13} - \mathrm{i}\mathrm{t}_{14}) + \widehat{\mathrm{h}}_{j2}(\mathrm{t}_{11} + \mathrm{i}\mathrm{t}_{12})) \\ \qquad \cdot(\cos\varphi - \mathrm{i}\sin\varphi), \quad 1 \le j \le 4. \end{cases}$$

Remark 2.2 Different from the case in S^4, where totally isotropic surfaces are all S-Willmore surfaces of finite uniton type, totally isotropic surfaces in S^6 can be even not Willmore in general. Moreover, for a totally isotropic Willmore surface in S^6, if the holomorphic 4-form $\Omega \mathrm{d}z^4 \neq 0$ (hence not S-Willmore), it is full in S^6 and is not of finite uniton type. Given the fact that such surfaces come from the twistor projection of holomorphic or anti-holomorphic curves of the twistor bundle $\mathfrak{T}S^6$ of S^6, they can be expressed by rational functions on the Riemann surface. Such harmonic maps which are not of finite uniton type are somewhat unexpected since they correspond to holomorphic or anti-holomorphic curves in the twistor bundle of S^6. And it will be an interesting topic to classify and/or to characterize such harmonic maps as

well as the corresponding Willmore surfaces, especially when the Riemann surface is a torus. As a consequence, it will be an interesting topic to generalize the work of Bohle on Willmore tori (see [2]) to Willmore tori in S^6.

We can use the DPW method to give another proof of Theorem 2.1.

Proof of Theorem 2.1 If y is non S-Willmore with $\langle\kappa,\kappa\rangle=0$, we claim that its normalized potential can only take the form of type 3. By Theorem 3.1 of Section 3, y is totally isotropic and its normal connection has the desired form.

Now let us prove the claim. [34, Theorem 2.8] and [16, Theorem 5.2] show that B_1 must be either of type 2 or of type 3 in [34, Theorem 2.8]. On the other hand, as we have seen before, the isotropy condition and the Willmore equation show $\langle\kappa,\kappa\rangle=\langle D_{\overline{z}}\kappa,\kappa\rangle=\langle D_{\overline{z}}\kappa,D_{\overline{z}}\kappa\rangle=0$. This yields that the Maurer-Cartan form of y satisfies $B_1B_1^t=0$. Then $\widetilde{B}_1$, the holomorphic part of B_1, also satisfies $\widetilde{B}_1\widetilde{B}_1^t=0$. As a consequence, we have $\widehat{B}_1\widehat{B}_1^t=K_1\widetilde{B}_1K_2^{-1}(K_2^{-1})^t\widetilde{B}_1^tK_1^t=K_1\widetilde{B}_1\widetilde{B}_1^tK_1^t=0$. If the normalized potential η of y is of type 2 in [35, Theorem 2.8], then

$$\widehat{B}_1=\begin{pmatrix}h_{11}&\mathrm{i}h_{11}&h_{12}&f_1h_{12}\\h_{21}&\mathrm{i}h_{21}&h_{12}&f_1h_{12}\\h_{31}&\mathrm{i}h_{31}&h_{32}&f_1h_{32}\\h_{41}&\mathrm{i}h_{41}&\mathrm{i}h_{32}&\mathrm{i}f_1h_{32}\end{pmatrix}.$$

So the condition $\widehat{B}_1\widehat{B}_1^t=0$ forces $f_1=\mathrm{i}$ or $f_1=-\mathrm{i}$. Hence η is of type 3 (up to a conjugation).

3 Construction of Totally Isotropic Willmore Two-spheres in S^6

This section is to describe geometric properties of Willmore surfaces of type 3 of [34, Theorem 3.3] We will provide an algorithm to derive a concrete construction of such Willmore surfaces in S^6 from the normalized potentials of type 3 of [34, Theorem 3.3] by a concrete Iwasawa decomposition. The geometric properties of this kind of Willmore surfaces are also revealed naturally. During this procedure, we will see that Willmore surfaces of this type will be the special kind of totally isotropic Willmore surfaces in S^6, which has been discussed in Section 2. This section has three parts. The main theorem and the new examples are stated first. The technical lemmas combining the proof of Theorem 3.1 are stated in the end. The concrete proofs and constructions of examples are postponed to two appendixes.

3.1 From potentials to surfaces

Theorem 3.1 (Case of [34, Theorem 3.3]) *Let y be a Willmore surface in S^6 with its normalized potential being of the form* (2.10)*. Then y is totally isotropic in S^6. Moreover, locally there exists an isotropic frame $\{E_1,E_2\}$ of the normal bundle $V_{\mathbb{C}}^{\perp}$ of y such that* (2.7) *holds.*

3.2 Examples of totally isotropic Willmore spheres in S^6

We have two kinds of examples to illustrate the algorithm presented in the proof of Theorem 3.1. The isotropic minimal surfaces in $\mathbb{R}^4$ are used to illustrate the algorithm with simpler computations. The new, totally isotropic, non S-Willmore, Willmore two-spheres in S^6 is constructed to answer Ejiri's question explicitly.

Theorem 3.2 *Let*

$$\eta = \lambda^{-1}\begin{pmatrix} 0 & \widehat{B}_1 \\ -\widehat{B}_1^t I_{1,3} & 0 \end{pmatrix}\mathrm{d}z, \quad \textit{with} \quad \widehat{B}_1 = \frac{1}{2}\begin{pmatrix} -\mathrm{i}f_2' & f_2' & 0 & 0 \\ \mathrm{i}f_2' & -f_2' & 0 & 0 \\ f_4' & \mathrm{i}f_4' & 0 & 0 \\ \mathrm{i}f_4' & -f_4' & 0 & 0 \end{pmatrix}. \tag{3.1}$$

Here f_2 and f_4 are (non-constant) meromorphic functions on $\mathbb{C}$. This $\widehat{B}_1$ is of both type 1 *and type* 3 *in* [34, Theorem 2.8]. *The corresponding associated family of Willmore surfaces is*

$$[Y_\lambda] = \left[\begin{pmatrix} (1+|f_2|^2) - \dfrac{\overline{f}_2 f_4 f_2'}{f_4'} - \dfrac{f_2\overline{f}_4\overline{f_2'}}{\overline{f_4'}} + \dfrac{|f_2'|^2(1+|f_4|^2)}{|f_4'|^2} \\ (1-|f_2|^2) + \dfrac{\overline{f}_2 f_4 f_2'}{f_4'} + \dfrac{f_2\overline{f}_4\overline{f_2'}}{\overline{f_4'}} - \dfrac{|f_2'|^2(1+|f_4|^2)}{|f_4'|^2} \\ -\dfrac{\mathrm{i}f_2'}{f_4'} + \dfrac{\mathrm{i}\overline{f_2'}}{\overline{f_4'}} \\ -\dfrac{f_2'}{f_4'} - \dfrac{\overline{f_2'}}{\overline{f_4'}} \\ -\mathrm{i}(\lambda^{-1}f_2 - \lambda\overline{f}_2) + \dfrac{\mathrm{i}\lambda^{-1}f_2' f_4}{f_4'} - \dfrac{\mathrm{i}\lambda\overline{f_2'}\overline{f}_4}{\overline{f_4'}} \\ (\lambda^{-1}f_2 + \lambda\overline{f}_2) - \dfrac{\lambda^{-1}f_2' f_4}{f_4'} - \dfrac{\lambda\overline{f_2'}\overline{f}_4}{\overline{f_4'}} \\ 0 \\ 0 \end{pmatrix}\right]. \tag{3.2}$$

Corollary 3.1 *The Willmore surface $[Y_\lambda]$ in Theorem* 3.2 *is conformal to the minimal surface*

$$x_\lambda = \begin{pmatrix} -\dfrac{\mathrm{i}f_2'}{f_4'} + \dfrac{\mathrm{i}\overline{f_2'}}{\overline{f_4'}} \\ -\dfrac{f_2'}{f_4'} - \dfrac{\overline{f_2'}}{\overline{f_4'}} \\ -\mathrm{i}(\lambda^{-1}f_2 - \lambda\overline{f}_2) + \dfrac{\mathrm{i}\lambda^{-1}f_2' f_4}{f_4'} - \dfrac{\mathrm{i}\lambda\overline{f_2'}\overline{f}_4}{\overline{f_4'}} \\ (\lambda^{-1}f_2 + \lambda\overline{f}_2) - \dfrac{\lambda^{-1}f_2' f_4}{f_4'} - \dfrac{\lambda\overline{f_2'}\overline{f}_4}{\overline{f_4'}} \end{pmatrix} \textit{ in } \mathbb{R}^4. \tag{3.3}$$

Note that λ is different from the usual parameter of the associated family of a minimal surface.

Theorem 3.3 (The case $p = 2$ in (1.2)) *Let*

$$\eta = \lambda^{-1}\begin{pmatrix} 0 & \widehat{B}_1 \\ -\widehat{B}_1^t I_{1,3} & 0 \end{pmatrix}\mathrm{d}z, \quad \textit{with } \widehat{B}_1 = \frac{1}{2}\begin{pmatrix} 2\mathrm{i}z & -2z & -\mathrm{i} & 1 \\ -2\mathrm{i}z & 2z & -\mathrm{i} & 1 \\ -2 & -2\mathrm{i} & -z & -\mathrm{i}z \\ 2\mathrm{i} & -2 & -\mathrm{i}z & z \end{pmatrix}. \tag{3.4}$$

The associated family of unbranched Willmore two-spheres x_λ, $\lambda \in S^1$, corresponding to η, is

$$x_\lambda = \frac{1}{\big(1+r^2+\frac{5r^4}{4}+\frac{4r^6}{9}+\frac{r^8}{36}\big)} \begin{pmatrix} \Big(1-r^2-\dfrac{3r^4}{4}+\dfrac{4r^6}{9}-\dfrac{r^8}{36}\Big) \\ -\mathrm{i}\Big(z-\overline{z}\Big)\Big(1+\dfrac{r^6}{9}\Big)\Big) \\ \Big(z+\overline{z}\Big)\Big(1+\dfrac{r^6}{9}\Big)\Big) \\ -\mathrm{i}\Big((\lambda^{-1}z^2-\lambda\overline{z}^2)\Big(1-\dfrac{r^4}{12}\Big)\Big) \\ \Big((\lambda^{-1}z^2+\lambda\overline{z}^2)\Big(1-\dfrac{r^4}{12}\Big)\Big) \\ -\mathrm{i}\dfrac{r^2}{2}(\lambda^{-1}z-\lambda\overline{z})\Big(1+\dfrac{4r^2}{3}\Big) \\ \dfrac{r^2}{2}(\lambda^{-1}z+\lambda\overline{z})\Big(1+\dfrac{4r^2}{3}\Big) \end{pmatrix} \quad \textit{with } r=|z|. \tag{3.5}$$

Moreover $x_\lambda : S^2 \to S^6$ is a Willmore immersion in S^6, which is full, not S-Willmore, and totally isotropic. Note that for all $\lambda \in S^1$, x_λ is isometric to each other in S^6.

3.3 Technical lemmas

3.3.1 The basic ideas

To begin with, we first explain our basic ideas, since the computations are very technical. We will divide the proof of Theorem 3.1 into two steps:

1. To derive the harmonic maps from the given normalized potentials.
2. To derive the geometric properties of the corresponding Willmore surfaces.

The main method in Step 1 is a concrete performing of Iwasawa decompositions. The main idea in Step 2 is to read off the Maurer-Cartan forms of the corresponding Willmore surfaces.

For Step 1, we first transform $SO^+(1,7,\mathbb{C})$ into $G(8,\mathbb{C})$ (see (3.6)) so that the normalized potentials in Theorem 3.1 are strictly upper-triangular in $\mathfrak{g}(8,\mathbb{C}) = \mathrm{Lie}(G(8,\mathbb{C}))$ (see Lemma 3.1). Then Lemma 3.2 provides the concrete expressions of the normalized potential and its meromorphic frame. Lemma 3.3 gives the Iwasawa decompositions of the meromorphic frame by the method of undetermined coefficients. This finishes Step 1. For Step 2, we first derive the forms of the Maurer-Cartan forms of the extended frame derived in Step 1. Then translating into the computations of moving frames, one will obtain the isotropic properties of the corresponding Willmore surfaces.

3.3.2 Step 1: Iwasawa decompositions

Set

$$G(8,\mathbb{C}) := \{A \in \mathrm{Mat}(8,\mathbb{C}) \mid A^t J_8 A = J_8, \det A = 1\} \tag{3.6}$$

with $J_n = (j_{k,l})_{n\times n}$, $j_{k,l} = \delta_{k+l,n+1}$ for all $1 \le k,l \le n$.

Lemma 3.1 *Let*

$$\begin{array}{rcl} \check{\mathcal{P}} : SO^+(1,7,\mathbb{C}) & \to & G(8,\mathbb{C}) \\ A & \mapsto & \check{P}^{-1}\widetilde{P}^{-1}A\widetilde{P}\check{P}, \end{array} \tag{3.7}$$

with

$$\check{P} = \begin{pmatrix} & J_2 & & \\ J_2 & & & \\ & & & J_2 \\ & & J_2 & \end{pmatrix}, \quad \widetilde{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & & & & -1 \\ 1 & & & & & & & 1 \\ & -\mathrm{i} & & & & & \mathrm{i} & \\ & 1 & & & & & 1 & \\ & & -\mathrm{i} & & & \mathrm{i} & & \\ & & 1 & & & 1 & & \\ & & & -\mathrm{i} & \mathrm{i} & & & \\ & & & 1 & 1 & & & \end{pmatrix}.$$

Then $\check{\mathcal{P}}$ is a Lie group isomorphism.

We also have that $\check{\mathcal{P}}(SO^+(1,7)) = \{F \in G(8,\mathbb{C}) \mid F = \check{S}_8^{-1}\overline{F}\check{S}_8\}$, with

$$\check{S}_8 = \overline{\check{P}}^{-1}\overline{\widetilde{P}}^{-1}\widetilde{P}\check{P} = \begin{pmatrix} 0 & 0 & J_2 \\ 0 & S_4 & 0 \\ J_2 & 0 & 0 \end{pmatrix}, \quad S_4 = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix}. \tag{3.8}$$

This induces an involution of $\Lambda G(8,\mathbb{C})$:

$$\begin{array}{rcl} \check{\tau} : \Lambda G(8,\mathbb{C}) & \to & \Lambda G(8,\mathbb{C}), \\ F & \mapsto & \check{S}_8^{-1}\overline{F}\check{S}_8 \end{array} \tag{3.9}$$

with $\check{P}(\Lambda SO^+(1,7)) = \{F \in \Lambda G(8,\mathbb{C}) \mid \check{\tau}(F) = F\}$ as its fixed point set.

The image of the subgroup $(SO^+(1,3) \times SO(4))^{\mathbb{C}}$ is

$$\check{\mathcal{P}}((SO^+(1,3) \times SO(4))^{\mathbb{C}}) = \{\check{F} \in G(8,\mathbb{C}) \mid \check{F} = \check{D}_0^{-1}\check{F}\check{D}_0\}$$

with

$$\check{D}_0 = \check{P}^{-1}\widetilde{P}^{-1}D\widetilde{P}\check{P} = -D_0 = \mathrm{diag}(1,1,-1,-1,-1,-1,1,1).$$

Set

$$\check{J}_8 = \check{S}_8 J_8 = J_8\check{S}_8 = \begin{pmatrix} I_2 & & \\ & \check{J}_4 & \\ & & I_2 \end{pmatrix} \quad \text{with} \quad \check{J}_4 = S_4 J_4 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}.$$

For any $F \in G(8,\mathbb{C})$, we have

$$\check{\tau}^{-1}(F) = \check{J}_8\overline{F}^t\check{J}_8. \tag{3.10}$$

Lemma 3.2 *Let η be the normalized potential of Theorem* 3.1. *Then*

$$\check{\mathcal{P}}(\eta) = \lambda^{-1}\begin{pmatrix} 0 & \check{f} & 0 \\ 0 & 0 & -\check{f}^\sharp \\ 0 & 0 & 0 \end{pmatrix}\mathrm{d}z, \quad \check{f}^\sharp := J_4\check{f}^t J_2,$$

with

$$\check{f} = \begin{pmatrix} -h_{32} - \mathrm{i}h_{42} & \mathrm{i}(h_{12} - h_{22}) & -\mathrm{i}(h_{12} + h_{22}) & h_{32} - \mathrm{i}h_{42} \\ -h_{31} - \mathrm{i}h_{41} & \mathrm{i}(h_{11} - h_{21}) & -\mathrm{i}(h_{11} + h_{21}) & h_{31} - \mathrm{i}h_{41} \end{pmatrix}. \tag{3.11}$$

Moreover, $H = I_8 + \lambda^{-1}H_1 + \lambda^{-2}H_2$ is a solution to

$$H^{-1}\mathrm{d}H = \check{\mathcal{P}}(\eta), \quad H|_{z=0} = I_8. \tag{3.12}$$

Here

$$H_1 = \begin{pmatrix} 0 & f & 0 \\ 0 & 0 & -f^\sharp \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & g \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f = \int_0^z \check{f}\mathrm{d}z, \quad g = -\int_0^z (f\check{f}^\sharp)\mathrm{d}z.$$

Lemma 3.3 *Retaining the assumptions and the notations of the previous lemmas, assume that $\check{\mathcal{P}}(\eta)$ is the normalized potential of some harmonic map, we obtain*:

Assume that the Iwasawa decomposition of H is $H = \check{F}\check{F}_+$, with $\check{F} \in \check{\mathcal{P}}(\Lambda SO^+(1,7)_\sigma) \subset \Lambda G(8,\mathbb{C})_\sigma$ and $\check{F}_+ \in \Lambda^+ G(8,\mathbb{C})_\sigma$. Then

$$\check{F} = H\check{\tau}(W)L_0^{-1}. \tag{3.13}$$

Here W, W_0 and L_0 are the solutions to the matrix equations $\check{\tau}(H)^{-1}H = WW_0\check{\tau}(W)^{-1}$, $W_0 = \check{\tau}(L_0)^{-1}L_0$, with $W = I_8 + \lambda^{-1}W_1 + \lambda^{-2}W_2$ and

$$W_1 = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & -u^\sharp \\ 0 & 0 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 0 & \check{g} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & d \end{pmatrix}, \quad L_0 = \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_0 & 0 \\ 0 & 0 & l_4 \end{pmatrix}.$$

Here the sub-matrices a, q, d and u are determined by the following equations:

$$d = I_2 + \overline{f}^{t\sharp}\check{J}_4 f^\sharp + \overline{g}^t g, \tag{3.14a}$$

$$u^\sharp d = f^\sharp - \check{J}_4\overline{f}^t g, \tag{3.14b}$$

$$q + u^\sharp \mathrm{d}\overline{u}^{\sharp t}\check{J}_4 = I_4 + \check{J}_4\overline{f}^t f, \tag{3.14c}$$

$$a + uq\check{J}_4\overline{u}^t + g(\overline{d}^t)^{-1}\overline{g}^t = I_2, \tag{3.14d}$$

$$uq - g\overline{u}^{\sharp t}\check{J}_4 = f. \tag{3.14e}$$

Moreover, $\check{F}$ can be expressed by these sub-matrices as below

$$\check{F} = H\check{\tau}(W)L_0^{-1} = \begin{pmatrix} (I - fS_4\overline{u}^\sharp J_2 + gJ_2\overline{gd^{-1}}J_2)l_1^{-1} & \lambda^{-1}(f + gJ_2\overline{u}S_4)l_0^{-1} & \lambda^{-2}gl_4^{-1} \\ -\lambda(S_4\overline{u}^\sharp J_2 + f^\sharp J_2\overline{gd^{-1}}J_2)l_1^{-1} & (I - f^\sharp J_2\overline{u}S_4)l_0^{-1} & -\lambda^{-1}f^\sharp l_4^{-1} \\ \lambda^2 J_2\overline{gd^{-1}}J_2 l_1^{-1} & \lambda J_2\overline{u}S_4 l_0^{-1} & l_4^{-1} \end{pmatrix}. \tag{3.15}$$

Remark 3.1 1. Since in Lemma 3.2, the matrices f and g are given, (3.14a) determines d, where d is invertible (true for z close to $z = 0$). Then (3.14b) determines $u^\sharp$, hence u. Inserting this into (3.14c) results in determining q. Inserting what we have so far into (3.14d) determines a. The last equation (3.14e) is a consequence of the previous equations. Therefore, the only condition for the solvability of (3.14) is the invertibility of d. If f and g are rational functions of z, the invertibility of d is satisfied on an open dense subset as a rational expression in $z, \overline{z}$.

2. For a general procedure for the computations of Iwasawa decompositions for algebraic loops, or more generally for rational loops, see [13, Section I.2].

3. In [14, 21], a different method is used to produce all harmonic maps of finite uniton type into $U(n)$, the complex Grassmannian $U(n+m)/(U(n)\times U(m))$ and G_2. The treatment of these papers basically follows the spirit of Wood [35], Uhlenbeck [33] and Segal [31], using some special unitons. In [32], the converse part of this procedure is also used for the computations of the Iwasawa decompositions of elements of the algebraic loop group $\lambda_{\mathrm{alg}}U(n)^{\mathbb{C}}$. It will be an interesting and very hard question to apply they results to detect the geometry of harmonic maps.

3.3.3 Step 2: Maurer-Cartan forms

Lemma 3.4 *Retaining the assumptions and the notations of the previous lemmas, the Maurer-Cartan form of* $\check{F}$ *in* (3.15) *is of the form*

$$\check{\alpha}'_{\mathfrak{k}} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_0 & 0 \\ 0 & 0 & a_4 \end{pmatrix} \mathrm{d}z, \quad \check{\alpha}'_{\mathfrak{p}} = \lambda^{-1} \begin{pmatrix} 0 & l_1\check{f}l_0^{-1} & 0 \\ 0 & 0 & -(l_1\check{f}l_0^{-1})^\sharp \\ 0 & 0 & 0 \end{pmatrix} \mathrm{d}z, \tag{3.16}$$

with

$$\begin{cases} a_1 = -l_1\check{f}S_4\overline{u}^\sharp J_2 l_1^{-1} - l_{1z}l_1^{-1}, \\ a_0 = -l_0(\check{f}^\sharp J_2\overline{u}S_4 - S_4\overline{u}^\sharp J_2\check{f})l_0^{-1} - l_{0z}l_0^{-1}, \\ a_4 = l_4 J_2\overline{u}S_4\check{f}^\sharp l_4^{-1} - l_{4z}l_4^{-1}. \end{cases} \tag{3.17}$$

Note that these three equations for a_1, a_0 and a_4 actually should be read as ordinary differential equations for l_1, l_0 and l_4, as initial conditions we may use $l_j(0) = I, j = 0, 1, 4$.

Lemma 3.5 *Let* $\mathcal{F} : M \to SO^+(1,7)/SO^+(1,3) \times SO(4)$ *be the conformal Gauss map of a Willmore surface* y*, with an extended frame* F*. If the Maurer-Cartan form of* $\check{F} = \check{\mathcal{P}}(F)$ *has the form* (3.16)*, then* y *is totally isotropic in* S^6*. Moreover, locally there exists an isotropic frame* $\{E_1, E_2\}$ *of the normal bundle* $V^\perp_{\mathbb{C}}$ *of* y *such that* (2.7) *holds.*

A combination of the above lemmas provides a complete proof of Theorem 3.1. Lemmas 3.1–3.2 can be verified by straightforward matrix computations since the concrete formulas are provided (compare also [34]). So we leave these computations to the readers. The proofs of the other lemmas will be contained in the following section.

4 Appendix A: Iwasawa Decompositions

4.1 Proof of Lemma 3.3

Firstly one computes

$$\begin{aligned} \check{\tau}^{-1}(H)H &= \begin{pmatrix} I_2 & 0 & 0 \\ \lambda\check{J}_4\overline{f}^t & I_4 & 0 \\ \lambda^2\overline{g}^t & -\lambda\overline{f}^{\sharp,t}\check{J}_4 & I_2 \end{pmatrix} \begin{pmatrix} I_2 & \lambda^{-1}f & \lambda^{-2}g \\ 0 & I_4 & -\lambda^{-1}f^\sharp \\ 0 & 0 & I_2 \end{pmatrix} \\ &= \begin{pmatrix} I_2 & \lambda^{-1}f & \lambda^{-2}g \\ \lambda\check{J}_4\overline{f}^t & I_4 + \check{J}_4\overline{f}^t f & \lambda^{-1}\check{J}_4\overline{f}^t g - \lambda^{-1}f^\sharp \\ \lambda^2\overline{g}^t & \lambda\overline{g}^t f - \lambda\overline{f}^{\sharp,t}\check{J}_4 & I_2 + \lambda\overline{f}^{\sharp,t}\check{J}_4 f^\sharp + \overline{g}^t g \end{pmatrix}. \end{aligned}$$

We write $\check{\tau}^{-1}(H)H = WW_0\check{\tau}^{-1}(W)$ with $W = I_8 + \lambda^{-1}W_1 + \lambda^{-2}W_2$ and

$$W_1 = \begin{pmatrix} 0 & u & 0 \\ -v^\sharp & 0 & -u^\sharp \\ 0 & v & 0 \end{pmatrix}, \quad W_0 = \begin{pmatrix} a & 0 & b \\ 0 & q & 0 \\ c & 0 & d \end{pmatrix}, \quad W_0^{-1} = \begin{pmatrix} \widehat{a} & 0 & \widehat{b} \\ 0 & q^{-1} & 0 \\ \widehat{c} & 0 & \widehat{d} \end{pmatrix}.$$

Hence we obtain

$$\begin{cases} W_2W_0 = H_2, \\ W_1W_0 + W_2W_0\check{J}_8\overline{W}_1^t\check{J}_8 = H_1 + \check{J}_8\overline{H}_1^t\check{J}_8H_2, \\ W_0 + W_1W_0\check{J}_8\overline{W}_1^t\check{J}_8 + W_2W_0\check{J}_8\overline{W}_2^t\check{J}_8 = I + \check{J}_8\overline{H}_1^t\check{J}_8H_1 + \check{J}_8\overline{H}_2^t\check{J}_8H_2. \end{cases} \tag{4.1}$$

Since

$$\check{J}_8\overline{W}_1^t\check{J}_8=\begin{pmatrix}0&-\overline{v}^{\sharp t}\check{J}_4&0\\ \check{J}_4\overline{u}^t&0&\check{J}_4\overline{v}^t\\ 0&-\overline{u}^{\sharp t}\check{J}_4&0\end{pmatrix},\quad W_1W_0=\begin{pmatrix}0&uq&0\\ -v^\sharp a-u^\sharp c&0&-v^\sharp b-u^\sharp d\\ 0&vq&0\end{pmatrix},$$

$$W_2W_0\check{J}_8\overline{W}_1^t\check{J}_8=H_2\check{J}_8\overline{W}_1^t\check{J}_8=\begin{pmatrix}0&-g\overline{u}^{\sharp t}\check{J}_4&0\\ 0&0&0\\ 0&0&0\end{pmatrix},$$

from the second matrix equation of (4.1), one derives easily that $vq=0,\ -v^\sharp a-u^\sharp c=0,\ uq-g\overline{u}^{\sharp t}\check{J}_4=f,\ -v^\sharp b-u^\sharp d=\check{J}_4\overline{f}^tg-f^\sharp$. Since q is invertible, $v=0$. Therefore we have

$$v=0,\quad u^\sharp c=0,\quad uq-g\overline{u}^{\sharp t}\check{J}_4=f,\quad u^\sharp d=f^\sharp-\check{J}_4\overline{f}^tg.$$

Next we consider the third matrix equation in (4.1). Since

$$W_1W_0\check{J}_8\overline{W}_1^t\check{J}_8=\begin{pmatrix}\cdots&0&0\\ 0&\cdots&0\\ 0&0&0\end{pmatrix},\quad W_2W_0\check{J}_8\overline{W}_2^t\check{J}_8=H_2\check{J}_8\overline{W}_2^t\check{J}_8=\begin{pmatrix}\cdots&0&0\\ 0&0&0\\ 0&0&0\end{pmatrix},$$

comparing with the λ-independent part of $\check{\tau}^{-1}(H)H$, we derive directly that $c=b=\widehat{c}=\widehat{b}=0$. Substituting these results into the matrix equations in (4.1), a straightforward computation yields (3.14).

In the end, let L_0 be of the form as in Lemma 3.3, it is easy to compute

$$\begin{aligned}\check{F}=H\check{\tau}(W)L_0^{-1}&=\begin{pmatrix}I&\lambda^{-1}f&\lambda^{-2}g\\ 0&I&-\lambda^{-1}f^\sharp\\ 0&0&I\end{pmatrix}\begin{pmatrix}I&&\\ -\lambda S_4\overline{u}^\sharp J_2&I&\\ \lambda^2J_2g\overline{d}^{-1}J_2&\lambda J_2\overline{u}S_4&I\end{pmatrix}L_0^{-1}\\ &=\begin{pmatrix}(I-fS_4\overline{u}^\sharp J_2+gJ_2\overline{gd^{-1}}J_2)l_1^{-1}&\lambda^{-1}(f+gJ_2\overline{u}S_4)l_0^{-1}&\lambda^{-2}gl_4^{-1}\\ -\lambda(S_4\overline{u}^\sharp J_2+f^\sharp J_2\overline{gd^{-1}}J_2)l_1^{-1}&(I-f^\sharp J_2\overline{u}S_4)l_0^{-1}&-\lambda^{-1}f^\sharp l_4^{-1}\\ \lambda^2J_2\overline{gd^{-1}}J_2l_1^{-1}&\lambda J_2\overline{u}S_4l_0^{-1}&l_4^{-1}\end{pmatrix}.\end{aligned}$$

4.2 The Maurer-Cartan form of $\check{F}$ and the geometry of Willmore surfaces

Proof of Lemma 3.4 We have $\check{F}^{-1}\mathrm{d}\check{F}=\lambda^{-1}\check{\alpha}'_{\mathfrak{p}}+\check{\alpha}_{\mathfrak{k}}+\lambda\check{\alpha}''_{\mathfrak{p}}$ with $\check{\alpha}'_{\mathfrak{p}}=L_0\check{\mathcal{P}}(\eta_{-1})L_0^{-1}\mathrm{d}z,\ \check{\alpha}'_{\mathfrak{k}}=L_0[\check{\mathcal{P}}(\eta_{-1}),\check{\tau}(W_1)]L_0^{-1}\mathrm{d}z+L_0(L_0^{-1})_z\mathrm{d}z$. Since

$$\check{\mathcal{P}}(\eta_{-1})=\begin{pmatrix}0&\check{f}&0\\ 0&0&-\check{f}^\sharp\\ 0&0&0\end{pmatrix},\quad \check{\tau}(W_1)=\begin{pmatrix}0&0&0\\ -S_4\overline{u}^\sharp J_2&0&0\\ 0&J_2\overline{u}S_4&0\end{pmatrix},$$

we obtain

$$L_0[\check{\mathcal{P}}(\eta_{-1}),\check{\tau}(W_1)]L_0^{-1}=\begin{pmatrix}-l_1f'S_4\overline{u}^\sharp J_2l_1^{-1}&&\\ &-l_0(f'^\sharp J_2\overline{u}S_4-S_4\overline{u}^\sharp Jf')l_0^{-1}&\\ &&l_4J_2\overline{u}S_4f'^\sharp l_4^{-1}\end{pmatrix}.$$

Proof of Lemma 3.5 By (3.16) in Lemma 3.4, there exists a frame $\check{F}$ such that the $(1,0)$-part $\check{\alpha}'$ of the Maurer-Cartan form of $\check{F}$ has the form

$$\begin{pmatrix} \check{c}_{11} & \check{c}_{12} & \check{b}_{11} & \check{b}_{12} & \check{b}_{13} & \check{b}_{14} & 0 & 0 \\ \check{c}_{21} & \check{c}_{22} & \check{b}_{21} & \check{b}_{22} & \check{b}_{23} & \check{b}_{24} & 0 & 0 \\ 0 & 0 & \check{s}_{11} & \check{s}_{12} & \check{s}_{13} & 0 & -\check{b}_{24} & -\check{b}_{14} \\ 0 & 0 & \check{s}_{21} & \check{s}_{22} & 0 & -\check{s}_{13} & -\check{b}_{23} & -\check{b}_{13} \\ 0 & 0 & \check{s}_{31} & 0 & \check{s}_{33} & -\check{s}_{12} & -\check{b}_{22} & -\check{b}_{12} \\ 0 & 0 & 0 & -\check{s}_{31} & -\check{s}_{21} & -\check{s}_{11} & -\check{b}_{21} & -\check{b}_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\check{c}_{22} & -\check{c}_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\check{c}_{21} & -\check{c}_{11} \end{pmatrix} \mathrm{d}z.$$

Set $F = \mathcal{P}^{-1}(\widetilde{F}) = (\phi_1, \phi_2, \phi_3, \phi_4, \psi_1, \psi_2, \psi_3, \psi_4)$. By (3.7) in Lemma 3.1, we derive that

$$\alpha' = F^{-1}F_z \mathrm{d}z = \check{\mathcal{P}}^{-1}(\widetilde{\alpha}') = \begin{pmatrix} A_1 & \lambda^{-1}B_1 \\ -\lambda^{-1}B_1^t I_{1,3} & A_2 \end{pmatrix} \mathrm{d}z$$

with

$$A_1 = \begin{pmatrix} 0 & s_{22} & s_{13} & s_{14} \\ s_{22} & 0 & s_{23} & s_{24} \\ s_{13} & -s_{23} & 0 & -\mathrm{i}s_{11} \\ s_{14} & -s_{24} & \mathrm{i}s_{11} & 0 \end{pmatrix}, \quad \begin{cases} 2s_{13} = -\mathrm{i}(\check{s}_{12} - \check{s}_{13}) - \mathrm{i}(\check{s}_{31} - \check{s}_{21}), \\ 2s_{14} = (\check{s}_{12} - \check{s}_{13}) + (\check{s}_{21} - \check{s}_{31}), \\ 2s_{23} = \mathrm{i}(\check{s}_{12} + \check{s}_{13} + \check{s}_{21} + \check{s}_{31}), \\ 2s_{24} = (\check{s}_{12} + \check{s}_{13}) + (\check{s}_{31} + \check{s}_{21}), \end{cases}$$

$$A_2 = \frac{1}{2}\begin{pmatrix} 0 & -2\mathrm{i}\check{c}_{22} & \check{c}_{21} - \check{c}_{12} & -\mathrm{i}(\check{c}_{12} + \check{c}_{21}) \\ 2\mathrm{i}\check{c}_{11} & 0 & \mathrm{i}(\check{c}_{12} + \check{c}_{21}) & \check{c}_{12} - \check{c}_{21} \\ \check{c}_{12} - \check{c}_{21} & -\mathrm{i}(\check{c}_{12} + \check{c}_{21}) & 0 & -2\mathrm{i}\check{c}_{11} \\ \mathrm{i}(\check{c}_{12} + \check{c}_{21}) & \check{c}_{21} - \check{c}_{12} & 2\mathrm{i}\check{c}_{11} & 0 \end{pmatrix}$$

and

$$\begin{aligned} B_1 &= \frac{1}{2}\begin{pmatrix} \mathrm{i}(\check{b}_{23} - \check{b}_{22}) & -(\check{b}_{23} - \check{b}_{22}) & \mathrm{i}(\check{b}_{13} - \check{b}_{12}) & -(\check{b}_{13} - \check{b}_{12}) \\ \mathrm{i}(\check{b}_{23} + \check{b}_{22}) & -(\check{b}_{23} + \check{b}_{22}) & \mathrm{i}(\check{b}_{13} + \check{b}_{12}) & -(\check{b}_{13} + \check{b}_{12}) \\ \check{b}_{24} - \check{b}_{21} & \mathrm{i}(\check{b}_{24} - \check{b}_{21}) & \check{b}_{14} - \check{b}_{11} & \mathrm{i}(\check{b}_{14} - \check{b}_{11}) \\ \mathrm{i}(\check{b}_{24} + \check{b}_{21}) & -(\check{b}_{24} + \check{b}_{21}) & \mathrm{i}(\check{b}_{14} + \check{b}_{11}) & -(\check{b}_{14} + \check{b}_{11}) \end{pmatrix} \\ &= \begin{pmatrix} h_{11} & \mathrm{i}h_{11} & h_{13} & \mathrm{i}h_{13} \\ h_{21} & \mathrm{i}h_{21} & h_{23} & \mathrm{i}h_{23} \\ h_{31} & \mathrm{i}h_{31} & h_{33} & \mathrm{i}h_{33} \\ h_{41} & \mathrm{i}h_{41} & h_{43} & \mathrm{i}h_{43} \end{pmatrix}. \end{aligned} \tag{4.2}$$

Therefore, one obtains

$$\begin{cases} \phi_{1z} = \lambda^{-1}(h_{11}(\psi_1 + \mathrm{i}\psi_2) + h_{13}(\psi_3 + \mathrm{i}\psi_4)) \mod \{\phi_1, \phi_2, \phi_3, \phi_4\} \\ \phi_{jz} = -\lambda^{-1}(h_{j1}(\psi_1 + \mathrm{i}\psi_2) + h_{j3}(\psi_3 + \mathrm{i}\psi_4)) \mod \{\phi_1, \phi_2, \phi_3, \phi_4\}, \quad j = 2, 3, 4. \end{cases} \tag{4.3}$$

Now assume that Y is a canonical lift of the Willmore surface y. Note that $\mathrm{Span}_{\mathbb{C}}\{Y, Y_z, Y_{\overline{z}}, N\} = \mathrm{Span}_{\mathbb{C}}\{\phi_1, \phi_2, \phi_3, \phi_4\}$. So Y_z is a linear combination of $\{\phi_1, \phi_2, \phi_3, \phi_4\}$. Then we compute the Hopf differential $\kappa = Y_{zz} \mod \{Y, Y_z, Y_{\overline{z}}, N\}$:

$$\kappa = \lambda^{-1}k_1(\psi_1 + \mathrm{i}\psi_2) + \lambda^{-1}k_2(\psi_3 + \mathrm{i}\psi_4) \quad \text{for some } k_1, k_2.$$

Hence $\langle \kappa, \kappa \rangle \equiv 0$, i.e., κ is isotropic. To show that Y is totally isotropic, we need only to verify that $D_z\kappa$ is isotropic. From the Maurer-Cartan form of F, we derive that

$$D_z\psi_1 = \mathrm{i}\check{c}_{22}\psi_2 + \frac{\check{c}_{12} - \check{c}_{21}}{2}\psi_3 + \frac{\mathrm{i}(\check{c}_{12} + \check{c}_{21})}{2}\psi_4,$$

$$D_z\psi_2 = -\mathrm{i}\check{c}_{22}\psi_1 + \frac{-\mathrm{i}(\check{c}_{12}+\check{c}_{21})}{2}\psi_3 + \frac{\check{c}_{12}-\check{c}_{21}}{2}\psi_4,$$
$$D_z\psi_3 = \frac{\check{c}_{21}-\check{c}_{12}}{2}\psi_1 + \frac{\mathrm{i}(\check{c}_{12}+\check{c}_{21})}{2}\psi_2 + \mathrm{i}\check{c}_{11}\psi_4,$$
$$D_z\psi_4 = \frac{-\mathrm{i}(\check{c}_{12}+\check{c}_{21})}{2}\psi_1 + \frac{\check{c}_{21}-\check{c}_{12}}{2}\psi_2 - \mathrm{i}\check{c}_{11}\psi_3.$$

So $D_z(\psi_1+\mathrm{i}\psi_2) = \check{c}_{22}(\psi_1+\mathrm{i}\psi_2)+\check{c}_{12}(\psi_3+\mathrm{i}\psi_4)$, $D_z(\psi_3+\mathrm{i}\psi_4) = \check{c}_{21}(\psi_1+\mathrm{i}\psi_2)+\check{c}_{11}(\psi_3+\mathrm{i}\psi_4)$. As a consequence, we obtain that $D_z\kappa = \lambda^{-1}(\delta_1(\psi_1+\mathrm{i}\psi_2)+\delta_2(\psi_3+\mathrm{i}\psi_4))$ for some complex valued function δ_1 and δ_2. This indicates that $D_z\kappa$ is also isotropic, i.e., Y as well as y is totally isotropic.

4.3 An Algorithm to derive Willmore surfaces from frames

This subsection is to derive an algorithm permitting to read off y from the frame F. Although the harmonic maps have been constructed in the above subsections, to obtain the Willmore surfaces from the harmonic maps needs more computations. We retain the notation, in the proof of Lemma 3.5.

Set $B_1 = (h_1, \mathrm{i}h_1, h_3, \mathrm{i}h_3)$ with $h_j = (h_{1j}, h_{2j}, h_{3j}, h_{4j})^t$, $j = 1,\ 3$. Since B_1 satisfies $B_1^t I_{1,3} B_1 = 0$, we have $h_j^t I_{1,3} h_l = 0$, $j,\ l = 1,\ 3$. Therefore h_1 and h_2 are contained in one of the following two subspaces (see also [34]):

$$\mathrm{Span}_{\mathbb{C}}\left\{\begin{pmatrix}1+\rho_1\rho_2\\1-\rho_1\rho_2\\\rho_1+\rho_2\\-\mathrm{i}(\rho_1-\rho_2)\end{pmatrix},\begin{pmatrix}\rho_1\\-\rho_1\\1\\\mathrm{i}\end{pmatrix}\right\},\ \text{or}\ \mathrm{Span}_{\mathbb{C}}\left\{\begin{pmatrix}1+\rho_1\rho_2\\1-\rho_1\rho_2\\\rho_1+\rho_2\\-\mathrm{i}(\rho_1-\rho_2)\end{pmatrix},\begin{pmatrix}\rho_2\\-\rho_2\\1\\-\mathrm{i}\end{pmatrix}\right\}.$$

Let Y be a canonical lift of y. Hence $Y \in \mathrm{Span}_{\mathbb{R}}\{\phi_1,\phi_2,\phi_3,\phi_4\}$. Since Y is real and lightlike, we may assume that

$$Y = \widehat{\rho}_0((1+|\widehat{\rho}_1|^2)\phi_1 + (1-|\widehat{\rho}_1|^2)\phi_2 + (\widehat{\rho}_1+\overline{\widehat{\rho}}_1)\phi_3 - \mathrm{i}(\widehat{\rho}_1-\overline{\widehat{\rho}}_1)\phi_4) \tag{4.4}$$

with $\widehat{\rho}_0 \neq 0$. A straightforward computation by use of (4.3) yields

$$\begin{aligned}Y_z = {}& \widehat{\rho}_0((1+|\widehat{\rho}_1|^2)h_{11} - (1-|\widehat{\rho}_1|^2)h_{21} + (\widehat{\rho}_1+\overline{\widehat{\rho}}_1)h_{31} - \mathrm{i}(\widehat{\rho}_1-\overline{\widehat{\rho}}_1)h_{41})(\psi_1+\mathrm{i}\psi_2)\\ &+ \widehat{\rho}_0((1+|\widehat{\rho}_1|^2)h_{13} - (1-|\widehat{\rho}_1|^2)h_{23} + (\widehat{\rho}_1+\overline{\widehat{\rho}}_1)h_{33} - \mathrm{i}(\widehat{\rho}_1-\overline{\widehat{\rho}}_1)h_{43})(\psi_3+\mathrm{i}\psi_4)\\ &\mod\{\phi_1,\phi_2,\phi_3,\phi_4\}.\end{aligned}$$

Hence, to ensure that $Y_z \in \mathrm{Span}_{\mathbb{C}}\{\phi_1,\phi_2,\phi_3,\phi_4\}$, $\widehat{\rho}_1$ needs to satisfy

$$\begin{cases}(1+|\widehat{\rho}_1|^2)h_{11} - (1-|\widehat{\rho}_1|^2)h_{21} + (\widehat{\rho}_1+\overline{\widehat{\rho}}_1)h_{31} - \mathrm{i}(\widehat{\rho}_1-\overline{\widehat{\rho}}_1)h_{41} = 0,\\ (1+|\widehat{\rho}_1|^2)h_{13} - (1-|\widehat{\rho}_1|^2)h_{23} + (\widehat{\rho}_1+\overline{\widehat{\rho}}_1)h_{33} - \mathrm{i}(\widehat{\rho}_1-\overline{\widehat{\rho}}_1)h_{43} = 0.\end{cases} \tag{4.5}$$

Without loss of generality, we assume that $h_1 = \rho_0(1+\rho_1\rho_2, 1-\rho_1\rho_2, \rho_1+\rho_2, -\mathrm{i}(\rho_1-\rho_2))^t$. If the maximal rank of B_1 is 1, then $h_3 = \rho_{01}(1+\rho_1\rho_2, 1-\rho_1\rho_2, \rho_1+\rho_2, -\mathrm{i}(\rho_1-\rho_2))^t$. So (4.5) is equivalent to $(\widehat{\rho}_1-\rho_1)(\rho_2-\overline{\widehat{\rho}}_1) = 0$. Hence $\widehat{\rho}_1 = \rho_1$ or $\widehat{\rho}_1 = \overline{\rho}_2$. These two solutions provide a pair of dual Willmore (therefore S-Willmore) surfaces y and $\widehat{y}$ with the same conformal Gauss map.

If the maximal rank of B_1 is 2, then

$$h_3 = \rho_{01}(1+\rho_1\rho_2, 1-\rho_1\rho_2, \rho_1+\rho_2, -\mathrm{i}(\rho_1-\rho_2))^t + \rho_{02}(\rho_1, -\rho_1, 1, \mathrm{i})^t,$$

or

$$h_3 = \rho_{01}(1+\rho_1\rho_2, 1-\rho_1\rho_2, \rho_1+\rho_2, -\mathrm{i}(\rho_1-\rho_2))^t + \rho_{02}(\rho_2, -\rho_2, 1, -\mathrm{i})^t.$$

For the first case, (4.5) is equivalent to $\widehat{\rho}_1 = \rho_1$. For the second case, (4.5) is equivalent to $\widehat{\rho}_1 = \overline{\rho}_2$. In both cases, we obtain a unique non-S-Willmore surface.

From the above discussions, clearly it is necessary to obtain the first four columns of F. By (3.15), $\check{F}$ can be derived from the Iwasawa decompositions. Set $\check{F} = (\mathrm{f}_{jl})$, $j, l = 1, \cdots, 8$, and $F = \check{\mathcal{P}}^{-1}(\check{F})$. Writing $F = (\phi_1, \phi_2, \phi_3, \phi_4, \psi_1, \psi_2, \psi_3, \psi_4)$, and setting $(\widehat{\phi}_1, \widehat{\phi}_2, \widehat{\phi}_3, \widehat{\phi}_4) = (\phi_1+\phi_2, \phi_1-\phi_2, \phi_3-\mathrm{i}\phi_4, \phi_3+\mathrm{i}\phi_4)$, one obtains straightforwardly from (3.7) that

$$(\widehat{\phi}_1, \widehat{\phi}_2, \widehat{\phi}_3, \widehat{\phi}_4) = \begin{pmatrix} (\mathrm{f}_{44}-\mathrm{f}_{54}) & -(\mathrm{f}_{45}-\mathrm{f}_{55}) & -\mathrm{i}(\mathrm{f}_{46}-\mathrm{f}_{56}) & \mathrm{i}(\mathrm{f}_{43}-\mathrm{f}_{53}) \\ (\mathrm{f}_{44}+\mathrm{f}_{54}) & -(\mathrm{f}_{45}+\mathrm{f}_{55}) & -\mathrm{i}(\mathrm{f}_{46}+\mathrm{f}_{56}) & \mathrm{i}(\mathrm{f}_{43}+\mathrm{f}_{53}) \\ -\mathrm{i}(\mathrm{f}_{34}-\mathrm{f}_{64}) & \mathrm{i}(\mathrm{f}_{35}-\mathrm{f}_{65}) & -(\mathrm{f}_{36}-\mathrm{f}_{66}) & (\mathrm{f}_{33}-\mathrm{f}_{63}) \\ (\mathrm{f}_{34}+\mathrm{f}_{64}) & -(\mathrm{f}_{35}+\mathrm{f}_{65}) & -\mathrm{i}(\mathrm{f}_{36}+\mathrm{f}_{66}) & \mathrm{i}(\mathrm{f}_{33}+\mathrm{f}_{63}) \\ -\mathrm{i}(\mathrm{f}_{24}-\mathrm{f}_{74}) & \mathrm{i}(\mathrm{f}_{25}-\mathrm{f}_{75}) & -(\mathrm{f}_{26}-\mathrm{f}_{76}) & (\mathrm{f}_{23}-\mathrm{f}_{73}) \\ (\mathrm{f}_{24}+\mathrm{f}_{74}) & -(\mathrm{f}_{25}+\mathrm{f}_{75}) & -\mathrm{i}(\mathrm{f}_{26}+\mathrm{f}_{76}) & \mathrm{i}(\mathrm{f}_{23}+\mathrm{f}_{73}) \\ -\mathrm{i}(\mathrm{f}_{14}-\mathrm{f}_{84}) & \mathrm{i}(\mathrm{f}_{15}-\mathrm{f}_{85}) & -(\mathrm{f}_{16}-\mathrm{f}_{86}) & (\mathrm{f}_{13}-\mathrm{f}_{83}) \\ (\mathrm{f}_{14}+\mathrm{f}_{84}) & -(\mathrm{f}_{15}+\mathrm{f}_{85}) & -\mathrm{i}(\mathrm{f}_{16}+\mathrm{f}_{86}) & \mathrm{i}(\mathrm{f}_{13}+\mathrm{f}_{83}) \end{pmatrix}. \tag{4.6}$$

5 Appendix B: Construction of Examples

5.1 Proof of Theorem 3.2

By the procedures in Subsection 4.3, to derive the expression of y, one needs to figure out B_1 of the Maurer-Cartan form and the first four columns of the frame F. Applying Lemmas 3.2–3.3 to $\check{\mathcal{P}}(\eta)$, F and the Maurer-Cartan form can be derived by solving (3.14) for the Iwasawa decompositions. Therefore we have three steps to derive y:

1. Computation of the first four columns of F.
2. Computation of the Maurer-Cartan form of F.
3. Computation of Y.

Step 1: Computation of the first four columns of F. By (3.11), it is straightforward to derive that

$$\check{\mathcal{P}}(\eta) = \lambda^{-1} \begin{pmatrix} 0 & \check{f} & 0 \\ 0 & 0 & -\check{f}^\sharp \\ 0 & 0 & 0 \end{pmatrix} \mathrm{d}z$$

with $\check{f} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_2' & 0 & f_4' \end{pmatrix}$ and $f = \int_0^z \check{f}\mathrm{d}z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_2 & 0 & f_4 \end{pmatrix}$. Since $f\check{f} = 0$, we obtain $g = 0$. And a straightforward computation yields

$$d = (d_{ij}) = \begin{pmatrix} 1+|f_4|^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad d^{-1} = \begin{pmatrix} \frac{1}{|d|} & 0 \\ 0 & 1 \end{pmatrix}, \text{ with } |d| = 1+|f_4|^2.$$

Since $W_0 \in G(8, \mathbb{C})$, we derive $a^t = Jd^{-1}J = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{|d|} \end{pmatrix} = a$. By (3.14b) and (3.14c), we have

$$u=\frac{1}{|d|}\begin{pmatrix}0&0&0&0\\0&f_2&0&f_4\end{pmatrix},\quad q=(q_{ij})=\begin{pmatrix}\frac{1}{|d|}&-\frac{\overline{f}_2f_4}{|d|}&0&0\\0&1&0&0\\-\frac{f_2\overline{f}_4}{|d|}&\frac{|f_2f_4|^2}{|d|}&1&\overline{f}_2f_4\\0&f_2\overline{f}_4&0&|d|\end{pmatrix}.$$

It is straightforward to verify that $q=\check{J}_4\overline{l}_0^t\check{J}_4l_0$ with $l_0\in G(4,\mathbb{C})$:

$$l_0=\begin{pmatrix}\frac{1}{\sqrt{|d|}}&-\frac{\overline{f}_2f_4}{\sqrt{|d|}}&0&0\\0&1&0&0\\0&0&1&\overline{f}_2f_4\\0&0&0&\sqrt{|d|}\end{pmatrix},\quad\text{and hence}\quad l_0^{-1}=\begin{pmatrix}\sqrt{|d|}&\overline{f}_2f_4&0&0\\0&1&0&0\\0&0&1&-\frac{\overline{f}_2f_4}{\sqrt{|d|}}\\0&0&0&\frac{1}{\sqrt{|d|}}\end{pmatrix}.$$

Moreover, by (3.15), we have

$$\begin{pmatrix}\mathrm{f}_{13}&\mathrm{f}_{14}&\mathrm{f}_{15}&\mathrm{f}_{16}\\\mathrm{f}_{23}&\mathrm{f}_{24}&\mathrm{f}_{25}&\mathrm{f}_{26}\\\mathrm{f}_{33}&\mathrm{f}_{34}&\mathrm{f}_{35}&\mathrm{f}_{36}\\\mathrm{f}_{43}&\mathrm{f}_{44}&\mathrm{f}_{45}&\mathrm{f}_{46}\\\mathrm{f}_{53}&\mathrm{f}_{54}&\mathrm{f}_{55}&\mathrm{f}_{56}\\\mathrm{f}_{63}&\mathrm{f}_{64}&\mathrm{f}_{65}&\mathrm{f}_{66}\\\mathrm{f}_{73}&\mathrm{f}_{74}&\mathrm{f}_{75}&\mathrm{f}_{76}\\\mathrm{f}_{83}&\mathrm{f}_{84}&\mathrm{f}_{85}&\mathrm{f}_{86}\end{pmatrix}=\begin{pmatrix}f+gJ\overline{u}S_4\\I-f^\sharp J\overline{u}S_4\\J\overline{u}S_4\end{pmatrix}l_0^{-1}=\begin{pmatrix}0&0&0&0\\0&f_2&0&\frac{f_4}{\sqrt{|d|}}\\\frac{1}{\sqrt{|d|}}&0&0&0\\0&1&0&0\\-\frac{f_2\overline{f}_4}{\sqrt{|d|}}&-|f_2|^2&1&-\frac{\overline{f}_2f_4}{\sqrt{|d|}}\\0&0&0&\frac{1}{\sqrt{|d|}}\\\frac{\overline{f}_4}{\sqrt{|d|}}&\overline{f}_2&0&0\\0&0&0&0\end{pmatrix}.$$

Step 2: Computation of the Maurer-Cartan form of F. Applying Lemma 3.4, the $l_1\check{f}l_0^{-1}$ part of the Maurer-Cartan form of $\check{F}$ is of the form

$$l_1\check{f}l_0^{-1}=\begin{pmatrix}0&0&0&0\\0&\frac{f_2'}{\sqrt{|d|}}&0&\frac{f_4'}{|d|}\end{pmatrix}\implies B_1=\frac{1}{2}\begin{pmatrix}-\frac{\mathrm{i}f_2'}{\sqrt{|d|}}&\frac{f_2'}{\sqrt{|d|}}&0&0\\\frac{\mathrm{i}f_2'}{\sqrt{|d|}}&-\frac{f_2'}{\sqrt{|d|}}&0&0\\\frac{f_4'}{|d|}&\frac{\mathrm{i}f_4'}{|d|}&0&0\\\frac{\mathrm{i}f_4'}{|d|}&-\frac{f_4'}{|d|}&0&0\end{pmatrix}.$$

Step 3: Computation of Y. Here we follow the discussions in Subsection 4.3. First from, the Maurer-Cartan form we have

$$D_z\phi_1=-\frac{\mathrm{i}f_2'}{2\sqrt{|d|}}(\psi_1+\mathrm{i}\psi_2),\quad D_z\phi_2=-\frac{\mathrm{i}f_2'}{2\sqrt{|d|}}(\psi_1+\mathrm{i}\psi_2),$$

$$D_z\phi_3 = -\frac{f_4'}{2|d|}(\psi_1 + \mathrm{i}\psi_2), \quad D_z\phi_4 = -\frac{\mathrm{i}f_4'}{2|d|}(\psi_1 + \mathrm{i}\psi_2).$$

Set $E_1 = \phi_1 - \phi_2,\ \widehat{E}_1 = \phi_1 + \phi_2,\ E_2 = \phi_3 - \mathrm{i}\phi_4$. Assume that $Y = \widehat{E}_1 + \overline{\mu}E_2 + \mu\overline{E}_2 + |\mu|^2E_1$ for some μ. We have that

$$D_zY = \big(\frac{-\mathrm{i}f_2'}{\sqrt{|d|}} - \overline{\mu}\frac{f_4'}{|d|}\big)(\psi_1 + \mathrm{i}\psi_2).$$

So $D_zY = 0$ if and only if $\overline{\mu} = -\frac{\mathrm{i}f_2'\sqrt{|d|}}{f_4'}$. This yields (3.2).

Remark 5.1 Note that the above Iwasawa decomposition only blows up at the poles of f_2 and f_4, showing that the above decomposition does not cross the boundary of an Iwasawa big cell.

5.2 Proof of Theorem 3.3

Here we have four steps:

1. Computation of the first four columns of F.
2. Computation of the Maurer-Cartan form of F.
3. Computation of Y.
4. Computation of metric of Y (to check the immersion properties of y).

Step 1: Computation of the first four columns of F. Since η is of the form stated in (3.4), by (3.11), it is easy to derive that

$$\check{\mathcal{P}}(\eta) = \lambda^{-1}\begin{pmatrix} 0 & \check{f} & 0 \\ 0 & 0 & -\check{f}^\sharp \\ 0 & 0 & 0 \end{pmatrix}\mathrm{d}z$$

with

$$\check{f} = \begin{pmatrix} 0 & 0 & -1 & -z \\ 2 & -2z & 0 & 0 \end{pmatrix}, \quad f = \int_0^z \check{f}\mathrm{d}z = \begin{pmatrix} 0 & 0 & -z & -\dfrac{z^2}{2} \\ 2z & -z^2 & 0 & 0 \end{pmatrix}.$$

Note now $g = -\int_0^z f(\check{f}^\sharp)\mathrm{d}z = \frac{z^3}{3}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Set $r = \sqrt{|z|^2}$. By (3.14a), we have

$$d = (d_{ij}) = \begin{pmatrix} 1 + 4r^2 + \dfrac{r^6}{9} & r^2\overline{z} \\ r^2 z & 1 + \dfrac{r^4}{4} + \dfrac{r^6}{9} \end{pmatrix}, \quad d^{-1} = \frac{1}{|d|}\begin{pmatrix} 1 + \dfrac{r^4}{4} + \dfrac{r^6}{9} & -r^2\overline{z} \\ -r^2 z & 1 + 4r^2 + \dfrac{r^6}{9} \end{pmatrix}$$

with $|d| = \big(1 + 4r^2 + \frac{r^4}{4} + \frac{r^6}{9}\big)\big(1 + \frac{r^6}{9}\big)$. So $a = (Jd^{-1}J)^t = \frac{1}{|d|}\begin{pmatrix} 1 + 4r^2 + \frac{r^6}{9} & -r^2\overline{z} \\ -r^2 z & 1 + \frac{r^4}{4} + \frac{r^6}{9} \end{pmatrix}$ and $a = \overline{l_1}^t l_1$ with $l_1 = \begin{pmatrix} \dfrac{\sqrt{d_{11}}}{\sqrt{|d|}} & -\dfrac{d_{12}}{\sqrt{|d|}\sqrt{d_{11}}} \\ 0 & \dfrac{1}{\sqrt{d_{11}}} \end{pmatrix} = \frac{1}{\sqrt{|d|}}\begin{pmatrix} \sqrt{1 + 4r^2 + \dfrac{r^6}{9}} & \dfrac{-r^2\overline{z}}{\sqrt{1 + 4r^2 + \dfrac{r^6}{9}}} \\ 0 & \dfrac{1}{\sqrt{1 + 4r^2 + \dfrac{r^6}{9}}} \end{pmatrix}$.

Moreover, by (3.14b), one computes

$$u^\sharp = \frac{1}{|d|}\begin{pmatrix} \frac{r^2z^3}{2}\Big(1+\frac{4r^2}{3}\Big) & -\frac{z^2}{2}\Big(1+\frac{4r^2}{3}\Big)\Big(1+4r^2+\frac{r^6}{9}\Big) \\ \frac{r^2z^2}{3}\Big(2-\frac{r^6}{9}-\frac{r^4}{4}\Big) & -z\Big(1+4r^2-\frac{2r^6}{9}\Big) \\ -z^2\Big(1+\frac{r^4}{4}+\frac{4r^6}{9}\Big) & \frac{r^4z}{3}\Big(4+4r^2+\frac{r^6}{9}\Big) \\ 2z\Big(1-\frac{r^4}{12}\Big)\Big(1+\frac{r^4}{4}+\frac{r^6}{9}\Big) & -2r^4\Big(1-\frac{r^4}{12}\Big) \end{pmatrix}.$$

Substituting these into (3.14c), we obtain $q = (q_{ij})$ with

$$\begin{aligned}
&|d|q_{11} = \Big(1+4r^2-\frac{2r^6}{9}\Big)^2, \quad |d|q_{12} = |d|\overline{q}_{31} = -2r^2z\Big(1-\frac{r^4}{12}\Big)\Big(1+4r^2-\frac{2r^6}{9}\Big),\\
&|d|q_{13} = |d|\overline{q}_{21} = -\frac{r^2z}{2}\Big(1+\frac{4r^2}{3}\Big)\Big(1+4r^2-\frac{2r^6}{9}\Big),\\
&|d|q_{14} = |d|\overline{q}_{41} = -r^4z^2\Big(1+\frac{4r^2}{3}\Big)\Big(1-\frac{r^4}{12}\Big),\\
&|d|q_{22} = |d|\overline{q}_{33} = \Big(1+4r^2-\frac{2r^6}{9}\Big)\Big(1+\frac{r^4}{4}+\frac{4r^6}{9}\Big), \quad |d|q_{23} = \frac{r^6}{4}\Big(1+\frac{4r^2}{3}\Big)^2,\\
&|d|q_{24} = |d|\overline{q}_{43} = \frac{r^2z}{2}\Big(1+\frac{4r^2}{3}\Big)\Big(1+\frac{r^4}{4}+\frac{4r^6}{9}\Big), \quad |d|q_{32} = 4r^6\Big(1-\frac{r^4}{12}\Big)^2,\\
&|d|q_{34} = |d|\overline{q}_{42} = 2r^2z\Big(1-\frac{r^4}{12}\Big)\Big(1+\frac{r^4}{4}+\frac{4r^6}{9}\Big), \quad |d|q_{44} = \Big(1+\frac{r^4}{4}+\frac{4r^6}{9}\Big)^2.
\end{aligned}$$

It is straightforward to check that $q = \check{J}_4\overline{l}_0^t\check{J}_4 l_0$ with

$$l_0 = \begin{pmatrix} \frac{\left(1+4r^2-\frac{2r^6}{9}\right)}{\sqrt{|d|}} & -\frac{2r^2z\left(1-\frac{r^4}{12}\right)}{\sqrt{|d|}} & -\frac{r^2z\left(1+\frac{4r^2}{3}\right)}{2\sqrt{|d|}} & -\frac{r^4z^2\left(1+\frac{4r^2}{3}\right)\left(1-\frac{r^4}{12}\right)}{\sqrt{|d|}\left(1+4r^2-\frac{2r^6}{9}\right)} \\ 0 & \sqrt{|d|} & 0 & \frac{r^2z\left(1+\frac{4r^2}{3}\right)\sqrt{|d|}}{2\left(1+4r^2-\frac{2r^6}{9}\right)} \\ 0 & 0 & \frac{1}{\sqrt{|d|}} & \frac{2r^2z\left(1-\frac{r^4}{12}\right)}{\sqrt{|d|}\left(1+4r^2-\frac{2r^6}{9}\right)} \\ 0 & 0 & 0 & \frac{\sqrt{|d|}}{\left(1+4r^2-\frac{2r^6}{9}\right)} \end{pmatrix}.$$

Assume that $\check{F}=(\mathrm{f}_{jl})=\frac{1}{|d|^{\frac{1}{2}}(1+4r^2-\frac{2r^6}{9})}(\widehat{\mathrm{f}}_{jl})$. By (3.15),

$$(\widehat{\mathrm{f}}_{jl})=\begin{pmatrix} -\frac{r^8}{6\lambda}\big(1+\frac{4r^2}{3}\big) & \frac{zr^4}{3\lambda} & -\frac{z(1+4r^2)|d|}{\lambda} & -\frac{z^2\widehat{\mathrm{f}}_{25}}{2}\\ \frac{2z}{\lambda}\widetilde{\mathrm{f}}_{23} & -\frac{z^2}{\lambda} & \frac{2z^2r^2|d|}{3\lambda} & \frac{z^3r^2\big(1+\frac{4r^2}{3}\big)}{2\lambda}\\ \widehat{\mathrm{f}}_{24} & 2zr^2 & -\frac{4zr^4|d|}{3} & -z^2r^4\big(1+\frac{4r^2}{3}\big)\\ -\frac{r^2\overline{z}}{2}\widehat{\mathrm{f}}_{25} & 1+4r^2+\frac{r^6}{9} & -r^2(1+4r^2)|d| & -\frac{zr^2}{2}\widetilde{\mathrm{f}}_{25}\\ \frac{r^6\overline{z}}{2}\big(1+\frac{4r^2}{3}\big) & -r^4 & \big(1+4r^2+\frac{4r^6}{9}\big)|d| & \frac{r^6\overline{z}}{2}\big(1+\frac{4r^2}{3}\big)\\ -r^4\overline{z}^2\big(1+\frac{4r^2}{3}\big) & 2\overline{z}r^2 & -\frac{4\overline{z}r^4|d|}{3} & \widehat{\mathrm{f}}_{24}\\ \frac{\lambda r^2\overline{z}^3}{2}\big(1+\frac{4r^2}{3}\big) & -\lambda\overline{z} & \frac{2\lambda\overline{z}^2r^2|d|}{3} & 2\lambda\overline{z}\widetilde{\mathrm{f}}_{23}\\ -\frac{\lambda\overline{z}^2}{2}\widetilde{\mathrm{f}}_{25} & \frac{\lambda\overline{z}r^4}{3} & -\lambda\overline{z}(1+4r^2)|d| & -\frac{\lambda r^8}{6}\big(1+\frac{4r^2}{3}\big) \end{pmatrix} \tag{5.1}$$

with $1\le j\le 8, 3\le l\le 6$, and

$$\begin{aligned}\widetilde{\mathrm{f}}_{23}&=1+4r^2+\frac{r^4}{6}-\frac{2r^6}{9}+\frac{r^{10}}{54},\\ \widetilde{\mathrm{f}}_{25}&=\big(1+\frac{4r^2}{3}\big)\big(1+4r^2+\frac{r^6}{9}\big),\\ \widehat{\mathrm{f}}_{24}&=1+4r^2-\frac{10r^6}{9}-\frac{8r^8}{9}-\frac{2r^{12}}{81}.\end{aligned}$$

Step 2: Computation of the Maurer-Cartan form of F. By (3.16)–(3.17) of Lemma 3.4, the Maurer-Cartan form of $\check{F}$ has the expression

$$\begin{aligned} l_1\check{f}l_0^{-1}&=\frac{1}{\sqrt{|d|}}\cdot l_1\\ &\quad\cdot\begin{pmatrix} 0 & 0 & -|d| & -z\big(1+2r^2-\frac{r^6}{18}\big)\\ \frac{2|d|}{\big(1+4r^2-\frac{2r^6}{9}\big)} & \frac{-2z\big(1+2r^2-\frac{r^6}{18}\big)}{\big(1+4r^2-\frac{2r^6}{9}\big)} & \frac{zr^2\big(1+\frac{4r^2}{3}\big)}{\big(1+4r^2-\frac{2r^6}{9}\big)} & \frac{z^2r^2\big(1+\frac{4r^2}{3}\big)\big(1+2r^2-\frac{r^6}{18}\big)}{\big(1+4r^2-\frac{2r^6}{9}\big)} \end{pmatrix}\\ &=\sqrt{|d|}\begin{pmatrix} \widetilde{w}_1 & -\widetilde{w}_1\rho & \widetilde{w}_2 & \widetilde{w}_2\rho\\ \widehat{w}_1 & -\widehat{w}_1\rho & \widehat{w}_2 & \widehat{w}_2\rho \end{pmatrix}\end{aligned}$$

with

$$\rho=\frac{z\big(1+2r^2-\frac{r^6}{18}\big)}{|d|},\qquad l_1=\frac{1}{\sqrt{|d|}}\begin{pmatrix} \sqrt{1+4r^2+\frac{r^6}{9}} & \frac{-r^2\overline{z}}{\sqrt{1+4r^2+\frac{r^6}{9}}}\\ 0 & \frac{1}{\sqrt{1+4r^2+\frac{r^6}{9}}} \end{pmatrix}=\begin{pmatrix} \widehat{l}_{11} & \widehat{l}_{12}\\ 0 & \widehat{l}_{22}\end{pmatrix},$$

$$\widetilde{w}_1=\frac{2\widehat{l}_{12}}{1+4r^2-\frac{2r^6}{9}},\quad \widetilde{w}_2=\frac{\widehat{l}_{12}zr^2\big(1+\frac{4r^2}{3}\big)}{1+4r^2-\frac{2r^6}{9}}-\widehat{l}_{11},$$

$$\widehat{w}_1=\frac{2\widehat{l}_{22}}{1+4r^2-\frac{2r^6}{9}},\quad \widehat{w}_2=\frac{\widehat{l}_{22}zr^2\Big(1+\frac{4r^2}{3}\Big)}{1+4r^2-\frac{2r^6}{9}}.$$

Transforming back to $\mathfrak{so}(1,7,\mathbb{C})$, we derive

$$\check{B}_1=\begin{pmatrix}\mathrm{i}(\widehat{w}_2+\widehat{w}_1\rho) & -(\widehat{w}_2+\widehat{w}_1\rho) & \mathrm{i}(\widetilde{w}_2+\widetilde{w}_1\rho) & -(\widetilde{w}_2+\widetilde{w}_1\rho)\\ \mathrm{i}(\widehat{w}_2-\widehat{w}_1\rho) & -(\widehat{w}_2-\widehat{w}_1\rho) & \mathrm{i}(\widetilde{w}_2-\widetilde{w}_1\rho) & -(\widetilde{w}_2-\widetilde{w}_1\rho)\\ (\widehat{w}_2\rho-\widehat{w}_1) & \mathrm{i}(\widehat{w}_2\rho-\widehat{w}_1) & (\widetilde{w}_2\rho-\widetilde{w}_1) & \mathrm{i}(\widetilde{w}_2\rho-\widetilde{w}_1)\\ \mathrm{i}(\widehat{w}_2\rho+\widehat{w}_1) & -(\widehat{w}_2\rho+\widehat{w}_1) & \mathrm{i}(\widetilde{w}_2\rho+\widetilde{w}_1) & -(\widetilde{w}_2\rho+\widetilde{w}_1)\end{pmatrix}=(h_1,\mathrm{i}h_1,h_3,\mathrm{i}h_3).$$

Step 3: Computation of Y. Here we follow the discussions in Subsection 4.3. It is easy to verify that h_1 and h_3 can be expressed as a (functional) linear combination of $(1,1,-\mathrm{i}\rho,\rho)^t$ and $(\rho,-\rho,\mathrm{i},1)^t$. Therefore one obtains easily that $\check{\rho}_1=-\mathrm{i}\rho$ is the unique solution to (4.5). Substituting $\check{\rho}_1$ into (4.4), we obtain

$$[Y]=[(1+|\rho|^2)\phi_1+(1-|\rho|^2)\phi_2-\mathrm{i}(\rho-\overline{\rho})\phi_3+(\rho+\overline{\rho})\phi_4]=[(\phi_1+\phi_2)+|\rho|^2(\phi_1-\phi_2)+\mathrm{i}\overline{\rho}(\phi_3-\mathrm{i}\phi_4)-\mathrm{i}\rho(\phi_3+\mathrm{i}\phi_4)].$$

Then by (5.1) and (4.6) we have that

$$[Y]=\Big[\frac{\big(1+4r^2-\frac{2r^6}{9}\big)}{|d|^{\frac{3}{2}}}\begin{pmatrix}\Big(1+r^2+\dfrac{5r^4}{4}+\dfrac{4r^6}{9}+\dfrac{r^8}{36}\Big)\\ \Big(1-r^2-\dfrac{3r^4}{4}+\dfrac{4r^6}{9}-\dfrac{r^8}{36}\Big)\\ -\mathrm{i}\Big((z-\overline{z})\Big(1+\dfrac{r^6}{9}\Big)\Big)\\ \Big((z+\overline{z})\Big(1+\dfrac{r^6}{9}\Big)\Big)\\ -\mathrm{i}\Big((\lambda^{-1}z^2-\lambda\overline{z}^2)\Big(1-\dfrac{r^4}{12}\Big)\Big)\\ \Big((\lambda^{-1}z^2+\lambda\overline{z}^2)\Big(1-\dfrac{r^4}{12}\Big)\Big)\\ -\mathrm{i}\frac{r^2}{2}(\lambda^{-1}z-\lambda\overline{z})\Big(1+\dfrac{4r^2}{3}\Big)\\ \frac{r^2}{2}(\lambda^{-1}z+\lambda\overline{z})\Big(1+\dfrac{4r^2}{3}\Big)\end{pmatrix}\Big].\tag{5.2}$$

Step 4: Let $[Y]$ be a global immersion. Let x_λ be of the form (3.5). Then $x_\lambda:S^2\to S^6$ is well-defined on S^2 with Y as its lift. Since

$$|x_{\lambda z}|^2|\mathrm{d}z|^2=\frac{2+8r^2+\frac{r^4}{2}+\frac{4r^6}{9}+\frac{8r^8}{9}+\frac{r^{10}}{18}+\frac{2r^{12}}{81}}{\big(1+r^2+\frac{5r^4}{4}+\frac{4r^6}{9}+\frac{r^8}{36}\big)^2}|\mathrm{d}z|^2,$$

x has no branch point at $z\in\mathbb{C}$. As to ∞, set $\widetilde{z}=\frac{1}{z}$ and $\widetilde{r}=\sqrt{|\widetilde{z}|}$, we derive that $|x_{\lambda\widetilde{z}}|^2|\mathrm{d}\widetilde{z}|^2=32|\mathrm{d}\widetilde{z}|^2$ at the point $\widetilde{z}=0$.

Remark 5.2 Note that in the above Iwasawa decomposition, there exists a circle $1+4r^2-\frac{2r^6}{9}=0$ such that the frame (5.1) obtained from the Iwasawa decomposition blows up. However, this blowing up can be avoided by a change of frames and hence the corresponding harmonic map is in fact globally well defined. This also means that the decomposition of the corresponding harmonic map does not cross the boundary of an Iwasawa big cell (compare [4, 24]).

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