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L^p Solutions for Multidimensional BDSDEs with Locally Weak Monotonicity Coefficients*

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Abstract In this paper, the authors establish the existence and uniqueness theorem of L^p $(1 < p \leq 2)$ solutions for multidimensional backward doubly stochastic differential equations (BDSDEs for short) under the p-order globally (locally) weak monotonicity conditions. Comparison theorem of L^p solutions for one-dimensional BDSDEs is also proved. These conclusions unify and generalize some known results.

Keywords Backward doubly stochastic differential equation, Locally monotonicity condition, L^p solution
2000 MR Subject Classification 60H10

1 Introduction

The following backward stochastic differential equation (BSDE for short) was first introduced by Pardoux and Peng [13]:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad 0 \leq t \leq T.$$

The existence and uniqueness result of L^2 solutions was proved under the Lipschitz condition. Since then, BSDEs have been developed rapidly and connected with other related fields, such as stochastic control and partial differential equations etc. Researchers have obtained several results under the weaker conditions on coefficients, such as [4–9, 15, 17, 19] etc.

For studying a probabilistic representation of certain quasilinear stochastic partial differential equations (SPDEs for short), Pardoux and Peng [14] first proposed the BDSDE and got the existence and uniqueness result of L^2 solutions under the Lipschitz condition. The BDSDE is given as bellow:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + \int_t^T g(s, Y_s, Z_s)\mathrm{d}B_s - \int_t^T Z_s \mathrm{d}W_s, \quad 0 \leq t \leq T, \tag{1.1}$$

where $\{B_t\}_{t\geq 0}$ and $\{W_t\}_{t\geq 0}$ are mutually independent standard Brownian motions with values in $\mathbb{R}^l$ and $\mathbb{R}^d$ respectively. The integral with respect to $W.$ is a standard forward Itô integral,

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while the integral with respect to $B.$ is a backward one. These two kinds of integrals are special cases of Itô-Skorohod integral. For each $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$, coefficients

$$f:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\mapsto\mathbb{R}^k,\quad g:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\mapsto\mathbb{R}^{k\times l}$$

are jointly measurable. BDSDE (f,g,T,ξ) denotes the BDSDE with parameters (f,g,T,ξ). The following lemma comes from Section 3 in Bihari [3] and definitions will be used in the remaining of the paper.

Lemma 1.1 *$\forall\,0\le t\le T$, suppose that $D(t)$ and $G(t)$ are continuous and positive functions. $\forall\,u\ge 0$, $q(u)$ is a non-negative and non-decreasing continuous function. Let $\delta\ge 0$, $M\ge 0$. If*

$$D(t)\le\delta+M\int_t^T G(s)q(D(s))\mathrm{d}s,\quad 0\le t\le T,$$

and $\forall u_0>0$, denote $Q(u)=\int_{u_0}^u\frac{\mathrm{d}s}{q(s)}, u\ge 0$, then

$$D(t)\le Q^{-1}\Big(Q(\delta)+M\int_t^T G(s)\mathrm{d}s\Big),\tag{1.2}$$

where $Q^{-1}(\cdot)$ is the inverse function of $Q(\cdot)$, and the boundary furnished by (1.2) is independent of u_0.

Definition 1.1 *A solution for BDSDE (1.1) is an $(\mathcal{F}_t)$-measurable process $(Y_t,Z_t)_{t\in[0,T]}$ with values in $\mathbb{R}^k\times\mathbb{R}^{k\times d}$ such that $\mathrm{d}P$-a.s., $t\mapsto Y_t$ continuous, $t\mapsto Z_t\in L^2(0,T)$, $t\mapsto f(t,Y_t,Z_t)\in L^1(0,T)$, $t\mapsto g(t,Y_t,Z_t)\in L^2(0,T)$. And for all $t\in[0,T]$, the solution satisfies BDSDE (1.1).*

Definition 1.2 *Let $(Y_t,Z_t)_{t\in[0,T]}$ be a solution for BDSDE (1.1) and $p>1$, and $(Y_t,Z_t)_{t\in[0,T]}\in S^p\times M^p$. Then $(Y_t,Z_t)_{t\in[0,T]}$ is an L^p solution for BDSDE (1.1).*

There have been various extensions of the BDSDEs to non-Lipschitz condition on coefficients or to L^p $(1<p\le 2)$ solutions, and we refer to some references, Shi, Gu and Liu [16] for linear growth condition, Lin [10] and Lin and Wu [11] for left-Lipschitz or uniformly continuous conditions, Owo [12] for stochastic Lipschitz condition. It is important for studying BDSDEs with weaker conditions, because BDSDEs have the closely connection to the theory of stochastic partial differential equations (SPDEs for short). For the relationship between BDSDEs and SPDEs, the readers can refer to [1–2, 14, 18, 20–21] etc.

Here, we would like to mention the following several results on multidimensional BDSDEs, which is related closely to our result. First of all, Wu and Zhang [20] investigated BDSDEs with locally monotone coefficients by adding the assumption that f satisfies γ-growth condition in (y,z). Second, Zong and Hu [24] considered the L^p solution to BDSDEs for the monotone coefficient f with linear growth condition on z in the infinite time horizon. Furthermore, in BSDEs, Fan [6] studied the existence and uniqueness of L^p $(p>1)$ solutions under the p-order weak monotonicity conditions.

Motivated by Wu and Zhang [20] and Fan [6], this paper is devoted to the results of them to the BDSDEs with p-order globally (locally) weak monotonicity condition. We will prove the existence and uniqueness result of BDSDEs in this weak assumption. The methods in this paper should be explained in two aspects: (i) Compared with BSDEs' situation, BDSDEs have two Brownian motions with two kinds of stochastic integrals. Conditional mathematical expectation can not make the integrals term w.r.t Brownian motions disappear simultaneously, and some standard approaches to deal with classical BSDEs can not be adapted effortless to the framework of BDSDEs, such as stopping time method. (ii) The existence and uniqueness result for solutions in p-order locally weak monotonicity situation is proved by the similar methods of Wu and Zhang [20] with a additional assumption $\rho_N(x) := \mu_N \cdot x + \rho(x)$. This assumption combines locally monotone and weak monotonicity conditions.

We present two main results in this paper. Theorem 4.1 deals with the existence and uniqueness result of L^p solutions for multidimensional BDSDEs under the p-order globally weak monotonicity conditions. A priori estimate and a truncation method are integrated together to derive the results. Theorem 5.1 investigates BDSDEs with p-order locally weak monotonicity conditions. Comparison theorem of L^p solutions for one-dimensional BDSDEs is also put forward and proved. As a byproduct of Theorem 5.1, Remark 5.1 extends the results of Zhu and Tian [22–23].

This paper is organized as follows. Some preliminaries are introduced in Section 2. We establish two priori estimates in Section 3. In Section 4, we give the existence and uniqueness theorem of L^p $(1 < p < 2)$ solutions for BDSDEs with p-order globally weak monotonicity coefficients. In Section 5, we present the existence and uniqueness theorem of L^p $(1 < p \le 2)$ solutions for BDSDEs with p-order locally weak monotonicity coefficients. In Section 6, we give the comparison theorem.

2 Preliminaries

Suppose that k and d are two positive integers. $|y|$ denotes the Euclidean norm of a vector $y \in \mathbb{R}^k$. $\langle x, y\rangle$ denotes the inner product of vectors $x, y \in \mathbb{R}^k$. For a $k \times d$ matrix z, $|z| := \sqrt{\mathrm{Tr}(zz^{\mathrm{T}})}$, where z^{T} is the transpose of z. $\mathbb{S}$ denotes the set of all nondecreasing and concave functions $\rho(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\rho(0) = 0$; $\forall\, x > 0$, $\rho(x) > 0$ and $\int_{0+} \frac{\mathrm{d}x}{\rho(x)} = +\infty$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. $(B_t)_{t\ge 0}$ and $(W_t)_{t\ge 0}$ are two mutually independent Brownian motions in this space. Suppose that $\mathcal{N}$ is the class of P-null sets of $\mathcal{F}$. $\forall\, t \in [0, T]$, $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$. For a process η,

$$\mathcal{F}_{s,t}^{\eta} := \sigma\{\eta_r - \eta_s,\ s \le r \le t\} \vee \mathcal{N}, \quad \mathcal{F}_t^{\eta} := \mathcal{F}_{0,t}^{\eta}, \quad s \in [0, t].$$

Especially, $\{\mathcal{F}_t, 0 \le t \le T\}$ is neither increasing nor decreasing, and it can not be a filtration. In this paper, a given real number $T > 0$ is the terminal time; random vector $\xi \in \mathbb{R}^k$ is $\mathcal{F}_T$-measurable; f, g are $(\mathcal{F}_t)$-measurable. $L^p(\mathcal{F}_T; \mathbb{R}^k)$ (or L^p) denotes the set of all $\mathbb{R}^k$-valued,

$\mathcal{F}_T$-measurable random vectors ξ such that $E[|\xi|^p] < \infty$. $S^p(0,T;\mathbb{R}^k)$ (or S^p) denotes the set of all $\mathbb{R}^k$-valued, $(\mathcal{F}_t)$-adapted and continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{S^p} := \Big(E\Big[\sup_{0\le t\le T}|Y_t|^p\Big]\Big)^{\frac{1}{p}} < \infty.$$

$M^p(0,T;\mathbb{R}^{k\times d})$ (or M^p) denotes the set of all $\mathbb{R}^{k\times d}$-valued, $(\mathcal{F}_t)$-progressively measurable processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{M^p} := \Big\{E\Big[\Big(\int_0^T |Z_t|^2 \mathrm{d}t\Big)^{\frac{p}{2}}\Big]\Big\}^{\frac{1}{p}} < \infty.$$

In the sequel, we introduce the following hypotheses.

(H1) $(g(t,0,0))_{t\in[0,T]} \in M^2$. And there exist constants $K>0$ and $0<\alpha<1$ such that for all $(y_i,z_i)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$, $i=1,2$, $\mathrm{d}P\times\mathrm{d}t$-a.e.,

$$|g(t,y_1,z_1)-g(t,y_2,z_2)|^2 \le K|y_1-y_2|^2+\alpha|z_1-z_2|^2.$$

(H2) For any given (ω,t), $f(\omega,t,\cdot,\cdot)$ is continuous.

(H3)$_p$ $E\big[|\xi|^p+\big(\int_0^T|f(t,0,0)|^2\mathrm{d}t\big)^{\frac{p}{2}}\big]<\infty.$

(H4) f satisfies γ-growth condition in (y,z), i.e., $\exists\ K>0$, $\gamma\in[0,1)$, such that $\forall\,(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$,

$$|f(t,y,z)|\le K(1+|y|^\gamma+|z|^\gamma).$$

Remark 2.1 According to inequality $|x|^\gamma\le 1+|x|$, $\gamma\in[0,1)$, (H4) implies that $|f(t,y,z)|\le K(3+|y|+|z|)$.

3 Priori Estimates

In this section, we establish two useful priori estimates. We first introduce the following assumptions.

(A1) $\mathrm{d}P\times\mathrm{d}t$-a.e., $\forall\,(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$,

$$\langle y, f(t,y,z)\rangle \le \theta|y|^2+\lambda|y||z|+|y|f_t+\upsilon_t,$$

where θ and λ are two non-negative constants, $(f_t)_{t\in[0,T]}$ and $(\upsilon_t)_{t\in[0,T]}$ are two non-negative, $(\mathcal{F}_t)$-progressively measurable processes with

$$E\Big[\Big(\int_0^T|f_t|^2\mathrm{d}t\Big)^{\frac{p}{2}}\Big]<+\infty\ ,\quad E\Big[\Big(\int_0^T\upsilon_t\mathrm{d}t\Big)^{\frac{p}{2}}\Big]<+\infty.$$

(A2) $\mathrm{d}P\times\mathrm{d}t$-a.e., $\forall\,(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$,

$$|y|^{p-1}\Big\langle\frac{y}{|y|}\mathbf{1}_{y\neq0}, f(t,y,z)\Big\rangle\le\varrho(|y|^p)+\lambda|y|^{p-1}|z|+|y|^{p-1}f_t,$$

where λ is a non-negative constant, $\varrho(\cdot) \in \mathbb{S}$, $(f_t)_{t\in[0,T]}$ is a non-negative $(\mathcal{F}_t)$-progressively measurable process and satisfies

$$E\Big[\Big(\int_0^T |f_t|^2 \mathrm{d}t\Big)^{\frac{p}{2}}\Big] < +\infty.$$

In the same way and steps of Proposition 3.1 in [23], it is not difficult to obtain the following proposition.

Proposition 3.1 *Suppose that $1 < p \le 2$, and* (A1), (H1) *and* $(\mathrm{H3})_p$ *hold. Let (Y, Z) be a solution of BDSDE* (1.1) *such that $Y \in S^p$. Then $Z \in M^p$. Moreover, $\forall\, t \in [0,T]$,*

$$\begin{aligned} E\Big[\Big(\int_t^T |Z_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] \le{}& d_{\theta,\lambda,K,p,T,\alpha} E\Big[\sup_{s\in[t,T]} |Y_s|^p\Big] + d_{\alpha,p} E\Big[\Big(\int_t^T \upsilon_s \mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\ &+ d_{\alpha,p} E\Big[\Big(\int_t^T |g(s,0,0)|^2 \mathrm{d}s\Big)^{\frac{p}{2}} + |\xi|^p\Big] \\ &+ d_{\alpha,p} E\Big[\Big(\int_t^T |f_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big], \end{aligned}$$

where $d_{\theta,\lambda,K,p,T,\alpha}$ is a non-negative constant depending on $(\theta, \lambda, K, p, T, \alpha)$, $d_{\alpha,p}$ is another non-negative constant depending on (α, p).

The proof of the following estimate is similar to that of Proposition 3.2 in [23], so we omit its partial proof.

Proposition 3.2 *Suppose that $1 < p \le 2$, and* (A2), (H1) *and* $(\mathrm{H3})_p$ *hold. Let (Y, Z) be an L^p solution of BDSDE* (1.1). *Then there exists a non-negative constant $d_{p,\lambda,\alpha,K,T}$ depending on $(p, \lambda, \alpha, K, T)$ such that $\forall\, t \in [0,T]$,*

$$\begin{aligned} E\Big[\sup_{s\in[t,T]} |Y_s|^p\Big] \le{}& d_{p,\lambda,\alpha,K,T} \Big\{ E\Big[\Big(\int_t^T |f_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] + \int_t^T \varrho(E[|Y_s|^p]) \mathrm{d}s \Big\} \\ &+ d_{p,\lambda,\alpha,K,T} E\Big[\int_t^T |Y_s|^{p-2} \mathbf{1}_{Y_s\neq 0} |g(s,0,0)|^2 \mathrm{d}s + |\xi|^p\Big]. \end{aligned}$$

Proof According to the proof of Proposition 3.2 in [23], we can easily get

$$\begin{aligned} E\Big[\sup_{s\in[t,T]} |Y_s|^p\Big] \le{}& h_{p,K,\alpha} E\Big[|\xi|^p + C_{\lambda,\alpha,p,K} \int_t^T |Y_s|^p \mathrm{d}s + p \int_t^T \varrho(|Y_s|^p) \mathrm{d}s\Big] \\ &+ \frac{p(p-1)}{2} \cdot \frac{1+\alpha}{1-\alpha} h_{p,K,\alpha} E\Big[\int_t^T |Y_s|^{p-2} \mathbf{1}_{Y_s\neq 0} |g(s,0,0)|^2 \mathrm{d}s\Big] \\ &+ h_{p,K,\alpha} E\Big[T^{1-\frac{p}{2}} \Big(\int_t^T |f_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big], \end{aligned}$$

where constant $h_{p,K,\alpha}$ depends on (p, K, α) and $C_{\lambda,\alpha,p,K} = \frac{2p\lambda^2}{(1-\alpha)(p-1)} + \frac{p(p-1)}{2} \cdot \frac{(1+\alpha)K}{2\alpha} + p - 1$.

Setting $h(t) = E\big[\sup_{s\in[t,T]} |Y_s|^p\big]$, using Fubini's theorem, the concavity of $\varrho(\cdot)$ and Jensen's inequality will derive that

$$h(t) \le h_{p,K,\alpha} E[|\xi|^p] + \frac{p(p-1)}{2} \cdot \frac{1+\alpha}{1-\alpha} h_{p,K,\alpha} E\Big[\int_t^T |Y_s|^{p-2} \mathbf{1}_{Y_s\neq 0} |g(s,0,0)|^2 \mathrm{d}s\Big]$$

$$+T^{1-\frac{p}{2}}h_{p,K,\alpha}E\Big[\Big(\int_t^T|f_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]+ph_{p,K,\alpha}\int_t^T\varrho(E[|Y_s|^p])\mathrm{d}s$$
$$+h_{p,K,\alpha}C_{\lambda,\alpha,p,K}\int_t^T h(s)\mathrm{d}s.$$

Applying Gronwall's inequality in the previous inequality, the proof of Proposition 3.2 can be completed.

4 BDSDEs with p-Order Globally Weak Monotonicity Coefficients

In this section, we study the BDSDEs with p-order globally weak monotonicity coefficients. The following assumptions will be used.

(H5) f is Lipschitz continuous in z, i.e., there exists a constant $K>0$ such that $\mathrm{d}P\times\mathrm{d}t$-a.e., $\forall\, y\in\mathbb{R}^k$, $z_1,z_2\in\mathbb{R}^{k\times d}$,

$$|f(t,y,z_1)-f(t,y,z_2)|\le K|z_1-z_2|.$$

$(\mathrm{H6})_p$ f satisfies p-order globally weak monotonicity condition in y, i.e., there exists a function $\rho(\cdot)\in\mathbb{S}$ s.t., $\mathrm{d}P\times\mathrm{d}t$-a.e., $\forall\, y_1,y_2\in\mathbb{R}^k$, $z\in\mathbb{R}^{k\times d}$,

$$|y_1-y_2|^{p-1}\Big\langle\frac{y_1-y_2}{|y_1-y_2|}\mathbf{1}_{|y_1-y_2|\neq0},f(t,y_1,z)-f(t,y_2,z)\Big\rangle\le\rho(|y_1-y_2|^p).$$

Remark 4.1 According to [6, Proposition 1], we can know that for any $1<p\le2$, $(\mathrm{H6})_p$ $\Rightarrow(\mathrm{H6})_2$.

Under the assumptions (H1)–(H2), $(\mathrm{H3})_2$, (H4)–(H5) and $(\mathrm{H6})_2$, Zhu and Tian [22] showed that BDSDE has a unique L^2 solution. The following theorem generalizes the result to L^p situation.

Theorem 4.1 *Let $1<p<2$. under the conditions* (H1)–(H2), $(\mathrm{H3})_p$, (H4)–(H5) *and* $(\mathrm{H6})_p$, *BDSDE* (1.1) *has a unique L^p solution.*

Proof We divide the proof into two steps.

Step 1 Let $1<p<2$. Under the conditions (H1)–(H2), $(\mathrm{H3})_p$, (H4)–(H5) and $(\mathrm{H6})_p$, we prove the existence of L^p solutions for BDSDE (1.1).

For any $n\ge1$, $x\in\mathbb{R}^k$, let $q_n(x):=\frac{xn}{|x|\vee n}$,

$$\xi^n:=q_n(\xi),\quad f^n(t,y,z):=f(t,y,z)-f(t,0,0)+q_n(f(t,0,0)).$$

Obviously, ξ^n,f^n satisfy (H2), $(\mathrm{H3})_2$, (H4)–(H5) and $(\mathrm{H6})_p$. Furthermore, by Remark 4.1, it implies that f^n also satisfies $(\mathrm{H6})_2$. Then by [22, Theorem 3.2], for each $n\ge1$, BDSDE (ξ^n,f^n,g,T) has a unique L^2 solution (Y^n,Z^n). Therefore, $(Y^n,Z^n)\in S^p\times M^p$.

On the other hand, by the definitions of ξ^n and $f^n(t,0,0)$ and the assumption $(\mathrm{H3})_p$, we know that

$$\lim_{n,m\to\infty}E\Big[|\xi^m-\xi^n|^p+\Big(\int_0^T|q_m(f(s,0,0))-q_n(f(s,0,0))|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]=0.\tag{4.1}$$

In the sequel, we will prove that $\{(Y_t^n, Z_t^n)_{t\in[0,T]}\}_{n=1}^{\infty}$ is a Cauchy sequence in $S^p \times M^p$. For any integrals $n, m \geq 1$, let (Y^{mn}, Z^{mn}) be a solution of the following BDSDE:

$$\begin{aligned} Y_t^{mn} = \xi^{mn} &+ \int_t^T f^{mn}(s, Y_s^{mn}, Z_s^{mn})\mathrm{d}s + \int_t^T g^{mn}(s, Y_s^{mn}, Z_s^{mn})\mathrm{d}B_s \\ &- \int_t^T Z_s^{mn}\mathrm{d}W_s, \end{aligned} \tag{4.2}$$

where $\xi^{mn} := \xi^m - \xi^n$, $Y_\cdot^{mn} := Y_\cdot^m - Y_\cdot^n$, $Z_\cdot^{mn} := Z_\cdot^m - Z_\cdot^n$,

$$\begin{aligned} f^{mn}(s, Y_s^{mn}, Z_s^{mn}) &:= f^m(s, Y_s^m, Z_s^m) - f^n(s, Y_s^n, Z_s^n), \\ g^{mn}(s, Y_s^{mn}, Z_s^{mn}) &:= g(s, Y_s^m, Z_s^m) - g(s, Y_s^n, Z_s^n). \end{aligned}$$

Assumptions (H5) and $(\mathrm{H6})_p$ yield that

$$\begin{aligned} &|Y_s^{mn}|^{p-1}\Big\langle \frac{Y_s^{mn}}{|Y_s^{mn}|}\mathbf{1}_{Y_s^{mn}\neq 0}, f^{mn}(s, Y_s^{mn}, Z_s^{mn})\Big\rangle \\ &\leq \rho(|Y_s^{mn}|^p) + K|Y_s^{mn}|^{p-1}|Z_s^{mn}| + |Y_s^{mn}|^{p-1}|q_m(f(s,0,0)) - q_n(f(s,0,0))|. \end{aligned}$$

Proposition 3.2 yields that there exists a constant $C > 0$ independent of n, m such that

$$\begin{aligned} E\Big[\sup_{r\in[t,T]}|Y_r^{mn}|^p\Big] &\leq CE\Big[|\xi^{mn}|^p + \Big(\int_0^T |q_m(f(s,0,0)) - q_n(f(s,0,0))|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\ &\quad + CE\Big[\int_0^T |Y_s^{mn}|^{p-2}\mathbf{1}_{Y_s^{mn}\neq 0}|g^{mn}(s,0,0)|^2\mathrm{d}s\Big] \\ &\quad + C\int_t^T \rho\Big(E\Big[\sup_{r\in[s,T]}|Y_r^{mn}|^p\Big]\Big)\mathrm{d}s. \end{aligned} \tag{4.3}$$

Noting that $g^{mn}(s,0,0) = 0$. Using (4.1), the fact that $\rho(\cdot)$ is of linear growth and Gronwall's inequality yields

$$\sup_{m,n\geq 1} E\Big[\sup_{r\in[0,T]}|Y_r^{mn}|^p\Big] < +\infty.$$

Thus, by taking limsup in (4.3) with respect to m and n, and by virtue of (4.1), Fatou's lemma, the continuity and monotonicity of $\rho(\cdot)$ and Bihari's inequality, we obtain

$$\lim_{m,n\to\infty} E\Big[\sup_{s\in[t,T]}|Y_s^m - Y_s^n|^p\Big] = 0.$$

On the other hand, f^n satisfies $(\mathrm{H6})_2$ and $\rho(\cdot)$ is of linear growth. Then for any $k \geq 1$, we have

$$\begin{aligned} &\langle Y_s^{mn}, f^{mn}(s, Y_s^{mn}, Z_s^{mn})\rangle \\ &\leq \rho(|Y_s^{mn}|^2) + K|Y_s^{mn}||Z_s^{mn}| + |q_m(f(s,0,0)) - q_n(f(s,0,0))||Y_s^{mn}| \\ &\leq K|Y_s^{mn}||Z_s^{mn}| + |q_m(f(s,0,0)) - q_n(f(s,0,0))||Y_s^{mn}| + \rho\Big(\frac{2A}{k+2A}\Big) \\ &\quad + (k+2A)|Y_s^{mn}|^2. \end{aligned}$$

Here, we use the estimate (see [6]): $\rho(x) \le (k+2A)x + \rho\big(\frac{2A}{k+2A}\big)$ for any $x \ge 0$ and $k \ge 1$, where A is the constant such that $\rho(x) \le A(1+x)$, $\forall x \ge 0$.

By Proposition 3.1, there exist constants $C_{k,K,p,T,\alpha}$ and $C_{\alpha,p}$ depending on (k,K,p,T,α) and (α,p) respectively such that

$$
\begin{aligned}
&E\Big[\Big(\int_0^T |Z_s^{mn}|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\
&\le C_{\alpha,p} E\Big[\Big(\int_0^T |q_m(f(s,0,0)) - q_n(f(s,0,0))|^2 \mathrm{d}s\Big)^{\frac{p}{2}} + |\xi^{mn}|^p\Big] \\
&\quad + C_{k,K,p,T,\alpha} E\Big[\sup_{t\in[0,T]} |Y_t^{mn}|^p\Big] + C_{\alpha,p} E\Big[\Big(\rho\Big(\frac{2A}{k+2A}\Big)\cdot T\Big)^{\frac{p}{2}}\Big] \\
&\quad + C_{\alpha,p} E\Big[\Big(\int_0^T |g^{mn}(s,0,0)|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big].
\end{aligned}
$$

In the previous inequality, first letting $n, m \to \infty$ and then $k \to \infty$, it implies

$$
\lim_{m,n\to\infty} E\Big[\Big(\int_0^T |Z_s^m - Z_s^n|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] = 0.
$$

Therefore, $\{(Y_t^n, Z_t^n)_{t\in[0,T]}\}_{n=1}^{\infty}$ is a Cauchy sequence in $S^p \times M^p$ with limit $(Y., Z.)$. The existence can be obtained.

Step 2 We prove the uniqueness. Let (Y^i, Z^i) be L^p solutions of BDSDE (1.1), $i = 1, 2$. (H5) and $(\mathrm{H6})_p$ imply that

$$
\begin{aligned}
&|Y_s^1 - Y_s^2|^{p-1}\Big\langle \frac{Y_s^1 - Y_s^2}{|Y_s^1 - Y_s^2|}\mathbf{1}_{Y_s^1 \ne Y_s^2}, f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2)\Big\rangle \\
&\le \rho(|Y_s^1 - Y_s^2|^p) + K|Y_s^1 - Y_s^2||Z_s^1 - Z_s^2|.
\end{aligned}
$$

According to Proposition 3.2, there exists a constant $C_{p,K,\alpha,T}$ depending on (p,K,α,T) such that

$$
E\Big[\sup_{s\in[t,T]} |Y_s^1 - Y_s^2|^p\Big] \le C_{p,K,\alpha,T} \int_t^T \rho\Big(E\Big[\sup_{r\in[s,T]} |Y_r^1 - Y_r^2|^p\Big]\Big)\mathrm{d}s.
$$

We get $Y^1 = Y^2$ by Bihari's inequality.

On the other hand, f satisfies $(\mathrm{H6})_2$, then

$$
\begin{aligned}
&\langle Y_s^1 - Y_s^2, f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2)\rangle \\
&\le \rho\Big(\frac{2A}{k+2A}\Big) + (k+2A)|Y_s^1 - Y_s^2|^2 + K|Y_s^1 - Y_s^2||Z_s^1 - Z_s^2|.
\end{aligned}
$$

Proposition 3.1 yields that there exist constants $C_{\alpha,p}$ and $C_{p,K,\alpha,T,k}$ depending on (α,p) and (p,K,α,T,k) respectively such that

$$
\begin{aligned}
&E\Big[\Big(\int_t^T |Z_s^1 - Z_s^2|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\
&\le C_{\alpha,p}\Big(\rho\Big(\frac{2A}{k+2A}\Big)\cdot T\Big)^{\frac{p}{2}} + C_{p,K,\alpha,T,k} E\Big[\sup_{s\in[t,T]} |Y_s^1 - Y_s^2|^p\Big]
\end{aligned}
$$

$$= C_{\alpha,p}\Big(\rho\Big(\frac{2A}{k+2A}\Big)\cdot T\Big)^{\frac{p}{2}}.$$

Letting $k \to \infty$ in previous inequality, we obtain the uniqueness of the solutions.

Now, we give an example of BDSDEs which satisfies the assumptions in Theorem 4.1.

Example 4.1 Assume that $k = 1$, $1 < p \le 2$, $\xi \in L^p$, $N \in \mathbb{N}$. Let $g(t,y,z) := y + 0.5z$, $f(t,y,z) := \sin|z| + h(|y|)$, where

$$h(y) = \begin{cases} \Big(1 - \dfrac{1}{y+1}\Big)^{\frac{1}{p}}, & y \ge 0, \\ 0, & y < 0. \end{cases}$$

Obviously, f, g satisfy (H1)–(H2) and $(\mathrm{H3})_p$. $\forall\, y, z$, $|f(t,y,z)| \le 3$, then f satisfies 0-growth condition in (y,z); $|f(t,y,z_1) - f(t,y,z_2)| \le |z_1 - z_2|$, then f satisfies Lipschitz condition in z;

$$(f(t,y_1,z) - f(t,y_2,z))\mathrm{sgn}(y_1 - y_2) \le |h(|y_1|) - h(|y_2|)| \le h(|y_1 - y_2|),$$

and $h(\cdot) \in \mathbb{S}$. Then f satisfies p-order globally weak monotonicity condition in y. Theorem 4.1 yields that BDSDE (1.1) with the above f, g, ξ has a unique L^p solution.

5 BDSDEs with p-Order Locally Weak Monotonicity Coefficients

In this section, we will extend the globally weak monotonicity condition to locally weak monotonicity condition. Let $g(t,0,0) \equiv 0$ for all $t \in [0,T]$ throughout this section. We first introduce the following assumptions.

(H5$'$) f satisfies locally Lipschitz condition in z, i.e., $\forall\, N \in \mathbb{N}$, $\exists\, L_N > 0$ such that for any y, z_1, z_2 with $|z_1|, |z_2|, |y| \le N$,

$$|f(t,y,z_1) - f(t,y,z_2)| \le L_N|z_1 - z_2|.$$

$(\mathrm{H6}')_p$ f satisfies p-order locally weak monotonicity condition in y, i.e., for any $N \in \mathbb{N}$, there exist functions $\rho_N(\cdot)$ such that for any z, y_1, y_2 with $|z|, |y_1|, |y_2| \le N$,

$$|y_1 - y_2|^{p-1}\Big\langle \frac{y_1 - y_2}{|y_1 - y_2|}\mathbf{1}_{y_1 \ne y_2}, f(t,y_1,z) - f(t,y_2,z)\Big\rangle \le \rho_N(|y_1 - y_2|^p),$$

where $\forall\, x \ge 0$, $\rho_N(x) := \mu_N \cdot x + \rho(x)$ with $\mu_N \in \mathbb{R}$, $\rho(\cdot) \in \mathbb{S}$.

The following lemma can be proved in a similar way as [20, Lemma 3.3], so we omit its proof.

Lemma 5.1 $\forall\, 1 < p \le 2$, *under the conditions* (H2), $(\mathrm{H3})_p$, (H4), (H5$'$) *and* $(\mathrm{H6}')_p$, *there exists a sequence* $\{f^n\}_{n=1}^{\infty}$ *such that*

(i) *for any given* n, ω, t, $f^n(t,\cdot,\cdot)$ *continuous*;

(ii) $\forall\, n$, $|f^n(t,y,z)| \le |f(t,y,z)| \le K(1 + |y|^\gamma + |z|^\gamma)$;

(iii) $\forall N$, $n\to\infty$, *then* $\Delta_N^p(f^n-f)\to 0$, *where*

$$\Delta_N^p(f):=E\Big[\Big(\int_0^T\sup_{|y|,|z|\le N}|f(t,y,z)|^2\mathrm{d}t\Big)^{\frac{p}{2}}\Big];$$

(iv) $\forall n$, f^n *satisfies* p*-order globally weak monotonicity condition in* y; *moreover, for any* n,N *with* $n\ge N$,

$$|y_1-y_2|^{p-1}\Big\langle\frac{y_1-y_2}{|y_1-y_2|}\mathbf{1}_{y_1\ne y_2},f^n(t,y_1,z)-f^n(t,y_2,z)\Big\rangle\le\rho_N(|y_1-y_2|^p),$$

where $\rho_N(\cdot)$ *is the same as the concave function in* $(\mathrm{H6}')_p$ *and* y_1,y_2,z *satisfy* $|y_1|,|y_2|,|z|\le N$;

(v) $\forall n$, f^n *satisfies globally Lipschitz condition in* z; *moreover, for any* n,N *with* $n\ge N$,

$$|f^n(t,y,z_1)-f^n(t,y,z_2)|\le L_N|z_1-z_2|,$$

where $L_N>0$ *and* y,z_1,z_2 *satisfy* $|y|,|z_1|,|z_2|\le N$.

The following two estimates are useful for dealing with the locally weak monotonicity coefficients.

Proposition 5.1 *Let* $1<p\le 2$. *Let* f_i,g *satisfy* (H1)–(H2), $(\mathrm{H3})_p$, (H4), (H5′), $(\mathrm{H6}')_p$. *Suppose that* (Y^i,Z^i) *is* L^p *solution of* BDSDE (ξ,f_i,g,T), $i=1,2$. *Then there exist non-negative constants* d_1,d_2 *independent of* N *such that*

$$\begin{aligned}&E\Big[\sup_{s\in[t,T]}|Y_s^1-Y_s^2|^p\Big]\\&\le d_1\times\Pi_1+d_2\times\Pi_2\times\int_t^T\overline{\rho}\Big(E\Big[\sup_{r\in[s,T]}|Y_r^1-Y_r^2|^p\Big]\Big)\mathrm{d}s,\end{aligned}\tag{5.1}$$

where

$$\Pi_1=N^{-p(1-\gamma)}+\Delta_N^p(f_1-f_2),\quad \Pi_2=1+\frac{2pL_N^2}{\varepsilon(p-1)}+p\mu_N^+$$

with $\overline{\rho}(x)=\rho(x)+x$, $x\ge 0$ *and* $0<\varepsilon<1-\alpha$.

Proof Note that $|f_i(t,y,z)|\le K(3+|y|+|z|)$ and $g(t,0,0)\equiv 0$. By the similar methods in Propositions 3.1–3.2, we can obtain that there exists a non-negative constant $C_{p,K,\alpha,T}$ depending on (p,K,α,T) such that

$$E\Big[\sup_{t\in[0,T]}|Y_t^i|^p+\Big(\int_0^T|Z_t^i|^2\mathrm{d}t\Big)^{\frac{p}{2}}\Big]\le C_{p,K,\alpha,T}E[1+|\xi|^p].\tag{5.2}$$

Let $(\overline{Y},\overline{Z})$ be an L^p solution of the following BDSDE:

$$\overline{Y}_t=\int_t^T\overline{f}(s,\overline{Y}_s,\overline{Z}_s)\mathrm{d}s+\int_t^T\overline{g}(s,\overline{Y}_s,\overline{Z}_s)\mathrm{d}B_s-\int_t^T\overline{Z}_s\mathrm{d}W_s,\quad 0\le t\le T,\tag{5.3}$$

where $\overline{Y}:=Y^1-Y^2$, $\overline{Z}:=Z^1-Z^2$,

$$\overline{f}(s,\overline{Y}_s,\overline{Z}_s):=f_1(s,Y_s^1,Z_s^1)-f_2(s,Y_s^2,Z_s^2),$$

$$\overline{g}(s,\overline{Y}_s,\overline{Z}_s) := g(s,Y_s^1,Z_s^1) - g(s,Y_s^2,Z_s^2).$$

Set

$$A := \{(\omega,s) \mid |Y_s^1| + |Y_s^2| + |Z_s^2| + |Z_s^2| \geq N\}, \quad \overline{A} := \Omega \backslash A.$$

Itô's formula yields that

$$\begin{aligned}
&E[|\overline{Y}_t|^p] + \frac{p(p-1)}{2} E\Big[\int_t^T |\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s\neq 0}|\overline{Z}_s|^2 \mathrm{d}s\Big] \\
\leq\ & \frac{p(p-1)}{2} K E\Big[\int_t^T |\overline{Y}_s|^p \mathrm{d}s\Big] + \frac{p(p-1)}{2}\alpha E\Big[\int_t^T |\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s\neq 0}|\overline{Z}_s|^2 \mathrm{d}s\Big] \\
&+ \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}_1 &= pE\Big[\int_t^T |\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s\neq 0}\langle \overline{Y}_s, f_1(s,Y_s^1,Z_s^1) - f_2(s,Y_s^2,Z_s^2)\rangle \mathbf{1}_A \mathrm{d}s\Big], \\
\mathrm{I}_2 &= pE\Big[\int_t^T |\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s\neq 0}\langle \overline{Y}_s, f_1(s,Y_s^1,Z_s^1) - f_1(s,Y_s^2,Z_s^1)\rangle \mathbf{1}_{\overline{A}} \mathrm{d}s\Big], \\
\mathrm{I}_3 &= pE\Big[\int_t^T |\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s\neq 0}\langle \overline{Y}_s, f_1(s,Y_s^2,Z_s^1) - f_1(s,Y_s^2,Z_s^2)\rangle \mathbf{1}_{\overline{A}} \mathrm{d}s\Big], \\
\mathrm{I}_4 &= pE\Big[\int_t^T |\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s\neq 0}\langle \overline{Y}_s, f_1(s,Y_s^2,Z_s^2) - f_2(s,Y_s^2,Z_s^2)\rangle \mathbf{1}_{\overline{A}} \mathrm{d}s\Big].
\end{aligned}$$

Applying Hölder's inequality and Young's inequality deduces

$$\begin{aligned}
\mathrm{I}_1 &\leq pE\Big[\Big(\int_t^T |\overline{Y}_s|^p \mathrm{d}s\Big)^{\frac{p-1}{p}}\Big(\int_t^T |\overline{f}(s,\overline{Y}_s,\overline{Z}_s)|^p \mathbf{1}_A \mathrm{d}s\Big)^{\frac{1}{p}}\Big] \\
&\leq (p-1)E\Big[\int_t^T |\overline{Y}_s|^p \mathrm{d}s\Big] + CN^{-p(1-\gamma)},
\end{aligned}$$

where $C > 0$ depends on $p, K, \alpha, T, \gamma, E[|\xi|^p]$. The above inequality comes from

$$|\overline{f}(s,\overline{Y}_s,\overline{Z}_s)|^p \leq C_{K,p}(1 + |Y_s^1|^{p\gamma} + |Z_s^1|^{p\gamma} + |Y_s^2|^{p\gamma} + |Z_s^2|^{p\gamma}),$$

$$\begin{aligned}
\mathbf{1}_A &\leq \Big(\frac{|Y_s^1| + |Y_s^2| + |Z_s^1| + |Z_s^2|}{N}\Big)^{p(1-\gamma)} \\
&\leq C_{p,\gamma}\cdot \frac{|Y_s^1|^{p(1-\gamma)} + |Z_s^1|^{p(1-\gamma)} + |Y_s^2|^{p(1-\gamma)} + |Z_s^2|^{p(1-\gamma)}}{N^{p(1-\gamma)}}
\end{aligned}$$

and (5.2), where $C_{K,p}$ and $C_{p,\gamma}$ are two positive constants related with (K,p) and (p,γ) respectively. By $(\mathrm{H6}')_p$, we obtain

$$\mathrm{I}_2 \leq pE\Big[\int_t^T \rho_N(|\overline{Y}_s|^p)\mathrm{d}s\Big] \leq pE\Big[\int_t^T \rho(|\overline{Y}_s|^p)\mathrm{d}s + \mu_N^+\int_t^T |\overline{Y}_s|^p \mathrm{d}s\Big].$$

From (H5′), there exists a constant $\varepsilon > 0$ such that

$$\mathrm{I}_3 \leq pE\Big[\Big(\int_t^T |\overline{Y}_s|^{\frac{p}{2}-1}\mathbf{1}_{\overline{Y}_s\neq 0}|\overline{Z}_s|\mathrm{d}s\Big)\Big(\int_t^T |\overline{Y}_s|^{\frac{p}{2}}\mathrm{d}s\Big)\Big]$$

$$\leq \frac{p(p-1)}{2}\varepsilon E\Big[\int_t^T |\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s\neq 0}|\overline{Z}_s|^2\mathrm{d}s\Big] + \frac{2pL_N^2}{\varepsilon(p-1)}E\Big[\int_t^T |\overline{Y}_s|^p\mathrm{d}s\Big].$$

In view of Hölder's inequality and Young's inequality again, it implies

$$\mathrm{I}_4 \leq (p-1)E\Big[\int_t^T |\overline{Y}_s|^p\mathrm{d}s\Big] + T^{1-\frac{p}{2}}\Delta_N^p(f_1-f_2).$$

Adding up the last five inequalities, we finally obtain that

$$\frac{p(p-1)}{2}(1-\alpha-\varepsilon)E\Big[\int_t^T |\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s\neq 0}|\overline{Z}_s|^2\mathrm{d}s\Big] \leq E[X_t],$$

where

$$\begin{aligned} X_t &= CN^{-p(1-\gamma)} + T^{1-\frac{p}{2}}\Delta_N^p(f_1-f_2) + pE\Big[\int_t^T \rho(|\overline{Y}_s|^p)\mathrm{d}s\Big] \\ &\quad + \Big(\frac{p(p-1)}{2}K + 2p - 2 + \frac{2pL_N^2}{\varepsilon(p-1)} + p\mu_N^+\Big)E\Big[\int_t^T |\overline{Y}_s|^p\mathrm{d}s\Big]. \end{aligned}$$

Therefore, by the method in Proposition 3.2, we obtain that there exists a non-negative constant $C_{\alpha,p,\varepsilon,K}$ depending on (α,p,ε,K) such that

$$E\Big[\sup_{s\in[t,T]}|\overline{Y}_s|^p\Big] \leq C_{\alpha,p,\varepsilon,K}E[X_t].$$

Setting $\overline{\rho}(x) = \rho(x) + x$, $x \geq 0$, the result (5.1) can be obtained by Jensen's inequality.

Proposition 5.2 *Let $1 < p \leq 2$. Let f_i, g satisfy* (H1)–(H2), (H3)$_p$, (H4), (H5′), (H6′)$_p$. *Suppose that (Y^i, Z^i) is L^p solutions of BDSDE (ξ, f_i, g, T), $i = 1, 2$. Then for any $k \geq 1$, there exists a constant $A > 0$ such that*

$$\begin{aligned} &E\Big[\Big(\int_t^T |Z_s^1 - Z_s^2|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\ &\leq d_3\Big\{N^{-p(1-\gamma)} + \Delta_N^p(f_1-f_2) + \Big(\rho\Big(\frac{2A}{k+2A}\Big)\Big)^{\frac{p}{2}}\Big\} \\ &\quad + d_4(1 + L_N^p + |\mu_N|^{\frac{p}{2}})E\Big[\sup_{s\in[t,T]}|Y_s^1 - Y_s^2|^p\Big], \end{aligned} \tag{5.4}$$

where non-negative constant d_3 is independent of N and k, d_4 is independent of N but depends on k.

Proof Let $(\overline{Y}, \overline{Z})$ be an L^p solution of (5.3). Set $A := \{(\omega, s) \mid |Y_s^1| + |Y_s^2| + |Z_s^2| + |Z_s^2| \geq N\}$, $\overline{A} := \Omega\backslash A$. Then (5.2) still holds. Itô's formula and (H1) yield

$$\begin{aligned} &(1-\alpha)^{\frac{p}{2}}\Big(\int_t^T |\overline{Z}_s|^2\mathrm{d}s\Big)^{\frac{p}{2}} \\ &\leq K^{\frac{p}{2}}\Big(\int_t^T |\overline{Y}_s|^2\mathrm{d}s\Big)^{\frac{p}{2}} + \Big|\int_t^T \langle\overline{Y}_s, \overline{Z}_s\mathrm{d}W_s\rangle\Big|^{\frac{p}{2}} \\ &\quad + \Big|\int_t^T \langle\overline{Y}_s, \overline{g}(s,\overline{Y}_s,\overline{Z}_s)\mathrm{d}B_s\rangle\Big|^{\frac{p}{2}} + \Big|\int_t^T \langle\overline{Y}_s, \overline{f}(s,\overline{Y}_s,\overline{Z}_s)\rangle\mathrm{d}s\Big|^{\frac{p}{2}}. \end{aligned} \tag{5.5}$$

$(\mathrm{H6}')_2$ holds for f_1 according to Remark 4.1. By the method similar to Proposition 5.1, we have

$$\begin{aligned}&E\Big[\Big|\int_t^T\langle\overline{Y}_s,\overline{f}(s,\overline{Y}_s,\overline{Z}_s)\rangle\mathrm{d}s\Big|^{\frac{p}{2}}\Big]\\ \le{}& T^{\frac{p}{2}}(|\mu_N|^{\frac{p}{2}}+2+\varepsilon^{-\frac{p}{2}}L_N^p)E\Big[\sup_{s\in[t,T]}|\overline{Y}_s|^p\Big]+CN^{-p(1-\gamma)}\\ &+\Delta_N^p(f_1-f_2)+\varepsilon^{\frac{p}{2}}E\Big[\Big(\int_t^T|\overline{Z}_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]+E\Big[\Big(\int_t^T\rho(|\overline{Y}_s|^2)\mathrm{d}s\Big)^{\frac{p}{2}}\Big].\end{aligned}$$

Note that $\rho(\cdot)$ is of linear growth. Then for any $k\ge 1$, there exists a constant $A>0$ such that

$$\Big(\int_t^T\rho(|\overline{Y}_s|^2)\mathrm{d}s\Big)^{\frac{p}{2}}\le((k+2A)T)^{\frac{p}{2}}\sup_{s\in[t,T]}|\overline{Y}_s|^p+\Big(\rho\Big(\frac{2A}{k+2A}\Big)\cdot T\Big)^{\frac{p}{2}}.$$

Therefore, $\forall\, 0<\varepsilon<\frac{1-\alpha}{9}$, BDG's inequality and Jensen's inequality yield

$$\begin{aligned}&[(1-\alpha)^{\frac{p}{2}}-3\varepsilon^{\frac{p}{2}}]E\Big[\Big(\int_t^T|\overline{Z}_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\ \le{}&\Big(\rho\Big(\frac{2A}{k+2A}\Big)\cdot T\Big)^{\frac{p}{2}}+\Delta_N^p(f_1-f_2)\\ &+C_{\alpha,p,K,T,\varepsilon,k}(1+L_N^p+|\mu_N|^{\frac{p}{2}})E\Big[\sup_{s\in[t,T]}|\overline{Y}_s|^p\Big]+CN^{-p(1-\gamma)},\end{aligned}$$

where $C_{\alpha,p,K,T,\varepsilon,k}>0$ depends on $(\alpha,p,K,T,\varepsilon,k)$.

In the sequel, motivated by Wu and Zhang [20, Theorem 3.2], we prove the following existence and uniqueness theorem by assuming a technical assumption (5.6).

Theorem 5.1 *Let* $1<p\le 2$, (H1)–(H2), $(\mathrm{H3})_p$, (H4), $(\mathrm{H5}')$, $(\mathrm{H6}')_p$ *hold and*

$$\lim_{N\to\infty}Q^{-1}(Q(d_1\times N^{-p(1-\gamma)})+d_2\times\Pi_2\times T)=0,\tag{5.6}$$

where $Q(u)=\int_{u_0}^u\frac{\mathrm{d}s}{\overline{\rho}(s)}$, $u_0>0$, $u\ge 0$ *with* $d_1,d_2,\overline{\rho}(\cdot)$ *and* Π_2 *in* (5.1). *Then BDSDE* (1.1) *has a unique solution in* $S^p\times M^p$.

Proof Let $(Y^i,Z^i),\ i=1,2$ are two solutions of BDSDE (1.1). By Lemma 1.1 and (5.1), we get

$$E\Big[\sup_{s\in[0,T]}|Y_s^1-Y_s^2|^p\Big]\le C_1\times Q^{-1}(Q(d_1\times N^{-p(1-\gamma)})+d_2\times\Pi_2\times T),$$

where $C_1>0$ is independent of N and $d_1,d_2,\Pi_2,Q(\cdot)$ are defined in (5.6). Letting $N\to\infty$ in previous inequality yields $Y^1=Y^2$. Furthermore,

$$\begin{aligned}&E\Big[\Big(\int_t^T|Z_s^1-Z_s^2|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\ \le{}& d_3\Big\{N^{-p(1-\gamma)}+\Big(\rho\Big(\frac{2A}{k+2A}\Big)\Big)^{\frac{p}{2}}\Big\}+d_4(1+L_N^p+|\mu_N|^{\frac{p}{2}})E\Big[\sup_{s\in[t,T]}|Y_s^1-Y_s^2|^p\Big]\end{aligned}$$

can be obtained by (5.4). Letting first $N \to \infty$ and then $k \to \infty$ in above inequality yields $Z^1 = Z^2$. The proof of uniqueness is completed.

In the sequel, we prove the existence. Let $\{f^n\}_{n\geq 1}$ be the approximation sequence in Lemma 5.1. Then for every n, f^n satisfies p-order globally weak monotonicity condition in y and globally Lipschitz condition in z. From Theorem 4.1, we have that BDSDE (ξ, f^n, g, T) has an L^p solution (Y^n, Z^n). For any $n \geq 1$, f^n satisfies (iv) and (v) in Lemma 5.1. According to Proposition 5.1 and Lemma 1.1, we can know that there exists a constant $C_2 > 0$ independent of m, n, N such that

$$\begin{aligned}
&E\Big[\sup_{s\in[0,T]} |Y_s^m - Y_s^n|^p\Big] \\
&\leq C_2 \times Q^{-1}(Q(d_1 \times (N^{-p(1-\gamma)} + \Delta_N^p(f^m - f^n))) + d_2 \times \Pi_2 \times T),
\end{aligned}$$

where $d_1, d_2, Q(\cdot), \Pi_2$ are defined in (5.6). First letting $m, n \to \infty$ and then $N \to \infty$ in previous inequality, by virtue of (5.6), we obtain

$$E\Big[\sup_{s\in[0,T]} |Y_s^m - Y_s^n|^p\Big] \to 0.$$

Furthermore, combined with (5.4), we have

$$\begin{aligned}
&E\Big[\Big(\int_t^T |Z_s^m - Z_s^n|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\
&\leq d_3\Big\{N^{-p(1-\gamma)} + \Delta_N^p(f^m - f^n) + \Big(\rho\Big(\frac{2A}{k+2A}\Big)\Big)^{\frac{p}{2}}\Big\} \\
&\quad + d_4(1 + L_N^p + |\mu_N|^{\frac{p}{2}})E\Big[\sup_{s\in[t,T]} |Y_s^m - Y_s^n|^p\Big].
\end{aligned}$$

Letting $m, n \to \infty$, $N \to \infty$, $k \to \infty$ respectively, we get

$$E\Big[\Big(\int_t^T |Z_s^n - Z_s^m|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] \to 0.$$

We have proved that $\{(Y_t^n, Z_t^n)_{t\in[0,T]}\}_{n=1}^{\infty}$ is a Cauchy sequence in $S^p \times M^p$ with limit (Y, Z).

In the sequel, we only need to prove that in $L^p(\Omega)$, as $n \to \infty$,

$$\int_t^T f^n(s, Y_s^n, Z_s^n)\mathrm{d}s \to \int_t^T f(s, Y_s, Z_s)\mathrm{d}s.$$

Let $A_n := \{(\omega, s) \mid |Y_s^n| + |Z_s^n| \geq N\}$, $\overline{A}_n = \Omega\backslash A_n$. Then

$$\begin{aligned}
&E\Big[\Big(\int_t^T |f^n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\
&\leq 2E\Big[\Big(\int_t^T |f^n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)|^2 (\mathbf{1}_{\overline{A}_n} + \mathbf{1}_{A_n})\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\
&\quad + 2E\Big[\Big(\int_t^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] = 2\mathrm{I}_1 + 2\mathrm{I}_2, \qquad (5.7)
\end{aligned}$$

where

$$
\begin{aligned}
\mathrm{I}_1 &= E\Big[\Big(\int_t^T |f^n(s,Y_s^n,Z_s^n)-f(s,Y_s^n,Z_s^n)|^2(\mathbf{1}_{\overline{A}_n}+\mathbf{1}_{A_n})\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\
&\leq \frac{C}{N^{p(1-\gamma)}}+\Delta_N^p(f^n-f)
\end{aligned} \tag{5.8}
$$

with a constant $C>0$ independent of n,N. Note that $\{(Y_t^n,Z_t^n)_{t\in[0,T]}\}_{n=1}^{\infty}$ is a Cauchy sequence in $S^p\times M^p$, then $Y_t^n\to Y_t, Z_t^n\to Z_t, \mathrm{d}P\times\mathrm{d}t$-a.s. As $n\to\infty$, the continuity of f and Lebesgue's dominated convergence theorem yield

$$
\mathrm{I}_2 = E\Big[\Big(\int_t^T |f(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\to 0.
$$

Hence, letting $n,N\to\infty$ in (5.7), it yields that

$$
E\Big[\Big(\int_t^T |f^n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\to 0.
$$

Therefore, the existence of L^p solutions can be obtained by taking limit in BDSDE (ξ,f^n,g,T).

Remark 5.1 In Theorem 5.1, $\rho_N(x)=\mu_N\cdot x+\rho(x)$, $x\geq 0$, $1<p\leq 2$. Assume that (H1)–(H2), $(\mathrm{H3})_p$, (H4), (H5′), $(\mathrm{H6}')_p$ hold. The existence and uniqueness result of L^p solutions in the following two special cases are corollaries of Theorem 5.1.

Case 1 $\rho_N(\cdot)=\rho(\cdot)$, $\mu_N=0$, $L_N=K$.

In this case, BDSDE (f,g,T,ξ) satisfies (H1)–(H2), $(\mathrm{H3})_p$, (H4)–(H5) and $(\mathrm{H6})_p$. Assumption (5.6) is reduced to

$$
\lim_{N\to\infty} Q_1^{-1}\Big(Q_1(d_1\times N^{-p(1-\gamma)})+d_2\times\Big(1+\frac{2pK^2}{\varepsilon(p-1)}\Big)\times T\Big)=0,
$$

where $Q_1(u)=\int_{u_0}^u\frac{\mathrm{d}s}{\rho(s)+s}$, $u_0>0$, $u\geq 0$. From $\int_{0+}\frac{\mathrm{d}s}{\rho(s)}=+\infty$, (5.6) holds.

This case is reduced to Theorem 4.1, and it generalizes the result of Zhu and Tian [22] to L^p situation.

Case 2 $\rho_N(x)=\mu_N\cdot x$, $\forall x>0$; $\rho(\cdot)=0$.

Assumption (5.6) is

$$
\lim_{N\to\infty} Q_2^{-1}(Q_2(d_1\times N^{-p(1-\gamma)})+d_2\times\Pi_2\times T)=0,
$$

where $Q_2(u)=\int_{u_0}^u\frac{\mathrm{d}s}{s}$, $u_0>0$, $u\geq 0$. Then $Q_2(u)=\ln\frac{u}{u_0}$ and $Q_2^{-1}(x)=u_0\exp(x)$.

Then (5.6) becomes

$$
\lim_{N\to\infty} u_0\exp\Big[\ln\frac{d_1\times N^{-p(1-\gamma)}}{u_0}+d_2\times\Pi_2\times T\Big]=0.
$$

In this situation, we can choose $d_2\times\Pi_2=K+2+\frac{2pL_N^2}{\varepsilon(p-1)}+p\mu_N^+$ in Proposition 5.1. Taking $\theta^{-1}=\frac{2p}{\varepsilon(p-1)}$, we equivalently have that

$$
\lim_{N\to\infty} N^{-p(1-\gamma)}\times\exp[(\theta^{-1}L_N^2+p\mu_N^+)\times T]=0.
$$

In fact, our condition is slightly weaker than [23, Theorem 5.1] since we only focus on Y in Proposition 5.1.

6 Comparison Theorem

We establish the comparison theorem of L^p solutions for one-dimensional BDSDE (1.1).

Theorem 6.1 *Let $1<p<2$, $\xi_1\le\xi_2$, and (Y^i,Z^i) be L^p solutions for* BDSDE (ξ_i,f_i,g,T), $i=1,2$. *Assume that one of the following conditions holds*:

(i) f_1 *satisfies* (H5), $(\text{H6})_p$, *and* $f_1(t,Y_t^2,Z_t^2)\le f_2(t,Y_t^2,Z_t^2)$, $\mathrm{d}P\times\mathrm{d}t$-a.e.;

(ii) f_2 *satisfies* (H5), $(\text{H6})_p$, *and* $f_1(t,Y_t^1,Z_t^1)\le f_2(t,Y_t^1,Z_t^1)$, $\mathrm{d}P\times\mathrm{d}t$-a.e.

Then, for any $t\in[0,T]$, $Y_t^1\le Y_t^2$, $\mathrm{d}P$-a.s.

Proof The comparison theorems under the above two conditions are proved in a same way, so we only give the proof under (i). Set $\overline{Y}:=Y^1-Y^2$, $\overline{Z}:=Z^1-Z^2$, $\overline{\xi}:=\xi_1-\xi_2$. Using Itô's formula and Tanaka's formula yield

$$\begin{aligned}
&(\overline{Y}_t^+)^p+\frac{p(p-1)}{2}\int_t^T|\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s>0}|\overline{Z}_s|^2\mathrm{d}s\\
\le{}&(\overline{\xi}^+)^p+\frac{p(p-1)}{2}\int_t^T|\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s>0}|g(s,Y_s^1,Z_s^1)-g(s,Y_s^2,Z_s^2)|^2\mathrm{d}s\\
&+p\int_t^T|\overline{Y}_s|^{p-1}\mathbf{1}_{\overline{Y}_s>0}[f_1(s,Y_s^1,Z_s^1)-f_2(s,Y_s^2,Z_s^2)]\mathrm{d}s\\
&+p\int_t^T|\overline{Y}_s|^{p-1}\mathbf{1}_{\overline{Y}_s>0}[g(s,Y_s^1,Z_s^1)-g(s,Y_s^2,Z_s^2)]\mathrm{d}B_s\\
&-p\int_t^T|\overline{Y}_s|^{p-1}\mathbf{1}_{\overline{Y}_s>0}\overline{Z}_s\mathrm{d}W_s.
\end{aligned}\tag{6.1}$$

According to $f_1(s,Y_s^2,Z_s^2)-f_2(s,Y_s^2,Z_s^2)\le0$, we have

$$\begin{aligned}
&f_1(s,Y_s^1,Z_s^1)-f_2(s,Y_s^2,Z_s^2)\\
\le{}&f_1(s,Y_s^1,Z_s^1)-f_1(s,Y_s^2,Z_s^1)+f_1(s,Y_s^2,Z_s^1)-f_1(s,Y_s^2,Z_s^2).
\end{aligned}$$

Note that f_1 satisfies (H5) and $(\text{H6})_p$. Then

$$\begin{aligned}
&p|\overline{Y}_s|^{p-1}\mathbf{1}_{\overline{Y}_s>0}[f_1(s,Y_s^1,Z_s^1)-f_2(s,Y_s^2,Z_s^2)]\\
\le{}&p\rho(\overline{Y}_s^+)^p+\frac{2pK^2}{(1-\alpha)(p-1)}(\overline{Y}_s^+)^p+\frac{p(1-\alpha)(p-1)}{8}|\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s>0}|\overline{Z}_s|^2.
\end{aligned}$$

From (6.1), $E[(\overline{\xi}^+)^p]=0$ and

$$\begin{aligned}
&|\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s>0}|g(s,Y_s^1,Z_s^1)-g(s,Y_s^2,Z_s^2)|^2\\
\le{}&K(\overline{Y}_s^+)^p+\frac{1+\alpha}{2}|\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s>0}|\overline{Z}_s|^2,
\end{aligned}$$

we have

$$E\Big[(\overline{Y}_t^+)^p+\frac{p(p-1)}{2}\int_t^T|\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s>0}|\overline{Z}_s|^2\mathrm{d}s\Big]$$

$$\leq \Big(\frac{p(1-\alpha)(p-1)}{8}+\frac{p(p-1)(1+\alpha)}{4}\Big)E\Big[\int_t^T |\overline{Y}_s|^{p-2}\mathbf{1}_{\overline{Y}_s>0}|\overline{Z}_s|^2\mathrm{d}s\Big]$$
$$+E\Big[\Big(\frac{p(p-1)}{2}K+\frac{2pK^2}{(1-\alpha)(p-1)}\Big)\int_t^T(\overline{Y}_s^+)^p\mathrm{d}s+p\int_t^T\rho((\overline{Y}_s^+)^p)\mathrm{d}s\Big].$$

By using the concavity of $\rho(\cdot)$ and Jensen's inequality, we obtain that there exists a constant $C_{p,K,\alpha}$ depending on (p,K,α) such that

$$E[(\overline{Y}_t^+)^p]\leq C_{p,K,\alpha}\int_t^T\overline{\rho}(E[(\overline{Y}_s^+)^p])\mathrm{d}s,$$

where $\overline{\rho}(x)=\rho(x)+x$. Using Bihari's inequality in previous inequality yields $E[(\overline{Y}_t^+)^p]=0$. Therefore, for any $t\in[0,T]$, $Y_t^1\leq Y_t^2$, $\mathrm{d}P$-a.s.

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