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Weighted Estimates of Variable Kernel Fractional Integral and Its Commutators on Vanishing Generalized Morrey Spaces with Variable Exponent*

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Abstract In this paper, the authors obtain the boundedness of the fractional integral operators with variable kernels on the variable exponent generalized weighted Morrey spaces and the variable exponent vanishing generalized weighted Morrey spaces. And the corresponding commutators generated by BMO function are also considered.

Keywords Fractional integral, Commutator, Variable kernel, Vanishing generalized weighted Morrey space with variable exponent, BMO space
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1 Introduction and Main Results

Let $\Omega(x,z')\in L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})$ $(1<d\le\infty)$ satisfying

$$\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}=\sup_{x\in\mathbb{R}^n}\Big(\int_{S^{n-1}}|\,\Omega(x,z')\,|^d\,\mathrm{d}z'\Big)^{\frac{1}{d}}<\infty; \tag{1.1}$$

$$\Omega(x,\iota z)=\Omega(x,z),\quad \int_{S^{n-1}}\Omega(x,z')\mathrm{d}z'=0\quad\text{for any } x,z\in\mathbb{R}^n,\ \iota>0, \tag{1.2}$$

where $S^{n-1}=\{x\in\mathbb{R}^n:|x|=1\}$ equipped with Lebesgue measure $\mathrm{d}z'$. For $0<\alpha<n$ and $d\ge1$, the fractional integral operator with variable kernel is defined by

$$T_{\Omega,\alpha}(f)(x)=\int_{\mathbb{R}^n}\frac{\Omega(x,x-y)}{|x-y|^{n-\alpha}}f(y)\mathrm{d}y. \tag{1.3}$$

In 1955, Calderón and Zygmund [1] investigated the L^2 boundedness of the singular integral operator with variable kernels. They found that these operators T_Ω are closely related to the problem about the second order linear elliptic equations with variable coefficients. They proved the following result.

Theorem A (see [1]) *Suppose that* $\Omega(x,z')\in L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})$ $\big(d>\frac{2(n-1)}{n}\big)$ *satisfies* (1.1)–(1.2). *Then there exists a constant* $C>0$ *independent of* f *such that*

$$\|T_\Omega f\|_{L^2}\le C\|f\|_{L^2}.$$

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In 1971, Muckenhoupt and Wheeden [2] gave the (L^p, L^q) boundedness of $T_{\Omega,\alpha}$.

Theorem B (see [2]) *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^d(S^{n-1})$ with $s > p'$. Then there exists a constant $C > 0$ independent of f such that*

$$\|T_{\Omega,\alpha} f\|_{L^q} \leq C\|f\|_{L^p}.$$

Suppose that $b \in L_{\rm loc}(\mathbb{R}^n)$, the corresponding m-order commutator generated by b and $T_{\Omega,\alpha}$ is defined by

$$T^m_{\Omega,\alpha,b}(f)(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n-\alpha}}[b(x) - b(y)]^m f(y){\rm d}y. \tag{1.4}$$

As it is known, in the last two decades there has been an increasing interest to the study of singular integral operators with variable kernels. For instance, Ding et al. [3] obtained the L^p boundedness of Marcinkiewicz integral operators μ_Ω with variable kernels; Chen and Ding [4] proved the L^p boundedness of Littlewood-Paley operator with variable kernel; Tao and Shao [5] proved the boundedness of Marcinkiewicz integral operator with variable kernel on the homogeneous Morrey-Herz spaces and the weak homogeneous Morrey-Herz spaces. Wang [6] proved the boundedness properties of singular integral operators T_Ω, fractional integral $T_{\Omega,\alpha}$ and parametric Marcinkiewicz integral μ^ρ_Ω with variable kernels on the Hardy spaces $H^p(\mathbb{R}^n)$ and weak Hardy spaces $WH^p(\mathbb{R}^n)$. Recently, Shao and Tao [7] obtained the boundedness of the fractional integral operators with variable kernels and its commutators on the variable exponent weak Morrey spaces as the infmum of exponent function $p(\cdot)$ equals 1. For further details on recent developments on this field, we refer the readers to [8–11].

After Kováčik and Rákosník [12] introduced the spaces $L^{p(x)}$ and $W^{k,p(x)}$ in high dimensional Euclidean spaces, many mathematicians have been involved in this field. The theory of function spaces with variable exponent has made great progress during the past 20 years. Due to their applications to PDE with nonstandard growth conditions and so on, we may refer to [13–16].

On the other hand, variable exponent Morrey spaces were introduced and studied in [17–18] in the Euclidean setting. Morrey type spaces have attracted considerable attention in recent years because the interesting norm includes explicitly both local and global information of the function. The authors of [19] established the boundedness of fractional integrals and oscillatory fractional integrals and their commutators on some generalized weighted Morrey spaces. Ho [20] gave some sufficient conditions for the boundedness of fractional integral operators and singular integral operators in Morrey space with variable exponent $\mathcal{M}_{p(\cdot),u}$, and he also obtained the weak type estimates of fractional integral operators on Morrey space with variable exponent (see [21]). In [22], Tao and Li proved the boundedness of Marcinkiewicz integral and its commutators on Morrey space with variable exponent. Guliyev [23] et al. obtained the boundedness of Riesz potential in the vanishing generalized weighted Morrey spaces with variable exponent. Long and Han [24] considered the boundedness of maximal operators, potential operators and singular integral operators on the vanishing generalized Morrey space with variable exponent.

Inspired by the statements above, in this paper, we continue to develop the results from [23–24]. The boundedness of the fractional integral operators with variable kernels and their commutators on the variable exponent generalized weighted Morrey spaces and the vanishing generalized weighted Morrey spaces with variable exponent were considered, where the smoothness condition on Ω has been removed.

Before stating the main results of this article, we first recall some necessary definitions and notations.

For any $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{z \in \mathbb{R}^n : |z-x| \le r\}$. $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^n$ and χ_E denotes its characteristic function.

Define $\mathcal{P}(\mathbb{R}^n)$ to be the set of $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ such that

$$1 < p_- := \operatorname{ess}\inf_{x\in\mathbb{R}^n} p(x), \quad p_+ := \operatorname{ess}\sup_{x\in\mathbb{R}^n} p(x) < \infty.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\Big\{\eta > 0 : \int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\eta}\Big)^{p(x)} \mathrm{d}x \le 1\Big\} < \infty. \tag{1.5}$$

It is easy to know that $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach function space when equipped with the Luxemburg-Nakano norm above.

Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$. The Hardy-Littlewood maximal operator is defined by

$$\mathrm{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\mathrm{d}y.$$

Given a measurable function b, the maximal commutator is defined by

$$\mathrm{M}_b f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x)-b(y)||f(y)|\mathrm{d}y.$$

For $0 \le \alpha < n$, fractional maximal operator with variable kernel is defined as

$$\mathrm{M}_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}} \int_{B(x,r)} |\Omega(x,x-y)||f(y)|\mathrm{d}y.$$

It is easy to see when $\Omega(x,y) = 1$, $\mathrm{M}_{\Omega,\alpha}$ is just the fractional maximal operator

$$\mathrm{M}_\alpha f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}} \int_{B(x,r)} |f(y)|\mathrm{d}y.$$

The sharp maximal function is defined by

$$\mathrm{M}^\sharp f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|\mathrm{d}y,$$

where $f_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\mathrm{d}y$.

Let $\mathcal{B}(\mathbb{R}^n)$ denote the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ which satisfies the following conditions

$$|p(x)-p(y)| \le \frac{C}{-\log(|x-y|)}, \quad |x-y| \le \frac{1}{2} \tag{1.6}$$

and

$$|p(x)-p(\infty)| \leq \frac{C}{\log(|x|+e)}, \quad x \in \mathbb{R}^n. \tag{1.7}$$

It is proved that the Hardy-Littlewood maximal operater M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ as $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ in [25].

Remark 1.1 For any $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $\lambda > 1$, by Jensen's inequality, we have $\lambda p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. See [26, Remark 2.13].

We say an order pair of variable exponents function $(p(\cdot), q(\cdot)) \in \mathcal{B}^{\alpha}(\mathbb{R}^n)$, if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < \alpha < \frac{n}{p_+}$ and

$$\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n} \tag{1.8}$$

with $\frac{q(\cdot)(n-\alpha)}{n} \in \mathcal{B}(\mathbb{R}^n)$.

Definition 1.1 *The space* $\mathrm{BMO}(\mathbb{R}^n)$ *consists of all functions* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *such that*

$$\|f\|_{\mathrm{BMO}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x) - f_{B(x,r)}| \mathrm{d}x < \infty,$$

where $f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\mathrm{d}y$.

Definition 1.2 *Define the* $\mathrm{BMO}_{p(\cdot),w}(\mathbb{R}^n)$ *space as the set of all functions* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *such that*

$$\|f\|_{\mathrm{BMO}_{p(\cdot),w}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{\|(f - f_{B(x,r)})\chi_{B(x,r)}\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}}.$$

Remark 1.2 Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and w be a Lebesgue measurable function. If $w \in A_{p(\cdot)}(\mathbb{R}^n)$, then the norms $\|\cdot\|_{\mathrm{BMO}_{p(\cdot),w}}$ and $\|\cdot\|_{\mathrm{BMO}}$ are mutually equivalent (see [27]).

Definition 1.3 (see [28]) *Let w be a positive, locally integrable function. We say that a weight function w belongs to the class $A_{p(\cdot)}(\mathbb{R}^n)$ if*

$$\sup_B \frac{1}{|B|} \|\chi_{B(x,r)} w\|_{L^{p(\cdot)}} \|\chi_{B(x,r)} w^{-1}\|_{L^{p'(\cdot)}} < \infty; \tag{1.9}$$

A weight function w belongs to the class $A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if

$$\sup_B |B|^{\frac{1}{p(x)} - \frac{1}{q(x)} - 1} \|\chi_{B(x,r)} w\|_{L^{q(\cdot)}} \|\chi_{B(x,r)} w^{-1}\|_{L^{p'(\cdot)}} < \infty. \tag{1.10}$$

Remark 1.3 Let $w \in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Then $w^{-1} \in A_{p'(\cdot),q'(\cdot)}(\mathbb{R}^n)$ (see [23]).

Definition 1.4 (see [23]) *Let $\lambda(\cdot) : \mathbb{R}^n \to (0, n)$ be a measurable function, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The Morrey space with variable exponents $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ and weighted Morrey space with variable exponents $\mathcal{L}^{p(\cdot),\lambda(\cdot)}_w(\mathbb{R}^n)$ are defined by*

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n) = \Big\{ f : \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}} = \sup_{z\in\mathbb{R}^n, r>0} r^{-\frac{\lambda(x)}{p(x)}} \|\chi_{B(x,r)} f\|_{L^{p(\cdot)}} < \infty \Big\},$$

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}_w(\mathbb{R}^n) = \Big\{ f : \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}_w} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda(x)}{p(x)}} \|\chi_{B(x,r)} f\|_{L^{p(\cdot)}_w(\mathbb{R}^n)} < \infty \Big\}.$$

Throughout this paper, $u(x,r)$, $u_1(x,r)$ and $u_2(x,r)$ are non-negative measurable functions on $\mathbb{R}^n \times (0,\infty)$.

Definition 1.5 (see [23]) *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $u(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. The variable exponent generalized Morrey space $\mathcal{M}^{p(\cdot),u}(\mathbb{R}^n)$ and variable exponent generalized weighted Morrey space $\mathcal{M}_w^{p(\cdot),u}(\mathbb{R}^n)$ are defined by*

$$\mathcal{M}^{p(\cdot),u} = \Big\{ f : \|f\|_{\mathcal{M}^{p(\cdot),u}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{u(x,r) r^{\theta_p(x,r)}} \|\chi_{B(x,r)} f\|_{L^{p(\cdot)}} < \infty \Big\},$$
$$\mathcal{M}_w^{p(\cdot),u} = \Big\{ f : \|f\|_{\mathcal{M}_w^{p(\cdot),u}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{u(x,r) \|\chi_{B(x,r)}\|_{L_w^{p(\cdot)}}} \|\chi_{B(x,r)} f\|_{L_w^{p(\cdot)}} < \infty \Big\},$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r \le 1, \\ \frac{n}{p(\infty)}, & r > 1. \end{cases}$

Remark 1.4 According to Definition 1.4, if $u(x,r) = r^{-\theta_p(x,r) + \frac{\lambda(x)}{p(x)}}$, then the variable exponent generalized Morrey space $\mathcal{M}^{p(\cdot),u}(\mathbb{R}^n)$ is exactly the Morrey space with variable exponent $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$.

Definition 1.6 (see [23]) *Let $u_1(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. The vanishing generalized weighted Morrey space with variable exponent $V\mathcal{M}_w^{p(\cdot),u}(\mathbb{R}^n)$ is defined as the space of functions $f \in \mathcal{M}_w^{p(\cdot),u}(\mathbb{R}^n)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{u_1(x,r)\|\chi_{B(x,r)} w\|_{L^{p(\cdot)}}} \|\chi_{B(x,r)} f\|_{L_w^{p(\cdot)}} = 0.$$

In this paper we assume that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{\inf\limits_{x\in\mathbb{R}^n} u(x,r)\|\chi_{B(x,r)}\|_{L_w^{p(\cdot)}}} = 0$$

and

$$\sup_{0<r<\infty} \frac{1}{\inf\limits_{x\in\mathbb{R}^n} u(x,r)\|\chi_{B(x,r)}\|_{L_w^{p(\cdot)}}} > 0,$$

which make the spaces $V\mathcal{M}_w^{p(\cdot),u}(\mathbb{R}^n)$ non-trivial.

The main results of this paper are stated as follows.

Theorem 1.1 *Suppose that $\Omega(x,z)$ satisfies* (1.1)–(1.2). *Let $0 < \alpha < n$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $p_+ < \frac{n}{\alpha}$, $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n}$. If $w \in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $u_1(x,t)$ and $u_2(x,t)$ satisfy the condition*

$$\int_t^\infty \frac{\operatorname{ess}\inf\limits_{s<r<\infty} u_1(x,r)\|\chi_{B(x,r)}\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}} \frac{\mathrm{d}s}{s} \le C u_2(x,t). \tag{1.11}$$

Then there exists a constant $C > 0$ such that for any $f \in \mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)$,

$$\|T_{\Omega,\alpha} f\|_{\mathcal{M}_w^{q(\cdot),u_2}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)}.$$

Theorem 1.2 *Suppose that* $\Omega(x,z)$ *satisfies* (1.1)–(1.2). *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \alpha < n$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $p_+ < \frac{n}{\alpha}$, $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n}$. *If* $w \in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $u_1(x,t)$ *and* $u_2(x,t)$ *satisfy the condition*

$$\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\frac{\operatorname{ess}\inf\limits_{s<r<\infty} u_1(x,r)\|\chi_{B(x,r)}\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s} \le Cu_2(x,t). \tag{1.12}$$

Then there exists a constant $C>0$ *such that for any* $f \in \mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)$,

$$\|T^m_{\Omega,\alpha,b}f\|_{\mathcal{M}_w^{q(\cdot),u_2}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)}.$$

Theorem 1.3 *Suppose that* $\Omega(x,z)$ *satisfies* (1.1)–(1.2). *Let* $0 < \alpha < n$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $p_+ < \frac{n}{\alpha}$, $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n}$. *If* $w \in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $u_1(x,t)$ *and* $u_2(x,t)$ *satisfy the condition*

$$\int_{\tau_0}^\infty \frac{\operatorname{ess}\inf\limits_{s<r<\infty} u_1(x,r)\|\chi_{B(x,r)}\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s} < \infty \tag{1.13}$$

for any $\tau_0 > 0$, *and*

$$\int_t^\infty \frac{\operatorname{ess}\inf\limits_{s<r<\infty} u_1(x,r)\|\chi_{B(x,r)}\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s} \le Cu_2(x,t). \tag{1.14}$$

Then there exists a constant $C>0$ *such that for any* $f \in V\mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)$,

$$\|T_{\Omega,\alpha}f\|_{V\mathcal{M}_w^{q(\cdot),u_2}(\mathbb{R}^n)} \le C\|f\|_{V\mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)}.$$

Theorem 1.4 *Suppose that* $\Omega(x,z)$ *satisfies* (1.1)–(1.2). *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \alpha < n$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $p_+ < \frac{n}{\alpha}$, $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n}$. *If* $w \in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $u_1(x,t)$ *and* $u_2(x,t)$ *satisfy the condition*

$$\int_\kappa^\infty \Big(1+\ln\frac{s}{t}\Big)\frac{\operatorname{ess}\inf\limits_{s<r<\infty} u_1(x,r)\|\chi_{B(x,r)}\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s} < \infty \tag{1.15}$$

for any $\kappa > 0$, *and*

$$\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\frac{\operatorname{ess}\inf\limits_{s<r<\infty} u_1(x,r)\|\chi_{B(x,r)}\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s} \le Cu_2(x,t). \tag{1.16}$$

Then there exists a constant $C>0$ *such that for any* $f \in V\mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)$,

$$\|T^m_{\Omega,\alpha,b}f\|_{V\mathcal{M}_w^{q(\cdot),u_2}(\mathbb{R}^n)} \le C\|f\|_{V\mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)}.$$

Throughout this paper, the letter C stands for a positive constant that is independent of the essential variables and not necessarily the same one in each occurrence.

2 Preliminaries Lemmas

In this section we shall give some lemmas which will be used in the proofs of our main theorems.

Lemma 2.1 (see [12]) (Generalized Hölder Inequality) *Let $q(\cdot):\mathbb{R}^n\to[1,\infty)$. If $f\in L^{q(\cdot)}(\mathbb{R}^n)$ and $g\in L^{q'(\cdot)}(\mathbb{R}^n)$, then f,g are integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n}|f(x)g(y)|\mathrm{d}x\le\ r_q\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)},$$

where $r_q=1+\frac{1}{q_-}-\frac{1}{q_+}$.

Lemma 2.2 *Suppose that $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $0<\alpha<n$, $\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\alpha}{n}$ and $p_+<\frac{n}{\alpha}$, $w\in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Then there exists a constant $C>0$ such that for any $f\in L_w^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|\mathrm{M}_\alpha f\|_{L_w^{q(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}.$$

By applying the similar method with the proof of [29], we can obtain the above result, the details are omitted here.

Lemma 2.3 (see [7]) *Suppose that $0<\theta<\min\{\alpha,n-\alpha\}$, $x\in\mathbb{R}^n$. Then*

$$|T_{\Omega,\alpha}f(x)|\le C(n,\alpha,\theta)[\mathrm{M}_{\Omega,\alpha-\theta}f(x)]^{\frac12}[\mathrm{M}_{\Omega,\alpha+\theta}f(x)]^{\frac12}.$$

Lemma 2.4 *Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $0<\alpha<n$, $\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\alpha}{n}$ and $p_+<\frac{n}{\alpha}$, $w\in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. If $\Omega(x,z)$ satisfies* (1.1)–(1.2)*, then there exists a constant $C>0$ such that for any $f\in L_w^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|\mathrm{M}_{\Omega,\alpha}f\|_{L_w^{q(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)},\tag{2.1}$$

$$\|T_{\Omega,\alpha}f\|_{L_w^{q(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}.\tag{2.2}$$

Proof We first prove (2.1). Let $f\in L_w^{p(\cdot)}(\mathbb{R}^n)$. Using Hölder's inequality, we obtain

$$\begin{aligned}|\mathrm{M}_{\Omega,\alpha}f(x)|&=\sup_{B\ni x}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B|\Omega(x,x-y)||f(y)|\mathrm{d}y\\&\le C\sup_{B\ni x}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\Big(\int_B|\Omega(x,x-y)|^d\mathrm{d}y\Big)^{\frac1d}\Big(\int_B|f(y)|^{d'}\mathrm{d}y\Big)^{\frac{1}{d'}}\\&\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\sup_{B\ni x}\frac{1}{|B|^{1-\frac{\alpha d'}{n}}}\Big(\int_B|f(y)|^{d'}\mathrm{d}y\Big)^{\frac{1}{d'}}\\&=C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}(\mathrm{M}_{\alpha d'}|f|^{d'}(x))^{\frac{1}{d'}},\end{aligned}$$

where $\frac1d+\frac{1}{d'}=1$.

According to above inequality, we have

$$\int_{\mathbb{R}^n}\Big(\frac{\mathrm{M}_{\Omega,\alpha}f}{\eta}\Big)^{q(x)}\mathrm{d}x\le C\int_{\mathbb{R}^n}\Big(\frac{\mathrm{M}_{\alpha d'}(|f|^{d'})}{\eta}\Big)^{\frac{q(x)}{d'}}\mathrm{d}x.$$

Noting that

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\alpha}{n},$$

so

$$\frac{d'}{p(x)}-\frac{d'}{q(x)}=\frac{\alpha d'}{n}.$$

Then we have

$$\Big\{\eta>0: C\int_{\mathbb{R}^n}\Big(\frac{\mathrm{M}_{\alpha d'}(|f|^{d'})}{\eta}\Big)^{\frac{q(x)}{d'}}\mathrm{d}x\le 1\Big\}\subseteq\Big\{\eta>0:\int_{\mathbb{R}^n}\Big(\frac{\mathrm{M}_{\Omega,\alpha}f}{\eta}\Big)^{q(x)}\mathrm{d}x\le 1\Big\}.$$

Therefore

$$\inf\Big\{\eta>0: C\int_{\mathbb{R}^n}\Big(\frac{\mathrm{M}_{\alpha d'}(|f|^{d'})}{\eta}\Big)^{\frac{q(x)}{d'}}\mathrm{d}x\le 1\Big\}\ge\inf\Big\{\eta>0:\int_{\mathbb{R}^n}\Big(\frac{\mathrm{M}_{\Omega,\alpha}f}{\eta}\Big)^{q(x)}\mathrm{d}x\le 1\Big\}.$$

By Lemma 2.2, we have

$$\begin{aligned}\|\mathrm{M}_{\Omega,\alpha}f(x)\|_{L_w^{q(\cdot)}(\mathbb{R}^n)}&\le C\|(\mathrm{M}_{\alpha d'}(|f|^{d'}))^{\frac{1}{d'}}w\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\le C\|\mathrm{M}_{\alpha d'}(|f|^{d'})w^{d'}\|^{\frac{1}{d'}}_{L^{\frac{q(\cdot)}{d'}}(\mathbb{R}^n)}\\&\le C\||f|^{d'}w^{d'}\|^{\frac{1}{d'}}_{L^{\frac{p(\cdot)}{d'}}(\mathbb{R}^n)}\\&\le C\|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Now we pay attention to the proof of (2.2).

It is enough to prove that the inequality

$$\Big\|wT_{\Omega,\alpha}\Big(\frac{f}{w}\Big)\Big\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

holds for every function f such that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C$.

Fix a θ with $0<\theta<\min\{\alpha,n-\alpha\}$ satisfying $1+\frac{\theta}{n}q_+<2$. Define $r(x)=\frac{2}{1+\frac{\theta q(x)}{n}}$. Thus, we have

$$\frac{1}{p(\cdot)}-\frac{1}{\frac{r(\cdot)q(\cdot)}{2}}=\frac{\alpha-\theta}{n},\quad \frac{1}{p(\cdot)}-\frac{1}{\frac{r'(\cdot)q(\cdot)}{2}}=\frac{\alpha+\theta}{n}. \tag{2.3}$$

By Lemmas 2.1 and 2.3, it has

$$\begin{aligned}\int_{\mathbb{R}^n}\Big|wT_{\Omega,\alpha}\Big(\frac{f(x)}{w}\Big)\Big|^{q(x)}\mathrm{d}x&\le C\int_{\mathbb{R}^n}\Big(w\Big[\mathrm{M}_{\Omega,\alpha-\theta}\Big(\frac{f(x)}{w}\Big)\Big]\Big)^{\frac{q(x)}{2}}\Big(w\Big[\mathrm{M}_{\Omega,\alpha+\theta}\Big(\frac{f(x)}{w}\Big)\Big]\Big)^{\frac{q(x)}{2}}\mathrm{d}x\\&\le C\Big\|w^{\frac{q(x)}{2}}\Big[\mathrm{M}_{\Omega,\alpha-\theta}\Big(\frac{f(x)}{w}\Big)\Big]^{\frac{q(x)}{2}}\Big\|_{L^{r(\cdot)}(\mathbb{R}^n)}\\&\quad\times\Big\|w^{\frac{q(x)}{2}}\Big[\mathrm{M}_{\Omega,\alpha+\theta}\Big(\frac{f(x)}{w}\Big)\Big]^{\frac{q(x)}{2}}\Big\|_{L^{r'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Without loss of generality, we may assume that the infimum is taken over values of η greater than 1. Since $\eta>1$ and $x\in\mathbb{R}^n$, $\eta^{\frac{2}{q(x)}}\ge\eta^{\frac{2}{q_+}}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(\frac{w^{\frac{q(x)}{2}}[\mathrm{M}_{\Omega,\alpha-\theta}(\frac{f(x)}{w})]^{\frac{q(x)}{2}}}{\eta}\Big)^{r(x)}\mathrm{d}x&=\int_{\mathbb{R}^n}\Big(\frac{w[\mathrm{M}_{\Omega,\alpha-\theta}(\frac{f(x)}{w})]}{\eta^{\frac{2}{q(x)}}}\Big)^{\frac{q(x)r(x)}{2}}\mathrm{d}x\\&\le\int_{\mathbb{R}^n}\Big(\frac{w[\mathrm{M}_{\Omega,\alpha-\theta}(\frac{f(x)}{w})]}{\eta^{\frac{2}{q_+}}}\Big)^{\frac{q(x)r(x)}{2}}\mathrm{d}x.\end{aligned}$$

Therefore, by (2.1) and (2.3), we can obtain

$$\Big\|w^{\frac{q(x)}{2}}\Big[\mathrm{M}_{\Omega,\alpha-\theta}\Big(\frac{f(x)}{w}\Big)\Big]^{\frac{q(x)}{2}}\Big\|_{L^{r(x)}(\mathbb{R}^n)}\le\Big\|w\Big[\mathrm{M}_{\Omega,\alpha-\theta}\Big(\frac{f(x)}{w}\Big)\Big]\Big\|_{L^{\frac{q(x)r(x)}{2}}(\mathbb{R}^n)}\le C\|f\|_{L^{p(x)}(\mathbb{R}^n)}.$$

Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(\frac{w^{\frac{q(x)}{2}}[\mathrm{M}_{\Omega,\alpha+\theta}(\frac{f(x)}{w})]^{\frac{q(x)}{2}}}{\eta}\Big)^{r'(x)}\mathrm{d}x&=\int_{\mathbb{R}^n}\Big(\frac{w[\mathrm{M}_{\Omega,\alpha+\theta}(\frac{f(x)}{w})]}{\eta^{\frac{2}{q(x)}}}\Big)^{\frac{q(x)r'(x)}{2}}\mathrm{d}x\\&\le\int_{\mathbb{R}^n}\Big(\frac{w[\mathrm{M}_{\Omega,\alpha+\theta}(\frac{f(x)}{w})]}{\eta^{\frac{2}{q_+}}}\Big)^{\frac{q(x)r'(x)}{2}}\mathrm{d}x.\end{aligned}$$

So it follows from (2.1) and (2.3) that

$$\Big\|w^{\frac{q(x)}{2}}\Big[\mathrm{M}_{\Omega,\alpha+\theta}\Big(\frac{f(x)}{w}\Big)\Big]^{\frac{q(x)}{2}}\Big\|_{L^{r'(x)}(\mathbb{R}^n)}\le\Big\|w\Big[\mathrm{M}_{\Omega,\alpha+\theta}\Big(\frac{f(x)}{w}\Big)\Big]\Big\|_{L^{\frac{q(x)r'(x)}{2}}(\mathbb{R}^n)}\le C\|f\|_{L^{p(x)}(\mathbb{R}^n)}.$$

Thus

$$\int_{\mathbb{R}^n}\Big|wT_{\Omega,\alpha}\Big(\frac{f(x)}{w}\Big)\Big|^{q(x)}\mathrm{d}x\le C\|f\|_{L^{p(x)}(\mathbb{R}^n)}.$$

Lemma 2.5 (see [30]) *Let v_1, v_2 and ϖ be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0}v_2(t)H_{\varpi}g(t)\le\sup_{t>0}v_1(t)g(t)$$

holds for some $C>0$ and all non-negative and non-decreasing g if and only if

$$\sup_{t>0}v_2(t)\int_t^{\infty}\frac{\varpi(s)\mathrm{d}s}{\sup\limits_{s<\sigma<\infty}v_1(\sigma)}<\infty,$$

where $H_{\varpi}g(t)=\int_t^{\infty}g(s)\varpi(s)\mathrm{d}s$.

Lemma 2.6 (see [31]) *Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$ and $w\in A_{p(\cdot)}(\mathbb{R}^n)$. Then there exists a constant $C>0$ independent of f such that*

$$\|f\|_{L_w^{p(\cdot)}}\le C\|\mathrm{M}^{\sharp}f\|_{L_w^{p(\cdot)}}.$$

Lemma 2.7 *Suppose that $\Omega(x,z)$ satisfies* (1.1)–(1.2). *If $1<s_1,s_2<\infty$, $b\in\mathrm{BMO}(\mathbb{R}^n)$, then there exists a constant $C>0$ independent of f such that, for any $f\in L^p(\mathbb{R}^n)$,*

$$\mathrm{M}^{\sharp}[T_{\Omega,\alpha,b}f(x)]\le C\|b\|_{\mathrm{BMO}}[(\mathrm{M}|T_{\Omega,\alpha}f(x)|^{s_1})^{\frac{1}{s_1}}+(\mathrm{M}_{s_2\alpha}|f(x)|^{s_2})^{\frac{1}{s_2}}].$$

With the similar argument in the proof of [32, Lemmas 2.4.1 and 3.5.1], it is easy to draw the above conclusion.

Lemma 2.8 (see [33]) *Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. Then $\mathrm{M}:L_w^{p(\cdot)}(\mathbb{R}^n)\to L_w^{p(\cdot)}(\mathbb{R}^n)$ if and only if $w\in A_{p(\cdot)}(\mathbb{R}^n)$.*

Lemma 2.9 *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \alpha < n$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $0 < \frac{\alpha}{n} \le \frac{1}{p_+}$. If $\Omega(x,z)$ satisfies (1.1)–(1.2), $q(\cdot)$ as defined in (1.8), $w \in A_{p(\cdot),q(\cdot)}$, then there exists a constant $C > 0$ such that for any $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|T_{\Omega,\alpha,b}^m f\|_{L_w^{q(\cdot)}} \le C\|f\|_{L_w^{p(\cdot)}}.$$

Proof Let $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$. Lemma 2.6 implies that

$$\|T_{\Omega,\alpha,b}^m f\|_{L_w^{q(\cdot)}} \le \|\mathrm{M}^\sharp(T_{\Omega,\alpha,b}^m f)\|_{L_w^{p(\cdot)}}.$$

By Lemma 2.7, we have

$$\begin{aligned}\|\mathrm{M}^\sharp(T_{\Omega,\alpha,b}^m f)\|_{L_w^{q(\cdot)}} &\le C\|b\|_{\mathrm{BMO}}^m \|(\mathrm{M}|T_{\Omega,\alpha} f|^{s_1})^{\frac{1}{s_1}} + (\mathrm{M}_{\alpha s_2}|f|^{s_2})^{\frac{1}{s_2}}\|_{L_w^{q(\cdot)}} \\ &\le C\|b\|_{\mathrm{BMO}}^m (\|(\mathrm{M}|T_{\Omega,\alpha} f|^{s_1})^{\frac{1}{s_1}}\|_{L_w^{q(\cdot)}} + \|(\mathrm{M}_{\alpha s_2}|f|^{s_2})^{\frac{1}{s_2}}\|_{L_w^{q(\cdot)}}).\end{aligned}$$

It follows from Lemmas 2.2 and 2.4 that

$$\begin{aligned}\|(\mathrm{M}|T_{\Omega,\alpha} f|^{s_1})^{\frac{1}{s_1}}\|_{L_w^{q(\cdot)}} &\le C\||T_{\Omega,\alpha} f|^{s_1}\|_{L_{s_1 w}^{\frac{q(\cdot)}{s_1}}}^{\frac{1}{s_1}} = \|T_{\Omega,\alpha} f\|_{L_w^{q(\cdot)}} \le C\|f\|_{L_w^{p(\cdot)}}, \\ \|(\mathrm{M}_{\alpha s_2}|f|^{s_2})^{\frac{1}{s_2}}\|_{L_w^{q(\cdot)}} &\le C\|f\|_{L_w^{p(\cdot)}}.\end{aligned}$$

Thus

$$\|T_{\Omega,\alpha,b}^m f\|_{L_w^{q(\cdot)}} \le C\|b\|_{\mathrm{BMO}}^m \|f\|_{L_w^{p(\cdot)}}.$$

Lemma 2.10 (see [23]) *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}(\mathbb{R}^n)$. Then M_b is bounded on $L_w^{p(\cdot)}(\mathbb{R}^n)$.*

Lemma 2.11 (see [34]) *Let v_1, v_2 and ϖ be weights on $(0,\infty)$ and v_1 be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_\varpi^\star g(t) \le \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g if and only if

$$\sup_{t>0} v_2(t) \int_t^\infty \Big(1 + \ln\frac{s}{t}\Big) \frac{\varpi(s)\mathrm{d}s}{\sup\limits_{s<\sigma<\infty} v_1(\sigma)} < \infty,$$

where $H_\varpi^\star g(t) = \int_t^\infty \big(1 + \ln\frac{s}{t}\big) g(s)\varpi(s)\mathrm{d}s$, $0 < t < \infty$.

3 Proofs of Theorems 1.1–1.4

Proof of Theorem 1.1 Let $f \in \mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)$. For any $t > 0$, write

$$f(x) = f_1(x) + f_2(x),$$

where $f_1 = f\chi_{B(x,2t)}$, $f_2 = f\chi_{B(x,2t)^c}$, and

$$\|\chi_{B(x,t)} T_{\Omega,\alpha} f(\cdot)\|_{L_w^{q(\cdot)}} = \|\chi_{B(x,t)} T_{\Omega,\alpha} f_1(\cdot)\|_{L_w^{q(\cdot)}} + \|\chi_{B(x,t)} T_{\Omega,\alpha} f_2(\cdot)\|_{L_w^{q(\cdot)}}$$

$$=: \mathrm{I}_1 + \mathrm{I}_2.$$

For I_1, Lemma 2.4 immediately implies that

$$\|\chi_{B(x,t)} T_{\Omega,\alpha} f_1\|_{L_w^{q(\cdot)}} \le \|T_{\Omega,\alpha} f_1\|_{L_w^{q(\cdot)}} \le C \|\chi_{B(x,2t)} f\|_{L_w^{p(\cdot)}}, \tag{3.1}$$

where the constant $C > 0$ is independent of f.

On the other hand, by (1.9)–(1.10) we have

$$\begin{aligned}
\|\chi_{B(x,2t)} f\|_{L_w^{p(\cdot)}} &\approx |B|^{-\frac{\alpha}{n}+1} \|\chi_{B(x,2t)} f\|_{L_w^{p(\cdot)}} \int_{2t}^{\infty} \frac{\mathrm{d}s}{s^{n-\alpha+1}} \\
&\le |B|^{-\frac{\alpha}{n}+1} \int_{2t}^{\infty} \|\chi_{B(x,s)} f\|_{L_w^{p(\cdot)}} \frac{\mathrm{d}s}{s^{n-\alpha+1}} \\
&\le C \|\chi_{B(x,t)} w\|_{L^{q(\cdot)}} \|\chi_{B(x,t)} w^{-1}\|_{L^{p'(\cdot)}} \int_t^{\infty} \frac{\|\chi_{B(x,s)} f\|_{L_w^{p(\cdot)}}}{s^{n-\alpha+1}} \mathrm{d}s \\
&\le C \|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}} \int_t^{\infty} \|\chi_{B(x,s)} w^{-1}\|_{L^{p'(\cdot)}} \|\chi_{B(x,s)} f\|_{L_w^{p(\cdot)}} \frac{\mathrm{d}s}{s^{n-\alpha+1}} \\
&\le C \|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}} \int_t^{\infty} \frac{\|\chi_{B(x,s)} f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}} \frac{\mathrm{d}s}{s}.
\end{aligned}$$

Taking into account that

$$\|\chi_{B(x,2t)} f\|_{L_w^{p(\cdot)}} \le C \|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}} \int_t^{\infty} \frac{\|\chi_{B(x,s)} f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}} \frac{\mathrm{d}s}{s}, \tag{3.2}$$

we have

$$\|\chi_{B(x,t)} T_{\Omega,\alpha} f_1\|_{L_w^{q(\cdot)}} \le C \|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}} \int_t^{\infty} \frac{\|\chi_{B(x,s)} f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}} \frac{\mathrm{d}s}{s}. \tag{3.3}$$

Now turn to estimate I_2. Note that $|x-z| \le t$, $|z-y| \ge 2t$, and $|z-y| \le 2|x-y| \le 3|z-y|$. By Hölder's inequality, we have

$$\begin{aligned}
|T_{\Omega,\alpha} f_2| &\le \int_{B(x,2t)^c} \frac{|\Omega(z,z-y)|}{|z-y|^{n-\alpha}} |f(y)| \mathrm{d}y \\
&\le C \int_{B(x,2t)^c} |\Omega(z,z-y)||f(y)| \Big(\int_{|z-y|}^{\infty} s^{\alpha-n-1} \mathrm{d}s \Big) \mathrm{d}y \\
&\le C \int_{2t}^{\infty} s^{\alpha-n-1} \int_{2t \le |z-y| \le s} |\Omega(z,z-y)||f(y)| \mathrm{d}y \mathrm{d}s \\
&\le C \int_{2t}^{\infty} s^{\alpha-n-1} \Big(\int_{2t \le |z-y| \le s} |\Omega(z,z-y)|^d \mathrm{d}y \Big)^{\frac{1}{d}} \|\chi_{B(x,s)} f\|_{L^{d'}} \mathrm{d}s.
\end{aligned}$$

Defined $p_1(\cdot)$ with $\frac{1}{p_1(\cdot)} + \frac{1}{p_1'(\cdot)} = 1$, and $\frac{1}{p_1(\cdot)} - \frac{1}{q_1(\cdot)} = \frac{d'\alpha}{n}$. (1.10) implies that

$$\|\chi_{B(x,s)} w^{-1}\|_{L^{p_1'(\cdot)}} \|\chi_{B(x,s)} w\|_{L^{q_1(\cdot)}} \le C |B(x,s)|^{1-\frac{d'\alpha}{n}}. \tag{3.4}$$

It follows from (3.4) and Lemma 2.1 that

$$
\begin{aligned}
|T_{\Omega,\alpha}f_2| &\le C\int_{2t}^{\infty} s^{\alpha-n-1}\Big(\int_{2t\le|z-y|\le s}|\Omega(z,z-y)|^d\mathrm{d}y\Big)^{\frac{1}{d}}\|\chi_{B(x,s)}f\|_{L^{d'}}\mathrm{d}s\\
&\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\\
&\quad\times\int_t^\infty s^{\alpha-n+\frac{n}{d}-1}\|\chi_{B(x,s)}|f|^{d'}w^{d'}\|_{L^{p_1(\cdot)}}^{\frac{1}{d'}}\|\chi_{B(x,s)}w^{-d'}\|_{L^{p_1'(\cdot)}}^{\frac{1}{d'}}\mathrm{d}s\\
&\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\\
&\quad\times\int_t^\infty s^{\alpha-\frac{n}{d'}-1}\|\chi_{B(x,s)}f\|_{L_w^{d'p_1(\cdot)}}\|\chi_{B(x,s)}w\|_{L^{d'q_1(\cdot)}}^{-1}|B(x,s)|^{\frac{1}{d'}-\frac{\alpha}{n}}\mathrm{d}s\\
&\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\int_t^\infty\|\chi_{B(x,s)}f\|_{L_w^{d'p_1(\cdot)}}\|\chi_{B(x,s)}w\|_{L^{d'q_1(\cdot)}}^{-1}\frac{\mathrm{d}s}{s}\\
&\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\int_t^\infty\|\chi_{B(x,s)}f\|_{L_w^{d'p_1(\cdot)}}\|\chi_{B(x,s)}\|_{L_w^{d'q_1(\cdot)}}^{-1}\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

Write $p(\cdot)=d'p_1(\cdot)$ and $q(\cdot)=d'q_1(\cdot)$, we have $\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\frac{\alpha}{n}$. Therefore

$$
\|\chi_{B(x,t)}T_{\Omega,\alpha}f_2\|_{L_w^{q(\cdot)}}\le C\|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}}\int_t^\infty\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}. \tag{3.5}
$$

From (3.3) and (3.5), we can obtain

$$
\|\chi_{B(x,t)}T_{\Omega,\alpha}f\|_{L_w^{q(\cdot)}}\le C\|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}}\int_t^\infty\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}. \tag{3.6}
$$

Let

$$
v_2(r)=\frac{1}{u_2(x,r)},\quad v_1(r)=u_1^{-1}(x,r)\|\chi_{B(x,r)}\|_{L_w^{q(\cdot)}}^{-1}
$$

and

$$
g(r)=\|\chi_{B(x,r)}f\|_{L_w^{p(\cdot)}},\quad \varpi(r)=r^{-1}\|\chi_{B(x,r)}\|_{L_w^{p(\cdot)}}^{-1}.
$$

Lemma 2.5 and (1.11) yield that

$$
\begin{aligned}
\|T_{\Omega,\alpha}f\|_{\mathcal{M}_w^{q(\cdot),u_2}} &\le C\sup_{x\in\mathbb{R}^n,t>0}\frac{1}{u_2(x,t)}\int_t^\infty\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\
&\le C\sup_{x\in\mathbb{R}^n,t>0}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{u_1(x,t)\|\chi_{B(x,t)}\|_{L_w^{p(\cdot)}}}\\
&=\|f\|_{\mathcal{M}_w^{p(\cdot),u_1}}.
\end{aligned}
$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2 Let $b\in\mathrm{BMO}(\mathbb{R}^n)$, $f\in\mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)$. As in the proof of Theorem 1.1, for any $t>0$, write

$$
f(x)=f\chi_{B(x,2t)}+f\chi_{B(x,2t)^c}=f_1(x)+f_2(x).
$$

Let us prove the following inequality:

$$\|\chi_{B(x,t)}T^m_{\Omega,\alpha,b}f\|_{L^{q(\cdot)}_w}\le C\|b\|^m_{\mathrm{BMO}}\|\chi_{B(x,t)}\|_{L^{q(\cdot)}_w}\int_t^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L^{p(\cdot)}_w}}{\|\chi_{B(x,s)}\|_{L^{q(\cdot)}_w}}\frac{\mathrm{d}s}{s}. \tag{3.7}$$

First we have that

$$\|\chi_{B(x,t)}T^m_{\Omega,\alpha,b}f\|_{L^{q(\cdot)}_w}\le\|\chi_{B(x,t)}T^m_{\Omega,\alpha,b}f_1\|_{L^{q(\cdot)}_w}+\|\chi_{B(x,t)}T^m_{\Omega,\alpha,b}f_2\|_{L^{q(\cdot)}_w}.$$

By Lemma 2.9, we can obtain

$$\|\chi_{B(x,t)}T^m_{\Omega,\alpha,b}f_1\|_{L^{q(\cdot)}_w}\le C\|b\|^m_{\mathrm{BMO}}\|f_1\|_{L^{p(\cdot)}_w}\le C\|b\|^m_{\mathrm{BMO}}\|\chi_{B(x,2t)}f\|_{L^{p(\cdot)}_w}, \tag{3.8}$$

where C is a constant independent of f.

From (3.5), we get

$$\|\chi_{B(x,2t)}f\|_{L^{p(\cdot)}_w}\le C\|\chi_{B(x,t)}\|_{L^{q(\cdot)}_w}\int_t^\infty\frac{\|\chi_{B(x,s)}f\|_{L^{p(\cdot)}_w}}{\|\chi_{B(x,s)}\|_{L^{q(\cdot)}_w}}\frac{\mathrm{d}s}{s}.$$

Then

$$\|\chi_{B(x,t)}T^m_{\Omega,\alpha,b}f_1\|_{L^{q(\cdot)}_w}\le C\|b\|^m_{\mathrm{BMO}}\|\chi_{B(x,t)}\|_{L^{q(\cdot)}_w}\int_t^\infty\frac{\|\chi_{B(x,s)}f\|_{L^{p(\cdot)}_w}}{\|\chi_{B(x,s)}\|_{L^{q(\cdot)}_w}}\frac{\mathrm{d}s}{s}. \tag{3.9}$$

Noting that $|x-z|\le t$, $|x-z|\ge 2t$, we have $|z-y|\le 2|x-y|\le 3|z-y|$, therefore

$$\begin{aligned}|T^m_{\Omega,\alpha,b}f_2|&\le\int_{B(x,2t)^c}\frac{|\Omega(z,z-y)|}{|z-y|^{n-\alpha}}|b(y)-b(z)|^m|f(y)|\mathrm{d}y\\&\le\int_{B(x,2t)^c}|\Omega(z,z-y)||b(y)-b(z)|^m|f(y)|\Big(\int_{|z-y|}^\infty s^{\alpha-n-1}\mathrm{d}s\Big)\mathrm{d}y\\&\le C\int_{2t}^\infty s^{\alpha-n-1}\Big(\int_{2t\le|z-y|\le s}|\Omega(z,z-y)||b(y)-b_{B(x,t)}|^m|f(y)|\mathrm{d}y\Big)\mathrm{d}s\\&\quad+C|b(z)-b_{B(x,t)}|^m\int_{2t}^\infty s^{\alpha-n-1}\Big(\int_{2t\le|z-y|\le s}|\Omega(z,z-y)||f(y)|\mathrm{d}y\Big)\mathrm{d}s\\&=:\mathrm{J}_1+\mathrm{J}_2.\end{aligned}$$

To estimate J_1, we have

$$\begin{aligned}\mathrm{J}_1&=C\int_{2t}^\infty s^{\alpha-n-1}\Big(\int_{2t\le|z-y|\le s}|\Omega(z,z-y)||b(y)-b_{B(x,t)}|^m|f(y)|\mathrm{d}y\Big)\mathrm{d}s\\&\le C\int_t^\infty s^{\alpha-n-1}\Big(\int_{B(x,s)}|b(y)-b_{B(x,s)}|^m|\Omega(z,z-y)||f(y)|\mathrm{d}y\Big)\mathrm{d}s\\&\quad+C\int_t^\infty s^{\alpha-n-1}|b_{B(x,t)}-b_{B(x,s)}|^m\Big(\int_{B(x,s)}|\Omega(z,z-y)||f(y)|\mathrm{d}y\Big)\mathrm{d}s\\&=\mathrm{J}_{11}+\mathrm{J}_{12}.\end{aligned}$$

For J_{11}, we can obtain by using Hölder's inequality and Lemma 2.1 that

$$\mathrm{J}_{11}\le C\int_t^\infty s^{\alpha-n-1}\Big(\int_{2t\le|z-y|\le s}|\Omega(z,z-y)|^d\mathrm{d}y\Big)^{\frac{1}{d}}$$

$$
\begin{aligned}
&\times\Big(\int_{2t\le|z-y|\le s}|b(y)-b_{B(x,s)}|^{d'm}|f(y)|^{d'}\mathrm{d}y\Big)^{\frac{1}{d'}}\mathrm{d}s\\
&\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\\
&\quad\times\int_t^\infty s^{\alpha-n+\frac{n}{d}-1}\|\chi_{B(x,s)}(b(\cdot)-b_{B(x,s)})^{d'm}w^{-d'}\|_{L^{p_1'(\cdot)}}^{\frac{1}{d'}}\|\chi_{B(x,s)}|f|^d w^{d'}\|_{L^{p_1(\cdot)}}^{\frac{1}{d'}}\mathrm{d}s\\
&\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\\
&\quad\times\int_t^\infty s^{\alpha-n+\frac{n}{d}-1}\|\chi_{B(x,s)}(b(\cdot)-b_{B(x,s)})^{m}\|_{L_{w^{-1}}^{d'p_1'(\cdot)}}\|\chi_{B(x,s)}f\|_{L_w^{d'p_1(\cdot)}}\mathrm{d}s\\
&\le C\|b\|_{\mathrm{BMO}}^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\\
&\quad\times\int_t^\infty s^{\alpha-n+\frac{n}{d}-1}\|\chi_{B(x,s)}\|_{L_{w^{-1}}^{d'p_1'(\cdot)}}\|\chi_{B(x,s)}f\|_{L_w^{d'p_1(\cdot)}}\mathrm{d}s.
\end{aligned}
$$

Noting that $p(\cdot)=d'p_1(\cdot)$, $q(\cdot)=d'q_1(\cdot)$ and $\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\frac{\alpha}{n}$, we have

$$
\begin{aligned}
\|\chi_{B(x,s)}\|_{L_{w^{-1}}^{d'p_1'(\cdot)}}&=\|\chi_{B(x,s)}w^{-1}\|_{L^{d'p_1'(\cdot)}}=\|\chi_{B(x,s)}w^{-d'}\|_{L^{p_1'(\cdot)}}^{\frac{1}{d'}}\\
&\le C\|\chi_{B(x,s)}w^{d'}\|_{L^{q_1(\cdot)}}^{-\frac{1}{d'}}|B(x,s)|^{(1-\frac{1}{p_1(\cdot)}+\frac{1}{q_1(\cdot)})\frac{1}{d'}}\\
&=C\|\chi_{B(x,s)}w\|_{L^{d'q_1(\cdot)}}^{-1}|B(x,s)|^{\frac{1}{d'}-(\frac{1}{d'p_1(\cdot)}-\frac{1}{d'q_1(\cdot)})}\\
&=Cs^{\frac{n}{d'}-\frac{\alpha}{n}}\|\chi_{B(x,s)}\|_{L_w^{d'q_1(\cdot)}}^{-1}.
\end{aligned}
$$

It follows from the above inequality that

$$
\begin{aligned}
\mathrm{J}_{11}&\le C\|b\|_{\mathrm{BMO}}^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\\
&\quad\times\int_t^\infty s^{\alpha-n+\frac{n}{d}-1}\|\chi_{B(x,s)}\|_{L_{w^{-1}}^{d'p_1'(\cdot)}}\|\chi_{B(x,s)}f\|_{L_w^{d'p_1(\cdot)}}\mathrm{d}s\\
&\le C\|b\|_{\mathrm{BMO}}^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\\
&\quad\times\int_t^\infty s^{\alpha-n+\frac{n}{d}-1}s^{\frac{n}{d'}-\alpha}\|\chi_{B(x,s)}\|_{L_w^{d'q_1(\cdot)}}^{-1}\|\chi_{B(x,s)}f\|_{L_w^{d'p_1(\cdot)}}\mathrm{d}s\\
&\le C\|b\|_{\mathrm{BMO}}^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\int_t^\infty\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}^{-1}\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

For J_{12}, the Hölder's inequality assures that

$$
\begin{aligned}
\mathrm{J}_{12}&=C\int_t^\infty s^{\alpha-n-1}|b_{B(x,t)}-b_{B(x,s)}|^m\Big(\int_{B(x,s)}|\Omega(z,z-y)||f(y)|\mathrm{d}y\Big)\mathrm{d}s\\
&\le C\|b\|_{\mathrm{BMO}}^m\int_t^\infty s^{\alpha-n-1}\int_{B(x,s)}|\Omega(z,z-y)||f(y)|\mathrm{d}yds\\
&\le C\|b\|_{\mathrm{BMO}}^m\int_t^\infty s^{\alpha-n-1}\Big(\int_{B(x,s)}|\Omega(z,z-y)|^d\mathrm{d}y\Big)^{\frac{1}{d}}\Big(\int_{B(x,s)}|f(y)|^{d'}\mathrm{d}y\Big)^{\frac{1}{d'}}\mathrm{d}s\\
&\le C\|b\|_{\mathrm{BMO}}^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\\
&\quad\times\int_t^\infty s^{\alpha-n+\frac{n}{d}-1}\Big(\int_{B(x,s)}|f(y)|^{d'}w^{d'}w^{-d'}\mathrm{d}y\Big)^{\frac{1}{d'}}\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|b\|_{\mathrm{BMO}}^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\\
&\quad\times\int_t^\infty s^{\alpha-n+\frac{n}{d}-1}\|\chi_{B(x,s)}w^{-d'}\|_{L^{p_1'(\cdot)}}^{\frac{1}{d'}}\|\chi_{B(x,s)}f\|_{L_w^{d'p_1(\cdot)}}\frac{\mathrm{d}s}{s}\\
&\leq C\|b\|_{\mathrm{BMO}}^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\int_t^\infty \ln\frac{s}{t}\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}^{-1}\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

By the estimates of J_{11} and J_{12}, we can get

$$
\begin{aligned}
\mathrm{J}_1&\leq C\|b\|_{\mathrm{BMO}}^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\int_t^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\
&\leq C\int_t^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

Now turn to estimate J_2. By Hölder's inequality,

$$
\begin{aligned}
\mathrm{J}_2&\leq C|b(z)-b_{B(x,t)}|^m\Big(\int_{2t}^\infty s^{\alpha-n-1}\int_{2t\leq|z-y|\leq s}|\Omega(z,z-y)||f(y)|\mathrm{d}y\Big)\mathrm{d}s\\
&\leq C\Big(\frac{1}{|B(x,t)|}\int_{B(x,t)}|b(z)-b(y)|\mathrm{d}y\Big)^m\\
&\quad\times\int_{2t}^\infty s^{\alpha-n-1}\Big(\int_{B(x,s)}|\Omega(z,z-y)|^d\mathrm{d}y\Big)^{\frac{1}{d}}\Big(\int_{B(x,s)}|f(y)|^{d'}\mathrm{d}y\Big)^{\frac{1}{d'}}\mathrm{d}s\\
&\leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\mathrm{M}_b^m\chi_{B(x,t)}(z)\\
&\quad\times\int_t^\infty s^{\alpha-n+\frac{n}{d}-1}\Big(\int_{B(x,s)}|f(y)|^{d'}w^{d'}w^{-d'}\mathrm{d}y\Big)^{\frac{1}{d'}}\mathrm{d}s\\
&\leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^d(S^{n-1})}\mathrm{M}_b^m\chi_{B(x,t)}(z)\\
&\quad\times\int_t^\infty s^{\alpha-n+\frac{n}{d}-1}\|\chi_{B(x,s)}w^{-d'}\|_{L^{p_1'(\cdot)}}^{\frac{1}{d'}}\|\chi_{B(x,s)}f\|_{L_w^{d'p_1(\cdot)}}\frac{\mathrm{d}s}{s}\\
&\leq C\mathrm{M}_b^m\chi_{B(x,t)}(z)\int_t^\infty\ln\frac{s}{t}\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}^{-1}\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}\frac{\mathrm{d}s}{s}\\
&\leq C\mathrm{M}_b^m\chi_{B(x,t)}(z)\int_t^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s},
\end{aligned}
$$

where $C>0$ is the constant independent of x and t.

Combining estimates of J_1 and J_2 yields that

$$
\begin{aligned}
\|\chi_{B(x,t)}T_{\Omega,\alpha,b}^m f_2\|_{L_w^{q(\cdot)}}&\leq\|\chi_{B(x,t)}\mathrm{J}_1\|_{L_w^{q(\cdot)}}+\|\chi_{B(x,t)}\mathrm{J}_2\|_{L_w^{q(\cdot)}}\\
&\leq C\|b\|_{\mathrm{BMO}}^m\|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}}\int_t^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\
&\quad+C\|\mathrm{M}_b^m\chi_{B(x,t)}\|_{L_w^{q(\cdot)}}\int_t^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

By Lemma 2.10, we have

$$
\|\mathrm{M}_b^m\chi_{B(x,t)}\|_{L_w^{q(\cdot)}}\leq C\|b\|_{\mathrm{BMO}}^m\|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}}.
$$

Hence

$$\|\chi_{B(x,t)}T^m_{\Omega,\alpha,b}f_2\|_{L^{q(\cdot)}_w} \le C\|b\|^m_{\mathrm{BMO}}\|\chi_{B(x,t)}\|_{L^{q(\cdot)}_w}\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L^{p(\cdot)}_w}}{\|\chi_{B(x,s)}\|_{L^{q(\cdot)}_w}}\frac{\mathrm{d}s}{s}.$$

This finishes the proof of (3.7).

Note that $w \in A_{p(\cdot),q(\cdot)}$. Let

$$v_2(r) = \frac{1}{u_2(x,r)}, \quad v_1(r) = u_1^{-1}(x,r)\|\chi_{B(x,r)}\|^{-1}_{L^{q(\cdot)}_w}$$

and

$$g(r) = \|\chi_{B(x,r)}f\|_{L^{p(\cdot)}_w}, \quad \varpi(r) = r^{-1}\|\chi_{B(x,r)}\|^{-1}_{L^{p(\cdot)}_w}.$$

Then by Lemma 2.11 and (1.12), we can obtain

$$\begin{aligned}\|T^m_{\Omega,\alpha,b}f\|_{\mathcal{M}^{q(\cdot),u_2}_w} &\le C\|b\|^m_{\mathrm{BMO}}\sup_{x\in\mathbb{R}^n,t>0}\frac{1}{u_2(x,t)}\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L^{p(\cdot)}_w}}{\|\chi_{B(x,s)}\|_{L^{q(\cdot)}_w}}\frac{\mathrm{d}s}{s}\\ &\le C\|b\|^m_{\mathrm{BMO}}\sup_{x\in\mathbb{R}^n,t>0}\frac{1}{u_1(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L^{p(\cdot)}_w}}{\|\chi_{B(x,t)}\|_{L^{p(\cdot)}_w}}\\ &= C\|b\|^m_{\mathrm{BMO}}\|f\|_{\mathcal{M}^{p(\cdot),u_1}_w}.\end{aligned}$$

The proof of Theorem 1.2 is completed.

Proof of Theorem 1.3 We have proved the following inequality in Theorem 1.1,

$$\|T_{\Omega,\alpha}f\|_{\mathcal{M}^{q(\cdot),u_2}_w} \le C\|f\|_{\mathcal{M}^{p(\cdot),u_1}_w},$$

so we only have to prove that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{1}{u_2(x,t)}\frac{\|\chi_{B(x,t)}T_{\Omega,\alpha}f\|_{L^{q(\cdot)}_w}}{\|\chi_{B(x,t)}\|_{L^{q(\cdot)}_w}} = 0$$

when

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{1}{u_1(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L^{p(\cdot)}_w}}{\|\chi_{B(x,t)}\|_{L^{p(\cdot)}_w}} = 0.$$

Next we prove, for any small $r > 0$, that

$$\sup_{x\in\mathbb{R}^n}\frac{1}{u_2(x,t)}\frac{\|\chi_{B(x,t)}T_{\Omega,\alpha}f\|_{L^{q(\cdot)}_w}}{\|\chi_{B(x,t)}\|_{L^{q(\cdot)}_w}} < \varepsilon.$$

We split the right-hand side of (3.6) as

$$\begin{aligned}\frac{1}{u_2(x,t)}\frac{\|\chi_{B(x,t)}T_{\Omega,\alpha}f\|_{L^{q(\cdot)}_w}}{\|\chi_{B(x,t)}\|_{L^{q(\cdot)}_w}} \le C'\frac{1}{u_2(x,t)}\int_t^{\tau_0}\frac{\|\chi_{B(x,s)}f\|_{L^{p(\cdot)}_w}}{\|\chi_{B(x,s)}\|_{L^{q(\cdot)}_w}}\frac{\mathrm{d}s}{s}\\ + C'\frac{1}{u_2(x,t)}\int_{\tau_0}^\infty\frac{\|\chi_{B(x,s)}f\|_{L^{p(\cdot)}_w}}{\|\chi_{B(x,s)}\|_{L^{q(\cdot)}_w}}\frac{\mathrm{d}s}{s}\end{aligned}$$

$$\le C'(\mathrm{K}_1+\mathrm{K}_2),$$

where $0<\tau_0<1$.

Since $f\in V\mathcal{M}_w^{p(\cdot),u_1}$, for all $0<t<\tau_0$, we can choose any fixed $\tau_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\frac{1}{u_1(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,t)}\|_{L_w^{p(\cdot)}}}<\frac{\varepsilon}{2CC'},$$

the constants C and C' come from (1.14) and the inequality above, respectively.

Then for $0<t<\tau_0$, by (1.14), we have

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}C'\mathrm{K}_1&=C'\sup_{x\in\mathbb{R}^n}\frac{1}{u_2(x,t)}\int_t^{\tau_0}\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\&\le CC'\sup_{x\in\mathbb{R}^n}\frac{1}{u_1(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,t)}\|_{L_w^{p(\cdot)}}}<\frac{\varepsilon}{2}.\end{aligned}$$

To the estimation of K_2, from (1.13),

$$\int_{\tau_0}^{\infty}\frac{\operatorname*{ess\,inf}_{s<r<\infty}u_1(x,r)\|w\chi_{B(x,r)}\|_{L^{p(\cdot)}}}{\|w\chi_{B(x,s)}\|_{L^{q(\cdot)}}}\frac{\mathrm{d}s}{s}<\infty.$$

Now we can choose t small enough. It follows from (1.14) that

$$\begin{aligned}\mathrm{K}_2&\le\frac{1}{u_2(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{u_1(x,t)\|w\chi_{B(x,t)}\|_{L^{p(\cdot)}}}\int_{\tau_0}^{\infty}\frac{u_1(x,r)\|w\chi_{B(x,r)}\|_{L^{p(\cdot)}}}{\|w\chi_{B(x,s)}\|_{L^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\&\le\frac{1}{u_2(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{u_1(x,t)\|w\chi_{B(x,t)}\|_{L^{p(\cdot)}}}\int_{t}^{\infty}\frac{u_1(x,r)\|w\chi_{B(x,r)}\|_{L^{p(\cdot)}}}{\|w\chi_{B(x,s)}\|_{L^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\&\le C\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{u_1(x,t)\|w\chi_{B(x,t)}\|_{L^{p(\cdot)}}}.\end{aligned}$$

Since $f\in V\mathcal{M}_w^{p(\cdot),u_1}$, by (1.13) it suffices to choose r small enough such that

$$\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{u_1(x,t)\|w\chi_{B(x,t)}\|_{L^{p(\cdot)}}}<\frac{\varepsilon}{2CC'}.$$

Thus

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}C'\mathrm{K}_2&=C'\sup_{x\in\mathbb{R}^n}\frac{1}{u_2(x,t)}\int_{\tau_0}^{\infty}\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\&\le CC'\sup_{x\in\mathbb{R}^n}\frac{1}{u_1(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,t)}\|_{L_w^{p(\cdot)}}}<\frac{\varepsilon}{2}.\end{aligned}$$

The proof of Theorem 1.3 is completed.

Proof of Theorem 1.4 We have proved the following inequality in Theorem 1.2,

$$\|T^m_{\Omega,\alpha,b}f\|_{\mathcal{M}_w^{q(\cdot),u_2}(\mathbb{R}^n)}\le C\|b\|^m_{\mathrm{BMO}}\|f\|_{\mathcal{M}_w^{p(\cdot),u_1}(\mathbb{R}^n)},$$

so we only have to prove that if

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{1}{u_1(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,t)}\|_{L_w^{p(\cdot)}}}=0,$$

then

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{1}{u_2(x,t)}\frac{\|\chi_{B(x,t)}T^m_{\Omega,\alpha,b}f\|_{L_w^{q(\cdot)}}}{\|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}}}=0.$$

Next, we show that

$$\sup_{x\in\mathbb{R}^n}\frac{1}{u_2(x,t)}\frac{\|\chi_{B(x,t)}T^m_{\Omega,\alpha,b}f\|_{L_w^{q(\cdot)}}}{\|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}}}<\varepsilon$$

for a sufficiently small r. We may decompose the right-hand side of (3.7) as

$$\begin{aligned}\frac{1}{u_2(x,t)}\frac{\|\chi_{B(x,t)}T^m_{\Omega,\alpha,b}f\|_{L_w^{q(\cdot)}}}{\|\chi_{B(x,t)}\|_{L_w^{q(\cdot)}}}&\le C''\|b\|^m_{\mathrm{BMO}}\frac{1}{u_2(x,t)}\int_t^\kappa\Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\&\quad+C''\|b\|^m_{\mathrm{BMO}}\frac{1}{u_2(x,t)}\int_\kappa^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\&\le C''\|b\|^m_{\mathrm{BMO}}(\mathrm{G}_1+\mathrm{G}_2).\end{aligned}$$

We estimate G_1. Since $f\in V\mathcal{M}_w^{p(\cdot),u_1}$, for all $0<t<\kappa$, we choose any fixed $\kappa>0$ such that

$$\sup_{x\in\mathbb{R}^n}\frac{1}{u_1(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,t)}\|_{L_w^{p(\cdot)}}}<\frac{\varepsilon}{2CC''\|b\|^m_{\mathrm{BMO}}}.$$

For $0<t<\kappa$, it follows from (1.16) that

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}C''\|b\|^m_{\mathrm{BMO}}\mathrm{G}_1&=\sup_{x\in\mathbb{R}^n}C''\|b\|^m_{\mathrm{BMO}}\frac{1}{u_2(x,t)}\int_t^\kappa\Big(1+\ln\frac{s}{t}\Big)\frac{\|\chi_{B(x,s)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,s)}\|_{L_w^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\&\le CC''\|b\|^m_{\mathrm{BMO}}\sup_{x\in\mathbb{R}^n}\frac{1}{u_1(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,t)}\|_{L_w^{p(\cdot)}}}<\frac{\varepsilon}{2}.\end{aligned}$$

To the estimation of G_2, we can choose t small enough. It follows from (1.15)–(1.16) that

$$\begin{aligned}\mathrm{G}_2&\le\frac{1}{u_2(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{u_1(x,t)\|w\chi_{B(x,t)}\|_{L^{p(\cdot)}}}\int_\kappa^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{u_1(x,r)\|w\chi_{B(x,r)}\|_{L^{p(\cdot)}}}{\|w\chi_{B(x,s)}\|_{L^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\&\le\frac{1}{u_2(x,t)}\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{u_1(x,t)\|w\chi_{B(x,t)}\|_{L^{p(\cdot)}}}\int_t^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{u_1(x,r)\|w\chi_{B(x,r)}\|_{L^{p(\cdot)}}}{\|w\chi_{B(x,s)}\|_{L^{q(\cdot)}}}\frac{\mathrm{d}s}{s}\\&\le C\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{u_1(x,t)\|w\chi_{B(x,t)}\|_{L^{p(\cdot)}}}.\end{aligned}$$

Since $f\in V\mathcal{M}_w^{p(\cdot),u_1}$, by (1.15) it suffices to choose r small enough such that

$$\frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{u_1(x,t)\|w\chi_{B(x,t)}\|_{L^{p(\cdot)}}}<\frac{\varepsilon}{2CC''}.$$

Hence we can get

$$\sup_{x\in\mathbb{R}^n} C''\|b\|_{\mathrm{BMO}}^m \mathrm{G}_2 \leq CC''\|b\|_{\mathrm{BMO}}^m \sup_{x\in\mathbb{R}^n} \frac{1}{u_1(x,t)} \frac{\|\chi_{B(x,t)}f\|_{L_w^{p(\cdot)}}}{\|\chi_{B(x,t)}\|_{L_w^{p(\cdot)}}} < \frac{\varepsilon}{2},$$

which completes the proof of Theorem 1.4.

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