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Time-Periodic Fitzhugh-Nagumo Equation and the Optimal Control Problems*

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Abstract In this work, the authors considered the periodic optimal control problem of Fitzhugh-Nagumo equation. They firstly prove the existence of time-periodic solution to Fitzhugh-Nagumo equation. Then they show the existence of optimal solution to the optimal control problem, and finally the first order necessary condition is obtained by constructing an appropriate penalty function.

Keywords Fitzhugh-Nagumo equation, Time-periodic, Optimal control
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1 Introduction and Main Results

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary $\partial\Omega$ ($N = 1, 2$ or 3), and let $T > 0$ be a finite number. Set $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$, and denote by $|\cdot|$ (resp. $\langle\cdot,\cdot\rangle$) the usual norm (resp. scalar product) in $L^2(\Omega)$. In the sequel, C denotes a generic positive constant.

Let ψ_1, ψ_2 and ψ_3 be three given functions in $L^\infty(Q)$, $\omega \subset \Omega$ be an open nonempty set. We consider in this work the following controlled time-periodic FitzHugh-Nagumo equation

$$\begin{cases} u_t - \triangle u + v + F_0(x,t;u) = \chi_\omega g, \\ v_t - \sigma u + \gamma v = 0, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u(x,T), \quad v(x,0) = 0, \end{cases} \tag{1.1}$$

where $g \in L^2(Q), \sigma > 0$ and $\gamma \geq 0$ are constants, and $F_0(x,t;u)$ is given by

$$F_0(x,t;u) = (u + \psi_1(x,t))(u + \psi_2(x,t))(u + \psi_3(x,t)).$$

In the above system, g is the control, and u, v are the state variables.

The FitzHugh-Nagumo model is a simplified version of the Hodgkin-Huxley model which models in a detailed manner activation and deactivation dynamics of a spiking neuron. This model plays an important role in physics, chemistry and mathematical biology. The variable

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u is the electrical potential across the axonal membrane; v is a recovery variable, associated to the permeability of the membrane to the principal ionic components of the transmembrane current; g is the medicine actuator (the control variable), see [8, 10] for more details.

Compared with standard semilinear elliptic or parabolic equations, the analysis of the FitzHugh-Nagumo system is more difficult. The analysis of optimal control problems for FitzHugh-Nagumo equations have been already considered in several works. In [5], the authors have investigated associated problems by the Dubovitskij-Milyutin optimality conditions. In [11], the time-optimal control problems for a linear version of the FitzHugh-Nagumo equations was studied. The sparse optimal control problems for FitzHugh-Nagumo equations have been investigated in [6–7].

In this work, we shall consider the periodic optimal control problem for the FitzHugh-Nagumo equations. To our best knowledge, the existence of the periodic solution to the FitzHugh-Nagumo equations is not known in the existed literatures. Therefore, we shall firstly apply the Leray-Schauder principle to prove the existence of the periodic solution to the FitzHugh-Nagumo equations. Then, the existence of the optimal solution and the first order necessary condition (maximum principle) will be given. Comparing the optimal control problems considered in the previous mentioned works, the periodic state constraint causes difficulties. We shall construct an appropriate penalty functional to deal with this type of state constraint. For time-periodic optimal control problems for other systems, we cite here (see [3–4, 13, 15]).

Now, we present the main results of this work. Concerning the existence and regularity of the periodic solution to (1.1), we have the the following result.

Theorem 1.1 *Assume that* $g \in L^2(Q)$. *Then* (1.1) *admits at least one solution* (u, v) *with*

$$u \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H_0^1(\Omega)), \quad u_t \in L^2(Q), \tag{1.2}$$

$$v \in C([0,T];H_0^1(\Omega)), \quad v_t \in L^2(0,T;H^2(\Omega)). \tag{1.3}$$

An equivalent formulation to (1.1) is

$$\begin{cases} u_t - \triangle u + \sigma \int_0^t \mathrm{e}^{-\gamma(t-s)} u(s)\mathrm{d}s + F_0(x,t;u) = \chi_\omega g, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u(x,T). \end{cases} \tag{1.4}$$

Our second goal in this work is to study an optimal control problem for (1.2). We will mainly deal with the cost functional

$$J(u,g) = \frac{1}{2} \int\int_Q |u - u_d|^2 \mathrm{d}x\mathrm{d}t + \frac{a}{2} \int\int_Q |g|^2 \mathrm{d}x\mathrm{d}t,$$

where u_d is a desired state, and the constant $a > 0$.

The second main result in this paper is as follows.

Theorem 1.2 *There exists at least one global optimal state-control* $(\overline{u}, \overline{g})$*. Moreover, there exists* $p \in H^{1,2}(Q)$ *satisfying*

$$\begin{cases} -p_t - \triangle p + \sigma \int_t^T \mathrm{e}^{-\gamma(s-t)} p(s)\mathrm{d}s + D_u F_0(x,t;\overline{u})p = \overline{u} - u_d, \\ p(x,t)|_\Sigma = 0, \\ p(x,0) = p(x,T) \end{cases} \tag{1.5}$$

and

$$p + a\overline{g} = 0 \quad \text{a.e. } in\ \omega \times (0,T). \tag{1.6}$$

About the main results above, we give here several notes.

(**i**) Theorem 1.1 claims the existence of periodic solution to the FitzHugh-Nagumo equations. Then it is natural to ask whether the solution is unique or not. Such kind of problem for nonlinear parabolic equations has been studied in [1], wherein the notations of "subsolution" and "supersolution" to (1.1) are introduced. In [1], using the comparison principle, which is based on the strong maximum principle, the author shows that the periodic solution exists between the subsolution and supersolution (see [1, Theorem 2.1]). Since the nonlinear term in FitzHugh-Nagumo equations is cubic, it is not difficult to see that, for certain specified $\psi_i, i = 1,2,3,$ (1.1) may possess two subsolutions $\underline{x}_1, \underline{x}_2$ and two supersolutions $\overline{x}_1, \overline{x}_2$ which obey $\underline{x}_1 < \overline{x}_1 < \underline{x}_2 < \overline{x}_2$. Thus, if we can prove similar results as those in [1] for the FitzHugh-Nagumo equations, then we can show that there may be multiple periodic solutions to (1.1). However, since the FitzHugh-Nagumo equation contains an integral term comparing with the classical parabolic equations, we find that it is not an easy job. Hence, we leave this problem for future research, and deal with the FitzHugh-Nagumo equations by assuming that it may possess multiple periodic solutions.

(**ii**) In our setting of the control problems, the control is distributedly plugged into the system, and the initial data is not specified but subject to periodic endpoints constraint. In this formulation, the control system might be a kind of multi-response system since the initial data is not specified and the periodic solution to (1.4) might not be unique. Corresponding to the optimal control, there might be multiple state functions satisfying (1.4), the optimal state function is the one such that the cost functional is minimized. An equivalent setting of the control problem is to treat the initial data as another control variable, which can be realized by impulsive control. Then, the state function is uniquely determined by the two control variables, and the periodic state constraint can be viewed as a mixed control-state constraint. This is somehow the standard formulation for optimal control problems with endpoints state constraint (see [12]). Nevertheless, in our formulation, we can approximate the optimal control problem by an optimization problem, and the optimality condition can be obtained by a very constructive way. It has been used in periodic optimal control problem governed by fluid flows and turns out to be efficient (see [3–4, 13, 15]).

(**iii**) Now that we obtain the optimality conditions presented in Theorem 1.2, we should try to see that whether we can apply these conditions to numerically approximate the optimal solution.

We can see from Theorem 1.2 that, with the optimality conditions, we can obtain a coupled periodic system with unknowns $\overline{u}$ and p, which seems not difficult to be solved. However, there are two essential difficulties remain to be overcomed. One is that the system is periodic. Unlike the classical forward-backward evolution systems, solving nonlinear coupled periodic systems is not easy. A monotone sequence method based on comparing principle has been applied in [9] to solve the periodic optimality systems for optimal control of parabolic Volterra-Lotka type equations. For other computing methods of periodic optimal control problems governed by ODEs, we refer to [2, 14] and references therein. Another difficulty is that the optimal control problem is nonlinear, and the optimal solution is not necessarily unique. The convergence of numerical approximation usually requires additional condition, such as the second order sufficient optimality condition for optimal solution (see [6]). These problems will be investigated in future work.

2 Existence and Regularity of Periodic Solution

Notice that (1.4) can be written in the form

$$\begin{cases} u_t - \triangle u + G(u) + F(u) = \chi_\omega g, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u(x,T), \end{cases} \tag{2.1}$$

where we have set

$$G(u)(x,t) = \sigma \int_0^t \mathrm{e}^{-\gamma(t-s)} u(s)\mathrm{d}s, \tag{2.2}$$

$$F(u)(x,t) = F_0(x,t;u). \tag{2.3}$$

We will first prove that (2.1) admits at least one solution $u \in H^{1,2}(Q)$ with the help of the Leray-Schauder's principle (see [16, Theorem 6.A]).

Thus, let us consider the auxiliary problem

$$\begin{cases} u_t - \triangle u + G(u) = \lambda(\chi_\omega g - F(u)), \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u(x,T) \end{cases} \tag{2.4}$$

for each $\lambda \in [0,1]$. Denote the space $Y := \{u \in L^6(Q) \cap C([0,T];L^2(\Omega)); u(0) = u(T)\}$, which is equipped with the norm $\|u\|_Y = \|u\|_{L^6(Q)} + \|u\|_{C([0,T];L^2(\Omega))}$. We also introduce the mapping $\Lambda : Y \times [0,1] \to Y$ with $u = \Lambda(w,\lambda)$ if and only if u is the unique solution to

$$\begin{cases} u_t - \triangle u + G(u) = \lambda(\chi_\omega g - F(w)), \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u(x,T). \end{cases} \tag{2.5}$$

We shall prove the following results.

Lemma 2.1 *The mapping $\Lambda : Y \times [0,1] \to Y$ is well-defined, continuous and compact.*

Lemma 2.2 *All functions u such that $u = \Lambda(u,\lambda)$ for some $\lambda \in [0,1]$ are uniformly bounded in Y.*

In view of the Leray-Schauder's principle, these will suffice to affirm that (1.2) admits at least one solution.

Proof of Lemma 2.1 Step 1 (Well-posedness of (2.5)) Let $u(0) = u_0 \in L^2(\Omega)$ be given. Define $J : L^2(\Omega) \to L^2(\Omega)$ by $J(u_0) = u(T)$, where $u(\cdot)$ is the solution to

$$\begin{cases} u_t - \triangle u + G(u) = \lambda(\chi_\omega g - F(w)), \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u_0. \end{cases} \tag{2.6}$$

We claim firstly that J is well-defined, which suffices to prove that the above equation admits a weak solution $u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$. Indeed, notice that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|G(u)|^2 = \Big\langle G(u), \frac{\mathrm{d}u}{\mathrm{d}t} \Big\rangle = \langle G(u), \sigma u - \gamma G(u) \rangle = \sigma \langle G(u), u \rangle - \gamma |G(u)|^2.$$

We can infer that

$$\langle G(u), u \rangle = \frac{1}{2\sigma}\frac{\mathrm{d}}{\mathrm{d}t}|G(u)|^2 + \frac{\gamma}{\sigma}|G(u)|^2.$$

Multiplying the first equation of (2.6) by u in the sense of $L^2(\Omega)$, and integrating on $(0,t)$, we can obtain by the above identity that, for any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$\begin{aligned} &\frac{1}{2}|u(t)|^2 + \int_0^t |\nabla u|^2 \mathrm{d}s + \frac{1}{2\sigma}|G(u)(t)|^2 + \frac{\gamma}{\sigma}\int_0^t |G(u)(s)|^2 \mathrm{d}s \\ &\le \varepsilon \int_0^t |u|^2 \mathrm{d}s + C_\varepsilon \int_0^t |\lambda(\chi_\omega g - F(w))|^2 \mathrm{d}s, \quad \forall\, t \in (0,T). \end{aligned}$$

Since $|\nabla w|^2 \ge c_1 |w|^2$ for some $c_1 > 0$, we can take $\varepsilon = \frac{c_1}{2}$, and by the above energy estimate, we can obtain that

$$\begin{aligned} &\frac{1}{2}|u(t)|^2 + \frac{1}{2}\int_0^t |\nabla u|^2 \mathrm{d}s + \frac{1}{2\sigma}|G(u)(t)|^2 + \frac{\gamma}{\sigma}\int_0^t |G(u)(s)|^2 \mathrm{d}s \\ &\le C\int_0^t |\lambda(\chi_\omega g - F(w))|^2 \mathrm{d}s \le C(\|g\|^2_{L^2(0,T;L^2(\Omega))} + \|w\|_Y^2 + 1), \quad \forall\, t \in (0,T). \end{aligned}$$

Together with the classical Galerkin's arguments, we can infer that (2.6) admits a weak solution, and it is unique. This subsequently implies the above claim.

Now, we show that the map J is contraction. Let u_0^1, u_0^2 be two different initial data, u^1, u^2 be the corresponding solutions. Then $\widetilde{w} = u^1 - u^2$ is the solution to

$$\begin{cases} \widetilde{w}_t - \triangle \widetilde{w} + G(\widetilde{w}) = 0, \\ \widetilde{w}(x,t)|_\Sigma = 0, \\ \widetilde{w}(x,0) = \widetilde{w}_0, \end{cases} \tag{2.7}$$

where $\widetilde{w}_0 = u_0^1 - u_0^2$.

Similarly as the above estimate, multiplying the first equation of system (2.7) by w, we get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\widetilde{w}|^2 + \frac{1}{2\sigma}\frac{\mathrm{d}}{\mathrm{d}t}|G(\widetilde{w})|^2 + |\nabla \widetilde{w}|^2 + \frac{\gamma}{\sigma}|G(\widetilde{w})|^2 = 0.$$

Since $|\nabla w|^2 \geq c_1|w|^2$ for some $c_1 > 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathrm{e}^{2c_0t}(|\widetilde{w}(t)|^2 + \frac{1}{\sigma}|G(\widetilde{w})(t)|)^2\Big] \leq 0, \quad \forall\, t > 0,$$

where $c_0 = \min\{c_1, \gamma\}$. This implies that

$$|\widetilde{w}(T)|^2 \leq \mathrm{e}^{-c_0T}|\widetilde{w}_0|^2. \tag{2.8}$$

Since $\mathrm{e}^{-c_0T} < 1$, we infer that the map J is contraction. This implies that (2.5) admits a unique periodic solution in $u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$.

Step 2 (The map Λ is well-defined and compact.) (2.5) can be equivalently written as

$$\begin{cases} u_t - \triangle u + v = \lambda(\chi_\omega g - F(w)), \\ v_t - \sigma u + \gamma v = 0, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u(x,T), \quad v(x,0) = 0. \end{cases} \tag{2.9}$$

Multiplying the first equation (resp. the second equation) of system (2.9) by u (resp. $\frac{1}{\sigma}v$) in the sense of $L^2(\Omega)$, and summing these two equations, we can obtain that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \frac{1}{2\sigma}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + |\nabla u|^2 + \frac{\gamma}{\sigma}|v|^2 \\ &= \langle \lambda(g - F(w)), u\rangle \\ &\leq C\lambda^2(|g(t)|^2 + \|w(t)\|^6_{L^6(\Omega)} + C_1) + \frac{c_1}{2}|u(t)|^2. \end{aligned}$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathrm{e}^{2c_0t}|u(t)|^2 + \frac{1}{\sigma}|v(t)|^2\Big] + |\nabla u|^2 + \frac{\gamma}{\sigma}|v|^2 \leq C\lambda^2(|g(t)|^2 + \|w(t)\|^6_{L^6(\Omega)} + 1). \tag{2.10}$$

Integrating the above inequality on $[0,T]$, and notice that $u(0) = u(T), v(0) = 0$, we infer that

$$(\mathrm{e}^{2c_0T} - 1)|u(0)|^2 + \frac{1}{\sigma}|v(T)|^2 + \int_0^T |\nabla u(t)|^2\mathrm{d}t + \int_0^T |v(t)|^2\mathrm{d}t \leq C\lambda^2(\|g\|^2_{L^2(Q)} + \|w\|^6_{L^6(Q)} + 1),$$

where C is a constant depending on Ω, T and $\psi_i, i = 1,2,3$. This implies that

$$|u(0)|^2 \leq C\lambda^2(\|g\|^2_{L^2(Q)} + \|w\|^6_{L^6(Q)} + 1). \tag{2.11}$$

Integrating (2.10) on $[0,t]$, using (2.11), we can obtain that

$$\sup_{0\leq t\leq T}(|u(t)|^2 + |v(t)|^2) + \int_0^T |\nabla u(t)|^2\mathrm{d}t \leq C\lambda^2(\|g\|^2_{L^2(Q)} + \|w\|^6_{L^6(Q)} + 1). \tag{2.12}$$

Multiplying the first equation of system (2.9) by $-t\triangle u$, we obtain that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(t|\nabla u|^2) - \frac{1}{2}|\nabla u|^2 + t|\triangle u|^2 \\ &= \langle -v + \lambda(g - F(w)), -t\triangle u\rangle \\ &\leq \frac{1}{2}t|\triangle u|^2 + C\lambda^2(|g(t)|^2 + \|w(t)\|^6_{L^6(\Omega)} + 1). \end{aligned}$$

This implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(t|\nabla u|^2)+\frac{1}{2}t|\triangle u|^2\le|\nabla u|^2+C\lambda^2(|g(t)|^2+\|w(t)\|_{L^6(\Omega)}^6+1). \tag{2.13}$$

Integrating on $[0,T]$, we infer from (2.12)–(2.13) that

$$T|\nabla u(T)|^2\le C\lambda^2(\|g\|_{L^2(Q)}^2+\|w\|_{L^6(Q)}^6+1).$$

Notice that $u(0)=u(T)$, hence

$$|\nabla u(0)|^2\le C\lambda^2(\|g\|_{L^2(Q)}^2+\|w\|_{L^6(Q)}^6+1). \tag{2.14}$$

Finally, multiplying the first equation of system (2.9) by $-\triangle u$, and integrating on $[0,t]$, we can infer by (2.12) and (2.14) that

$$\|u\|_{H^{1,2}(Q)}\le C\lambda^2(\|g\|_{L^2(Q)}^2+\|w\|_{L^6(Q)}^6+1). \tag{2.15}$$

By Aubin-Lions lemma, we know that $H^{1,2}(Q)$ is compact imbedded in Y. This implies that Λ is well-defined and compact.

Step 3 (The map Λ is continuous) Let (w_n,λ_n) be a sequence in $Y\times[0,1]$, and $w_n\to w$ strongly in $Y,\lambda_n\to\lambda$ as $n\to\infty$. Let

$$u_n=\Lambda(w_n,\lambda_n),\quad u=\Lambda(w,\lambda).$$

By the result of Step 2, we see that

$$\|u_n\|_{H^{1,2}(Q)}\le C\lambda_n^2(\|g\|_{L^2(Q)}^2+\|w_n\|_{L^6(Q)}^6+1)\le C.$$

Hence, there exists at least a subsequence of u_n which will be denoted by itself, such that

$$u_n\to\widetilde{u}\quad\text{weakly in } H^{1,2}(Q)$$

for some $\widetilde{u}\in H^{1,2}(Q)$. Since $H^{1,2}(Q)$ is compact imbedded in Y, it follows that

$$u_n\to\widetilde{u}\quad\text{strongly in } Y,$$

so $\widetilde{u}(0)=\widetilde{u}(T)$. Moreover,

$$w_n\to w,\quad u_n\to\widetilde{u}\quad\text{a.e. in } Q.$$

Since $F(w)$ is continuous in w, we see that

$$F(w_n)\to F(w)\quad\text{a.e. in } Q.$$

Then, we can pass to limit in the equation satisfied by w_n and u_n to get that

$$u=\widetilde{u}.$$

Hence, we proved that $\Lambda(w_n,\lambda_n)\to\Lambda(w,\lambda)$ strongly in Y.

Proof of Lemma 2.2 (2.4) can be written as

$$\begin{cases} u_t - \triangle u + v = \lambda(\chi_\omega g - F(u)), \\ v_t - \sigma u + \gamma v = 0, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u(x,T). \end{cases} \tag{2.16}$$

Multiplying the first equation (resp. the second equation) of system (2.16) by u (resp. $\frac{1}{\sigma}v$), integrating on Ω, and adding the resulting identities, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \frac{1}{2\sigma}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + |\nabla u|^2 + \frac{\gamma}{\sigma}|v|^2 + \langle \lambda F(u), u\rangle = \lambda\langle g, u\rangle.$$

Notice that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$\langle \lambda F(u), u\rangle \geq (1-\varepsilon)\|u\|^4_{L^4(\Omega)} - C_\varepsilon.$$

It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \frac{1}{2\sigma}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + |\nabla u|^2 + \frac{\gamma}{\sigma}|v|^2 + \lambda(1-\varepsilon)\|u\|^4_{L^4(\Omega)} \leq C\lambda(|g|^2+1).$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathrm{e}^{2c_0 t}|u(t)|^2 + \frac{1}{\sigma}|v(t)|^2\Big] \leq C\lambda(|g(t)|^2+1).$$

Integrating on $[0,T]$, we can infer that

$$|u(0)| = |u(T)| \leq C\lambda(1+\|g\|^2_{L^2(Q)}). \tag{2.17}$$

This in turn implies that

$$\sup_{0\leq t\leq T}(|u(t)|^2 + |v(t)|^2) + \|u\|_{L^2(0,T;H^1_0(\Omega))} + \lambda\|u\|_{L^4(Q)} \leq C(\|g\|_{L^2(Q)}+1). \tag{2.18}$$

Now multiply the first equation of (2.16) by tu_t, and integrating on Ω, we get that

$$t|u_t|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(t|\nabla u|^2) - \frac{1}{2}|\nabla u|^2 + \langle v, tu_t\rangle + \langle \lambda F(u), tu_t\rangle = \lambda\langle g, tu_t\rangle. \tag{2.19}$$

Notice that $t \leq T$ and $\lambda \leq 1$, we can check by Cauchy-Schwaz inequality and Young's inequality that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$\begin{aligned} \langle \lambda F(u), tu_t\rangle &\geq \frac{\lambda}{4}\frac{\mathrm{d}}{\mathrm{d}t}(t\|u(t)\|^4_{L^4(\Omega)}) - \varepsilon t|u_t(t)|^2 - C_\varepsilon(\lambda\|u\|^4_{L^4(\Omega)}+1), \\ \langle v, tu_t\rangle &\leq \varepsilon t|u_t|^2 + C_\varepsilon T|v|^2, \\ \lambda\langle g, tu_t\rangle &\leq \varepsilon t|u_t|^2 + C_\varepsilon T\lambda^2|g|^2. \end{aligned}$$

This together with (2.18)–(2.19) imply that

$$(1-3\varepsilon)t|u_t|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(t|\nabla u|^2) + \frac{\lambda}{4}\frac{\mathrm{d}}{\mathrm{d}t}(t\|u(t)\|^4_{L^4(\Omega)}) \leq C_\varepsilon(1+|g|^2).$$

Taking ε small enough, and integrating on $[0,T]$, we get that

$$|\nabla u(0)| = |\nabla u(T)| \leq C(1+\|g\|^2_{L^2(Q)}). \tag{2.20}$$

This in turn implies that

$$\|u\|_{C([0,T];H_0^1(\Omega))} + \|u_t\|_{L^2(Q)} \le C(\|g\|_{L^2(Q)} + 1). \tag{2.21}$$

It follows that

$$\|u\|_{L^2(0,T;H^2(\Omega))} \le C(\|g\|_{L^2(Q)} + 1). \tag{2.22}$$

By Aubin-Lions lemma, we know that $H^{1,2}(Q)$ is compact imbedded in Y. Hence, all functions u such that $u = \Lambda(u, \lambda)$ for some $\lambda \in [0, 1]$ are uniformly bounded in Y. This completes the proof.

Proof of Theorem 1.1 By Leray-Schauder's principle, we can see from Lemmas 2.1–2.2 that (1.1) admits at least one periodic solution. The regularity properties (1.2)–(1.3) follows from the proof of Lemma 2.2.

3 Time Periodic Optimal Control Problems

We prove the existence of optimal solution firstly.

Proof of Theorem 1.2 (Part I: The existence of optimal solution.) Let (u_n, g_n) be a minimizing sequence in problem (P), i.e.,

$$\begin{aligned} &\inf(\mathrm{P}) \le J(u_n, g_n) \le \inf(\mathrm{P}) + \frac{1}{n}, \\ &\partial_t u_n - \triangle u_n + G(u_n) + F(u_n) = \chi_\omega g_n, \quad u_n(0) = u_n(T). \end{aligned} \tag{3.1}$$

By the definition of J, $\{g_n\}$ is bounded in $L^2(Q)$, and therefore, on a subsequence, again denoted by n, we have

$$g_n \to \overline{g} \quad \text{weakly in } L^2(Q). \tag{3.2}$$

By Theorem 1.1, we see that

$$\|u_n\|_{H^{1,2}(Q)} \le C. \tag{3.3}$$

Selecting further subsequences, if necessary, we have

$$\begin{aligned} u_n \to u \quad &\text{strongly in } L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; H); \\ \partial_t u_n \to \partial_t \overline{u}, \quad -\triangle u_n \to -\triangle \overline{u} \quad &\text{weakly in } L^2(Q). \end{aligned}$$

Moreover, it is not difficult to see that

$$F(u_n) \to F(\overline{u}) \quad \text{strongly in } L^2(Q).$$

Then letting n goes to ∞ in (3.1), we infer that $(\overline{u}, \overline{g})$ satisfies the system (1.2) and $J(\overline{u}, \overline{g}) = \inf(\mathrm{P})$.

Now, given an optimal solution $(\overline{u},\overline{g})$, to prove the second part of Theorem 1.2, that is, the first order necessary optimality condition, we need to firstly consider the approximate control problem as follows:

$$(\mathrm{P}_\varepsilon) \quad \inf J_\varepsilon(u,g) \text{ over all } (u,g)\in X\times L^2(Q),$$

where $X=\{u\in H^{1,2}(Q); u(0)=u(T)\}$ and

$$\begin{aligned} J_\varepsilon(u,g) = J(y,u) &+ \frac{1}{2}\int\int_Q |u-\overline{u}|^2\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int\int_Q |g-\overline{g}|^2\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{2\varepsilon}\int\int_Q |u_t - \triangle u + G(u) + F(u) - \chi_\omega g|^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.4}$$

Remark 3.1 Although we have shown that for each control $g\in L^2(Q)$, there exists at leat one periodic solution to the state equation, there are still other problems to apply variation directly for the original optimal control problem. The essential problem is that the solution map might be multi-valued and might not be Fréchet differentiable. Indeed, the linearized equation of the nonlinear equation (1.4) may do not have periodic solution. Therefore, we define an optimization problem to approximate the original control problem. We view the state and control as two independent variables, and view the state equation as constraint. The last term in (3.4) is defined to penalize this constraint. Notice that the optimal solution to the control problem is not necessarily unique, the second and third terms in the right-hand side of (3.4) are introduced to make sure that the optimal solutions for the approximate control problem converge to the specified optimal solution $(\overline{u},\overline{g})$.

Similar to the first part of the proof of Theorem 1.2, we have the following result.

Lemma 3.1 *For each $\varepsilon>0$, problem (P_ε) has at least one solution.*

Moreover, for the relation between the optimal solution for (P_ε) and the optimal solution $(\overline{u},\overline{g})$ for the original optimal control problem, we have the following lemma.

Lemma 3.2 *Let $(u_\varepsilon,g_\varepsilon)\in X\times L^2(Q)$ be optimal for problem (P_ε). Then,*

$$\begin{aligned} &u_\varepsilon \to \overline{u} \quad \text{strongly in } X, \\ &g_\varepsilon \to \overline{g} \quad \text{strongly in } L^2(Q). \end{aligned} \tag{3.5}$$

Proof Denote $v_\varepsilon = \partial_t u_\varepsilon - \triangle_\varepsilon u + G(u_\varepsilon) + F(u_\varepsilon) - \chi_\omega g_\varepsilon$. Since $J_\varepsilon(u_\varepsilon,g_\varepsilon)\le J_\varepsilon(\overline{u},\overline{g}) = J(\overline{u},\overline{g})$, it follows that

$$\int_0^T \Big(|u_\varepsilon|^2 + \frac{1}{\varepsilon}|v_\varepsilon|^2 + |g_\varepsilon|^2\Big)\mathrm{d}t \le C.$$

By the proof of Theorem 1.1, we see that

$$\|u_\varepsilon\|_{H^{1,2}(Q)} \le C. \tag{3.6}$$

Hence, on a subsequence, again denoted by ε, we have

$$u_\varepsilon \to \widetilde{u} \quad \text{strongly in } C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega));$$

$$
\begin{aligned}
&u_\varepsilon \to \widetilde{u} \quad \text{weakly in } H^{1,2}(Q);\\
&v_\varepsilon \to 0, \quad g_\varepsilon \to \widetilde{g} \quad \text{weakly in } L^2(Q)
\end{aligned}
\tag{3.7}
$$

for some $(\widetilde{u}, \widetilde{g}) \in X \times L^2(Q)$. Moreover, we can get that

$$F(u_\varepsilon) \to F(\widetilde{u}) \quad \text{strongly in } L^2(Q).$$

Therefore, $(\widetilde{u}, \widetilde{g})$ is a solution to (1.2). By (3.7) and the weak lower-semicontinuous of cost functional, we can obtain that

$$
\begin{aligned}
&J(\widetilde{u}, \widetilde{g}) + \frac{1}{2}\int\int_Q |\widetilde{u} - \overline{u}|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int\int_Q |\widetilde{g} - \overline{g}|^2 \mathrm{d}x\mathrm{d}t\\
&\le J_\varepsilon(u_\varepsilon, g_\varepsilon) \le J(\overline{u}, \overline{g}) \le J(\widetilde{u}, \widetilde{g}).
\end{aligned}
\tag{3.8}
$$

Hence, $\widetilde{u} = \overline{u}$ and $\widetilde{g} = \overline{g}$. Moreover, we can see from (3.8) that $g_\varepsilon \to \overline{g}$ strongly in $L^2(Q)$.

In the space $L^2(Q)$, we introduce the operators

$$\mathcal{A}_\varepsilon \varphi = \varphi_t - \triangle\varphi + G(\varphi) + F'_u(u_\varepsilon)\varphi, \quad \forall\, \varphi \in D(\mathcal{A}_\varepsilon) = X \tag{3.9}$$

and

$$\mathcal{A}^*_\varepsilon \varphi = -\varphi_t - \triangle\varphi + G^*(\varphi) + F'_u(u_\varepsilon)\varphi, \quad \forall\, \varphi \in D(\mathcal{A}^*_\varepsilon) = X. \tag{3.10}$$

It is readily seen that

$$\int_0^T \langle \mathcal{A}^*_\varepsilon q, \varphi\rangle \mathrm{d}t = \int_0^T \langle \mathcal{A}_\varepsilon \varphi, q\rangle \mathrm{d}t, \quad \forall\, \varphi, q \in X. \tag{3.11}$$

The operators $\mathcal{A}$ and $\mathcal{A}^*$ are defined by the same formulas (3.10) and (3.11), where $u_\varepsilon = \overline{u}$.

To obtain the first order optimality condition, we need to use some properties of operators $\mathcal{A}_\varepsilon, \mathcal{A}^*_\varepsilon, \mathcal{A}$ and $\mathcal{A}^*$, which can be stated as follows. (Similar properties for the 2-D and 3-D Navier-Stokes equations has been obtained in [3] and [15] respectively, and here we shall apply the same method to prove these properties for the FitzHugh-Nagumo equation.)

Lemma 3.3 *The operators $\mathcal{A}_\varepsilon, \mathcal{A}^*_\varepsilon, \mathcal{A}$ and $\mathcal{A}^*$ are closed, densely defined, and have closed ranges in $L^2(Q)$. Moreover, $\dim N(\mathcal{A}_\varepsilon) \le n_0$, independent on ε, and the following estimates hold:*

$$
\begin{aligned}
&\|\mathcal{A}_\varepsilon^{-1} g\|_{H^{1,2}(Q)} \le C\|g\|_{L^2(Q)}, \quad \forall\, g \in R(\mathcal{A}_\varepsilon);\\
&\|(\mathcal{A}^*_\varepsilon)^{-1} g\|_{H^{1,2}(Q)} \le C\|g\|_{L^2(Q)}, \quad \forall\, g \in R(\mathcal{A}^*_\varepsilon).
\end{aligned}
\tag{3.12}
$$

Similarly, the operators $\mathcal{A}$ and $\mathcal{A}^*$ are mutually adjoint and estimates (3.12) *remain true for $\mathcal{A}$ and $\mathcal{A}^*$. Here we use the symbols N and R to denote the null space and the range of the corresponding operators.*

Proof Consider the linear equation

$$
\begin{cases}
\varphi_t - \triangle\varphi + G(\varphi) + F'_u(u_\varepsilon)\varphi = g,\\
\varphi(x,t)|_\Sigma = 0,\\
\varphi(x,0) = \varphi_0(x).
\end{cases}
\tag{3.13}
$$

We claim that system (3.13) has a unique solution $\varphi = \varphi_\varepsilon(t;\varphi_0,g) \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, for each $\varphi_0 \in L^2(\Omega), g \in L^2(Q)$. Indeed, to get the energy estimate, we multiply the first equation of (3.13) by φ, and it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\varphi|^2 + |\nabla\varphi|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|G(\varphi)|^2 + \gamma|G(\varphi)|^2 = \langle -F_u'(u_\varepsilon)\varphi + g, \varphi\rangle. \tag{3.14}$$

Notice that $\|u_\varepsilon\|_{H^{1,2}(Q)} \le C$, and $H^{1,2}(Q) \subset C([0,T];H_0^1(\Omega))$, we can infer that

$$\|F_u'(u_\varepsilon)\|_{L^\infty(0,T;L^3(\Omega))} \le C(\|u_\varepsilon\|_{L^\infty(0,T;L^6(\Omega))} + 1) \le C(\|u_\varepsilon\|_{L^\infty(0,T;H_0^1(\Omega))} + 1) \le C. \tag{3.15}$$

This implies that

$$\begin{aligned}
&\int_0^T \langle F_u'(u_\varepsilon)\varphi,\varphi\rangle \mathrm{d}t \\
&\le C\int_0^T \Big(\int_\Omega \varphi^3 \mathrm{d}x\mathrm{d}t\Big)^{\frac{2}{3}} \\
&\le C_\eta \int_0^T |\varphi|^2 \mathrm{d}t + \eta\int_0^T \|\varphi\|_{L^6(\Omega)}^2 \mathrm{d}t \\
&\le C_\eta \int_0^T |\varphi|^2 \mathrm{d}t + C_1\eta\int_0^T |\nabla\varphi|^2 \mathrm{d}t.
\end{aligned} \tag{3.16}$$

Taking η small enough, and integrating (3.14) from 0 to T, we can obtain by (3.15) that $\varphi = \varphi_\varepsilon(t;\varphi_0,g)$ satisfies the following energy estimate

$$\sup_{0\le t\le T}(|\varphi_\varepsilon(t)|^2 + |G(\varphi_\varepsilon)(t)|^2) + \int_0^T |\nabla\varphi_\varepsilon|^2 \mathrm{d}t \le C(\|g\|_{L^2(Q)} + |\varphi_0|). \tag{3.17}$$

This indicates the existence and uniqueness of solution to (3.13). Moreover, if $\varphi_0 \in H_0^1(\Omega)$, it is not difficult to show that the solution $\varphi_\varepsilon \in H^{1,2}(Q) \subset C([0,T];H_0^1(\Omega))$.

Multiplying (3.13) by $-t\partial_t\varphi_\varepsilon$, integrating on Q, and using (3.17), we can get by the similar approach applied to obtain (2.20) that $\varphi_\varepsilon(T) \in H_0^1(\Omega)$, and

$$\|\varphi_\varepsilon(T)\|_{H_0^1(\Omega)} \le C(|\varphi_0|^2 + \|g\|_{L^2(Q)}^2), \quad \forall\,\varepsilon > 0. \tag{3.18}$$

We define $G_\varepsilon : L^2(Q) \to L^2(\Omega)$ by $G_\varepsilon(g) = \varphi_\varepsilon(t;0,g)$, and define $\Gamma_\varepsilon : L^2(\Omega) \to L^2(\Omega)$ by $\Gamma_\varepsilon\varphi_0 = \varphi_\varepsilon(t;\varphi_0,0)$. It is clear that

$$\varphi_\varepsilon(T;\varphi_0,g) = \Gamma_\varepsilon\varphi_0 + G_\varepsilon(g), \tag{3.19}$$

and the estimate (3.18) yields that

$$\|\Gamma_\varepsilon\|_{\mathcal{L}(L^2(\Omega),H_0^1(\Omega))} + \|G_\varepsilon\|_{\mathcal{L}(L^2(Q),H_0^1(\Omega))} \le C, \quad \forall\,\varepsilon > 0. \tag{3.20}$$

Since the injection of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, we infer that Γ_ε is completely continuous. Let $g \in \mathcal{R}(\mathcal{A}_\varepsilon)$, i.e., there exists $u \in D(\mathcal{A}_\varepsilon)$, such that $\mathcal{A}_\varepsilon u = g$. We have $u(t) = \varphi_\varepsilon(t;\varphi_0,g)$, where $(I-\Gamma_\varepsilon)\varphi_0 = G_\varepsilon g$. By the Fredholm-Riesz theory, we know that $R(I-\Gamma_\varepsilon)$ is closed and dim $N(I-\Gamma_\varepsilon)\le\infty$. Hence, $R(\mathcal{A}_\varepsilon)$ is closed in $L^2(Q)$, and $N(\mathcal{A}_\varepsilon)$ is finite dimensional. Moreover, if $(\varphi_n,g_n)\in\mathcal{A}_\varepsilon$, and $(\varphi_n,g_n)\to(\varphi,g)$ strongly in $X\times L^2(Q)$, then we have

$$\partial_t\varphi_n-\triangle\varphi_n+G(\varphi_n)+F'_u(u_\varepsilon)\varphi_n=g_n. \tag{3.21}$$

Similarly as the proof of Theorem 1.1, we can get that $(\varphi,g)\in\mathcal{A}_\varepsilon$, i.e., $\mathcal{A}_\varepsilon$ is closed.

Now, let $\Gamma\in\mathcal{L}(L^2(\Omega),L^2(\Omega))$ be defined by $\Gamma\varphi_0=\varphi(T,\varphi_0,0)$, where φ is the solution to

$$\begin{cases}\varphi_t-\triangle\varphi+G(\varphi)+F'_u(\overline{u})\varphi=0,\\ \varphi(x,t)|_\Sigma=0,\\ \varphi(x,0)=\varphi_0(x).\end{cases} \tag{3.22}$$

As seen earlier, $\Gamma\in\mathcal{L}(L^2(\Omega),H_0^1(\Omega))$, and so Γ is completely continuous from $L^2(\Omega)$ into itself. Moreover, it is not difficult to show that

$$\Gamma_\varepsilon\to\Gamma\quad\text{in } \mathcal{L}(L^2(\Omega),L^2(\Omega)) \text{ as } \varepsilon\to 0. \tag{3.23}$$

Since $\dim N(I-\Gamma)<\infty$, (3.23) implies that there exists $n_0>0$, such that $\dim N(I-\Gamma_\varepsilon)\le n_0$ for all $\varepsilon>0$. Hence $\dim N(\mathcal{A}_\varepsilon)\le n_0$ for all $\varepsilon>0$, as claimed. Moreover, we claim that

$$|(I-\Gamma_\varepsilon)^{-1}g_0|\le C|g_0|,\quad \forall\, g_0\in R(I-\Gamma_\varepsilon). \tag{3.24}$$

Indeed, otherwise, there exist $\varphi_{0\varepsilon}\in N(I-\Gamma_\varepsilon)^\perp=R((I-\Gamma_\varepsilon)^*)$, $f_\varepsilon\in R(I-\Gamma_\varepsilon)$ such that $(I-\Gamma_\varepsilon)\varphi_{0\varepsilon}=f_\varepsilon$ and $|f_\varepsilon|=1,|\varphi_{0\varepsilon}|\to\infty$. Let $\widetilde{\varphi}_{0\varepsilon}=\frac{\varphi_{0\varepsilon}}{|\varphi_{0\varepsilon}|}$ and $\widetilde{f}_\varepsilon=\frac{f_\varepsilon}{|\varphi_{0\varepsilon}|}$. Then, $\widetilde{f}_\varepsilon\to 0$, and $(I-\Gamma_\varepsilon)\widetilde{\varphi}_{0\varepsilon}=\widetilde{f}_\varepsilon$. We can see from (3.18) that $\|\Gamma_\varepsilon\widetilde{\varphi}_{0\varepsilon}\|_{H_0^1(\Omega)}\le C|\widetilde{\varphi}_{0\varepsilon}|\le C$, so $\{\Gamma_\varepsilon\widetilde{\varphi}_{0\varepsilon}\}$ has a subsequence which converges strongly in $L^2(\Omega)$. Since $\widetilde{\varphi}_{0\varepsilon}=\Gamma_\varepsilon\widetilde{\varphi}_{0\varepsilon}+\widetilde{f}_\varepsilon$, we infer that there exists a subsequence of $\{\widetilde{\varphi}_{0\varepsilon}\}$ such that $\widetilde{\varphi}_{0\varepsilon}\to\varphi_0$ in $L^2(\Omega)$, and $|\varphi_0|=1$. Moreover, we can see that $\varphi_0\in R((I-\Gamma)^*)\cap N(I-\Gamma)$, which contradicts the fact that $R((I-\Gamma)^*)\oplus N(I-\Gamma)=L^2(\Omega)$.

By (3.24), we see that

$$|\varphi_\varepsilon(0)|\le|(I-\Gamma_\varepsilon)^{-1}(G_\varepsilon g)|\le C|(G_\varepsilon g)|. \tag{3.25}$$

Finally, we have that

$$\|\varphi_\varepsilon\|_{H^{1,2}(Q)}\le C\|g\|_{L^2(Q)}. \tag{3.26}$$

This implies the first inequality of (3.12). The corresponding properties of the operator $\mathcal{A}_\varepsilon^*$ follow from the same arguments as above. The corresponding results of the operators of $\mathcal{A}$ and $\mathcal{A}^*$ follow similarly.

Proof of Theorem 1.2 (Part II: The necessary condition of optimality) Let $(u_\varepsilon,g_\varepsilon)$ be optimal for (P_ε). For any $w\in X,h\in L^2(Q)$ fixed, we set $u_\varepsilon^\rho=u_\varepsilon+\rho w$ and $g_\varepsilon^\rho=g_\varepsilon+\rho h$. Then $(u_\varepsilon^\rho,g_\varepsilon^\rho)\in X\times L^2(0,T;Q)$. Then,

$$0\le\int_0^T[\langle u_\varepsilon-u_d,w\rangle+\langle g_\varepsilon,h\rangle]\mathrm{d}t$$

$$+\int_0^T \langle u_\varepsilon-\overline{u},w\rangle \mathrm{d}t+\int_0^T \langle g_\varepsilon-\overline{g},h\rangle \mathrm{d}t$$
$$+\int_0^T \langle p_\varepsilon, w_t-\triangle w+G(w)+F'_u(u_\varepsilon)w-\chi_\omega v\rangle \mathrm{d}t, \tag{3.27}$$

where $p_\varepsilon=\frac{1}{\varepsilon}[\partial_t u_\varepsilon-\triangle u_\varepsilon+G(u_\varepsilon)+F(u_\varepsilon)-\chi_\omega g_\varepsilon]$.

By taking $h=0$ in (3.27), we get that

$$\int_0^T \langle u_\varepsilon-u_d,w\rangle \mathrm{d}t+\int_0^T \langle u_\varepsilon-\overline{u},w\rangle \mathrm{d}t+\int_0^T \langle p_\varepsilon,\mathcal{A}_\varepsilon w\rangle \mathrm{d}t=0,\quad \forall\, w\in X. \tag{3.28}$$

Hence, $p_\varepsilon\in D(\mathcal{A}_\varepsilon^*)=X$ and

$$\mathcal{A}_\varepsilon^* p_\varepsilon=-(u_\varepsilon-u_d)-u_\varepsilon-\overline{u}. \tag{3.29}$$

By taking $w=0$ in (3.27), we get that

$$\int_0^T [\langle g_\varepsilon,h\rangle-\langle \chi_\omega p_\varepsilon,h\rangle+\langle g_\varepsilon-\overline{g},h\rangle]\mathrm{d}t=0,\quad \forall\, h\in L^2(Q). \tag{3.30}$$

This yields that

$$\chi_\omega p_\varepsilon=2g_\varepsilon-\overline{g}\quad \text{a.e. in } Q. \tag{3.31}$$

Define the linear bounded operator $D:L^2(Q)\to L^2(Q)$ by $Dp=\chi_\omega p$. (3.31) implies that

$$\|Dp_\varepsilon\|_{L^2(Q)}\le C,\quad \forall\,\varepsilon>0. \tag{3.32}$$

Now, by Lemma 3.3 and the closed range theorem, we may write

$$p_\varepsilon=p_\varepsilon^1+p_\varepsilon^2,\quad p_\varepsilon^1\in R(\mathcal{A}_\varepsilon)(=N(\mathcal{A}_\varepsilon^*)^\perp),\quad p_\varepsilon^2\in N(\mathcal{A}_\varepsilon^*).$$

Then, by (3.29) and Lemma 3.3 again, we get that

$$\|p_\varepsilon^1\|_{H^{1,2}(Q)}\le C,\quad \forall\,\varepsilon>0. \tag{3.33}$$

On the other hand, since the space $N(\mathcal{A}_\varepsilon^*)$ is finite dimensional, we infer that the restriction of D to $N(\mathcal{A}_\varepsilon^*)$, still denoted by D, has closed range. Then, we may write

$$p_\varepsilon^2=p_\varepsilon^3+p_\varepsilon^4,\quad p_\varepsilon^3\in R(D^*),\quad p_\varepsilon^4\in N(D).$$

Then, by (3.32), we see that $\{p_\varepsilon^3\}$ is bounded in $L^2(Q)$. Since $\{p_\varepsilon^3\}\subset N(\mathcal{A}_\varepsilon^*)$ and $\dim N(\mathcal{A}_\varepsilon^*)\le n_0$, there exist $p^3\in L^2(Q)$ and a subsequence of $\{p_\varepsilon^3\}$, still denoted by itself, such that

$$p_\varepsilon^3\to p^3\quad \text{strongly in } L^2(Q) \text{ as } \varepsilon\to 0. \tag{3.34}$$

By (3.29), we know that there exist $p^1\in L^2(Q)$ and a subsequence of $\{p_\varepsilon^1\}$, still denoted by itself, such that

$$p_\varepsilon^1\to p^1\quad \text{strongly in } L^2(Q) \text{ as } \varepsilon\to 0. \tag{3.35}$$

Since $p_\varepsilon^3 + p_\varepsilon^4 = p_\varepsilon^2 \in N(\mathcal{A}_\varepsilon^*)$, we may rewrite (3.29) as

$$\mathcal{A}_\varepsilon^*(p_\varepsilon^1 + p_\varepsilon^3) = -(u_\varepsilon - u_d) - u_\varepsilon - \overline{u}, \tag{3.36}$$

which is equivalent to

$$\int_0^T \langle u_\varepsilon - u_d, w\rangle \mathrm{d}t + \int_0^T \langle u_\varepsilon - \overline{u}, w\rangle \mathrm{d}t + \int_0^T \langle p_\varepsilon^1 + p_\varepsilon^3, \mathcal{A}_\varepsilon w\rangle \mathrm{d}t = 0, \quad \forall\, w \in X. \tag{3.37}$$

Passing to the limit for $\varepsilon \to 0$, we get

$$\int_0^T \langle \overline{u} - u_d, w\rangle \mathrm{d}t + \int_0^T \langle p^1 + p^3, \mathcal{A} w\rangle \mathrm{d}t = 0, \quad \forall\, w \in X. \tag{3.38}$$

This shows that $p^1 + p^3 \in D(\mathcal{A}^*)$, and

$$\mathcal{A}^*(p^1 + p^3) = \overline{u} - u_d. \tag{3.39}$$

Passing to the limit in (3.31) for $\varepsilon \to 0$, we get

$$\chi_\omega(p^1 + p^3) = \overline{g} \quad \text{a.e. in } Q. \tag{3.40}$$

Let $p = p^1 + p^3 \in X$. Then $p \in X$. By (3.39)–(3.40), we derive (1.5)–(1.6). This competes the proof.

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