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On Blow-up of Regular Solutions to the Isentropic Euler and Euler-Boltzmann Equations with Vacuum*

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Abstract In this paper, the authors study the Cauchy problem of n-dimensional isentropic Euler equations and Euler-Boltzmann equations with vacuum in the whole space. They show that if the initial velocity satisfies some condition on the integral J in the "isolated mass group" (see (1.13)), then there will be finite time blow-up of regular solutions to the Euler system with $J \le 0$ $(n \ge 1)$ and to the Euler-Boltzmann system with $J < 0$ $(n \ge 1)$ and $J = 0$ $(n \ge 2)$, no matter how small and smooth the initial data are. It is worth mentioning that these blow-up results imply the following: The radiation is not strong enough to prevent the formation of singularities caused by the appearance of vacuum, with the only possible exception in the case $J = 0$ and $n = 1$ since the radiation behaves differently on this occasion.

Keywords Euler and Euler-Boltzmann equations, Finite time blow-up, Multi-dimensional, Regular solutions, Vacuum
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1 Introduction

It is well-known that the motion of isentropic inviscid fluid can be described by Euler equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p_m = 0, \end{cases} \tag{1.1}$$

where $t \ge 0$ is the time variable, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is the space variable, $\rho(t,x)$ is the mass density, $u(t,x) = (u^1, \cdots, u^n)^{\mathrm{T}}$ is the fluid velocity, p_m is the material pressure satisfying the equation of state

$$p_m = A\rho^\gamma, \tag{1.2}$$

where $A > 0$ is the gas constant, $\gamma > 1$ is the adiabatic exponent. As is known to all, the radiation effects become remarkable in some physical problems as the temperature increases, for example, in the high-temperature plasma physics [24] and various astrophysical contexts [15]. Thus, one needs to consider radiation effects when describing the fluid motion. For the isentropic

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fluid, the coupling of fluid field and radiation field involves momentum source depending on the specific radiation intensity driven by the so called radiation transfer integro-differential equation (see [24]), and the equations of radiation hydrodynamics result from the balances of particles and momentum. More precisely, the mass density $\rho(t,x)$, the fluid velocity $u(t,x)$ and the specific radiation intensity $I(v,\Omega,t,x)$ are governed by the following Euler-Boltzmann equations

$$\begin{cases} \frac{1}{c}I_t + \Omega\cdot\nabla I = A_r, \\ \rho_t + \mathrm{div}(\rho u) = 0, \\ \Big(\rho u + \frac{1}{c^2}F_r\Big)_t + \mathrm{div}(\rho u\otimes u + P_r) + \nabla p_m = 0, \end{cases} \tag{1.3}$$

where v and Ω are radiation variables, $v\in\mathbb{R}^+$ is the frequency of photon and $\Omega\in S^{n-1}$ is the travel direction of photon, S^{n-1} stands for the unit sphere in $\mathbb{R}^n$, and c is the light speed. The impact of radiation on the dynamical properties of the fluid is described by the radiation flux F_r and the radiation pressure tensor P_r:

$$F_r = \int_0^\infty\int_{S^{n-1}} I(v,\Omega,t,x)\Omega\mathrm{d}\Omega\mathrm{d}v, \quad P_r = \frac{1}{c}\int_0^\infty\int_{S^{n-1}} I(v,\Omega,t,x)\Omega\otimes\Omega\mathrm{d}\Omega\mathrm{d}v.$$

The collision term on the right-hand side of the radiative transfer equation is

$$A_r = \sigma_e - \sigma_a I + \int_0^\infty\int_{S^{n-1}}\Big(\frac{v}{v'}\sigma_s I' - \sigma_s' I\Big)\mathrm{d}\Omega'\mathrm{d}v',$$

which involves emission, absorption and scattering of energy, where $I = I(v,\Omega,t,x)$, $I' = I(v',\Omega',t,x)$; $\sigma_e = \sigma_e(v,\Omega,t,x)\geq 0$ is the rate of energy emission due to spontaneous process; $\sigma_a = \sigma_a(v,t,x,\rho)\geq 0$ denotes the absorption coefficient that may depend on the mass density ρ; Similar to absorption, a photon can undergo scattering interactions with matter, and the probability of a photon being scattered from v' to v contained in $\mathrm{d}v$, from Ω' to Ω contained in $\mathrm{d}\Omega$, and traveling a distance $\mathrm{d}s$ is given by the "differential scattering coefficient" $\sigma_s(v'\to v,\Omega'\cdot\Omega)\mathrm{d}v\mathrm{d}\Omega\mathrm{d}s$ (see [17, 24]). Moreover, the time rates of outscattering and inscattering within a unit volume element are

$$\begin{aligned} \text{outscattering} &= \int_0^\infty\int_{S^{n-1}} \sigma_s(v\to v',\Omega\cdot\Omega',\rho)I(v,\Omega,t,x)\mathrm{d}\Omega'\mathrm{d}v', \\ \text{inscattering} &= \int_0^\infty\int_{S^{n-1}} \sigma_s(v'\to v,\Omega'\cdot\Omega,\rho)I(v',\Omega',t,x)\mathrm{d}\Omega'\mathrm{d}v', \end{aligned}$$

and σ_s, σ_s' behave like

$$\sigma_s = \sigma_s(v'\to v,\Omega'\cdot\Omega,\rho) = O(\rho), \quad \sigma_s' = \sigma_s(v\to v',\Omega\cdot\Omega',\rho) = O(\rho).$$

Noticing that, unlike Euler-Poisson or Euler-Maxwell systems, where Euler equations are coupled with an elliptic or a parabolic equation, (1.3) is a system that Euler equations are coupled with a hyperbolic equation. Thus the study of Euler-Boltzmann equations is challenging due to the high complexity and mathematical difficulty of the system itself.

As has been shown in [17, 24], from the assumptions of "induced process" and local thermal equilibrium, σ_e and σ_a can be written as

$$\begin{cases} \sigma_e(v,t,x,\rho) = K_a\overline{B}(v)\Big(1+\dfrac{c^2 I}{2hv^3}\Big), \\ \sigma_a(v,t,x,\rho) = K_a\Big(1+\dfrac{c^2}{2hv^3}\overline{B}(v)\Big), \end{cases}$$

where $\overline{B}(v) \in L^2(\mathbb{R}^+)$ is actually a simplification of Planck function that represents the energy density of black-body radiation, and the black-body holds the smallest radiation; h is the Planck constant, and K_a satisfies

$$K_a = K_a(v,t,x,\rho) = \rho\overline{K}_a(v,t,x,\rho) = o(\rho) \geq 0,$$

in which, $\overline{K}_a \in C^\infty$ for (v,t,x,ρ) and

$$\lim_{\rho\to 0} \overline{K}_a(v,t,x,\rho) = 0.$$

Thus, when $\sigma_s = 0$, the radiative transfer equation can be written as

$$\frac{1}{c}I_t + \Omega\cdot\nabla I = -K_a(I-\overline{B}(v)), \tag{1.4}$$

and the isentropic Euler-Boltzmann system (1.3) is reduced to

$$\begin{cases} \dfrac{1}{c}I_t + \Omega\cdot\nabla I = -K_a(I-\overline{B}(v)), \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u\otimes u) + \nabla p_m = \dfrac{1}{c}\displaystyle\int_0^\infty\int_{S^{n-1}} K_a(I-\overline{B}(v))\Omega \mathrm{d}\Omega\mathrm{d}v. \end{cases} \tag{1.5}$$

One of the motivations that we study the radiation system (1.5) lies on the fact that the isentropic Euler system (1.1) can be, to some extent, regarded as the non-radiation limit of the isentropic Euler-Boltzman system (1.5). In fact, since the radiation of the black-body is the smallest one, so $I \equiv \overline{B}(v)$ implies that the radiation effect is ignored, and formally system (1.5) is reduced to system (1.1). For the rigorous justification of this type of limit, we refer to Ducomet-Nečasová [7] for the diffusion limit of the Navier-Stokes-Boltzmann system, when the radiative intensity is driven to equilibrium or non-equilibrium, see also Lowrie-Morel-Hittinger [21], Buet-Després [2] for more results in this direction. While, as far as we know, the rigorous diffusion limit of system (1.5) is still unknown, which is worth considering in the future work.

In this paper, we consider the singularity formation of regular solutions to the Cauchy problems of (1.1) and (1.5) with initial data

$$(\rho,u)|_{t=0} = (\rho_0,u_0)(x), \quad \rho_0(x) \geq 0 \tag{1.6}$$

and

$$(I,\rho,u)|_{t=0} = (I_0,\rho_0,u_0)(x), \quad \rho_0(x) \geq 0, \tag{1.7}$$

respectively.

For the classical well-posedness of Euler equations, there are rich literatures on the existence of strong or classical solutions with vacuum. For the Cauchy problem of non-isentropic Euler equations with compactly supported initial density and velocity, Makino-Ukai-Kawashima [22] established the local existence of regular solutions in 3D space and proved that the life span is finite for any non-trivial solution, see also Liu-Yang [20] for similar results with damping for non-isentropic Euler equations in 3D space and isentropic Euler equations (1.1) in 1D space. For some small initial density with compact support and smooth initial velocity satisfying

$$\mathrm{dist}(S_p(\nabla u_0(x)), \mathbb{R}^-) \geq \delta > 0, \quad \forall x \in \mathbb{R}^n (n \geq 1),$$

where $S_p(\nabla u_0)$ stands for the spectrum of the matrix ∇u_0 and δ is a constant, Grassin [8] and Serre [25] obtained the global existence of smooth solutions for both isentropic and non-isentropic Euler equations (see also [9]). For the Euler equations with radiation, when the radiation fluid is inviscid and the initial density is away from vacuum, Jiang-Zhong [14] obtained the local existence of C^1 solutions to the Cauchy problem of non-isentropic fluid in multi-dimensional space, see also Jiang-Wang [13] for the initial-boundary value problem in multi-dimensional space and Blanc-Ducomet [1] for the global existence of weak solutions in 1D space. When the initial vacuum is allowed, Jiang-Wang [12] obtained the global existence of weak entropy solutions to the Cauchy problem of (1.5) in 1D space, see also Jiang [10] for isothermal fluids. Li-Zhu [19] proved the local existence of Makino-Ukai-Kawashima type (see [22]) regular solutions to the Cauchy problem of (1.5) in 3D space. For the Navier-Stokes-Boltzmann equations of viscous radiation fluid, we refer to Ducomet-Feireisl-Nečasová [3], Ducomet-Nečasová [3–6], Li-Zhu [17–18] and references therein for related existence results.

In this paper, we are interested in the finite time blow-up of regular solutions with initial density containing vacuum state. For the case without initial vacuum, Sideris [27] proved two types of finite time blow-up of C^1 solutions to the Cauchy problem of Euler equations in 3D space. The first one is for the non-isentropic Euler equations, if the initial data are "large", where the "large" essentially means that the initial flow velocity must be supersonic in some region, then the singularity formation is detected as a disturbance that overtakes the wave front forcing the front to propagate with supersonic speed. This blow-up result has been generalized to the multi-dimensional non-isentropic Euler-Boltzmann equations by Jiang-Zhong [14] and to the 3D isentropic version (1.5) by Jiang-Wang [11], with the initial radiation satisfying

$$I_0(v, \Omega, x) \geq \overline{B}(v) \quad \text{and} \quad I_0(v, \Omega, x) \equiv \overline{B}(v) \quad \text{if } |x| \geq R, \tag{1.8}$$

for some positive constant R. The second one shows that singularity will develop for both isentropic and non-isentropic Euler equations if the fluid, on average, is slightly compressed and out-going near the wave front, which has been generalized to the 3D isentropic Euler-Boltzmann equations (1.5) by Jiang-Wang [11] with the initial radiation I_0 satisfying (1.8) and in addition

$$I_0(v, \Omega, x) \equiv \overline{B}(v) \quad \text{for } x \cdot \Omega \leq 0. \tag{1.9}$$

For the case with initial vacuum, Serre [26] proved that the regular solution to the Cauchy problem of Euler system (1.1) will blow up in finite time if the initial density is compactly supported in some bounded region $V \subset \mathbb{R}^n (n \geq 1)$ and the initial velocity satisfies some integral condition, i.e.,

$$\mathrm{supp}\rho_0 \subset V, \quad \int_V \det\nabla u_0(x)\mathrm{d}x \leq 0. \tag{1.10}$$

For the Euler-Boltzmann system (1.5), inspired by the "isolated mass group" introduced by Xin-Yan [30] for Navier-Stokes equations and the global existence established by Grassin and Serre [8–9, 25] for Euler equations, Li-Zhu [19] identified the following two classes of initial data which contain local vacuum states such that the regular solutions of Cauchy problem blow up in finite time:

(i) (Isolated Mass Group) For some bounded open sets $A_0 \subset B_0 \subset B_{R_0} \subset \mathbb{R}^3$,

$$\begin{aligned} &\rho_0(x) = u_0(x) = 0, \quad \forall x \in B_0 \backslash A_0, \ \int_{A_0} \rho_0(x)\mathrm{d}x > 0, \\ &I_0(v,\Omega,x) \equiv \overline{B}(v), \quad \forall x \in B^C_{R_0}, \end{aligned} \tag{1.11}$$

where R_0 is a positive constant and B_{R_0} is the ball centered at the origin with radius R_0, $B^C_{R_0}$ is the complementary set of B_{R_0} in $\mathbb{R}^3$. This class removed the key assumption (1.9) for radiation field in [11].

(ii) (Hyperblic Singularity Set) In some smooth open set $\mathbb{V} \subset \mathbb{R}^3$,

$$\rho_0(x) = 0, \quad S_p(\nabla u_0(x)) \cap \mathbb{R}^- \neq \emptyset, \quad \forall\ x \in \mathbb{V}. \tag{1.12}$$

They also proved that this blow-up mechanism also holds for the corresponding non-radiation version, i.e., the Euler system (1.1).

For the finite time formation of singularities on the Navier-Stokes-Boltzmann equations of viscous fluid, we refer to [17] and the references cited therein for details.

As have been observed in Li-Zhu [19], according to the definition of regular solutions (see Definition 1.1), one has

$$\partial_t u + u \cdot \nabla u = 0, \quad \text{when } \rho = 0,$$

which implies that in the vacuum domain, the behavior of the velocity is controlled by a positive and symmetric hyperbolic system, i.e., the so called multi-dimensional Burgers equations. Thus, in general, the velocity may not be zero in vacuum region. In this paper, based on the similar idea, we will present a scenario (see Definition 1.2) for finite time singularity formation of regular solutions to the Cauchy problems (1.1) with (1.6) and (1.5) with (1.7) for both Euler and Euler-Boltzmann equations.

Before stating our main results, we give some related definitions. The first one is the regular solution of system (1.1) and system (1.5) that we consider in this paper.

Definition 1.1 (Regular Solutions) *For $1 < \gamma \leq 3$, let $T > 0$ be a finite constant. If*

(1) $(\rho, u)(t,x) \in C^1([0,T) \times \mathbb{R}^n)$, $\rho^{\frac{\gamma-1}{2}} \in C^1([0,T) \times \mathbb{R}^n)$, $\rho \geq 0$;

(2) $u_t + u \cdot \nabla u = 0$ *holds when* $\rho(t,x) = 0$, *then* $(\rho, u)(t,x)$ *is called a regular solution to the Cauchy problem* (1.1) *and* (1.6) *in* $[0,T) \times \mathbb{R}^n$;

If, in addition;

(3) $I(v,\Omega,t,x) \in L^2(\mathbb{R}^+ \times S^{n-1}, C^1([0,T) \times \mathbb{R}^n))$, *then* $(I, \rho, u)(t,x)$ *is called a regular solution to the Cauchy problem* (1.5) *and* (1.7) *in* $[0,T) \times \mathbb{R}^n$.

Remark 1.1 It is worth pointing out that the local existence of regular solutions to (1.1) and (1.5) in 3D space have been established by Makino-Ukai-Kawashima [22] and Li-Zhu [19], respectively. Actually, these frameworks in [22] and [19] are applicable to arbitrary space dimension with some minor modifications.

Now we give the definition of "isolated mass group" in this paper.

Definition 1.2 (Isolated Mass Group) *Let A_0, B_0 be two smooth, bounded and connected open sets in $\mathbb{R}^n$, and $\overline{A}_0 \subset B_0 \subseteq B_{R_0}$, where $\overline{A}_0$ is the closure of A_0 under standard Euclidean norm, R_0 is a positive constant and B_{R_0} is the ball centered at the origin with radius R_0. If*

$$\begin{cases} \rho_0(x) = 0, \quad \forall x \in B_0 \backslash A_0, \quad \displaystyle\int_{A_0} \rho_0(x)\mathrm{d}x > 0, \\ J := \displaystyle\int_{A_0} \det \nabla u_0(x)\mathrm{d}x \le 0, \end{cases} \tag{1.13}$$

we say that $(\rho_0, u_0)(x)$ has an isolated mass group (A_0, B_0).

Remark 1.2 This definition is inspired by Li-Zhu [19] and Serre [26], we replace the condition on u_0 in [19] (i.e., $u_0 = 0$ when $\rho_0 = 0$ in (1.11)) by the one introduced in [26] (i.e., the condition on the integral J in (1.13)), where the integral J depends only on the value of u_0 on the boundary ∂A_0. In fact, by direct calculation, one has

$$\det \nabla u_0(x) = \mathrm{div}(u_0^j(U_{1j}, \cdots, U_{nj}))$$

for any $j = 1, \cdots, n$, where $U_{ij} (i = 1, \cdots, n)$ are cofactors of the matrix

$$U = (\nabla u_0^1, \cdots, \nabla u_0^{j-1}, \mathbf{1}, \nabla u_0^{j+1}, \cdots, \nabla u_0^n),$$

in which, $\mathbf{1} = (1, \cdots, 1)^{\mathrm{T}} \in \mathbb{R}^n$. Thus

$$J = \int_{A_0} \det \nabla u_0 \mathrm{d}x = \int_{A_0} \mathrm{div}(u_0^j(U_{1j}, \cdots, U_{nj}))\mathrm{d}x = \int_{\partial A_0} u_0^j(U_{1j}, \cdots, U_{nj}) \cdot \nu \mathrm{d}S,$$

where ν is the unit outward normal vector of ∂A_0, this implies that J only depends on the value of u_0^j on the boundary ∂A_0. For example, when $n = 3$, one has

$$\begin{aligned} J &= \int_{A_0} \det \nabla u_0 \mathrm{d}x = \int_{\partial A_0} u_0^1(U_{11}, U_{21}, U_{31}) \cdot \nu \mathrm{d}S \\ &= \int_{\partial A_0} u_0^1(\nabla u_0^2 \times \nabla u_0^3) \cdot \nu \mathrm{d}S = \int_{\partial A_0} u_0^1 \det(\nu, \nabla u_0^2, \nabla u_0^3)\mathrm{d}S, \end{aligned}$$

when $n = 1$, one has

$$J = \int_{A_0} (u_0)_x \mathrm{d}x = u_0|_{\partial A_0}.$$

Now we state main theorems in this paper, the first one is the finite time blow-up for Euler equations.

Theorem 1.1 *Let $1 < \gamma \le 3$. Assume that the initial data $(\rho_0, u_0)(x)$ has an isolated mass group (A_0, B_0), and $(\rho, u)(t, x)$ is the corresponding regular solution to the Cauchy problem* (1.1) *and* (1.6) *in* $[0, T_m) \times \mathbb{R}^n$ *with maximal existence time* T_m. *Then for all* $n \geq 1$, *under either* $J < 0$ *or*

$$J = 0 \quad \text{with } 1 < \gamma \leq 1 + \frac{2}{n},$$

we both have $T_m < +\infty$.

The second one is the finite time blow-up for Euler-Boltzmann equations.

Theorem 1.2 *Let* $1 < \gamma \leq 3$. *Assume that the initial data* $(\rho_0, u_0)(x)$ *has an isolated mass group* (A_0, B_0), $|u_0(x)|_{L^\infty(\mathbb{R}^n)} < c$, I_0 *satisfies*

$$I_0 \equiv \overline{B}(v), \quad \forall (v, \Omega, x) \in \mathbb{R}^+ \times S^{n-1} \times B^C_{R_0}, \tag{1.14}$$

where $B^C_{R_0}$ *is the complementary set of* B_{R_0} *in* $\mathbb{R}^n$, *and* $(I, \rho, u)(t, x)$ *is the corresponding regular solution to the Cauchy problem* (1.5) *and* (1.7) *in* $[0, T_m) \times \mathbb{R}^n$ *with maximal existence time* T_m. *Then under either* $J < 0$ *for all* $n \geq 1$ *or*

$$J = 0 \quad \text{with } 1 < \gamma \leq 1 + \frac{2}{n} \text{ and } n \geq 2,$$

we both have $T_m < +\infty$.

Remark 1.3 Compared with the blow-up result obtained in [26] for system (1.1), we remove the assumption that the initial density is compactly supported, i.e., the vacuum can appear in local domain. Compared with the two blow-up results obtained in [19] for system (1.5), we remove the assumption in the first result (see (1.11)) that the initial velocity vanishes where initial density vanishes, and we only need the information of initial velocity on the local vacuum boundary $\partial A(t)$ instead of the vacuum domain (see (1.12)) assumed in the second result of [19].

Remark 1.4 For the case without radiation, Theorem 1.1 shows that the appearance of vacuum will cause finite time blow-up of regular solutions to (1.1) in arbitrary dimensional space. Moreover, even with the radiation effect, Theorem 1.2 shows that the vacuum still leads to finite time blow-up of regular solutions to (1.5) in arbitrary dimensional space when $J < 0$ or in multi-dimensional space ($n \geq 2$) when $J = 0$, which implies that the radiation is not strong enough to prevent the formation of singularities caused by vacuum in multi-dimensional space.

In the rest section, we first provide some preliminary lemmas required for the proofs of Theorems 1.1–1.2, then we prove Theorem 1.1 and Theorem 1.2, respectively.

2 Finite Time Blow-up

Hereinafter, it is always assumed that initial data $(\rho_0, u_0)(x)$ and $(I_0, \rho_0, u_0)(x)$ satisfy assumptions in Theorems 1.1–1.2, $(\rho, u)(t, x)$ and $(I, \rho, u)(t, x)$ are regular solutions to the corresponding Cauchy problems of (1.1) and (1.5) in $[0, T_m) \times \mathbb{R}^n$, respectively.

2.1 Preliminaries

Now, we introduce some necessary quantities and preliminary lemmas. First, we define the particle path generated by the velocity $u(t, x)$.

Definition 2.1 (Particle Path) *Let $X(t;0,x_0)$ be the particle path starting from x_0 at $t=0$, i.e.,*

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t;0,x_0)=u(t,X(t;0,x_0)),\quad X(t;0,x_0)=x_0,\tag{2.1}$$

and denote $A(t),B(t),B(t)\backslash A(t)$ as the images of $A_0,B_0,B_0\backslash A_0$ under (2.1), respectively, then,

$$\begin{aligned}A(t)&=\{X(t;0,x_0)\mid x_0\in A_0\},\quad B(t)=\{X(t;0,x_0)\mid x_0\in B_0\},\\ B(t)\backslash A(t)&=\{X(t;0,x_0)\mid x_0\in B_0\backslash A_0\}.\end{aligned}$$

This definition implies the evolution of A_0. Moreover, for any $t\in[0,T_m)$, according to the mass equation $(1.1)_1$ or $(1.5)_2$, one has

$$\rho(t,X(t;0,x))=\rho_0(x)\exp\Big(\int_0^t\mathrm{div}u(s,X(s;0,x))\mathrm{d}s\Big),$$

which, along with the condition in (1.13): $\rho_0(x)=0$ in $B_0\backslash A_0$, we have $\rho(t,x)=0$ in $B(t)\backslash A(t)$. Then, together with the definition of regular solutions, one has

$$u_t+u\cdot\nabla u=0\quad\text{in }B(t)\backslash A(t).$$

Thus for each $x\in B_0\backslash A_0$, it yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,X(t;0,x))=u_t(t,X(t;0,x))+u(t,X(t;0,x))\cdot\nabla u(t,X(t;0,x))=0,$$

which implies that

$$u(t,X(t;0,x))=u_0(x),\quad\forall x\in B_0\backslash A_0,\ t\in[0,T_m).$$

This, with the aid of (2.1), gives

$$X(t;0,x)=x+tu_0(x),\quad\forall x\in\partial A_0,\ t\in[0,T_m),$$

and it immediately has

$$\partial A(t)=\{x=x_0+tu_0(x_0)\mid x_0\in\partial A_0\}.\tag{2.2}$$

Here, (2.2) shows the evolution of the set A_0.

To show the finite time formation of singularities, in the region $A(t)$, we define the following physical quantities:

$$\begin{aligned}M(t)&=\int_{A(t)}\rho|x|^2\mathrm{d}x\text{ (second momentum)},\\ F(t)&=\int_{A(t)}\rho u\cdot x\mathrm{d}x\text{ (first moment of momentum density)},\\ \varepsilon(t)&=\int_{A(t)}\Big(\frac{1}{2}\rho|u|^2+\frac{p_m}{\gamma-1}\Big)\mathrm{d}x\text{ (total energy)},\end{aligned}$$

which is used to define the following functional (see also [16, 29]):

$$H(t)=M(t)-2(t+1)F(t)+2(t+1)^2\varepsilon(t).\tag{2.3}$$

We also denote by

$$m(t)=\int_{A(t)}\rho \mathrm{d}x$$

the total mass in $A(t)$, then it is clear that $m(0)>0$ according to (1.13).

Now, we give some useful lemmas. The first one is the Reynolds transport lemma (see [28]).

Lemma 2.1 *Considering any part of the fluid in $\tau(t)$ and with velocity u, for any $G(t,x)\in C^1(\mathbb{R}^+\times\mathbb{R}^n)$, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tau(t)}G(t,x)\mathrm{d}x=\int_{\tau(t)}G_t\mathrm{d}x+\int_{\partial\tau(t)}Gu\cdot\nu\mathrm{d}S,$$

where ν is the unit outward normal vector and $\mathrm{d}S$ is the surface elements of $\partial\tau(t)$.

With the help of Lemma 2.1, the second lemma implies the conservation of mass in $A(t)$.

Lemma 2.2 *The mass is conserved in $A(t)$ for both Euler and Euler-Boltzmann systems, i.e.,*

$$m(t)=m(0),\quad t\in[0,T_m).$$

Proof According to $(1.1)_2$ or $(1.5)_2$ and Lemma 2.1, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t)=\int_{A(t)}\rho_t\mathrm{d}x+\int_{\partial A(t)}\rho u\cdot\nu\mathrm{d}S=\int_{A(t)}(\rho_t+\mathrm{div}(\rho u))\mathrm{d}x=0,$$

which implies that $m(t)=m(0)$ for any $t\in[0,T_m)$.

The third lemma implies that the volume of the region $A(t)$ is a polynomial function of t with degree no more than n (see [23 Theorem 1, 26, Lemma 2.1]).

Lemma 2.3 (Polynomial Growth of Volume) *In $[0,T_m)$, the volume of $A(t)$ is a polynomial with degree no more than n, and the coefficient of the highest order term t^n in the polynomial $|A(t)|$ is exactly J.*

Moreover, the volume of $A(t)$ has strictly positive lower bound.

Lemma 2.4 *For both Euler and Euler-Boltzmann systems, the volume of $A(t)$ satisfies*

$$|A(t)|\geq C_0(1+t)^k,\quad t\in[0,T_m)$$

for some nonnegative integer $k\leq n$ and positive constant C_0 that depends on u_0. More precisely, if $J=0$, then $k\leq n-1$.

Proof First, if $|A(t)|=0$ for some $t_0\in[0,T_m)$, then $m(t_0)=0$, which contradicts to the conservation of mass in $A(t)$, i.e., $m(t_0)=m(0)>0$ (see Lemma 2.2). Thus

$$|A(t)|>0,\quad \forall t\in[0,T_m).$$

Moreover, there exists a constant $C_0>0$ such that

$$|A(t)|\geq C_0,\quad \forall t\in[0,T_m), \tag{2.4}$$

otherwise, for any $T^* < T_m$, one can find a sequence $\{t_j\}$ $(j = 1, 2, \cdots)$, $t_j \in [0, T^*]$ satisfying

$$|A(t_j)| \le \frac{C_0}{j},$$

thus there exists a limit $t_1 \in [0, T_m)$ of the bounded sequence $\{t_j\}$, such that $|A(t_1)| = 0$, which contradicts to Lemma 2.2.

Second, according to Lemma 2.3, $|A(t)|$ is a polynomial with degree no more than n, thus with the aid of (2.4), there exists a integer $0 \le k \le n$, such that

$$|A(t)| \ge C_0(1+t)^k, \quad t \in [0, T_m) \tag{2.5}$$

for some positive constant C_0 that depends on u_0.

Especially, when $J = 0$, $|A(t)|$ is a polynomial with degree no more than $n-1$, thus (2.5) holds for some nonnegative integer $k \le n-1$.

In the following lemma, we give some basic estimates on $H(t)$.

Lemma 2.5 *For $1 < \gamma \le 3$ and $t \in [0, T_m)$, it holds for both Euler and Euler-Boltzmann systems that*

$$H(t) \ge \frac{2(t+1)^2}{\gamma-1} \int_{A(t)} p_m \mathrm{d}x \ge C(1+t)^{2-k(\gamma-1)} \tag{2.6}$$

for $J = 0$ and some nonnegative integer $k \le n-1$, where C is a positive constant that depends on $C_0, m(0), A$ and γ.

Proof First, according to the definition of $H(t)$, one has

$$\begin{aligned} H(t) &= \int_{A(t)} \Big(\rho|x|^2 - 2(t+1)\rho u \cdot x + 2(t+1)^2\Big(\frac{1}{2}\rho|u|^2 + \frac{p_m}{\gamma-1}\Big)\Big)\mathrm{d}x \\ &= \int_{A(t)} \Big(\rho|x-(t+1)u|^2 + 2(t+1)^2\frac{p_m}{\gamma-1}\Big)\mathrm{d}x, \end{aligned} \tag{2.7}$$

thus by the non-negativity of the first term on the right-hand side of (2.7), it is easy to show that

$$\begin{aligned} H(t) &\ge \frac{2(t+1)^2}{\gamma-1}\int_{A(t)} p_m \mathrm{d}x = \frac{2A(t+1)^2}{\gamma-1}\int_{A(t)} \rho^\gamma \mathrm{d}x \\ &\ge \frac{C(t+1)^2}{\gamma-1}|A(t)|^{1-\gamma} m(0)^\gamma \ge C(1+t)^{2-k(\gamma-1)}, \end{aligned}$$

where we have used Lemma 2.4 and the following estimate

$$m(0) = \int_{A(t)} \rho \mathrm{d}x \le \Big(\int_{A(t)} \rho^\gamma \mathrm{d}x\Big)^{\frac{1}{\gamma}} |A(t)|^{\frac{\gamma-1}{\gamma}},$$

and C is a positive constant that depends on $C_0, m(0), A$ and γ.

2.2 Blow-up of Euler equations

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 We divide the proof into two steps.

Step 1 The case of $J = 0$. We will control $H(t)$ by using Gronwall's inequality for $t \in [0, T_m)$. On one hand, from the definition of $H(t)$, one has

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} = \frac{\mathrm{d}M(t)}{\mathrm{d}t} - 2(t+1)\frac{\mathrm{d}F(t)}{\mathrm{d}t} + 2(t+1)^2\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} - 2F(t) + 4(t+1)\varepsilon(t), \tag{2.8}$$

in which, from the definition of $M(t)$ and $(1.1)_1$, one has

$$\begin{aligned}\frac{\mathrm{d}M(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{A(t)} \rho|x|^2\mathrm{d}x = \int_{A(t)} \rho_t|x|^2\mathrm{d}x + \int_{\partial A(t)} \rho u|x|^2 \cdot \nu\mathrm{d}S \\ &= -\int_{A(t)} \mathrm{div}(\rho u)|x|^2\mathrm{d}x = 2\int_{A(t)} \rho u \cdot x\mathrm{d}x,\end{aligned} \tag{2.9}$$

where we have used the fact that $\rho = 0$ on $\partial A(t)$. Similarly,

$$\begin{aligned}-\frac{\mathrm{d}F(t)}{\mathrm{d}t} &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{A(t)} \rho u \cdot x\mathrm{d}x = -\int_{A(t)} (\rho u)_t \cdot x\mathrm{d}x \\ &= \int_{A(t)} (\mathrm{div}(\rho u \otimes u) \cdot x + \nabla p_m \cdot x)\mathrm{d}x \\ &= -\int_{A(t)} (\rho|u|^2 + np_m)\mathrm{d}x,\end{aligned} \tag{2.10}$$

where we have used $(1.1)_2$, integrating by parts and $\rho = 0$ on $\partial A(t)$. For

$$\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} = \int_{A(t)} \Big(\frac{1}{2}\rho|u|^2 + \frac{p_m}{\gamma-1}\Big)_t\mathrm{d}x,$$

noticing that

$$\begin{aligned}\frac{1}{2}(\rho|u|^2)_t &= \rho u_t \cdot u + \frac{1}{2}\rho_t|u|^2 = \rho u_t \cdot u - \frac{1}{2}\mathrm{div}(\rho u)|u|^2, \\ \frac{(p_m)_t}{\gamma-1} &= -p_m\mathrm{div}u - \frac{\mathrm{div}(p_m u)}{\gamma-1},\end{aligned}$$

with the help of integrating by parts and $(1.1)_2$, it arrives at

$$\begin{aligned}\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} &= \int_{A(t)} \Big(\rho u_t \cdot u - \frac{1}{2}\mathrm{div}(\rho u)|u|^2 + \nabla p_m \cdot u\Big)\mathrm{d}x \\ &= \int_{A(t)} \Big((\rho u)_t \cdot u + \mathrm{div}(\rho u \otimes u) \cdot u + \nabla p_m \cdot u - \frac{1}{2}\mathrm{div}(\rho u|u|^2)\Big)\mathrm{d}x = 0,\end{aligned} \tag{2.11}$$

where we have used the equalities

$$-\rho_t|u|^2 - \frac{1}{2}\mathrm{div}(\rho u)|u|^2 = \frac{1}{2}\mathrm{div}(\rho u)|u|^2 = \mathrm{div}(\rho u \otimes u) \cdot u - \frac{1}{2}\mathrm{div}(\rho u|u|^2).$$

Thus, submitting (2.9)–(2.11) into (2.8), it arrives at

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} = \frac{2(2 - n(\gamma-1))(t+1)}{\gamma-1}\int_{A(t)} p_m\mathrm{d}x$$

$$= \frac{2-n(\gamma-1)}{t+1}\ \frac{2(t+1)^2}{\gamma-1}\int_{A(t)} p_m \mathrm{d}x. \tag{2.12}$$

Then, for $2-n(\gamma-1)\geq 0$, i.e., $1<\gamma\leq 1+\dfrac{2}{n}$, combination of (2.12) and (2.6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t)\leq \frac{2-n(\gamma-1)}{t+1}H(t) \quad \text{for } t\in[0,T_m). \tag{2.13}$$

With the help of (2.13) and the Gronwall's inequality, we immediately have

$$H(t)\leq (t+1)^{2-n(\gamma-1)}H(0).$$

This, together with (2.6), gives

$$(1+t)^{2-k(\gamma-1)}\leq C(t+1)^{2-n(\gamma-1)} \tag{2.14}$$

for some constant $C>1$ that depends on $C_0, m(0), A$ and γ. Since for $k\leq n-1$ and $n\geq 1$, it holds that

$$2-k(\gamma-1)>2-n(\gamma-1),$$

thus (2.14) implies $T_m<+\infty$, otherwise, there is a contradiction to (2.14).

Step 2 The case of $J<0$. According to Lemma 2.3, the polynomial $|A(t)|$ can be expressed as

$$|A(t)| = Jt^n + \text{l.o.t}, \tag{2.15}$$

where l.o.t denotes the lower order terms in the polynomial. When $J<0$, for any $n\geq 1$ and $t>0$ large enough, we have $|A(t)|\leq 0$, then $m(t)=0$, which contradicts to the conservation of mass in Lemma 2.2, thus $T_m<+\infty$.

The proof of Theorem 1.1 is finished.

2.3 Blow-up of Euler-Boltzmann equations

Proof of Theorem 1.2 We divide the proof into two steps.

Step 1 When $J<0$, the proof is the same as the case of the Euler system, see Step 2 in the proof of Theorem 1.1. This implies that the radiation has no effect when $J<0$.

Step 2 When $J=0$, we need to consider the behavior of radiation effect. First, one knows that

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} = \frac{\mathrm{d}M(t)}{\mathrm{d}t} - 2(t+1)\frac{\mathrm{d}F(t)}{\mathrm{d}t} + 2(t+1)^2\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} - 2F(t) + 4(t+1)\varepsilon(t), \tag{2.16}$$

in which, by using $(1.5)_2$ and the fact that $\rho=0$ on $\partial A(t)$, one has

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{A(t)}\rho|x|^2\mathrm{d}x = 2\int_{A(t)}\rho u\cdot x\mathrm{d}x. \tag{2.17}$$

Similarly, with the help of $(1.5)_3$ and integrating by parts, one has

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{A(t)}\rho u\cdot x\mathrm{d}x = \int_{A(t)}(\rho u)_t\cdot x\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{A(t)} \Big(- \operatorname{div}(\rho u \otimes u) - \nabla p_m + \frac{1}{c}\int_0^\infty \int_{S^{n-1}} K_a(I - \overline{B}(v))\Omega \mathrm{d}\Omega \mathrm{d}v \Big) \cdot x \mathrm{d}x \\
&= \int_{A(t)} (\rho |u|^2 + n p_m) \mathrm{d}x + \frac{1}{c}\int_{A(t)} \int_0^\infty \int_{S^{n-1}} K_a(I - \overline{B}(v))\Omega \mathrm{d}\Omega \mathrm{d}v \cdot x \mathrm{d}x \qquad (2.18)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} &= \int_{A(t)} \Big(\rho u_t \cdot u - \frac{1}{2}\operatorname{div}(\rho u)|u|^2 + \nabla p_m \cdot u\Big)\mathrm{d}x \\
&= \int_{A(t)} ((\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p_m) \cdot u \mathrm{d}x \\
&= \frac{1}{c}\int_{A(t)} \Big(\int_0^\infty \int_{S^{n-1}} K_a(I - \overline{B}(v))\Omega \mathrm{d}\Omega \mathrm{d}v\Big) \cdot u \mathrm{d}x \qquad (2.19)
\end{aligned}
$$

for all $t \in [0, T_m)$.

Now, we need to consider the radiation effect.

We first claim the following: In the multi-dimensional space, the assumption on I_0 in (1.14) results in the phenomena that the impact of radiation on dynamical properties of the fluid in $A(t)$ vanishes after some time T_b, i.e., for $n \geq 2$, one has

$$
I(v, \Omega, t, x) \equiv \overline{B}(v), \quad \forall (v, \Omega, t, x) \in \mathbb{R}^+ \times S^{n-1} \times [T_b, T_m) \times A(t). \qquad (2.20)
$$

Noticing that, if $T_m \leq T_b$, then T_m is finite and Theorem 1.2 follows immediately, thus we only consider the case that $T_b < T_m$.

Second, combining the claim (2.20) with (2.18)–(2.19), it arrives at

$$
\frac{\mathrm{d}F(t)}{\mathrm{d}t} = \int_{A(t)} (\rho |u|^2 + n p_m) \mathrm{d}x, \quad \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} = 0 \quad \text{for } T_b \leq t < T_m. \qquad (2.21)
$$

Submitting (2.17) and (2.21) into (2.16), one has

$$
\frac{\mathrm{d}H(t)}{\mathrm{d}t} = \frac{2 - n(\gamma - 1)}{t+1} \frac{2(t+1)^2}{\gamma - 1} \int_{A(t)} p_m \mathrm{d}x \quad \text{for } T_b \leq t < T_m, \qquad (2.22)
$$

thus, for $2 - n(\gamma - 1) \geq 0$, i.e., $1 < \gamma \leq 1 + \dfrac{2}{n}$, combination of (2.22) and (2.6) immediately implies

$$
\frac{\mathrm{d}}{\mathrm{d}t} H(t) \leq \frac{2 - n(\gamma - 1)}{t+1} H(t) \quad \text{for } T_b \leq t < T_m. \qquad (2.23)
$$

With the help of (2.23) and the Gronwall's inequality, one has

$$
H(t) \leq (t+1)^{2-n(\gamma-1)} H(T_b),
$$

where $H(T_b)$ is independent of T_m. This, together with (2.6) in Lemma 2.5, gives

$$
(t+1)^{2-k(\gamma-1)} \leq C(t+1)^{2-n(\gamma-1)} \qquad (2.24)
$$

for some constant $C > 1$ that depends on $C_0, m(0), A, \gamma$ and T_b. Then for $1 < \gamma \leq 1 + \frac{2}{n}$, and any nonnegative integer $k \leq n - 1$, (2.24) implies $T_m < +\infty$.

Now it remains to prove the claim (2.20) to finish the proof of Theorem 1.2, which will be proved in the following lemma.

Lemma 2.6 (Behavior of Radiation Effect) *Let $n \geq 2$. For the regular solution to the Cauchy problem* (1.5) *and* (1.7)*, there exists a time $T_b < +\infty$, such that*

$$I(v,\Omega,t,x) \equiv \overline{B}(v), \quad \forall (v,\Omega,t,x) \in \mathbb{R}^+ \times S^{n-1} \times [T_b, T_m) \times A(t).$$

Proof Since $\overline{B}(v)$ is independent of t and x, for $n \geq 2$, according to $(1.5)_1$, one has

$$\frac{1}{c}(I-\overline{B}(v))_t + \Omega \cdot \nabla (I-\overline{B}(v)) = -K_a(I-\overline{B}(v)). \tag{2.25}$$

If we denote by $y(t;y_0)$ the photon path starting from y_0 at $t=0$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t;y_0) = c\Omega, \quad y(0;y_0) = y_0,$$

then it is easy to show that $y_0 = y - c\Omega t$. According to (2.25), along the photon path, we have

$$(I-\overline{B}(v))(t,y(t;y_0)) = (I_0 - \overline{B}(v))(y_0)\exp\Big(\int_0^t -cK_a(v,s,y(s;y_0),\rho)\mathrm{d}s\Big). \tag{2.26}$$

Considering the photon positioned in $A(t)$, since the light speed $c > |u_0|_{L^\infty(\mathbb{R}^n)}$, thus taking

$$T_b = \frac{2R_0}{c - |u_0|_{L^\infty(\mathbb{R}^n)}}, \tag{2.27}$$

and combining with (2.2), $\overline{A}_0 \subset B_{R_0}$ and $|\Omega| = 1$ for $n \geq 2$, one has

$$|y_0| = |y - c\Omega t| \geq |R_0 - (c - |u_0|_{L^\infty(\mathbb{R}^n)})t| \geq R_0 \quad \text{for } t \in [T_b, T_m),\ y \in A(t), \tag{2.28}$$

which implies that after the time T_b, the photon positioned in $A(t)$ comes from $B_{R_0}^C$. Thus, together with (2.26) and the assumption on I_0 in (1.14), it arrives at

$$I(v,\Omega,t,x) = \overline{B}(v), \quad \forall (v,\Omega,t,x) \in \mathbb{R}^+ \times S^{n-1} \times [T_b, T_m) \times A(t).$$

The proof of this lemma is completed.

Remark 2.1 We emphasize that for 1D case, the Euler-Boltzmann system (see [3, 6, 17]) is deduced from the multi-dimensional case by considering only one single space variable. More precisely, one can consider the three-dimensional case with specific radiation intensity $I = I(v,\Omega,t,x)$ that depends only on the single spatial coordinate x_3 and the single angular coordinate ϕ, the angle between Ω and x_3 axis. Introducing $\omega = \cos\phi$, since $I = I(v,\omega,t,x_3)$, we have

$$\Omega \cdot \nabla I(v,\omega,t,x_3) = \Omega_3 \partial_{x_3} I(v,\omega,t,x_3) = \omega \partial_{x_3} I(v,\omega,t,x_3).$$

So, the one-dimensional radiation hydrodynamics equations read as (see [17, 24])

$$\begin{cases} \dfrac{1}{c} I_t + \omega I_x = -K_a(I - \overline{B}(v)), \\ \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u \otimes u)_x + (p_m)_x = \dfrac{1}{c}\displaystyle\int_0^\infty \int_{S^0} K_a(I-\overline{B}(v))\omega \mathrm{d}\omega \mathrm{d}v, \end{cases} \tag{2.29}$$

where $\omega \in S^0 = [-1, 1]$ stands for the angular variable (we emphasize here that $|\omega|$ is not just equal to 1, which is different from the multi-dimensional case, where $|\Omega| = 1$ for $n \geq 2$).

Moreover, according to $(2.29)_1$, under the assumptions in this paper, the radiative effect in $A(t)$ will never disappear, in fact, due to $|\omega| \leq 1$, we are not able to find a uniform time T_b such that (2.28) holds. Thus the radiation related terms on right-hand side of (2.18)–(2.19) will never disappear, and we are not able to deduce finite time blow-up for 1D case under the framework in this paper.

Remark 2.2 We also emphasize that, for 1D space, if we denote $A_0 = (a_1, a_2)$, $B_0 = (b_1, b_2)$ for some constants $b_1 < a_1 < a_2 < b_2$, then the assumption $J \leq 0$ is equivalent to

$$J = \int_{A_0} (u_0)_x \mathrm{d}x = \int_{a_1}^{a_2} (u_0)_x \mathrm{d}x = u_0(a_2) - u_0(a_1) \leq 0, \quad \text{i.e., } u_0(a_2) \leq u_0(a_1). \tag{2.30}$$

Compared with the two blow-up results in [19], the first one needs $u_0(x) = 0$ on $(b_1, a_1] \cup [a_2, b_2)$, which plays an important role in proving the finite time blow-up of regular solutions to (2.29), and is much stronger than our condition (2.30); The second one needs $Sp(\nabla u_0) \cap \mathbb{R}^- \neq \emptyset$ in the vacuum domain, which is equivalent to $(u_0)_x < 0$ on $(b_1, a_1] \cup [a_2, b_2)$, here, we only need the condition on the boundary points a_1, a_2.

Remark 2.3 Based on Theorems 1.1–1.2, a natural question that we are working on is to consider whether there exists a global regular solution to the Euler system or the Euler-Boltzmann system for all dimensions $n \geq 1$ when $J > 0$, as well as to the Euler-Boltzmann system for dimension $n = 1$ when $J = 0$.

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