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Some Gradient Estimates and Liouville Properties of the Fast Diffusion Equation on Riemannian Manifolds*

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Abstract In the paper, the authors provide a new proof and derive some new elliptic type (Hamilton type) gradient estimates for fast diffusion equations on a complete noncompact Riemannian manifold with a fixed metric and along the Ricci flow by constructing a new auxiliary function. These results generalize earlier results in the literature. And some parabolic type Liouville theorems for ancient solutions are obtained.

Keywords Gradient estimate, Fast diffusion equation, Ricci flow, Liouville theorem
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1 Introduction and Main Results

In this paper, we continue to consider the fast diffusion equation (FDE for short)

$$u_t = \Delta_{g(t)} u^{\alpha}, \quad 0 < \alpha < 1, \tag{1.1}$$

on a family of Riemannian manifolds $(M, g(t))$ for two cases: The one is that $g(t)$ is some fixed metric, and the other one is $g(t)$ deformed by the Ricci flow:

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}(g(t)).$$

Li and Yau [16] established a famous space-time gradient estimate for positive solutions to the heat equation. In 1993, Hamilton [9] proved the space-only gradient estimate for closed manifolds, which was extended by Souplet and Zhang in [21] to the complete noncompact manifolds. Bailesteanu, Cao and Pulemotov [1] generalized the Hamilton's gradient estimates to the Ricci flow. For the developments, see [4, 10, 12, 17–18, 20, 22–24, 27]. In 2009, Lu, Ni, Vázquez and Villani [19] studied the FDE (1.1) on Riemannian manifolds, and derived a local space-time gradient estimates. Later in [28], Zhu studied the FDE (1.1) on complete

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noncompact Riemannian manifolds, and derived the following space-only gradient estimate (Hamilton type gradient estimate) and Liouville type theorem.

Theorem A (see [28]) *Let (M^n, g) be a Riemannian manifold with $n \geq 2$ and $\mathrm{Ric}(M^n) \geq -k$ for some $k \geq 0$. Suppose that u is an arbitrary positive solution to the FDE (1.1) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Let $v = \frac{\alpha}{1-\alpha} u^{\alpha-1}$ and $v \leq M$. Then for $1 - \frac{2}{n} < \alpha < 1$,*

$$\frac{|\nabla v|}{v^{\frac{1}{2}}} \leq CM^{\frac{1}{2}}\Big(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\Big). \tag{1.2}$$

Recently, Xu [26], Huang and Ma [11] improved the result of Zhu [28]. Huang and Ma [10] proved a gradient estimate and Liouville theorem for FDE (1.1) with $1 - \frac{1}{-3+2\sqrt{n+3}} < \alpha < 1 - \frac{3}{n+3}, n \neq 6$. Xu [26] derived the gradient estimate for the FDE (1.1) with $1 - \frac{4}{n+3} < \alpha < 1$. Cao and Zhu [3] proved Li-Yau-Hamilton type differential Harnack estimates for positive solutions of the FDE (1.1). Li, Bai and Zhang [13] proved Hamilton type gradient estimates for the fast diffusion equations under the Ricci flow.

Our results of this paper are encouraged by the work in [1, 3, 8, 10–11, 13, 21–22, 25, 28–29]. We consider the FDE (1.1), and derive some elliptic type (Hamilton-Souplet-Zhang type) gradient estimates.

To prove the propertity of the positive solution of the FDE (1.1), we will use the following transformation: Let $v = \frac{\alpha}{1-\alpha} u^{\alpha-1}$. Then

$$v_t = (1-\alpha)v\Delta v - |\nabla v|^2. \tag{1.3}$$

Our paper is organized as follows: We show our main results in Section 1. We will give some lemmas and the proof of the main results on Riemannian manifolds with a fixed metric in Section 2. The proof of the main results on Riemannian manifolds along the Ricci flow will be given in Section 3.

Theorem 1.1 *Let $(\mathrm{M^n}, g)$ be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}(\mathrm{M^n}) \geq -K$ for some $K \geq 0$ in $B_{x_0,R}$, which is a geodesic ball centered at some fixed point x_0 in $\mathrm{M^n}$ with radius R. Assume that v is any positive solution to (1.3) in $Q_{R,T} = B_{x_0,R} \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$ with $0 < \delta \leq v \leq A$ for some constants δ and A.*

(1) If $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$, then

$$\frac{|\nabla v|^2}{v^{2-\beta}} \leq C\delta^\beta\Big(K + \frac{\sqrt{1+\delta^{2\beta}}}{R^2} + \frac{C}{\delta T}\Big) \quad in\ Q_{\frac{R}{2},\frac{T}{2}}. \tag{1.4}$$

(2) If $1 - \frac{4}{n+4} < \alpha < 1$, then

$$\frac{|\nabla v|^2}{v^2} \leq C\Big(K + \frac{1}{R^2} + \frac{1}{\delta T}\Big) \quad in\ Q_{\frac{R}{2},\frac{T}{2}}. \tag{1.5}$$

Here $\beta = -\frac{\alpha}{2(1-\alpha)}$ and $C = C(n, \alpha)$ is a positive constant.

By applying Theorem 1.1, we deduce the following Liouville type theorem.

Theorem 1.2 *Let* $(\mathrm{M}^{\mathrm{n}}, g)$ *be an* n*-dimensional complete, noncompact manifold with non-negative Ricci curvature. Let* u *be a positive solution to* (1.1) *and* $d(x)$ *be the geodesic distance of* g*.*

(1) *If* $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$ *and* $\frac{1}{u(x,t)} = o\big([d(x) + |t|]^{\frac{1}{1-\alpha}}\big)$ *near infinity, then* u *is a constant.*

(2) *If* $1 - \frac{4}{n+4} < \alpha < 1$ *and* $\frac{1}{u(x,t)} = o\big([d(x) + |t|]^{\frac{1}{1-\alpha}}\big)$ *near infinity, then* u *is a constant.*

Remark 1.1 (1) When $n \geq 2$, we have

$$\begin{aligned}&\frac{3+\sqrt{16+2n}}{7+2n} - \frac{1}{-3+2\sqrt{n+3}}\\ &= \frac{6n - 12 + (4n+3)\sqrt{16+2n} - (4n+14)\sqrt{n+3}}{(7+2n)(4n+3)} > 0,\end{aligned}$$

that is

$$1 - \frac{3+\sqrt{16+2n}}{7+2n} < 1 - \frac{1}{-3+2\sqrt{n+3}}.$$

(2) When $n \geq 4$, we have

$$\frac{3+\sqrt{16+2n}}{7+2n} - \frac{2}{n} = \frac{n\sqrt{16+2n} - n - 14}{n(7+2n)} > 0,$$

that is

$$1 - \frac{3+\sqrt{16+2n}}{7+2n} < 1 - \frac{2}{n}.$$

(3) When $n \geq 7$, we have

$$\frac{3+\sqrt{16+2n}}{7+2n} - \frac{4}{n+3} = \frac{(n+3)\sqrt{16+2n} - 5n - 19}{(n+3)(7+2n)} > 0,$$

that is

$$1 - \frac{3+\sqrt{16+2n}}{7+2n} < 1 - \frac{4}{n+3}.$$

Hence, Theorems 1.1–1.2 generalize some known results in [11, 26, 28].

Remark 1.2 Our proof is a little different from the proofs of Zhu [28], Huang, Ma [11] and Xu [26]. We derive the evolution equation of quantity $\log \frac{A}{v}$ with $v \leq A$.

Remark 1.3 When $n \geq 4$, we have

$$1 - \frac{4}{n+4} \leq 1 - \frac{2}{n}.$$

When $3 \leq n \leq 29$, we have

$$\frac{4}{n+4} - \frac{1}{-3+2\sqrt{n+3}} = \frac{13n - 2(n+4)\sqrt{n+3}}{(n+4)(4n+3)} > 0,$$

that is

$$1 - \frac{4}{n+4} < 1 - \frac{1}{-3+2\sqrt{n+3}}.$$

So, (1.5) and Theorem 1.2 generalize the results of Zhu [28], Huang and Ma [11].

Remark 1.4 The upper bound of the gradient estimate (1.5) does not contain the upper bound of v.

Theorem 1.3 *Let* $(\mathrm{M^n}, g)$ *be an* n*-dimensional complete Riemannian manifold with* $\mathrm{Ric}(\mathrm{M^n}) \geq -K$ *for some* $K \geq 0$ *in* $B_{x_0,R}$, *which is a geodesic ball centered at some fixed point* x_0 *in* $\mathrm{M^n}$ *with radius* R. *Assume that* v *is any positive solution to* (1.3) *in* $Q_{R,T} = B_{x_0,R} \times [t_0 - T, t_0] \subset \mathrm{M^n} \times (-\infty, \infty)$ *with* $v \leq A$. *Let* $1 - \frac{2}{n+4} < \alpha < 1$. *Then there exist a constant* $C = C(n, \alpha)$ *such that*

$$\frac{|\nabla v|^2}{v} \leq CA\Big(K + \frac{1}{R^2}\Big) + \frac{C}{T} \tag{1.6}$$

in $Q_{\frac{R}{2},\frac{T}{2}}$.

Moreover, if $(\mathrm{M^n}, g)$ *has nonnegative Ricci curvature and* u *is any positive solution to* (1.1) *on* $\mathrm{M^n} \times (0, \infty)$, *then there exists a constant* $C = C(n, \alpha)$ *such that*

$$\frac{|\nabla v|^2}{v} \leq \frac{C}{T}. \tag{1.7}$$

By applying Theorem 1.3, we deduce the following Liouville type theorem.

Theorem 1.4 *Let* $(\mathrm{M^n}, g)$ *be an* n*-dimensional complete, noncompact manifold with nonnegative Ricci curvature. Let* u *be a positive solution to* (1.1) *with* $1 - \frac{2}{n+4} < \alpha < 1$ *such that* $\frac{1}{u(x,t)} = o\big([d(x) + |t|]^{\frac{1}{1-\alpha}}\big)$ *near infinity, where* $d(x)$ *is the geodesic distance of* g. *Then* u *is a constant.*

When $u(x,t)$ is independent on t, by (1.6), we can derive the following Liouville type theorem.

Theorem 1.5 *Let* $(\mathrm{M^n}, g)$ *be an* n*-dimensional complete, noncompact manifold with nonnegative Ricci curvature. Let* u *be a positive solution to the equation*

$$\Delta u^m = 0, \quad 1 - \frac{2}{n+4} < \alpha < 1. \tag{1.8}$$

Assume that $v = \frac{\alpha}{1-\alpha} u^{\alpha-1}$ *with* $0 < v \leq A$ *for some constant* A. *Then* u *is a constant.*

Daskalopoulos et al. [6–7] observed that the metric $g = u^{\frac{4}{n-2}} \mathrm{d}y^2$ satisfies the Yamabe flow (see [2])

$$\frac{\partial g}{\partial t} = -Rg$$

on $\mathbb{R}^n$, $n \geq 3$, for $0 < t < T$, where R is the scalar curvature of the metric g, if and only if u satisfies

$$u_t = \frac{(n-1)(n+2)}{n-2} \Delta u^{\frac{n-2}{n+2}}.$$

When $n \geq 17$, we have $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \frac{n-2}{n+2} < 1$. Therefore, we obtain the following theorem.

Theorem 1.6 *Let* $(\mathrm{M^n}, g)$ *be an* n*-dimensional complete Riemannian manifold with* $\mathrm{Ric}(\mathrm{M^n}) \geq -K$ *for some* $K \geq 0$ *in* $B_{x_0,R}$, *which is a geodesic ball centered at some fixed point* x_0 *in* $\mathrm{M^n}$ *with radius* R. *Assume that* u *is any positive solution to the equation*

$$u_t = \Delta u^{\frac{n-2}{n+2}}, \quad n \geq 17 \tag{1.9}$$

in $Q_{R,T} = B_{x_0,R} \times [t_0 - T, t_0] \subset \mathrm{M}^{\mathrm{n}} \times (-\infty, \infty)$. *Assume also that* $v = \frac{n-2}{4} u^{-\frac{4}{n+2}}$ *with* $0 < \delta \le v \le A$ *for some constants* δ *and* A. *Then there exists a constant* $C = C(n, \alpha)$ *such that*

$$\frac{|\nabla v|^2}{v^{2-\beta}} \le C\delta^{\beta}\Big(K + \frac{\sqrt{1+\delta^{2\beta}}}{R^2} + \frac{C}{\delta T}\Big) \tag{1.10}$$

in $Q_{\frac{R}{2},\frac{T}{2}}$, *where* $\beta = -\frac{\alpha}{2(1-\alpha)}$.

By using Theorem 1.6, we deduce the following Liouville type theorem.

Theorem 1.7 *Let* $(\mathrm{M}^{\mathrm{n}}, g)$ *be an n-dimensional complete, noncompact manifold with non-negative Ricci curvature. Let u be a positive solution to* (1.9) *such that* $\frac{1}{u(x,t)} = o\big([d(x)+|t|]^{\frac{n+2}{4}}\big)$ *near infinity, where* $d(x)$ *is the geodesic distance of* g. *If* $n \ge 17$, *then* u *is a constant.*

Next, we state our estimates for the FDE coupled with the Ricci flow (see [5, 9]), which are similar to the fixed metric case.

Let $(\mathrm{M}^{\mathrm{n}}, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}(g(t)). \tag{1.11}$$

Theorem 1.8 *Let* $(\mathrm{M}^{\mathrm{n}}, g(x,t))_{t\in[0,T]}$ *be a complete solution to* (1.11). *Suppose that* $|\mathrm{Ric}(x,t)| \le K$ *for some* $K \ge 0$ *and all* $(x,t) \in B_{R,T} = B(x_0, R) \times (0, T]$ *for some fixed* $x_0 \in \mathrm{M}^{\mathrm{n}}$. *Assume that v is any positive solution to the equation*

$$v_t = (1-\alpha)v\Delta_{g(t)}v - |\nabla^{g(t)} v|^2.$$

Assume also that $v = \frac{\alpha}{1-\alpha}u^{\alpha-1}$ *with* $0 < \delta \le v \le A$.

(1) *If* $1 - \frac{4}{n+4} < \alpha < 1$, *then*

$$\frac{|\nabla^{g(t)} v|^2}{v^2} \le C\Big([(1-\alpha)A+1]\frac{K}{\delta} + \frac{1}{R^2} + \frac{1}{\delta t}\Big) \quad in\ B_{\frac{R}{2},T}. \tag{1.12}$$

(2) *If* $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$, *then*

$$\frac{|\nabla^{g(t)} v|^2}{v^{2-\beta}} \le C\delta^{\beta}\Big([(1-\alpha)A+1]\frac{K}{\delta} + \frac{1}{R^2} + \frac{1}{\delta t}\Big) \quad in\ B_{\frac{R}{2},T}. \tag{1.13}$$

Here $\beta = -\frac{\alpha}{2(1-\alpha)}$ *and* $C = C(n,\alpha)$ *is a positive constant.*

Remark 1.5 Since

$$1 - \frac{4}{n+4} \le 1 - \frac{4}{n+8}.$$

So, Theorem 1.8 generalizes the one of Li, Bai and Zhang [13].

Theorem 1.9 *Let* $(\mathrm{M}^{\mathrm{n}}, g(x,t))_{t\in[0,T]}$ *be a complete solution to* (1.11). *Suppose that* $|\mathrm{Ric}(x,t)| \le K$ *for some* $K \ge 0$ *and all* $(x,t) \in B_{R,T} = B(x_0, R) \times (0, T]$ *for some fixed* $x_0 \in \mathrm{M}^{\mathrm{n}}$. *Assume that v is any positive solution to the equation*

$$v_t = (1-\alpha)v\Delta_{g(t)}v - |\nabla^{g(t)} v|^2.$$

Assume also that $v = \frac{\alpha}{1-\alpha}u^{\alpha-1}$ *with* $v \le A$. *Let* $1 - \frac{2}{n+4} < \alpha < 1$. *Then there exist a constant* $C = C(n, \alpha)$ *such that*

$$\frac{|\nabla^{g(t)}v|^2}{v} \le CA\Big([(1-\alpha)A+1]K + \frac{1}{R^2}\Big) + \frac{C}{t} \tag{1.14}$$

in $B_{\frac{R}{2},T}$.

When $n \ge 17$, we have $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \frac{n-2}{n+2} < 1$. Therefore, we obtain the following theorem.

Theorem 1.10 *Let* $(\mathrm{M}^{\mathrm{n}}, g(x,t))_{t\in[0,T]}$ *be a complete solution to* (1.11). *Suppose that* $|\mathrm{Ric}(x,t)| \le K$ *for some* $K \ge 0$ *and all* $(x,t) \in B_{R,T} = B(x_0, R) \times (0, T]$ *for some fixed* $x_0 \in \mathrm{M}^{\mathrm{n}}$. *Assume that u is any positive solution to the equation*

$$u_t = \Delta^{g(t)}u^{\frac{n-2}{n+2}}, \quad n \ge 17$$

in $B_{R,T}$. *Assume also that* $v = \frac{n-2}{4}u^{-\frac{4}{n+2}}$ *with* $0 < \delta \le v \le A$ *for some constants* δ *and* A. *Then there exists a constant* $C = C(n, \alpha)$ *such that*

$$\frac{|\nabla^{g(t)}v|^2}{v^{2-\beta}} \le C\delta^\beta\Big(K + \frac{\sqrt{1+\delta^{2\beta}}}{R^2} + \frac{C}{\delta t}\Big) \tag{1.15}$$

in $B_{\frac{R}{2},T}$, *where* $\beta = -\frac{\alpha}{2(1-\alpha)}$.

2 FED Under the Fixed Metric

2.1 Basic lemmas

Before proving the main theorems, we need some lemmas. Consider the equation

$$v_t = (1-\alpha)v\Delta v - |\nabla v|^2 \tag{2.1}$$

on a complete Riemannian manifold $(\mathrm{M}^{\mathrm{n}}, g)$. Let $v(x,t)$ be a solution of (2.1) and $0 < v < A$ for some constant A in the cylinder

$$Q_{R,T} :\equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty),$$

here $t_0 \in \mathbb{R}$ and $T > 0$. We first introduce a new smooth function

$$g = \log\frac{A}{v}$$

in $Q_{R,T}$. Then $v = A\cdot \mathrm{e}^{-g}$,

$$\begin{aligned} v_t &= -A\mathrm{e}^{-g}g_t = -vg_t,\\ \nabla v &= -A\mathrm{e}^{-g}\nabla g = -v\nabla g,\\ \Delta v &= -A\mathrm{e}^{-g}\Delta g + A\mathrm{e}^{-g}|\nabla g|^2 = -v\Delta g + v|\nabla g|^2. \end{aligned} \tag{2.2}$$

From (2.1), we have

$$\begin{aligned} g_t &= -\frac{1}{A\mathrm{e}^{-g}}v_t = -\frac{1}{v}[(1-\alpha)v\Delta v - |\nabla v|^2]\\ &= -(1-\alpha)[-v\Delta g + v|\nabla g|^2] + v|\nabla g|^2\\ &= (1-\alpha)v\Delta g + \alpha v|\nabla g|^2. \end{aligned} \tag{2.3}$$

By utilizing the above equation (2.3), we can derive the following lemma.

Lemma 2.1 *Let $\omega = |\nabla g|^2$. Then for any $(x,t) \in Q_{R,T}$,*

$$(1-\alpha)v\Delta\omega - \omega_t = bv\omega^2 - 2K(1-\alpha)v\omega - 2\alpha v\langle\nabla\omega, \nabla g\rangle, \tag{2.4}$$

where $\alpha > 1 - \frac{4}{n+4}$ and $b = 2\alpha - \frac{n(1-\alpha)}{2} > 0$.

Proof By using the Bochner-Weitzenböck formula

$$\Delta|\nabla g|^2 = 2|\nabla^2 g|^2 + 2\mathrm{Ric}(\nabla g, \nabla g) + 2\langle\nabla\Delta g, \nabla g\rangle, \tag{2.5}$$

we have

$$\begin{aligned}(1-\alpha)v\Delta\omega - \omega_t = {} & 2(1-\alpha)v|\nabla^2 g|^2 + 2(1-\alpha)v\mathrm{Ric}(\nabla g, \nabla g) \\ & + 2(1-\alpha)v\langle\nabla\Delta g, \nabla g\rangle - \omega_t.\end{aligned}$$

By (2.3), we obtain

$$\begin{aligned}&(1-\alpha)v\Delta\omega - \omega_t \\ &= 2(1-\alpha)v|\nabla^2 g|^2 + 2(1-\alpha)v\mathrm{Ric}(\nabla g, \nabla g) + 2v\Big\langle\nabla\Big(\frac{g_t}{v} - \alpha|\nabla g|^2\Big), \nabla g\Big\rangle - \omega_t \\ &= 2(1-\alpha)v|\nabla^2 g|^2 + 2(1-\alpha)v\mathrm{Ric}(\nabla g, \nabla g) + 2\langle\nabla g_t, \nabla g\rangle \\ &\quad - \frac{2g_t}{v}\langle\nabla v, \nabla g\rangle - 2\alpha v\langle\nabla\omega, \nabla g\rangle - \omega_t \\ &= 2(1-\alpha)v|\nabla^2 g|^2 + 2(1-\alpha)v\mathrm{Ric}(\nabla g, \nabla g) \\ &\quad - \frac{2g_t}{v}\langle\nabla v, \nabla g\rangle - 2\alpha v\langle\nabla\omega, \nabla g\rangle.\end{aligned} \tag{2.6}$$

By applying (2.2)–(2.3) and the Cauchy inequality, we have

$$\begin{aligned}&2(1-\alpha)v|\nabla^2 g|^2 - \frac{2g_t}{v}\langle\nabla v, \nabla g\rangle \\ &= 2(1-\alpha)v|\nabla^2 g|^2 + 2g_t|\nabla g|^2 \\ &\geq 2(1-\alpha)v\frac{(\Delta g)^2}{n} + 2|\nabla g|^2[(1-\alpha)v\Delta g + \alpha v|\nabla g|^2] \\ &\geq -\frac{n(1-\alpha)}{2}v|\nabla g|^4 + 2\alpha v|\nabla g|^4.\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6) and noting that $\mathrm{Ric} \geq -K$, we have

$$\begin{aligned}&(1-\alpha)v\Delta\omega - \omega_t \\ &\geq 2(1-\alpha)v\mathrm{Ric}(\nabla g, \nabla g) - \frac{n(1-\alpha)}{2}v|\nabla g|^4 + 2\alpha v|\nabla g|^4 - 2\alpha v\langle\nabla\omega, \nabla g\rangle \\ &= \Big[2\alpha - \frac{n(1-\alpha)}{2}\Big]v\omega^2 - 2K(1-\alpha)v\omega - 2\alpha v\langle\nabla\omega, \nabla g\rangle \\ &= bv\omega^2 - 2K(1-\alpha)v\omega - 2\alpha v\langle\nabla\omega, \nabla g\rangle,\end{aligned}$$

where $\alpha > 1 - \frac{4}{n+4}$ and $b = 2\alpha - \frac{n(1-\alpha)}{2} > 0$. The proof is completed.

Lemma 2.2 *Let $\varpi = v^\beta|\nabla g|^2$ with $\beta = -\frac{\alpha}{2(1-\alpha)}$. Then*

$$(1-\alpha)v\Delta\varpi - \varpi_t \geq -2K(1-\alpha)v\varpi + \gamma v^{1-\beta}\varpi^2 - \alpha v\langle\nabla\varpi, \nabla g\rangle, \tag{2.8}$$

where $\gamma = \frac{\alpha^2}{4(1-\alpha)} + 2\alpha - \frac{n(1-\alpha)}{2} > 0$ and $1 - \frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$.

Proof Applying (2.5), we have

$$\begin{aligned}(1-\alpha)v\Delta\varpi-\varpi_t &= (1-\alpha)v^\beta v\Delta|\nabla g|^2+(1-\alpha)v|\nabla g|^2\Delta v^\beta\\&\quad+2(1-\alpha)v\nabla|\nabla g|^2\nabla v^\beta-\varpi_t\\&=2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2+2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g)\\&\quad+2(1-\alpha)v^{\beta+1}\langle\nabla\Delta g,\nabla g\rangle+(1-\alpha)v|\nabla g|^2\Delta v^\beta\\&\quad+2(1-\alpha)v\langle\nabla|\nabla g|^2,\nabla v^\beta\rangle-\varpi_t.\end{aligned}$$

By utilizing (2.1) and (2.3), we have

$$\begin{aligned}&(1-\alpha)v\Delta\varpi-\varpi_t\\&=2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2+2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g)\\&\quad+2v^{\beta+1}\Big\langle\nabla\Big(\frac{g_t}{v}-\alpha|\nabla g|^2\Big),\nabla g\Big\rangle-2(1-\alpha)\beta v^{\beta+1}\langle\nabla|\nabla g|^2,\nabla g\rangle\\&\quad+(1-\alpha)\beta(\beta-1)v^{\beta-1}|\nabla v|^2|\nabla g|^2+\beta(1-\alpha)v^\beta|\nabla g|^2\Delta v-\varpi_t\\&=2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2+2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g)\\&\quad+2v^\beta\langle\nabla g_t,\nabla g\rangle-2v^{\beta-1}g_t\langle\nabla v,\nabla g\rangle-2\alpha v^{\beta+1}\langle\nabla|\nabla g|^2,\nabla g\rangle\\&\quad-2(1-\alpha)\beta v^{\beta+1}\langle\nabla|\nabla g|^2,\nabla g\rangle+(1-\alpha)\beta(\beta-1)v^{\beta+1}|\nabla g|^4\\&\quad+\beta v^{\beta-1}|\nabla g|^2(v_t+|\nabla v|^2)-\varpi_t\\&=2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2+2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g)\\&\quad-2v^{\beta-1}g_t\langle\nabla v,\nabla g\rangle-2\alpha v^{\beta+1}\langle\nabla|\nabla g|^2,\nabla g\rangle\\&\quad-2(1-\alpha)\beta v^{\beta+1}\langle\nabla|\nabla g|^2,\nabla g\rangle+(1-\alpha)\beta(\beta-1)v^{\beta+1}|\nabla g|^4\\&\quad+\beta v^{\beta+1}|\nabla g|^4.\end{aligned}\tag{2.9}$$

By the Cauchy inequality, we have

$$\begin{aligned}&2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2-2v^{\beta-1}g_t\langle\nabla v,\nabla g\rangle\\&=2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2+2v^\beta g_t|\nabla g|^2\\&\ge\frac{2}{n}(1-\alpha)v^{\beta+1}(\Delta g)^2+2(1-\alpha)v^{\beta+1}|\nabla g|^2\Delta g+2\alpha v^{\beta+1}|\nabla g|^4\\&\ge-\frac{n(1-\alpha)}{2}v^{\beta+1}|\nabla g|^4+2\alpha v^{\beta+1}|\nabla g|^4.\end{aligned}\tag{2.10}$$

Combining (2.9) and (2.10), we have

$$\begin{aligned}&(1-\alpha)v\Delta\varpi-\varpi_t\\&\ge-2K(1-\alpha)v^{\beta+1}|\nabla g|^2\\&\quad+\Big[(1-\alpha)\beta(\beta-1)+\beta+2\alpha-\frac{n(1-\alpha)}{2}-2\alpha\beta-2\beta^2(1-\alpha)\Big]v^{\beta+1}|\nabla g|^4\\&\quad-[2\alpha+2\beta(1-\alpha)]v\langle\nabla\varpi,\nabla g\rangle,\end{aligned}\tag{2.11}$$

where we use the fact that

$$\langle\nabla\varpi,\nabla g\rangle=v^\beta\langle\nabla|\nabla g|^2,\nabla g\rangle-\beta v^\beta|\nabla g|^4.$$

In order to obtain the gradient estimates, we need to require the coefficient $f(\beta)$ of $|\nabla g|^4$ to be positive. In fact,

$$\begin{aligned} f(\beta) &= (1-\alpha)\beta(\beta-1)+\beta+2\alpha-\frac{n(1-\alpha)}{2}-2\alpha\beta-2\beta^2(1-\alpha) \\ &= -(1-\alpha)\Big[\beta+\frac{\alpha}{2(1-\alpha)}\Big]^2+\frac{\alpha^2}{4(1-\alpha)}+2\alpha-\frac{n(1-\alpha)}{2}, \end{aligned}$$

we choose $\beta=-\frac{\alpha}{2(1-\alpha)}$, then $f(\beta)>0$ when $1-\frac{3+\sqrt{16+2n}}{7+2n}<\alpha<1$. Therefore, (2.11) can be written as

$$(1-\alpha)v\Delta\varpi-\varpi_t\geq -2K(1-\alpha)v\varpi+\gamma v^{1-\beta}\varpi^2-\alpha v\langle\nabla\varpi,\nabla g\rangle,$$

where $\gamma=\frac{\alpha^2}{4(1-\alpha)}+2\alpha-\frac{n(1-\alpha)}{2}>0$. The proof is completed.

Taking $\beta=1$ in (2.11), the following lemma is derived.

Lemma 2.3 *Let* $\widetilde{\omega}=v|\nabla g|^2$. *Then*

$$(1-\alpha)v\Delta\widetilde{\omega}-\partial_t\widetilde{\omega}\geq \epsilon\widetilde{\omega}^2-2K(1-\alpha)v\widetilde{\omega}-2v\langle\nabla\widetilde{\omega},\nabla g\rangle, \tag{2.12}$$

where $\epsilon=2\alpha-1-\frac{n(1-\alpha)}{2}>0$ *with* $1-\frac{2}{n+4}<\alpha<1$.

We next introduce a smooth cut-off function (see [10, 17, 24]), which will be used in the proof of our main theorems.

Lemma 2.4 (see [16, 21, 28]) *We use the geodesic polar coordinate here. Assume that a function* $\varphi=\varphi(x,t)$ *is a smooth cut-off function supported in* $Q_{R,T}$, *satisfying the following properties*:

(1) $\varphi=\varphi(d(x,x_0),t)\equiv\varphi(r,t)$; $\varphi(r,t)=1$ *in* $Q_{\frac{R}{2},\frac{T}{2}}$, $0\leq\varphi\leq 1$.
(2) φ *is decreasing as a radial function in the spatial variables.*
(3) $\frac{|\partial_r\varphi|}{\varphi^a}\leq\frac{C_a}{R}$, $\frac{|\partial_r^2\varphi|}{\varphi^a}\leq\frac{C_a}{R^2}$ *when* $0<a<1$.
(4) $\frac{|\partial_t\varphi|}{\varphi^{\frac12}}\leq\frac{C}{T}$.

2.2 The proof of theorems

In this section, we will prove our main theorems by Lemma 2.4.

Proof of Theorem 1.1 Part 1: Assume that the maximum of $\varphi\varpi$ is arrived at a point (x_1,t_1). By [16], we can suppose, without loss of generality, that x_1 is not on the cut-locus of $\mathrm{M^n}$. Therefore, at (x_1,t_1), it yields $\Delta(\varphi\varpi)\leq 0$, $(\varphi\varpi)_t\geq 0$ and $\nabla(\varphi\varpi)=0$. By

$$0=\nabla(\varphi\varpi)=\varpi\nabla\varphi+\varphi\nabla\varpi,$$

then

$$\nabla\varpi=-\frac{\nabla\varphi}{\varphi}\varpi.$$

Hence, by (2.8) and a straightforward calculation, it yields that

$$\begin{aligned} 0 &\geq (1-\alpha)v\Delta(\varphi\varpi)-(\varphi\varpi)_t \\ &= \varphi[(1-\alpha)v\Delta\varpi-\varpi_t]+(1-\alpha)v\varpi\Delta\varphi+2(1-\alpha)v\nabla\varphi\nabla\varpi-\varpi\varphi_t \end{aligned}$$

$$
\begin{aligned}
&\geq \gamma v^{1-\beta}\varphi\varpi^2 - 2K(1-\alpha)v\varphi\varpi - \alpha v\varphi\langle\nabla\varpi, \nabla g\rangle \\
&\quad + (1-\alpha)v\varpi\Delta\varphi + 2(1-\alpha)v\nabla\varphi\nabla\varpi - \varpi\varphi_t \\
&= \gamma v^{1-\beta}\varphi\varpi^2 - 2K(1-\alpha)v\varphi\varpi + \alpha v\varpi\langle\nabla\varphi, \nabla g\rangle \\
&\quad + (1-\alpha)v\varpi\Delta\varphi - 2(1-\alpha)v\varpi\frac{|\nabla\varphi|^2}{\varphi} - \varpi\varphi_t. \qquad (2.13)
\end{aligned}
$$

This implies

$$
\begin{aligned}
2\varphi\varpi^2 \leq{}& \frac{4}{\gamma}K(1-\alpha)v^\beta\varphi\varpi - \frac{2\alpha}{\gamma}\langle\nabla\varphi, \nabla g\rangle v^\beta\varpi \\
&- \frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Delta\varphi + \frac{4(1-\alpha)}{\gamma}\frac{|\nabla\varphi|^2}{\varphi}v^\beta\varpi + \frac{2}{\gamma}v^{\beta-1}\varpi\varphi_t. \qquad (2.14)
\end{aligned}
$$

We next estimate upper bounds for each term of the right hand side of (2.14). Applying the Young inequality, we have

$$
\frac{4}{\gamma}K(1-\alpha)v^\beta\varphi\varpi \leq \frac{1}{5}\varphi\varpi^2 + C\varphi K^2\delta^{2\beta} \leq \frac{1}{5}\varphi\varpi^2 + CK^2\delta^{2\beta}, \qquad (2.15)
$$

$$
-\frac{2\alpha}{\gamma}\langle\nabla\varphi, \nabla g\rangle v^\beta\varpi \leq \frac{2\alpha}{\gamma}|\nabla\varphi|\cdot\varpi^{\frac{3}{2}}v^\beta \leq \frac{1}{5}\varphi\varpi^2 + C\frac{|\nabla\varphi|^4}{\varphi^3}\delta^{4\beta} \leq \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{4\beta}}{R^4}, \qquad (2.16)
$$

$$
\begin{aligned}
-\frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Delta\varphi &= -\frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Big(\partial_r^2\varphi + (n-1)\frac{\partial_r\varphi}{r} + \partial_r\varphi\cdot\partial_r(\log\sqrt{g})\Big) \\
&\leq Cv^\beta\varpi\Big(|\partial_r^2\varphi| + (n-1)\frac{|\partial_r\varphi|}{r} + \sqrt{K}|\partial_r\varphi|\Big) \\
&\leq C\delta^\beta\varpi\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}} + (n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}} + \sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big| \\
&\leq \frac{1}{5}\varphi\varpi^2 + C\delta^{2\beta}\Big(\frac{1}{R^4} + \frac{K}{R^2}\Big), \qquad (2.17)
\end{aligned}
$$

$$
\frac{4(1-\alpha)}{\gamma}\frac{|\nabla\varphi|^2}{\varphi}v^\beta\varpi \leq C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}v^\beta\varpi \leq \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{2\beta}}{R^4} \qquad (2.18)
$$

and

$$
\frac{2}{\gamma}v^{\beta-1}\varpi\varphi_t \leq \frac{C}{\gamma}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\varphi^{\frac{1}{2}}v^{\beta-1}\varpi \leq \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{2\beta-2}}{T^2}. \qquad (2.19)
$$

We substitute (2.15)–(2.19) into (2.14), and have

$$
\varphi\varpi^2 \leq C\delta^{2\beta}\Big(K^2 + \frac{1}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2T^2}\Big) \qquad (2.20)
$$

at (x_1, t_1). Therefore, for all $(x,t) \in Q_{R,T}$, we obtain

$$
\begin{aligned}
(\varphi\varpi)^2(x,t) &\leq (\varphi\varpi)^2(x_1,t_1) \leq \varphi\varpi^2(x_1,t_1) \\
&\leq C\delta^{2\beta}\Big(K^2 + \frac{1+\delta^{2\beta}}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2T^2}\Big). \qquad (2.21)
\end{aligned}
$$

Notice that $\varphi(x,t) = 1$ in $Q_{\frac{R}{2},\frac{T}{2}}$ and $\varpi = v^\beta|\nabla g|^2 = v^\beta\frac{|\nabla v|^2}{v^2}$, we get that

$$
\frac{|\nabla v|^2}{v^{2-\beta}} \leq C\delta^\beta\Big(K + \frac{\sqrt{1+\delta^{2\beta}}}{R^2} + \frac{1}{\delta T}\Big).
$$

This proves part 1 of the theorem.

Part 2 Assume that the maximum of $\varphi\omega$ is arrived at a point (x_1, t_1). By [16], we can suppose, without loss of generality, that x_1 is not on the cut-locus of $\mathrm{M^n}$. Therefore, at (x_1, t_1), it yields $\Delta(\varphi\omega) \le 0$, $(\varphi\omega)_t \ge 0$ and $\nabla(\varphi\omega) = 0$. By $0 = \nabla(\varphi\omega) = \omega\nabla\varphi + \varphi\nabla\omega$, then $\nabla\omega = -\frac{\nabla\varphi}{\varphi}\omega$. Hence, by (2.4) and a straightforward calculation, it yields that

$$\begin{aligned}
0 &\ge (1-\alpha)v\Delta(\varphi\omega) - (\varphi\omega)_t \\
&= \varphi[(1-\alpha)v\Delta\omega - \omega_t] + (1-\alpha)v\omega\Delta\varphi + 2(1-\alpha)v\nabla\varphi\nabla\omega - \omega\varphi_t \\
&\ge bv\varphi\omega^2 - 2K(1-\alpha)v\varphi\omega - 2\alpha v\varphi\langle\nabla\omega, \nabla g\rangle \\
&\quad + (1-\alpha)v\omega\Delta\varphi + 2(1-\alpha)v\nabla\varphi\nabla\omega - \omega\varphi_t \\
&= bv\varphi\omega^2 - 2K(1-\alpha)v\varphi\omega + 2\alpha v\omega\langle\nabla\varphi, \nabla g\rangle \\
&\quad + (1-\alpha)v\omega\Delta\varphi - 2(1-\alpha)v\omega\frac{|\nabla\varphi|^2}{\varphi} - \omega\varphi_t. \qquad (2.22)
\end{aligned}$$

This implies

$$\begin{aligned}
2\varphi\omega^2 \le{}& \frac{4}{b}K(1-\alpha)\varphi\omega - \frac{4\alpha}{b}\langle\nabla\varphi, \nabla g\rangle\omega \\
&- \frac{2(1-\alpha)}{b}\omega\Delta\varphi + \frac{4(1-\alpha)}{b}\frac{|\nabla\varphi|^2}{\varphi}\omega + \frac{2}{bv}\omega\varphi_t. \qquad (2.23)
\end{aligned}$$

We next estimate upper bounds for each term of the right hand side of (2.23). Applying the Young inequality, we have

$$\frac{4}{b}K(1-\alpha)\varphi\omega \le \frac{1}{5}\varphi\omega^2 + C\varphi K^2 \le \frac{1}{5}\varphi\omega^2 + CK^2, \qquad (2.24)$$

$$\begin{aligned}
-\frac{4\alpha}{b}\langle\nabla\varphi, \nabla g\rangle\omega &\le \frac{4\alpha}{b}|\nabla\varphi| \cdot \omega^{\frac{3}{2}} \\
&\le \frac{1}{5}\varphi\omega^2 + C\frac{|\nabla\varphi|^4}{\varphi^3} \le \frac{1}{5}\varphi\omega^2 + \frac{C}{R^4}, \qquad (2.25)
\end{aligned}$$

$$\begin{aligned}
-\frac{2(1-\alpha)}{b}\omega\Delta\varphi &= -\frac{2(1-\alpha)}{b}\omega\Big(\partial_r^2\varphi + (n-1)\frac{\partial_r\varphi}{r} + \partial_r\varphi \cdot \partial_r(\log\sqrt{g})\Big) \\
&\le C\omega\Big(|\partial_r^2\varphi| + (n-1)\frac{|\partial_r\varphi|}{r} + \sqrt{K}|\partial_r\varphi|\Big) \\
&\le C\omega\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}} + (n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}} + \sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big| \\
&\le \frac{1}{5}\varphi\omega^2 + C\Big(\frac{1}{R^4} + \frac{K}{R^2}\Big), \qquad (2.26)
\end{aligned}$$

$$\frac{4(1-\alpha)}{b}\frac{|\nabla\varphi|^2}{\varphi}\omega \le C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}\omega \le \frac{1}{5}\varphi\omega^2 + \frac{C}{R^4} \qquad (2.27)$$

and

$$\frac{2}{bv}\omega\varphi_t \le \frac{C}{\delta}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\varphi^{\frac{1}{2}}\omega \le \frac{1}{5}\varphi\omega^2 + \frac{C}{\delta^2T^2} \qquad (2.28)$$

We substitute (2.24)–(2.28) into (2.23), and have

$$\varphi\omega^2 \le C\Big(K^2 + \frac{1}{R^4} + \frac{K}{R^2}\Big) + \frac{C}{\delta^2T^2} \qquad (2.29)$$

at (x_1,t_1). Therefore, for all $(x,t)\in Q_{R,T}$, we obtain

$$\begin{aligned}(\varphi\omega)^2(x,t)&\le(\varphi\omega)^2(x_1,t_1)\le\varphi\omega^2(x_1,t_1)\\&\le C\Big(K^2+\frac{1}{R^4}+\frac{K}{R^2}\Big)+\frac{C}{\delta^2T^2}.\end{aligned}\tag{2.30}$$

Notice that $\varphi(x,t)=1$ in $Q_{\frac{R}{2},\frac{T}{2}}$ and $\omega=|\nabla g|^2=\frac{|\nabla v|^2}{v^2}$, we get that

$$\frac{|\nabla v|^2}{v^2}\le C\Big(K+\frac{1}{R^2}+\frac{1}{\delta T}\Big).$$

The proof is completed.

Proof of Theorem 1.2 Part 1 From (1.4), we know that, when v is a positive ancient solution to (2.1) such that $v(x,t)=o([d(x,x_0)+|t|])$, then v is a constant. Notice that $v=\frac{\alpha}{1-\alpha}u^{\alpha-1}$, so when u is a positive ancient solution to (1.1) such that $\frac{1}{u(x,t)}=o([d(x,x_0)+|t|]^{\frac{1}{1-\alpha}})$, then u is a constant. This ends the part 1.

Part 2 From (1.5), we know that, when v is a positive ancient solution to (2.1) such that $v(x,t)=o([d(x,x_0)+|t|])$, then v is a constant. Notice that $v=\frac{\alpha}{1-\alpha}u^{\alpha-1}$, so when u is a positive ancient solution to (1.1) such that $\frac{1}{u(x,t)}=o([d(x,x_0)+|t|]^{\frac{1}{1-\alpha}})$, then u is a constant. This ends the proof of Theorem 1.2.

Proof of Theorem 1.3 Assume that the maximum of $\varphi\widetilde{\omega}$ is arrived at a point (x_1,t_1). By [16], we can suppose, without loss of generality, that x_1 is not on the cut-locus of $\mathrm{M^n}$. Therefore, at (x_1,t_1), it yields $\Delta(\varphi\widetilde{\omega})\le 0$, $(\varphi\widetilde{\omega})_t\ge 0$ and $\nabla(\varphi\widetilde{\omega})=0$. By $0=\nabla(\varphi\widetilde{\omega})=\widetilde{\omega}\nabla\varphi+\varphi\nabla\widetilde{\omega}$, then $\nabla\widetilde{\omega}=-\frac{\nabla\varphi}{\varphi}\widetilde{\omega}$. Hence, by (2.12) and a straightforward calculation, it yields that

$$\begin{aligned}0&\ge(1-\alpha)v\Delta(\varphi\widetilde{\omega})-(\varphi\widetilde{\omega})_t\\&=\varphi[(1-\alpha)v\Delta\widetilde{\omega}-\widetilde{\omega}_t]+(1-\alpha)v\widetilde{\omega}\Delta\varphi+2(1-\alpha)v\nabla\varphi\nabla\widetilde{\omega}-\widetilde{\omega}\varphi_t\\&\ge\epsilon\varphi\widetilde{\omega}^2-2K(1-\alpha)v\varphi\widetilde{\omega}-2v\varphi\langle\nabla\widetilde{\omega},\nabla g\rangle\\&\quad+(1-\alpha)v\widetilde{\omega}\Delta\varphi+2(1-\alpha)v\nabla\varphi\nabla\widetilde{\omega}-\widetilde{\omega}\varphi_t\\&=\epsilon\varphi\widetilde{\omega}^2-2K(1-\alpha)v\varphi\widetilde{\omega}+2v\widetilde{\omega}\langle\nabla\varphi,\nabla g\rangle\\&\quad+(1-\alpha)v\widetilde{\omega}\Delta\varphi-2(1-\alpha)v\widetilde{\omega}\frac{|\nabla\varphi|^2}{\varphi}-\widetilde{\omega}\varphi_t.\end{aligned}\tag{2.31}$$

This implies

$$\begin{aligned}2\varphi\widetilde{\omega}^2\le{}&\frac{4}{\epsilon}K(1-\alpha)\varphi v\widetilde{\omega}-\frac{4}{\epsilon}\langle\nabla\varphi,\nabla g\rangle v\widetilde{\omega}\\&-\frac{2(1-\alpha)}{\epsilon}v\widetilde{\omega}\Delta\varphi+\frac{4(1-\alpha)}{\epsilon}\frac{|\nabla\varphi|^2}{\varphi}v\widetilde{\omega}+\frac{2}{\epsilon}\widetilde{\omega}\varphi_t.\end{aligned}\tag{2.32}$$

We next estimate upper bounds for each term of the right hand side of (2.32). Applying the Young inequality, we have

$$\frac{4}{\epsilon}K(1-\alpha)\varphi v\widetilde{\omega}\le\frac{1}{5}\varphi\widetilde{\omega}^2+C\varphi A^2K^2\le\frac{1}{5}\varphi\widetilde{\omega}^2+CA^2K^2,\tag{2.33}$$

$$-\frac{4}{\epsilon}\langle\nabla\varphi,\nabla g\rangle v\widetilde{\omega}\le\frac{4}{\epsilon}|\nabla\varphi|\cdot\widetilde{\omega}^{\frac{3}{2}}\sqrt{A}$$

$$\leq \frac{1}{5}\varphi\widetilde{\omega}^2 + C\frac{|\nabla\varphi|^4}{\varphi^3}A^2 \leq \frac{1}{5}\varphi\widetilde{\omega}^2 + \frac{CA^2}{R^4}, \tag{2.34}$$

$$\begin{aligned}-\frac{2(1-\alpha)}{\epsilon}v\widetilde{\omega}\Delta\varphi &= -\frac{2(1-\alpha)}{\epsilon}v\widetilde{\omega}\Big(\partial_r^2\varphi + (n-1)\frac{\partial_r\varphi}{r} + \partial_r\varphi\cdot\partial_r(\log\sqrt{g})\Big)\\ &\leq CA\widetilde{\omega}\Big(|\partial_r^2\varphi| + (n-1)\frac{|\partial_r\varphi|}{r} + \sqrt{K}|\partial_r\varphi|\Big)\\ &\leq CA\widetilde{\omega}\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}} + (n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}} + \sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big|\\ &\leq \frac{1}{5}\varphi\widetilde{\omega}^2 + CA^2\Big(\frac{1}{R^4} + \frac{K}{R^2}\Big),\end{aligned} \tag{2.35}$$

$$\frac{4(1-\alpha)}{\epsilon}\frac{|\nabla\varphi|^2}{\varphi}v\widetilde{\omega} \leq C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}A\widetilde{\omega} \leq \frac{1}{5}\varphi\widetilde{\omega}^2 + \frac{CA^2}{R^4} \tag{2.36}$$

and

$$\frac{2}{\epsilon}\widetilde{\omega}\varphi_t \leq \frac{2}{\epsilon}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\varphi^{\frac{1}{2}}\widetilde{\omega} \leq \frac{1}{5}\varphi\widetilde{\omega}^2 + \frac{C}{T^2}. \tag{2.37}$$

We substitute (2.33)–(2.37) into (2.32), and have

$$\varphi\widetilde{\omega}^2 \leq CA^2\Big(K^2 + \frac{1}{R^4} + \frac{K}{R^2}\Big) + \frac{C}{T^2} \tag{2.38}$$

at (x_1, t_1). Therefore, for all $(x,t) \in Q_{R,T}$, we obtain

$$\begin{aligned}(\varphi\widetilde{\omega})^2(x,t) &\leq (\varphi\widetilde{\omega})^2(x_1,t_1) \leq \varphi\widetilde{\omega}^2(x_1,t_1)\\ &\leq CA^2\Big(K^2 + \frac{1}{R^4} + \frac{K}{R^2}\Big) + \frac{C}{T^2}.\end{aligned} \tag{2.39}$$

Notice that $\varphi(x,t) = 1$ in $Q_{\frac{R}{2},\frac{T}{2}}$ and $\widetilde{\omega} = v|\nabla g|^2 = \frac{|\nabla v|^2}{v}$, we get that

$$\frac{|\nabla v|^2}{v} \leq CA\Big(K + \frac{1}{R^2} + \frac{\sqrt{K}}{R}\Big) + \frac{C}{T}.$$

The proof is completed.

Proof of Theorem 1.4 From Theorem 1.3, we know that, when v is a positive ancient solution to (2.1) such that $v(x,t) = o([d(x,x_0) + |t|])$, then v is a constant. Notice that $v = \frac{\alpha}{1-\alpha}u^{\alpha-1}$, so when u is a positive ancient solution to (1.1) such that $\frac{1}{u(x,t)} = o([d(x,x_0)+|t|]^{\frac{1}{1-\alpha}})$, then u is a constant. This ends the proof of Theorem 1.4.

3 FDE along the Ricci Flow

The Ricci flow (1.11) was first introduced by Hamilton [9], and was an important tool of analyzing the structure of manifolds. In 2010, Bailesteanu, Cao and Pulemotov [1] generalized the Hamiltons gradient estimates for the heat equation on Riemannian manifolds with a fixed metric to the Ricci flow, and proved the theorem below.

Theorem B (see [1]) *Let $(\mathrm{M}^n, g(x,t))_{t\in(0,T]}$ be a complete solution along the Ricci flow. Let $|\mathrm{Ric}(x,t)| \leq K$ for some $K > 0$ and all $(x,t) \in B_{R,T} :\equiv B(x_0,R)\times[0,T]$. Suppose that u is a smooth positive solution to the heat equation*

$$u_t = \Delta_{g_t}u.$$

If $u \le A$ for some $A > 0$ and all $(x,t) \in B_{R,T}$, then there exists a constant $C = C(n)$ such that

$$\frac{|\nabla^{g(t)} u|}{u} \le \Big(\frac{1}{R} + \frac{1}{\sqrt{t}} + \sqrt{K}\Big)\Big(1 + \log\frac{A}{u}\Big). \tag{3.1}$$

In this section, we will derive some Hamilton type gradient estimates for fast diffusion equations (1.1) on a Riemannian manifold evolved by the Ricci flow.

3.1 Basic lemmas

Before the proof of the main theorems, we need some lemmas. Consider the equation

$$v_t = (1-\alpha)v\Delta_{g(t)}v - |\nabla^{g(t)}v|^2 \tag{3.2}$$

on a complete Riemannian manifold $(\mathrm{M}^{\mathrm{n}}, g)$ along the Ricci flow. Let $v(x,t)$ be a solution of (3.2) and $0 < v < A$ for some constant A in the cylinder

$$B_{R,T} :\equiv B(x_0, R) \times (0, T] \subset \mathrm{M}^{\mathrm{n}} \times (-\infty, \infty),$$

here $T > 0$.

Now, in order to simplify writing, we all set $\Delta = \Delta^{g(t)}$ and $\nabla = \nabla^{g(t)}$.

We first introduce a new smooth function

$$g = \log\frac{A}{v}$$

in $B_{R,T}$. From (3.2), we have

$$g_t = (1-\alpha)v\Delta g + \alpha v|\nabla g|^2. \tag{3.3}$$

By utilizing the above equation (3.3), we can derive the following lemma.

Lemma 3.1 *Let $\omega = |\nabla g|^2$. Then for any $(x,t) \in B_{R,T}$,*

$$(1-\alpha)v\Delta\omega - \omega_t \ge bv\omega^2 - 2[(1-\alpha)A + 1]K\omega - 2\alpha v\langle\nabla\omega, \nabla g\rangle, \tag{3.4}$$

where $\alpha > 1 - \frac{4}{n+4}$ and $b = 2\alpha - \frac{n(1-\alpha)}{2} > 0$.

Proof The Ricci flow equation (3.1) implies

$$\partial_t|\nabla g|^2 = 2\langle\nabla g, \nabla g_t\rangle + 2\mathrm{Ric}(\nabla g, \nabla g). \tag{3.5}$$

By further using the Bochner-Weitzenböck formula (2.5), we have

$$\begin{aligned}(1-\alpha)v\Delta\omega - \omega_t = {} & 2(1-\alpha)v|\nabla^2 g|^2 + [2(1-\alpha)v - 2]\mathrm{Ric}(\nabla g, \nabla g)\\ & + 2(1-\alpha)v\langle\nabla\Delta g, \nabla g\rangle - 2\langle\nabla g, \nabla g_t\rangle.\end{aligned}$$

By (3.3), we obtain

$$\begin{aligned}(1-\alpha)v\Delta\omega - \omega_t = {} & 2(1-\alpha)v|\nabla^2 g|^2 + [2(1-\alpha)v - 2]\mathrm{Ric}(\nabla g, \nabla g)\\ & + 2v\Big\langle\nabla\Big(\frac{g_t}{v} - \alpha|\nabla g|^2\Big), \nabla g\Big\rangle - 2\langle\nabla g, \nabla g_t\rangle\end{aligned}$$

$$
\begin{aligned}
&= 2(1-\alpha)v|\nabla^2 g|^2 + [2(1-\alpha)v-2]\mathrm{Ric}(\nabla g,\nabla g) + 2\langle \nabla g_t,\nabla g\rangle \\
&\quad - \frac{2g_t}{v}\langle \nabla v,\nabla g\rangle - 2\alpha v\langle \nabla\omega,\nabla g\rangle - 2\langle \nabla g,\nabla g_t\rangle \\
&= 2(1-\alpha)v|\nabla^2 g|^2 + [2(1-\alpha)v-2]\mathrm{Ric}(\nabla g,\nabla g) \\
&\quad - \frac{2g_t}{v}\langle \nabla v,\nabla g\rangle - 2\alpha v\langle \nabla\omega,\nabla g\rangle. \qquad (3.6)
\end{aligned}
$$

Substituting (2.7) into (3.6) and noting that $|\mathrm{Ric}|\le K$, we have

$$
\begin{aligned}
&(1-\alpha)v\Delta\omega - \omega_t \\
&\ge +[2(1-\alpha)v-2]\mathrm{Ric}(\nabla g,\nabla g) - \frac{n(1-\alpha)}{2}v|\nabla g|^4 + 2\alpha v|\nabla g|^4 - 2\alpha v\langle \nabla\omega,\nabla g\rangle \\
&= \Big[2\alpha - \frac{n(1-\alpha)}{2}\Big]v\omega^2 - 2[(1-\alpha)A+1]K\omega - 2\alpha v\langle \nabla\omega,\nabla g\rangle \\
&= bv\omega^2 - 2[(1-\alpha)A+1]K\omega - 2\alpha v\langle \nabla\omega,\nabla g\rangle,
\end{aligned}
$$

where $\alpha > 1-\frac{4}{n+4}$ and $b = 2\alpha - \frac{n(1-\alpha)}{2} > 0$. The proof is completed.

Lemma 3.2 *Let $\varpi = v^\beta|\nabla g|^2$ with $\beta = -\frac{\alpha}{2(1-\alpha)}$. Then*

$$
(1-\alpha)v\Delta\varpi - \varpi_t \ge -2[(1-\alpha)A+1]K\varpi + \gamma v^{1-\beta}\varpi^2 - \alpha v\langle \nabla\varpi,\nabla g\rangle, \qquad (3.7)
$$

where $\gamma = \frac{\alpha^2}{4(1-\alpha)} + 2\alpha - \frac{n(1-\alpha)}{2} > 0$ and $1-\frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$.

Proof Applying (2.5), we have

$$
\begin{aligned}
(1-\alpha)v\Delta\varpi - \varpi_t &= (1-\alpha)v^\beta v\Delta|\nabla g|^2 + (1-\alpha)v|\nabla g|^2\Delta v^\beta \\
&\quad + 2(1-\alpha)v\nabla|\nabla g|^2\nabla v^\beta - \varpi_t \\
&= 2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2 + 2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g) \\
&\quad + 2(1-\alpha)v^{\beta+1}\langle \nabla\Delta g,\nabla g\rangle + (1-\alpha)v|\nabla g|^2\Delta v^\beta \\
&\quad + 2(1-\alpha)v\langle \nabla|\nabla g|^2,\nabla v^\beta\rangle - \varpi_t.
\end{aligned}
$$

By utilizing (2.1), (2.3) and (3.5), we have

$$
\begin{aligned}
&(1-\alpha)v\Delta\varpi - \varpi_t \\
&= 2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2 + 2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g) \\
&\quad + 2v^{\beta+1}\Big\langle \nabla\Big(\frac{g_t}{v} - \alpha|\nabla g|^2\Big),\nabla g\Big\rangle - 2(1-\alpha)\beta v^{\beta+1}\langle \nabla|\nabla g|^2,\nabla g\rangle \\
&\quad + (1-\alpha)\beta(\beta-1)v^{\beta-1}|\nabla v|^2|\nabla g|^2 + \beta(1-\alpha)v^\beta|\nabla g|^2\Delta v - \varpi_t \\
&= 2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2 + 2(1-\alpha)v^{\beta+1}\mathrm{Ric}(\nabla g,\nabla g) \\
&\quad + 2v^\beta\langle \nabla g_t,\nabla g\rangle - 2v^{\beta-1}g_t\langle \nabla v,\nabla g\rangle - 2\alpha v^{\beta+1}\langle \nabla|\nabla g|^2,\nabla g\rangle \\
&\quad - 2(1-\alpha)\beta v^{\beta+1}\langle \nabla|\nabla g|^2,\nabla g\rangle + (1-\alpha)\beta(\beta-1)v^{\beta+1}|\nabla g|^4 \\
&\quad + \beta v^{\beta-1}|\nabla g|^2(v_t + |\nabla v|^2) - \varpi_t \\
&= 2(1-\alpha)v^{\beta+1}|\nabla^2 g|^2 + \big[2(1-\alpha)v-2\big]v^\beta\mathrm{Ric}(\nabla g,\nabla g) \\
&\quad - 2v^{\beta-1}g_t\langle \nabla v,\nabla g\rangle - 2\alpha v^{\beta+1}\langle \nabla|\nabla g|^2,\nabla g\rangle \\
&\quad - 2(1-\alpha)\beta v^{\beta+1}\langle \nabla|\nabla g|^2,\nabla g\rangle + (1-\alpha)\beta(\beta-1)v^{\beta+1}|\nabla g|^4
\end{aligned}
$$

$$+ \beta v^{\beta+1}|\nabla g|^4. \tag{3.8}$$

Therefore, by (2.10) we have

$$\begin{aligned}(1-\alpha)v\Delta\varpi - \varpi_t \geq & -2[(1-\alpha)A+1]v^\beta|\nabla g|^2 K \\ & + \Big[(1-\alpha)\beta(\beta-1)+\beta+2\alpha-\frac{n(1-\alpha)}{2}-2\alpha\beta-2\beta^2(1-\alpha)\Big]v^{\beta+1}|\nabla g|^4 \\ & - [2\alpha+2\beta(1-\alpha)]v\langle\nabla\varpi,\nabla g\rangle,\end{aligned} \tag{3.9}$$

where we use the fact that

$$\langle\nabla\varpi,\nabla g\rangle = v^\beta\langle\nabla|\nabla g|^2,\nabla g\rangle - \beta v^\beta|\nabla g|^4.$$

In order to obtain the gradient estimates, we need to require the coefficient $f(\beta)$ of $|\nabla g|^4$ to be positive. In fact,

$$\begin{aligned}f(\beta) &= (1-\alpha)\beta(\beta-1)+\beta+2\alpha-\frac{n(1-\alpha)}{2}-2\alpha\beta-2\beta^2(1-\alpha) \\ &= -(1-\alpha)\Big[\beta+\frac{\alpha}{2(1-\alpha)}\Big]^2+\frac{\alpha^2}{4(1-\alpha)}+2\alpha-\frac{n(1-\alpha)}{2}.\end{aligned}$$

We choose $\beta = -\frac{\alpha}{2(1-\alpha)}$, then $f(\beta) > 0$ when $1-\frac{3+\sqrt{16+2n}}{7+2n} < \alpha < 1$. Therefore, (3.9) can be written as

$$(1-\alpha)v\Delta\varpi - \varpi_t \geq -2[(1-\alpha)A+1]K\varpi + \gamma v^{1-\beta}\varpi^2 - \alpha v\langle\nabla\varpi,\nabla g\rangle,$$

where $\gamma = \frac{\alpha^2}{4(1-\alpha)}+2\alpha-\frac{n(1-\alpha)}{2} > 0$. The proof is completed.

Taking $\beta = 1$ in (3.9),the following lemma is derived.

Lemma 3.3 *Let $\omega_1 = v|\nabla g|^2$. Then*

$$(1-\alpha)v\Delta\omega_1 - \partial_t\omega_1 \geq \epsilon\omega_1^2 - 2[(1-\alpha)A+1]K\omega_1 - 2v\langle\nabla\omega_1,\nabla g\rangle, \tag{3.10}$$

where $\epsilon = 2\alpha-1-\frac{n(1-\alpha)}{2} > 0$ with $1-\frac{2}{n+4} < \alpha < 1$.

We next introduce a smooth cut-off function (see [1, 16]), which will be used in the proof of our main theorems.

Lemma 3.4 (see [1, 16]) *We use the geodesic polar coordinate here. Given $\tau \in (0,T]$, there exists a smooth function $\overline{\Psi} : [0,\infty)\times[0,T] \to \mathrm{R}$ satisfying the following requirements:*

1. *The support of $\overline{\Psi}(r,t)$ is a subset of $[0,R]\times[0,T]$, and $0 \leq \overline{\Psi}(r,t) \leq 1$ in $[0,R]\times[0,T]$.*
2. *The equalities $\overline{\Psi}(r,t) = 1$ and $\frac{\partial\overline{\Psi}}{\partial r}(r,t) = 0$ hold in $[0,\frac{R}{2}]\times[\tau,T]$ and $[0,\frac{R}{2}]\times[0,T]$, respectively.*
3. *The estimate $\left|\frac{\partial\overline{\Psi}}{\partial t}\right| \leq \frac{\overline{C}\,\overline{\Psi}^{\frac{1}{2}}}{\tau}$ is satisfied on $[0,\infty)\times[0,T]$ for some $\overline{C} > 0$, and $\overline{\Psi}(r,0) = 0$ for all $r \in [0,\infty)$.*
4. *The inequalities $\frac{-C_a\overline{\Psi}^a}{R} \leq \frac{\partial\overline{\Psi}}{\partial r} \leq 0$ and $\left|\frac{\partial^2\overline{\Psi}}{\partial r^2}\right| \leq \frac{C_a\overline{\Psi}^a}{R^2}$ hold on $[0,\infty)\times[0,T]$ for every $a \in (0,1)$ with some constant C_a dependent on a.*

3.2 The proof of theorems

In this section, we will prove our main theorems by Lemma 3.4. Let $\mathrm{dist}(x, x_0, t)$ be the distance between $x \in \mathrm{M^n}$ and x_0 with respect to the metric $g(x, t)$.

Proof of Theorem 1.8 Part 1: In order to derive the result, we also need a cut-off function φ by Li-Yau [16] on $B_{R,T}$. Define a smooth function $\varphi : \mathrm{M^n} \times [0, T] \to \mathrm{R}$ by $\varphi(x, t) = \overline{\Psi}(\mathrm{dist}(x, x_0, t), t)$ supported in $B_{R,T}$, where $\overline{\Psi}$ satisfies Lemma 3.4.

Let $\omega = |\nabla g|^2$. Assume that the function $\varphi\omega$ arrives its maximum at a point (x_1, t_1) and x_1 is not in the cut-locus of $\mathrm{M^n}$ by [15]. Therefore, at (x_1, t_1), it yields $\Delta(\varphi\omega) \le 0$, $(\varphi\omega)_t \ge 0$ and $\nabla(\varphi\omega) = 0$.

By $0 = \nabla(\varphi\omega) = \omega\nabla\varphi + \varphi\nabla\omega$, then we have $\nabla\omega = -\frac{\nabla\varphi}{\varphi}\omega$. Hence, by (3.4) and a straightforward calculation, it yields that

$$\begin{aligned} 0 &\ge (1-\alpha)v\Delta(\varphi\omega) - (\varphi\omega)_t \\ &= \varphi[(1-\alpha)v\Delta\omega - \omega_t] + (1-\alpha)v\omega\Delta\varphi + 2(1-\alpha)v\langle\nabla\varphi, \nabla\omega\rangle - \omega\varphi_t \\ &\ge bv\varphi\omega^2 - 2[(1-\alpha)A + 1]K\varphi\omega - 2\alpha v\varphi\langle\nabla\omega, \nabla g\rangle \\ &\quad + (1-\alpha)v\omega\Delta\varphi + 2(1-\alpha)v\langle\nabla\varphi, \nabla\omega\rangle - \omega\varphi_t \\ &= bv\varphi\omega^2 - 2[(1-\alpha)A + 1]K\varphi\omega + 2\alpha v\omega\langle\nabla\varphi, \nabla g\rangle \\ &\quad + (1-\alpha)v\omega\Delta\varphi - 2(1-\alpha)v\omega\frac{|\nabla\varphi|^2}{\varphi} - \omega\varphi_t. \end{aligned} \tag{3.11}$$

This implies

$$\begin{aligned} 2\varphi\omega^2 \le{}& \frac{4}{bv}[(1-\alpha)A + 1]K\varphi\omega - \frac{4\alpha}{b}\langle\nabla\varphi, \nabla g\rangle\omega \\ &- \frac{2(1-\alpha)}{b}\omega\Delta\varphi + \frac{4(1-\alpha)}{b}\frac{|\nabla\varphi|^2}{\varphi}\omega + \frac{2}{bv}\omega\varphi_t. \end{aligned} \tag{3.12}$$

We next estimate upper bounds for each term of the right hand side of (3.12). Applying the Young inequality, we have

$$\begin{aligned} \frac{4}{bv}[(1-\alpha)A + 1]K\varphi\omega &\le \frac{1}{5}\varphi\omega^2 + C[(1-\alpha)A + 1]^2\varphi\frac{K^2}{\delta^2} \\ &\le \frac{1}{5}\varphi\omega^2 + C[(1-\alpha)A + 1]^2\frac{K^2}{\delta^2}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} -\frac{4\alpha}{b}\langle\nabla\varphi, \nabla g\rangle\omega &\le \frac{4\alpha}{b}|\nabla\varphi| \cdot \omega^{\frac{3}{2}} \\ &\le \frac{1}{5}\varphi\omega^2 + C\frac{|\nabla\varphi|^4}{\varphi^3} \le \frac{1}{5}\varphi\omega^2 + \frac{C}{R^4}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} -\frac{2(1-\alpha)}{b}\omega\Delta\varphi &= -\frac{2(1-\alpha)}{b}\omega\Big(\partial_r^2\varphi + (n-1)\frac{\partial_r\varphi}{r} + \partial_r\varphi \cdot \partial_r(\log\sqrt{g})\Big) \\ &\le C\omega\Big(|\partial_r^2\varphi| + (n-1)\frac{|\partial_r\varphi|}{r} + \sqrt{K}|\partial_r\varphi|\Big) \\ &\le C\omega\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}} + (n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}} + \sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big| \\ &\le \frac{1}{5}\varphi\omega^2 + C\Big(\frac{1}{R^4} + \frac{K}{R^2}\Big) \end{aligned} \tag{3.15}$$

and

$$\frac{4(1-\alpha)}{b}\frac{|\nabla\varphi|^2}{\varphi}\omega \le C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}\omega \le \frac{1}{5}\varphi\omega^2 + \frac{C}{R^4}. \tag{3.16}$$

For the last term, by [1], we have

$$\frac{2}{bv}\omega\varphi_t \le \frac{C}{\delta}\Big|\frac{\partial\varphi}{\partial t}\Big|\omega + \frac{C}{\delta}\Big|\frac{\partial\varphi}{\partial r}\Big|\Big|\frac{\partial}{\partial t}\mathrm{dist}\Big|\omega \le \frac{1}{5}\varphi\omega^2 + \frac{C}{\delta^2 t^2}. \tag{3.17}$$

We substitute (3.13)–(3.17) into (3.12), and have

$$\varphi\omega^2 \le C\Big([(1-\alpha)A+1]^2\frac{K^2}{\delta^2} + \frac{1}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2 t^2}\Big) \tag{3.18}$$

at (x_1, t_1). Therefore, for all $(x,t) \in B_{\frac{R}{2},T}$, we obtain

$$\begin{aligned}(\varphi\omega)^2(x,t) &\le (\varphi\omega)^2(x_1,t_1) \le \varphi\omega^2(x_1,t_1)\\ &\le C\Big([(1-\alpha)A+1]^2\frac{K^2}{\delta^2} + \frac{1}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2 t^2}\Big).\end{aligned} \tag{3.19}$$

Notice that $\varphi(x,t) = 1$ in $B_{\frac{R}{2},T}$ and $\omega = |\nabla g|^2 = \frac{|\nabla v|^2}{v^2}$, we get that

$$\frac{|\nabla v|^2}{v^2} \le C\Big([(1-\alpha)A+1]\frac{K}{\delta} + \frac{1}{R^2} + \frac{1}{\delta t}\Big).$$

Part 2: Define a smooth function

$$\varphi : \mathrm{M}^{\mathrm{n}} \times [0,T] \to \mathrm{R}$$

by $\varphi(x,t) = \overline{\Psi}(\mathrm{dist}(x,x_0,t),t)$ supported in $B_{R,T}$, where $\overline{\Psi}$ satisfies Lemma 3.4.

Let $\varpi = v^\beta|\nabla g|^2$. Assume that the function $\varphi\overline{\omega}$ arrives its maximum at a point (x_1,t_1) and x_1 is not in the cut-locus of M^{n} by [16]. Therefore, at (x_1,t_1), it yields $\Delta(\varphi\overline{\omega}) \le 0$, $(\varphi\overline{\omega})_t \ge 0$ and $\nabla(\varphi\overline{\omega}) = 0$.

By $0 = \nabla(\varphi\varpi) = \varpi\nabla\varphi + \varphi\nabla\varpi$, then we have $\nabla\varpi = -\frac{\nabla\varphi}{\varphi}\varpi$. Hence, by (3.7) and a straightforward calculation, it yields that

$$\begin{aligned}0 &\ge (1-\alpha)v\Delta(\varphi\varpi) - (\varphi\varpi)_t\\ &= \varphi[(1-\alpha)v\Delta\varpi - \varpi_t] + (1-\alpha)v\varpi\Delta\varphi + 2(1-\alpha)v\langle\nabla\varphi,\nabla\varpi\rangle - \varpi\varphi_t\\ &\ge \gamma v^{1-\beta}\varphi\varpi^2 - 2[(1-\alpha)A+1]K\varphi\varpi - \alpha v\varphi\langle\nabla\varpi,\nabla g\rangle\\ &\quad + (1-\alpha)v\varpi\Delta\varphi + 2(1-\alpha)v\nabla\varphi\nabla\varpi - \varpi\varphi_t\\ &= \gamma v^{1-\beta}\varphi\varpi^2 - 2[(1-\alpha)A+1]K\varphi\varpi + \alpha v\varpi\langle\nabla\varphi,\nabla g\rangle\\ &\quad + (1-\alpha)v\varpi\Delta\varphi - 2(1-\alpha)v\varpi\frac{|\nabla\varphi|^2}{\varphi} - \varpi\varphi_t.\end{aligned} \tag{3.20}$$

This implies

$$\begin{aligned}2\varphi\varpi^2 \le{}& \frac{4}{\gamma}[(1-\alpha)A+1]Kv^{\beta-1}\varphi\varpi - \frac{2\alpha}{\gamma}\langle\nabla\varphi,\nabla g\rangle v^\beta\varpi\\ &- \frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Delta\varphi + \frac{4(1-\alpha)}{\gamma}\frac{|\nabla\varphi|^2}{\varphi}v^\beta\varpi + \frac{2}{\gamma}v^{\beta-1}\varpi\varphi_t.\end{aligned} \tag{3.21}$$

We next estimate upper bounds for each term of the right hand side of (3.21). Applying the Young inequality, we have

$$\begin{aligned} \frac{4[(1-\alpha)A+1]}{\gamma}Kv^{\beta-1}\varphi\varpi &\le \frac{1}{5}\varphi\varpi^2 + C\varphi K^2\delta^{2\beta-2}[(1-\alpha)A+1]^2 \\ &\le \frac{1}{5}\varphi\varpi^2 + CK^2\delta^{2\beta-2}[(1-\alpha)A+1]^2, \qquad (3.22) \\ -\frac{2\alpha}{\gamma}\langle\nabla\varphi,\nabla g\rangle v^\beta\varpi &\le \frac{2\alpha}{\gamma}|\nabla\varphi|\cdot\varpi^{\frac{3}{2}}v^{\frac{\beta}{2}} \\ &\le \frac{1}{5}\varphi\varpi^2 + C\frac{|\nabla\varphi|^4}{\varphi^3}\delta^{2\beta} \le \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{2\beta}}{R^4}, \qquad (3.23) \\ -\frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Delta\varphi &= -\frac{2(1-\alpha)}{\gamma}v^\beta\varpi\Big(\partial_r^2\varphi + (n-1)\frac{\partial_r\varphi}{r} + \partial_r\varphi\cdot\partial_r(\log\sqrt{g})\Big) \\ &\le Cv^\beta\varpi\Big(|\partial_r^2\varphi| + (n-1)\frac{|\partial_r\varphi|}{r} + \sqrt{K}|\partial_r\varphi|\Big) \\ &\le C\delta^\beta\varpi\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}} + (n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}} + \sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big| \\ &\le \frac{1}{5}\varphi\varpi^2 + C\delta^{2\beta}\Big(\frac{1}{R^4} + \frac{K}{R^2}\Big) \qquad (3.24) \\ \frac{4(1-\alpha)}{\gamma}\frac{|\nabla\varphi|^2}{\varphi}v^\beta\varpi &\le C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}v^\beta\varpi \le \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{2\beta}}{R^4}, \qquad (3.25) \end{aligned}$$

and

$$\frac{2}{\gamma}v^{\beta-1}\overline{\varpi}\varphi_t \le \frac{C}{\gamma}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\varphi^{\frac{1}{2}}v^{\beta-1}\varpi \le \frac{1}{5}\varphi\varpi^2 + \frac{C\delta^{2\beta-2}}{t^2}. \qquad (3.26)$$

We substitute (3.22)–(3.26) into (3.21), and have

$$\varphi\varpi^2 \le C\delta^{2\beta}\Big([(1-\alpha)A+1]^2\frac{K^2}{\delta^2} + \frac{1}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2t^2}\Big) \qquad (3.27)$$

at (x_1,t_1). Therefore, for all $(x,t)\in B_{\frac{R}{2},T}$, we obtain

$$\begin{aligned} (\varphi\varpi)^2(x,t) &\le (\varphi\varpi)^2(x_1,t_1) \le \varphi\varpi^2(x_1,t_1) \\ &\le C\delta^{2\beta}\Big([(1-\alpha)A+1]^2\frac{K^2}{\delta^2} + \frac{1}{R^4} + \frac{K}{R^2} + \frac{1}{\delta^2t^2}\Big). \qquad (3.28) \end{aligned}$$

Notice that $\varphi(x,t)=1$ in $B_{\frac{R}{2},T}$ and $\overline{\varpi} = v^\beta|\nabla g|^2 = \frac{|\nabla v|^2}{v^{2-\beta}}$, we get that

$$\frac{|\nabla v|^2}{v^{2-\beta}} \le C\delta^\beta\Big([(1-\alpha)A+1]\frac{K}{\delta} + \frac{1}{R^2} + \frac{1}{\delta t}\Big).$$

So, we prove Theorem 1.8.

Proof of Theorem 1.9 Define a smooth function $\varphi : \mathrm{M}^{\mathrm{n}}\times[0,T]\to\mathrm{R}$ by $\varphi(x,t) = \overline{\Psi}(\mathrm{dist}(x,x_0,t),t)$ supported in $B_{R,T}$, where $\overline{\Psi}$ satisfies Lemma 3.4.

Let $\omega_1 = v|\nabla g|^2$. Assume that the function $\varphi\omega_1$ arrives its maximum at a point (x_1,t_1) and x_1 is not in the cut-locus of M^{n} by [16]. Therefore, at (x_1,t_1), it yields $\Delta(\varphi\omega_1)\le 0$, $(\varphi\omega_1)_t \ge 0$

and $\nabla(\varphi\omega_1)=0$. By $0=\nabla(\varphi\omega_1)=\omega_1\nabla\varphi+\varphi\nabla\omega_1$, then we have $\nabla\omega_1=-\frac{\nabla\varphi}{\varphi}\omega_1$. Hence, by (3.10) and a straightforward calculation, it yields that

$$\begin{aligned}
0 &\geq (1-\alpha)v\Delta(\varphi\omega_1)-(\varphi\omega_1)_t \\
&= \varphi[(1-\alpha)v\Delta\omega_1-(\omega_1)_t]+(1-\alpha)v\omega_1\Delta\varphi+2(1-\alpha)v\langle\nabla\varphi,\nabla\omega_1\rangle-\omega_1\varphi_t \\
&\geq \epsilon\varphi\omega_1^2-2[(1-\alpha)A+1]K\varphi\omega_1-2v\varphi\langle\nabla\omega_1,\nabla g\rangle \\
&\quad +(1-\alpha)v\omega_1\Delta\varphi+2(1-\alpha)v\nabla\varphi\nabla\omega_1-\omega_1\varphi_t \\
&= \epsilon\varphi\omega_1^2-2[(1-\alpha)A+1]K\varphi\omega_1+2v\omega_1\langle\nabla\varphi,\nabla g\rangle \\
&\quad +(1-\alpha)v\omega_1\Delta\varphi-2(1-\alpha)v\omega_1\frac{|\nabla\varphi|^2}{\varphi}-\omega_1\varphi_t
\end{aligned} \tag{3.29}$$

This implies

$$\begin{aligned}
2\varphi\omega_1^2 \leq{}& \frac{4}{\epsilon}[(1-\alpha)A+1]K\varphi\omega_1-\frac{4}{\epsilon}\langle\nabla\varphi,\nabla g\rangle v\omega_1 \\
&-\frac{2(1-\alpha)}{\epsilon}v\omega_1\Delta\varphi+\frac{4(1-\alpha)}{\epsilon}\frac{|\nabla\varphi|^2}{\varphi}v\omega_1+\frac{2}{\epsilon}\omega_1\varphi_t.
\end{aligned} \tag{3.30}$$

We next estimate upper bounds for each term of the right hand side of (3.30). Applying the Young inequality, we have

$$\begin{aligned}
\frac{4}{\epsilon}[(1-\alpha)A+1]K\varphi\omega_1 &\leq \frac{1}{5}\varphi\omega_1^2+C\varphi[(1-\alpha)A+1]^2K^2 \\
&\leq \frac{1}{5}\varphi\omega_1^2+C[(1-\alpha)A+1]^2K^2, && (3.31)\\
-\frac{4}{\epsilon}\langle\nabla\varphi,\nabla g\rangle v\omega_1 &\leq \frac{4}{\epsilon}|\nabla\varphi|\cdot\omega_1^{\frac{3}{2}}\sqrt{A} \\
&\leq \frac{1}{5}\varphi\omega_1^2+C\frac{|\nabla\varphi|^4}{\varphi^3}A^2\leq\frac{1}{5}\varphi\omega_1^2+\frac{CA^2}{R^4}, && (3.32)\\
-\frac{2(1-\alpha)}{\epsilon}v\omega_1\Delta\varphi &= -\frac{2(1-\alpha)}{\epsilon}v\omega_1\Big(\partial_r^2\varphi+(n-1)\frac{\partial_r\varphi}{r}+\partial_r\varphi\cdot\partial_r(\log\sqrt{g})\Big) \\
&\leq CA\omega_1\Big(|\partial_r^2\varphi|+(n-1)\frac{|\partial_r\varphi|}{r}+\sqrt{K}|\partial_r\varphi|\Big) \\
&\leq CA\omega_1\varphi^{\frac{1}{2}}\Big|\frac{|\partial_r^2\varphi|}{\varphi^{\frac{1}{2}}}+(n-1)\frac{|\partial_r\varphi|}{K\varphi^{\frac{1}{2}}}+\sqrt{K}\frac{|\partial_r\varphi|}{\varphi^{\frac{1}{2}}}\Big| \\
&\leq \frac{1}{5}\varphi\omega_1^2+CA^2\Big(\frac{1}{R^4}+\frac{K}{R^2}\Big), && (3.33)\\
\frac{4(1-\alpha)}{\epsilon}\frac{|\nabla\varphi|^2}{\varphi}v\omega_1 &\leq C\frac{|\nabla\varphi|^2}{\varphi^{\frac{3}{2}}}\varphi^{\frac{1}{2}}A\omega_1\leq\frac{1}{5}\varphi\omega_1^2+\frac{CA^2}{R^4} && (3.34)
\end{aligned}$$

and

$$\frac{2}{\epsilon}\omega_1\varphi_t\leq\frac{2}{\epsilon}\frac{|\varphi_t|}{\varphi^{\frac{1}{2}}}\varphi^{\frac{1}{2}}\omega_1\leq\frac{1}{5}\varphi\omega_1^2+\frac{C}{t^2}. \tag{3.35}$$

We substitute (3.31)–(3.35) into (3.30), and have

$$\varphi\omega_1^2\leq CA^2\Big([(1-\alpha)A+1]^2K^2+\frac{1}{R^4}+\frac{K}{R^2}\Big)+\frac{C}{t^2} \tag{3.36}$$

at (x_1, t_1). Therefore, for all $(x,t) \in B_{\frac{R}{2},T}$, we obtain

$$\begin{aligned}(\varphi\omega_1)^2(x,t) &\le (\varphi\omega_1)^2(x_1,t_1) \le \varphi\omega_1^2(x_1,t_1)\\ &\le CA^2\Big(\frac{1}{R^4}+\frac{K}{R^2}\Big)+C[(1-\alpha)A+1]^2K^2+\frac{C}{t^2}.\end{aligned} \tag{3.37}$$

Notice that $\varphi(x,t)=1$ in $B_{\frac{R}{2},T}$ and $\omega_1 = v|\nabla g|^2 = \frac{|\nabla v|^2}{v}$, we get that

$$\frac{|\nabla v|^2}{v} \le CA\Big([(1-\alpha)A+1]K+\frac{1}{R^2}+\frac{\sqrt{K}}{R}\Big)+\frac{C}{t}.$$

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