Chin. Ann. Math. Ser. B
42(4), 2021, 551–560
DOI: 10.1007/s11401-021-0277-2

Chinese Annals of Mathematics, Series B

Computations of the Adams-Novikov E_2-term*

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Abstract The author introduces the notion of a minimal resolution for BP_*BP-comodules, and gives an effective algorithm to produce minimal resolutions. This produces the data needed in the work [3] for studying motivic stable stems up to stem 90 .

Keywords Stable homotopy, Adams-Novikov spectral sequence, Hopf algebroid
2000 MR Subject Classification 55-04, 55Q45

1 Introduction

The Adams-Novikov spectral sequence, constructed in [5], is an important tool for the study of stable homotpy groups. See [6] for a detailed treatment of its construction and computational techniques. One of the difficulty for this method is that the computations of the E_2 term is very hard. This paper gives an efficient algorithm for stemwise computations of the Adams-Novikov E_2 term using computers. Together with techniques introduced in [2], this has made breakthroughs in the stemwise computations of classical and motivic stable homotopy groups (see [3] and forthcoming works).

Usually people compute the Adams-Novikov E_2 term by the following three methods: The algebraic Novikov spectral sequence, the Bockstein spectral sequence, and the chromatic spectral sequence. The chromatic spectral sequence, introduced in [4], separates informations of different heights (or periods), which are computed individually. This gives many global structure theorems for the Adams-Novikov spectral sequence (see [6] for details). However, for stem-wise computations, especially for small primes, it is hard to get complete informations using the chromatic methods alone. For example, to get to the stem 126 at prime 2, informations up to chromatic level 6 are involved, which are beyond our current knowledge.

The Bockstein spectral sequence is often used for stem-wise computations. For example, this method is illustrated in [6] for the computation of the first 25 stems. The method used in [6] to the computation the differentials in the Bockstein spectral sequence is to do the computation in the cobar complex. As the cobar complex grows very fast, direct computations in the cobar complex become impossible very quickly, even for computers.

The algebraic Novikov spectral sequence is also often used in stem-wise computations. This spectral sequence is very important from a theoretic as well as a practical point of view. It is proved in [2] that the algebraic Novikov spectral sequence for the sphere spectrum is isomorphic

Manuscript received July 15, 2020. Revised December 11, 2020.
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*This work was supported by the National Natural Science Foundation of China (No. 11801082) and the Shanghai Rising-Star Program (No. 20QA1401600).

to the motivic Adams spectral sequence for the cofiber of tau, giving us a very powerful method for stemwise computations provided we know the structure of the algebraic Novikov spectral sequence. As in the case of the Bockstein spectral sequence, the method of computing the algebraic Novikov differentials using the cobar complex becomes too complicated very quickly.

The main technique of this paper is the observation that, the Bockstein and the algebraic Novikov filtrations are defined on any resolution of BP_*BP-comodules. So we can replace the cobar complex with any cofree resolution. In particular we introduce the notion of a minimal resolution which is smallest among cofree resolutions.

Recall that the maximal ideal I of BP_* is an invariant ideal, with the quotient BP_*BP/I isomorphic to the even part of the dual Steenrod algebra. We define a cofree resolution to be minimal, if its modulo I reduction is a minimal resolution as BP_*BP/I-comodules. Then we use the Bockstein and the algebraic Novikov filtrations on the minimal resolution to compute the Bockstein and the algebraic Novikov spectral sequences respectively.

To construct the minimal resolution, we first construct a minimal resolution for the modulo I reduction. Then an arbitrary lift of this resolution almost gives us what we want, except that the compositions of consecutive maps are only zero modulo I. Then we do adjustments using some kind of Gaussian elimination. One such algorithm is given in Section 4. The naive method can be optimized by observing that, for maps between cofree comodules, we only need to know the projection to the cogenerators. This reduces the sizes of the matrices to be computed by one order. This optimization is given in Section 5.

Implementations

The algorithm is implemented with C++ code (using C++11 standard), using the library GMP (GNU Multiple Precision Arithmetic Library) to deal with large integers, and the library OpenMP (Open Multi-Processing) to deal with parallelism. The source code is available at: https://github.com/pouiyter/MinimalResolution.

The program has been performed on the MiG (minimum intrusion grid) at University of Copenhagen, the Wayne State University Grid high performance computing cluster, and the high performance computer at Shanghai Center for Mathematical Sciences. The (current) outputs are available at: https://github.com/pouiyter/morestablestems/raw/master/algNovikov-machine.csv.

2 Preliminaries

Let BP be the Brown-Peterson spectrum. It is a complex oriented ring spectrum whose associated formal group law is the universal p-typical formal group law over $\mathbb{Z}_{(p)}$. We have

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots],$$
$$BP_*BP = BP_*[t_1, t_2, \cdots].$$

Define

$$I = (p, v_1, v_2, \cdots),$$
$$P = BP_*BP/I = \mathbb{F}_p[t_1, t_2, \cdots].$$

The pair (BP_*, BP_*BP) forms a Hopf algebroid which represents the moduli stack of formal groups over $\mathbb{Z}_{(p)}$. See [6] for a detailed treatment of Brown-Peterson theory and the theory of Hopf algebroids.

Reduction modulo I gives the Hopf algebra P, which is a sub-Hopf algebra of the dual Steenrod algebra. For $p = 2$, P is isomorphic to the dual Steenrod algebra with degrees doubled.

A BP_*BP-comodule F is cofree if F is a direct product of copies of BP_*BP (and its degree shifts).

For a BP_*BP-comodule M, let $\mathrm{Prim}(M)$ denote the primitive elements of M.

We carry out all constructions in the graded sense. In particular, a direct product of objects which are finite in each degree is also a direct sum.

Definition 2.1 *A graded $\mathbb{Z}_{(p)}$-module is locally finite if it is bounded below and finitely generated in each degree.*

For example, BP_*, BP_*BP and P are all locally finite. In this paper, we implicitly assume that all modules are locally finite.

3 Minimal Resolutions of BP_*BP-Comodules

In this section we introduce the notion of a minimal resolution for BP_*BP-comodules, which are lifts of minimal resolutions of P-comodules.

Let M be a BP_*BP-comodule which is free as a BP_*-module. Then M/I becomes a P-comodule.

Definition 3.1 *A comodule map is a strong injection (resp., strong surjection) if it is a split injection (resp., split surjection) of underlying BP_*-modules.*

An exact sequence $0 \to M \xrightarrow{f} F \xrightarrow{g} N \to 0$ of BP_*BP-comodules is strongly exact if f is strongly injective and g is strongly surjective.

Remark 3.1 A map is strongly injective (resp., surjective), if its associated matrix over BP_* can be transformed into the form $(\begin{pmatrix} id \\ 0 \end{pmatrix})$ (resp., $(id \quad 0)$) by row (resp., column) transformations.

For a comodule map $f : M \to N$, let $\widetilde{f}$ be the reduction $M/I \to N/I$ of f modulo I. If f is strongly injective (resp., strongly surjective), the cokernel (resp., kernel) of $\widetilde{f}$ is the reduction of $\mathrm{coker}(f)$ (resp., $\ker(f)$) modulo I.

Proposition 3.1 *Let M and N be BP_*BP-comodules which are locally finite and free as BP_*-modules. Then a comodule map*

$$f : M \to N$$

is strongly injective (resp., strongly surjective) if and only if the reduction

$$\widetilde{f} : M/I \to N/I$$

modulo I is injective (resp., surjective).

Proof If $\widetilde{f}$ is injective (resp., surjective), then by lifting the row and column transformations, the matrix for f can be transformed into the one which is equivalent modulo I to $\begin{pmatrix} id \\ 0 \end{pmatrix}$ (resp., $(id \quad 0)$. Since I is a maximal ideal in BP_*, one can further transform the matrix into $\begin{pmatrix} id \\ 0 \end{pmatrix}$ (resp., $(id \quad 0)$) by row and column transformations.

Corollary 3.1 *Suppose that we have a sequence* $0 \to M \xrightarrow{f} F \xrightarrow{g} N \to 0$ *of* BP_*BP*-comodule maps such that* M, F *and* N *are free over* BP_*, *and* $g \circ f = 0$. *The sequence is strongly exact if and only if its reduction modulo* I *is exact.*

Recall that the data of a long exact sequence $0 \to M \to F_0 \to F_1 \to \cdots$ is equivalent to the data of a collection

$$\begin{gathered} 0 \to M \to F_0 \to M_1 \to 0 \\ 0 \to M_1 \to F_1 \to M_2 \to 0 \\ 0 \to M_2 \to F_2 \to M_3 \to 0 \\ \vdots \end{gathered}$$

of short exact sequences.

Definition 3.2 *A long exact sequence* $0 \to M \to F_0 \to F_1 \to \cdots$ *of* BP_*BP*-comodules is a cofree resolution of* M *if each* F_i *is cofree, and each short exact sequence* $0 \to M_i \to F_i \to M_{i+1} \to 0$ *is strongly exact.*

Recall that we have the following notion of minimal resolutions of P-comodules.

Definition 3.3 *A cofree resolution*

$$\begin{gathered} 0 \to \widetilde{M} \to \widetilde{F_0} \to \widetilde{M_1} \to 0 \\ 0 \to \widetilde{M_1} \to \widetilde{F_1} \to \widetilde{M_2} \to 0 \\ 0 \to \widetilde{M_2} \to \widetilde{F_2} \to \widetilde{M_3} \to 0 \\ \vdots \end{gathered}$$

of a P*-comodule* $\widetilde{M}$ *is called minimal if for all* $i \geq 0$, *the induced map*

$$\mathrm{Prim}(\widetilde{M_i}) \to \mathrm{Prim}(\widetilde{F_i})$$

is bijective. (*By convention,* $M_0 = M$.)

Remark 3.2 Since Prim is a left exact functor from P-comodules to $\mathbb{F}_p$-modules, it follows that the induced map

$$\mathrm{Prim}(\widetilde{F_i}) \to \mathrm{Prim}(\widetilde{F_{i+1}})$$

is trivial.

For BP_*BP-comodules, we define minimal resolutions in terms of reductions modulo I.

Definition 3.4 *Let* M *be a* BP_*BP*-comodule which is free over* BP_*. *A cofree resolution of* M *is called a minimal resolution if its reduction modulo* I *is a minimal resolution of* M/I.

One can see that a minimal resolution has the smallest size among all cofree resolutions.

4 Construction of Minimal Resolutions

We will construct minimal resolutions of BP_*BP-comodules by lifting minimal resolutions of reductions modulo I.

First we introduce the notion of cogenerators dual to the notion of generators. Let M be a locally finite BP_*BP-comodule which is free as BP_*-module, and let X be a locally finite free BP_*-module. Let $f : M \to X$ be a BP_*-module map. Recall that the adjoint map of f is the composite

$$M \xrightarrow{\psi_M} BP_*BP \otimes_{BP_*} M \xrightarrow{1\otimes f} BP_*BP \otimes_{BP_*} X.$$

We say that f exhibits X as cogenerators of M if the adjoint map of f is strongly injective. We will also abbreviate to say that X is the cogenerators of M.

In this case, if $g : N \to M$ is a comodule map, then its corestriction to the cogenerators is defined to be the composite map $N \xrightarrow{g} M \xrightarrow{f} X$. One finds that the adjoint of the above map factors through g, and we have a commutative diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & BP_*BP \otimes_{BP_*} N \arrow[d] \\ M \arrow[r] & BP_*BP \otimes_{BP_*} X \end{tikzcd}.$$

It follows that g is determined by its corestriction to the cogenerators of M.

Proposition 4.1 *A map $f : M \to X$ exhibits X as cogenerators of M if and only if its reduction*

$$\widetilde{f} : M/I \to X/I$$

modulo I exhibits X/I as cogenerators of M/I as a P-comodule.

Proof The adjoint map

$$M/I \to P \otimes_{\mathbb{F}_p} X/I$$

for $\widetilde{f}$ is the reduction of the adjoint map for f modulo I, so the proposition follows from Proposition 3.1.

Now we construct minimal resolutions as follows. Suppose that M is a locally finite BP_*BP-comodule with free underlying BP_*-module. We will construct strongly exact sequences

$$0 \to M \to F_0 \to M_1 \to 0$$

$$0 \to M_1 \to F_1 \to M_2 \to 0$$

$$\vdots$$

such that each F_i is cofree.

Set $M_0 = M$. Suppose that we have constructed a locally finite BP_*BP-comodule M_n which is free as a BP_*-module. We do the following to construct M_{n+1}:

1. Find a minimal cogenerator $\widetilde{f}_n : M_n/I \to \widetilde{X}_n$ for M_n/I, i.e., an $\mathbb{F}_p$-module $\widetilde{X}_n$ such that the adjoint map $M_n/I \to P \otimes \widetilde{X}_n$ is injective and the map $\mathrm{Prim}(M_n/I) \to \widetilde{X}_n$ is bijective.

2. Take a free BP_*-module X_n such that $X_n/I \cong \widetilde{X}_n$. By the freeness of M_n as a BP_*-module, we can lift $\widetilde{f}_n$ to $f_n : M_n \to X_n$.

3. Take F_n to be $BP_*BP \otimes_{BP_*} X_n$, and M_{n+1} to be the cokernel of the adjoint map $g_n : M_n \to F_n$ of f_n, so we have the quotient map $h_n : F_n \to M_{n+1}$.

In this way, we construct a minimal resolution inductively.

Remark 4.1 The first step is the standard one for computing Adams E_2 terms using minimal resolutions. In the second step, in addition to constructing the map f_n, we also need to do a Gaussian elimination for its adjoint map g_n, in order to find the quotient matrix needed in Step 3. And in Step 3, we need to compose the quotient map with the coaction map on F_n to get the coaction map on M_{n+1}.

Remark 4.2 The formulas for coactions of cofree comodules can be computed beforehand, e.g. save a comultiplication table for BP_*BP in the disk.

5 Optimization of the Process

The complexity of computing a minimal resolution of M/I has smaller order than computing a minimal resolution of M. So the process can be optimized by computing a minimal resolution of M/I first, and then using it as a model for a resolution of M.

Once we know the structure of the minimal resolution of M/I, the structures of the cofree comodules F_i are already known. The problem with an arbitrary lift of the minimal resolution of M/I is that the compositions of consecutive maps are not guaranteed to be zero.

Once we know the structure of F_n, we already know a set of cogenerators for it. Moreover, these also make a set of cogenerators for M_n. So the data for g_n and h_n (in Step 3 of Section 4) are determined by their corestrictions to cogenerators. This reduces the order of the size of the matrices for the data of f_n and g_n. So the optimized process is as follows:

1. This is the same as Step 1 of Section 4, but it is computed beforehand.
2. This is the same as Step 2 of Section 4, but it is computed beforehand.
3. Compute the matrix for the composite map

$$\widetilde{h}_n : F_n \to M_{n+1} \to X_{n+1}$$

by first taking an arbitrary lift of the map $F_n/I \to X_{n+1}/I$ constructed in Step 1 and using a Gaussian elimination process for the map $g_n : M_n \to F_n$ to modify $\widetilde{h}_n$ so that the composite $\widetilde{h}_n \circ g_n$ is 0. Since h_n is a comodule map, this suffices to imply that $h_n \circ g_n = 0$.

4. In order to compute g_{n+1}, we only need to know the composite

$$M_{n+1} \xrightarrow{\psi} BP_*BP \otimes M_{n+1} \to BP_*BP \otimes X_{n+1}$$

instead of the full formula for the coaction of M_n. We do this by composing the coaction of F_n with the composite

$$F_n \to M_{n+1} \to X_{n+1}.$$

Remark 5.1 In Step 3, instead of solving the linear equations $h_n \circ g_n = 0$, we solve the equations $\widetilde{h}_n \circ g_n = 0$, which only involve the cogenerators. This reduces the complexity of Step 3 by one order.

Similarly, in Step 4, only the coactions of cogenerators are computed. This reduces the complexity of Step 4 by one order.

6 Computations of Homology

With the minimal resolution constructed, our next step is to compute the homology of the chain complex of its primitives. We will modify the algorithm of [1].

Suppose $X_0 \to X_1 \to \cdots$ is a complex of locally finite $\mathbb{Z}_{(p)}$-modules.

Suppose each X_i is given a maximal filtration. This means that, there is an ordinal α_0 such that for each ordinal number $\alpha < \alpha_0$, there is a submodule $\mathrm{Fil}^{\geq\alpha} X_i$ of X_i, decreasing with respect to α. Moreover, the maximality means that, $\mathrm{Fil}^{\geq\alpha} X_i/\mathrm{Fil}^{\geq\alpha+1} X_i \cong \mathbb{F}_p$, and when α is a limit ordinal, $\mathrm{Fil}^{\geq\alpha} X_i = \bigcap\limits_{\beta<\alpha} \mathrm{Fil}^{\geq\beta} X_i$.

For each i, we take a set A_i consisting of an $\mathbb{F}_p$ basis for the graded pieces of X_i. Then each A_i has a canonical order. For any $x \in X_i$, we say $a \in A_i$ is the leading term of x, if the projection of x to the graded pieces of X_i is a nontrivial multiple of a.

Remark 6.1 Here we allow the generality that the filtration is indexed by any ordinal number. In actual computations we will use appropriate truncations to make the filtration finite.

Remark 6.2 We do not require that the differentials in the complex respect the filtrations.

A Curtis table is a list with entries of the form:

- a, where $a \in A_i$ for some i, or
- $a \to b$, where $a \in A_i$ and $b \in A_{i+1}$ for some i;

such that

1. An entry a is in the table if and only if a is the leading term of a cycle, and no boundary has leading term a.

2. An entry $a \to b$ is in the table if and only if

(a) there is an element $x \in X_i$ with leading term a, and $d(x)$ has leading term b, and

(b) for any element x' such that $d(x')$ has leading term b, the leading term of x' is at least a (i.e., x' has filtration at most that of a).

The following proposition is proved in [7].

Proposition 6.1 *Suppose that $X_0 \to X_1 \to \cdots$ is a complex of locally finite $\mathbb{Z}_{(p)}$-modules such that each X_i is maximally filtered, and let A_i be an $\mathbb{F}_p$ basis for the graded pieces of X_i. Then a* Curtis *table exists and is unique. And in addition, all the elements of A_i appears in the table exactly once.*

Now we suppose that there is an additional filtration $\mathcal{F}_i$ on each X_i which is preserved by the differentials. Moreover, we suppose that the maximal filtration associated to A_i is a refinement of $\mathcal{F}_i$. Then the Curtis table describes the structure of the spectral sequence defined by $\mathcal{F}_i$ (see [7] for details).

Proposition 6.2 *The entries of the form a in the* Curtis *table correspond bijectively to the surviving permanent cycles in the spectral sequence defined by the filtration $\mathcal{F}_i$.*

The entries of the form $a \to b$ correspond bijectively to the differentials in the spectral sequence, where the length of the differential is the difference of the $\mathcal{F}$-filtration degrees of a and b. (We include all the differentials from d_0.)

7 The Algebraic Adams-Novikov Spectral Sequence

Let

$$M \to F_0 \to F_1 \to \cdots$$

be a cofree resolution of a BP_*BP-comodule M, which is free over BP_*.

Then $\mathrm{Ext}(M)$ is computed by the complex

$$\mathrm{Prim}(F_0) \to \mathrm{Prim}(F_1) \to \cdots .$$

By construction, $\mathrm{Prim}(F_i)$ has the structure of a free BP_*-module when a set of cogenerators for F_i is given. (Note that this BP_*-module structure is not preserved by the differentials.)

The algebraic Adams-Novikov filtration is given by filtering $\mathrm{Prim}(F_i)$ by powers of the augmentation ideal I. To be precise, let G_i be a set of BP_*-generators for $\mathrm{Prim}(F_i)$. Then all elements of $\mathrm{Prim}(F_i)$ are linear combinations of expressions of the form $v_0^{i_0}v_1^{i_1}\cdots v_k^{i_k}a$ with $a \in G_i$. By convention, $v_0 = p$. Then we define the decreasing algebraic Adams-Novikov filtration by setting $v_0^{i_0}v_1^{i_1}\cdots v_k^{i_k}a$ to have filtration $i_0 + i_1 + \cdots + i_k$.

Let A_i be the set

$$\{v_0^{i_0}v_1^{i_1}\cdots v_k^{i_k}a : i_j \geq 0, a \in G_i\}.$$

We order A_i by the following rules. We first order them by using the Adams-Novikov filtration. Then, amongst elements with the same Adams-Novikov filtration, we use a lexicographic order.

By Proposition 6.2, we have the following propositon.

Proposition 7.1 *The* Curtis *table associated to the above A_i gives the structure of the algebraic* Adams-Novikov *spectral sequence for M.*

Remark 7.1 We can also introduce an order, by taking the lexicographic order directly. This gives results in the Bockstein spectral sequence.

8 The Atiyah-Hirzebruch Spectral Sequence and Multiplicative Structure

We fix a cofree resolution

$$BP_* \to F_0 \to F_1 \to \cdots$$

for BP_*.

Let M be a BP_*BP comodule. Recall that the tensor product of two BP_*BP-comodules (over BP_*) has the structure of a BP_*BP-comodule (see [6, A1.1.2]). BP_* is the unit for this tensor product, i.e.,

$$M \otimes BP_* \cong M.$$

Since F_i is cofree, $M \otimes F_i$ is relatively injective, and

$$\mathrm{Prim}(M \otimes F_i) \cong M \otimes_{BP_*} \mathrm{Prim}(F_i).$$

So $\mathrm{Ext}(M)$ can be computed by the complex $M \otimes_{BP_*} \mathrm{Prim}(F_i)$.

Remark 8.1 The formula for the above isomorphism involves the coaction on M, because we used the diagonal coaction.

Now suppose that M is filtered as a BP_*BP-comodule. Then there is an associated algebraic Atiyah-Hirzebruch spectral sequence computing $\mathrm{Ext}(M)$. We have the following standard fact for Atiyah-Hirzebruch differentials.

Proposition 8.1 *Let M have a two-step filtration as shown in the short exact sequence*

$$0 \to BP_* \to M \to \Sigma^j BP_* \to 0.$$

Let $h \in \mathrm{Ext}^{j,1}(BP_*)$ be the element corresponding to this extension. Then the Atiyah-Hirzebruch *differentials for M corresponds to multiplication by h.*

So we can compute multiplication by elements in homological degree 1 by computing the Atiyah-Hirzebruch differentials. To do this, we do the following.

Suppose in general that M is a filtered BP_*BP-comodule. We select a set K of BP_*-generators for M, with order refining the filtration on M. Then an element in

$$\mathrm{Prim}(M \otimes F_i) \cong M \otimes_{BP_*} \mathrm{Prim}(F_i)$$

is a linear combination of expressions of the form

$$k \otimes v_0^{i_0} \cdots v_k^{i_k} a$$

with $k \in K$ and $a \in G_i$. Here G_i is a set of BP_*-generators for $\mathrm{Prim}(F_i)$, as in Section 7.

We order these elements as follows. We first consider the filtration of k, then the Adams-Novikov filtration of $v_0^{i_0} \cdots v_k^{i_k}$, and finally the lexicographic order. By Proposition 6.2, the Curtis table for this order gives the structure of the Atiyah-Hirzebruch spectral sequence for M.

Remark 8.2 More generally, if the comodule M has a three step filtration, then the Atiyah-Hirzebruch spectral sequence gives data for the corresponding Massy products. The same generalizes to higher Massy products. One needs to be careful that the choice of M fixes some part of the indeterminacies of the Massey products.

Remark 8.3 Once we have a cofree resolution of BP_*, we always get a relative injective resolution for any M, which needs not to be free over BP_*. In particular, we can take M to be an extension of BP_*/p. This allows us to compute multiplications and Massey products involving p as well as the β families, the latter living in homological degree 1 on the top cell of BP_*/p.

Acknowledgements The author thanks Dan Isaksen who has read through the paper and made many modifications. The author also thanks Mark Behrens, Robert Bruner, Jesper Grodal, Lars Hesselholt, Haynes Miller, Doug Ravenel, John Rognes and Zhouli Xu for many useful discussions.

References

[1] Curtis, E. B., Goerss, P., Mahowald, M. and Milgram, J. R., Calculations of unstable Adams E_2 terms for spheres, Algebraic Topology (Seattle, Wash., 1985), 208–266, Lecture Notes in Math., **1286**, Springer-Verlag, Berlin, 1987.

[2] Gheorghe, B., Wang, G. Z. and Xu, Z. L., The special fiber of the motivic deformation of the stable homotopy category is algebraic, Acta Mathematica, to appear.

[3] Isaksen, D. C., Wang, G. Z. and Xu, Z. L., More stable stems, arXiv:2001.04511, 2020.

[4] Miller, H. R., Ravenel, D. C. and Wilson, W. S., Periodic phenomena in the Adams-Novikov spectral sequence, *Annals of Mathematics*, **106**, 1977, 469–516.

[5] Novikov, S. P., The methods of algebraic topology from viewpoint of cobordism theory, *Izv. Akad. Nauk SSSR, Ser. Mat.,* **31**(4), 1967.

[6] Ravenel, D. C., Complex cobordism and stable homotopy groups of spheres, Pure and Applied Mathematics, **121**, Academic Press, Inc, Orlando, FL, 1986.

[7] Wang, G. Z. and Xu, Z. L., The algebraic Atiyah-Hirzebruch spectral sequence of real projective spectra, arXiv:1601.02185, 2016.