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Hermitian-Poisson Metrics on Flat Bundles over Complete Hermitian Manifolds

Changpeng PAN[1]

Abstract In this paper, the author solves the Dirichlet problem for Hermitian-Poisson metric equation $\sqrt{-1}\Lambda_\omega G_H = \lambda \mathrm{Id}$ and proves the existence of Hermitian-Poisson metrics on flat bundles over a class of complete Hermitian manifolds. When $\lambda = 0$, the Hermitian-Poisson metric is a Hermitian harmonic metric.

Keywords Flat bundle, Hermitian harmonic metric, Hermitian-poisson metric, Complete Hermitian manifolds
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1 Introduction

Let (X, g) be a Hermitian manifold, ω be the Kähler form related to g. Let (V, D) be a flat bundle of rank r over X, i.e., the connection satisfies $D^2 = 0$. For any Hermitian metric H on V, we have the following unique decomposition:

$$D = D_H + \psi_H, \tag{1.1}$$

where D_H is compatible with H and $\psi_H \in \Omega_X^1(\mathrm{End}(E))$ is self-adjoint with respect to H. Set

$$\partial_H = D_H^{1,0}, \quad \overline{\partial}_H = D_H^{0,1}, \quad \theta_H = \psi_H^{1,0}, \quad \overline{\theta}_H = \psi_H^{0,1}. \tag{1.2}$$

Define

$$D''_H = \overline{\partial}_H + \theta_H, \quad D'_H = \partial_H + \overline{\theta}_H \tag{1.3}$$

and $G_H = (D''_H)^2$. The harmonic metric equation is

$$D_H^* \psi_H = 0. \tag{1.4}$$

We say H is a harmonic metric if it satisfies the harmonic metric equation. When (X, g) is a Kähler manifold, Kähler identity implies that $D_H^* \psi_H = 2\sqrt{-1}\Lambda_\omega G_H$. So the harmonic metric equation is equivalent to $\sqrt{-1}\Lambda_\omega G_H = 0$.

It is well known that there is a correspondence between flat bundles and representations of fundamental group. Let $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ be the representation related to (V, D). The Hermitian metric H induces a ρ-equivariant map

$$f_H : \widetilde{X} \to \mathrm{GL}(r, \mathbb{C})/U(r), \tag{1.5}$$

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[1]School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China.
E-mail: pcp1995@mail.ustc.edu.cn

where $\widetilde{X}$ is the universal covering of X. f_H is a harmonic map if and only if H is a harmonic metric. The existence of harmonic metrics has an important application in non-abelian Hodge theory, see [6, 13] for detail.

When the rank of the bundle equals to 1, (1.4) becomes a Poisson equation. For the high rank case, Donaldson [4] and Corlette [2] proved the existence of harmonic metrics on semi-simple flat bundles over compact Riemannian manifolds. Jost-Zuo proved the existence of harmonic metrics on semi-simple flat bundles over quasi-compact Kähler manifolds in [8–9]. In [1], Collins, Jacob and Yau considered the following Poisson metric equation over non-compact curves:

$$D_H^* \psi_H = \lambda \mathrm{Id}. \tag{1.6}$$

This is a deformation of the harmonic metric equation. They proved the existence of Poisson metric on polystable parabolic flat bundle.

In this paper, we are interested in the following Hermitian-Poisson metric equation:

$$\sqrt{-1}\Lambda_\omega G_H = \lambda \mathrm{Id} \tag{1.7}$$

over complete Hermitian manifolds. We call H on V is a Hermitian-Poisson metric if H satisfies (1.7). When $\lambda = 0$, we call H is a Hermitian harmonic metric.

We first prove the Dirichlet problem for Hermitian-Poisson metric equation over compact Hermitian manifolds with smooth boundary.

Theorem 1.1 *Assume that (X, g) is a compact Hermitian manifold with non-empty smooth boundary ∂X. Let (V, D) be a flat bundle over X. Then there is a unique Hermitian-Poisson metric H on X such that $H|_{\partial X} = \varphi$, where φ is a Hermitian metric on $V|_{\partial X}$.*

For a non-compact complete Hermitian manifold (X, g), take a compact sub-domains exhausting sequence $\{\Omega_i\}_{i=1}^{\infty}$ of X. Then the Poisson metric equation can be solved on Ω_i for every i. Suppose that the manifold (X, g) satisfies some suitable conditions and there exists a good background Hermitian metric on V. Then we can deform these Poisson metrics on Ω_i into a Poisson metric on X.

Theorem 1.2 *Let (X, g) be a complete non-compact Hermitian manifold of dimension n and $\widetilde{\Delta} = 2\sqrt{-1}\Lambda_\omega \partial\overline{\partial}$. Let (V, D) be a flat bundle over X and K be a background metric. Then*

(i) *if $\widetilde{\Delta}$ has the positive first eigenvalue $\widetilde{\lambda}_1(X)$, and $\|\sqrt{-1}\Lambda_\omega G_K - \lambda \mathrm{Id}\|_{L^p} < +\infty$ for some $p \geq 2$ and real number λ, then there exists a Poisson metric H on V.*

(ii) *if $\widetilde{\Delta}$ satisfies the L^2-Sobolev inequality, and $\|\sqrt{-1}\Lambda_\omega G_K - \lambda \mathrm{Id}\|_{L^p} < +\infty$ for some $p \in [2, n)$ and real number λ, then there exists a Poisson metric H on V.*

Remark 1.1 Suppose that X is a Kähler manifold and $\lambda = 0$. Then Theorem 1.1 is a special case of Dirichlet problem for harmonic map equation from compact manifolds with smooth boundary to complete Riemannian manifold with nonpositive sectional curvature. Theorem 1.2 should be a special case of harmonic map equation from complete noncompact manifolds to complete Riemannian manifold with nonpositive sectional curvature. See [3] for detail (see [7, 10] for Hermitian harmonic map).

This paper is organized as follows. In Section 2, we introduce a heat flow about Poisson metric equation and prove the long time existence of its solution. In Section 3, we prove Theorem 1.1, and in Section 4, we prove Theorem 1.2.

2 Heat Flow on Compact Manifolds with Boundary

Let X be a complex manifold with non-empty smooth boundary ∂X. Define the holomorphic Laplace operator for functions as $\widetilde{\Delta} f = 2\sqrt{-1}\Lambda_\omega \partial\overline{\partial} f$. Let (V, D) be a flat bundle over X. Consider the following flow on V,

$$\begin{cases} H^{-1}(t)\dfrac{\partial H(t)}{\partial t} = 4(\sqrt{-1}\Lambda_\omega G_{H(t)} - \lambda \mathrm{Id}), \\ H(t)|_{t=0} = H_0, \quad H|_{\partial X} = H_0|_{\partial X}, \end{cases} \tag{2.1}$$

where H_0 is a background metric on V. It is not hard to check that this is a nonlinear parabolic equation, so the solution exists for short time.

Proposition 2.1 *Let $H(t)$ be the solution of* (2.1). *Then*

$$\Big(\frac{\partial}{\partial t} - \widetilde{\Delta}\Big)|\Phi(H(t))|^2_{H(t)} \le 0, \tag{2.2}$$

where $\Phi(H(t)) = \sqrt{-1}\Lambda_\omega G_{H(t)} - \lambda \mathrm{Id}$, *and*

$$\sup_X |\Phi(H(t))|_{H(t)} \le \sup_X |\Phi(H_0)|_{H_0}. \tag{2.3}$$

Proof When there is no confusion, we omit the parameter t in the computations for simplicity. Under the local flat basis of (V, D), we have

$$\overline{\partial}_H = \overline{\partial} + \frac{1}{2}H^{-1}\overline{\partial}H, \quad \theta_H = -\frac{1}{2}H^{-1}\partial H. \tag{2.4}$$

A direct computation implies

$$\frac{\partial}{\partial t}\overline{\partial}_H = \frac{1}{2}\overline{\partial}_H\Big(H^{-1}\frac{\partial H}{\partial t}\Big) - \frac{1}{2}\Big[\overline{\theta}_H, H^{-1}\frac{\partial H}{\partial t}\Big] \tag{2.5}$$

and

$$\frac{\partial}{\partial t}\theta_H = -\frac{1}{2}\partial_H\Big(H^{-1}\frac{\partial H}{\partial t}\Big) + \frac{1}{2}\Big[\theta_H, H^{-1}\frac{\partial H}{\partial t}\Big]. \tag{2.6}$$

So

$$\begin{aligned} \frac{\partial \Phi(H)}{\partial t} &= \sqrt{-1}\Lambda_\omega D''_H\Big(\frac{\partial D''_H}{\partial t}\Big) \\ &= \frac{1}{2}\sqrt{-1}\Lambda_\omega D''_H\Big\{D''_H\Big(H^{-1}\frac{\partial H}{\partial t}\Big) - D'_H\Big(H^{-1}\frac{\partial H}{\partial t}\Big)\Big\} \\ &= -\frac{1}{2}\sqrt{-1}\Lambda_\omega D''_H D'_H\Big(H^{-1}\frac{\partial H}{\partial t}\Big), \end{aligned} \tag{2.7}$$

where in the last equality we have used the fact that $\big[\sqrt{-1}\Lambda_\omega G_H, H^{-1}\frac{\partial H}{\partial t}\big] = 0$. On the other hand, notice that $\Phi(H)^{*H} = \Phi(H)$, so

$$-\sqrt{-1}\Lambda_\omega \overline{\partial}\partial|\Phi(H)|^2_H = -\sqrt{-1}\Lambda_\omega \overline{\partial}\partial \mathrm{tr}(\Phi(H) \circ \Phi(H))$$

$$= -\sqrt{-1}\Lambda_\omega\overline{\partial}\mathrm{tr}\{D'_H\Phi(H)\circ\Phi(H)+\Phi(H)\circ D'_H\Phi(H)\}$$
$$= -\sqrt{-1}\Lambda_\omega\mathrm{tr}\{D''_H D'_H\Phi(H)\circ\Phi(H) - D'_H\Phi(H)\wedge D''_H\Phi(H) + D''_H\Phi(H)\wedge D'_H\Phi(H) + \Phi(H)\circ D''_H D'_H\Phi(H)\}. \tag{2.8}$$

Combining all the above, we have

$$\begin{aligned}&\Big(\frac{\partial}{\partial t} - 2\sqrt{-1}\Lambda_\omega\partial\overline{\partial}\Big)|\Phi(H)|^2_H\\ &= 2\mathrm{tr}\Big(\frac{\partial\Phi(H)}{\partial t}\circ\Phi(H)\Big) - 2\sqrt{-1}\Lambda_\omega\partial\overline{\partial}|\Phi(H)|^2_H\\ &= -4|D'_H\Phi(H)|^2_H.\end{aligned} \tag{2.9}$$

On the boundary ∂X we know $\Phi(H(t))=0$. By the maximum principle, (2.3) holds.

Definition 2.1 *For any two Hermitian metrics H and K on V, define*

$$\sigma(H,K) = \mathrm{tr}(H^{-1}K) + \mathrm{tr}(K^{-1}H) - 2\mathrm{rank}(V). \tag{2.10}$$

Let $h = K^{-1}H$ and $D^c_K = D''_K - D'_K$. Then we find

$$D''_H - D''_K = \frac{1}{2}h^{-1}D^c_K h, \quad \sqrt{-1}\Lambda_\omega(G_H - G_K) = \frac{1}{4}\sqrt{-1}\Lambda_\omega D(h^{-1}D^c_K h). \tag{2.11}$$

Lemma 2.1 *Let H and K be two Hermitian-Poisson metrics. Then we have*

$$\widetilde{\Delta}\sigma(H,K) \geq 0. \tag{2.12}$$

Proof From (2.11), one can see that

$$h(\Phi(H)-\Phi(K)) = -\frac{1}{4}\sqrt{-1}\Lambda_\omega Dhh^{-1}D^c_K h + \frac{1}{4}\sqrt{-1}\Lambda_\omega DD^c_K h. \tag{2.13}$$

Taking trace of both sides, we get

$$\frac{1}{4}\sqrt{-1}\Lambda_\omega dd^c\mathrm{tr}(h) - \frac{1}{4}|Dhh^{-\frac{1}{2}}|^2_K \geq -\mathrm{tr}(h)(|\Phi(H)|_H + |\Phi(K)|_K). \tag{2.14}$$

Also, we can derive

$$\frac{1}{4}\sqrt{-1}\Lambda_\omega dd^c\mathrm{tr}(h^{-1}) - \frac{1}{4}|Dh^{-1}h^{\frac{1}{2}}|^2_K \geq -\mathrm{tr}(h^{-1})(|\Phi(H)|_H + |\Phi(K)|_K). \tag{2.15}$$

Since $\Phi(H)=\Phi(K)=0$, this lemma follows.

Lemma 2.2 *Let $H(t)$ and $K(t)$ be two solutions of* (2.1). *Then*

$$\Big(\frac{\partial}{\partial t} - \widetilde{\Delta}\Big)\sigma(H(t),K(t)) \leq 0. \tag{2.16}$$

Proof Let $h(t) = K^{-1}(t)H(t)$. Note that

$$\frac{\partial}{\partial t}\mathrm{tr}(h(t)) = 4\mathrm{tr}(h(t)(\Phi(H(t)) - \Phi(K(t)))) \tag{2.17}$$

and

$$\frac{1}{4}\sqrt{-1}\Lambda_\omega dd^c\mathrm{tr}(h(t)) - \frac{1}{4}|Dh(t)h^{-\frac{1}{2}}(t)|^2_{K(t)} = \mathrm{tr}(h(t)(\Phi(H(t)) - \Phi(K(t)))). \tag{2.18}$$

This finishes the proof.

We will show the long-time existence of the solution in the following.

Proposition 2.2 *If $H(t)$ is a solution of the parabolic equation* (2.1) *defined for* $0 \le t < T$, *then $H(t)$ approaches a continuous non-degenerate limit H_T in C^0-norm as $t \to T$.*

Proof Given $\epsilon > 0$, by the continuity at $t = 0$ we can find a δ such that

$$\sup_X \sigma(H(t_0), H(t_1)) \le \epsilon \tag{2.19}$$

for $0 < t_0, t_1 < \delta$. Then Lemma 2.2 and the maximum principle imply that

$$\sup_X \sigma(H(s), H(t)) \le \epsilon \tag{2.20}$$

for all $s, t > T - \delta$. Then $H(t)$ are uniform Cauchy sequence and converge to a continuous limiting metric H_T. Set $h(t) = K^{-1}H(t)$. A direct calculation shows

$$\Big|\frac{\partial}{\partial t}\log\operatorname{tr}(h(t))\Big| \le 4|\Phi(H(t))|_{H(t)} \tag{2.21}$$

and

$$\Big|\frac{\partial}{\partial t}\log\operatorname{tr}(h^{-1}(t))\Big| \le 4|\Phi(H(t))|_{H(t)}. \tag{2.22}$$

This together with Proposition 2.1 means that H_T is non-degenerate.

Following Simpson's argument (see [12, Lemma 6.4]), we can conclude the following lemma.

Lemma 2.3 *Suppose that $H(t)$ is a family of metrics on V over X with $H(t) \to H_T$ in C^0-norm. If $H(t)$ satisfy Dirichlet boundary conditions, and if $\sup_X |\Lambda_\omega G_{H(t)}|_{H(t)}$ is bounded uniformly in t, then $H(t)$ are bounded in L_2^p uniformly in t.*

Corollary 2.1 *The parabolic equation* (2.1) *has a unique solution $H(t)$ which exists for $0 \le t < +\infty$.*

3 Proof of Theorem 1.1

In this section we will consider the Dirichlet boundary problem for Hermitian-Poisson metric equation and use the heat equation method to deform an arbitrary initial metric to the desired one. The main points in the discussion are similar to that in [5] or [12]. Let X be a compact Hermitian manifolds with smooth boundary ∂X. For any Hermitian metric φ on $V|_{\partial X}$ over ∂X. We can extend it to V over X, denoted by H_0. Let $H(t)$ be the solution of (2.1).

Proof of Theorem 1.1 By a direct calculation, one can check that

$$|d|\gamma|_{H(t)}|^2 \le |D_{H(t)}\gamma|^2_{H(t)} \tag{3.1}$$

for any section $\gamma \in \Gamma(X, V)$. According to Proposition 2.1, we have

$$\Big(\frac{\partial}{\partial t} - \widetilde{\Delta}\Big)|\Phi(H(t))|_{H(t)} \le 0. \tag{3.2}$$

Let v be a solution of the following equation (see [14, Chapter 5, Proposition 1.8] for the existence of v):

$$\begin{cases} \widetilde{\Delta} v = -|\Phi(K)|_K, \\ v|_{\partial X} = 0. \end{cases} \tag{3.3}$$

Set $w(x,t)=\int_0^t|\Phi(H)|_H(x,s)\mathrm{d}s-v(x)$. Since $\Phi(H(t))=0$ on ∂X, we have

$$\begin{cases}\Big(\dfrac{\partial}{\partial t}-\widetilde{\Delta}\Big)w\geq 0,\\ w|_{t=0}=-v,\\ w|_{\partial X}=0.\end{cases}\tag{3.4}$$

The maximum principle implies

$$\int_0^t|\Phi(H)|_H(x,s)\mathrm{d}s\leq\sup_X v\tag{3.5}$$

for any $0\leq t<+\infty$. Let $0\leq t_1\leq t$ and $\overline{h}(t)=H^{-1}(t_1)H(t)$. Then

$$\overline{h}^{-1}(t)\frac{\partial\overline{h}(t)}{\partial t}=4\Phi(H(t))\tag{3.6}$$

and

$$\frac{\partial}{\partial t}\log\mathrm{tr}(\overline{h}(t))\leq 4|\Phi(H(t))|_{H(t)}.\tag{3.7}$$

From the above formula, we see

$$\mathrm{tr}(H^{-1}(t_1)H(t))\leq r\exp\Big(4\int_{t_1}^t|\Phi(H)|_H\mathrm{d}s\Big).\tag{3.8}$$

We have a similar estimate for $\mathrm{tr}(H^{-1}(t)H(t_1))$. This gives us that

$$\sigma(H(t),H(t_1))\leq 2r\Big\{\exp\Big(4\int_{t_1}^t|\Phi(H)|_H\mathrm{d}s\Big)-1\Big\}.\tag{3.9}$$

Combining (3.5) and (3.9), we deduce that $H(t)$ converge in the C^0 topology to some continuous metric H_∞ as $t\to+\infty$. Using Lemma 2.3 again, one can find that $H(t)$ are bounded in L_2^p uniformly in t. By the heat equation, $|H^{-1}\frac{\partial H}{\partial t}|$ is bounded. Then, the standard elliptic regularity implies that there exists a subsequence $H(t)\to H_\infty$ in C^∞ topology. Due to formula (3.5), we know that H_∞ satisfies

$$\sqrt{-1}\Lambda_\omega G_{H_\infty}=\lambda\mathrm{Id}.\tag{3.10}$$

From Lemma 2.1 and the maximum principle, it is easy to conclude the uniqueness of solution.

4 Proof of Theorem 1.2

In this section, we study the existence of the Hermitian-Poisson metrics on some complete Hermitian manifolds. The argument is similar to that used by Zhang in [15] (also in [11, 16]).

Suppose that X is a complete noncompact Hermitian manifold. Let $\{\Omega_i\}_{i=1}^\infty$ be an exhausting sequence of compact sub-domains of X, and K be a Hermitian metric on V. In Section 3, we have already shown that the following Dirichlet problem is solvable on Ω_i, i.e., there exists a Hermitian metric H_i such that

$$\begin{cases}\sqrt{-1}\Lambda_\omega G_{H_i}=\lambda\mathrm{Id},\\ H_i|_{\partial\Omega_i}=K|_{\partial\Omega_i}.\end{cases}\tag{4.1}$$

In order to prove that we can pass to limit and eventually obtain a solution on the whole manifold X, we need to establish some estimates. Denote $h_i = K^{-1}H_i$ and $\widetilde{\sigma}_i = \widetilde{\sigma}(H_i, K) = \log(\mathrm{tr}(h_i) + \mathrm{tr}(h_i^{-1})) - \log(2r)$. Then we have

$$\begin{cases} \widetilde{\Delta}\widetilde{\sigma}_i \geq -4|\Phi(K)|_K, \\ \widetilde{\sigma}_i|_{\partial\Omega_i} = 0. \end{cases} \tag{4.2}$$

Definition 4.1 *We say that $\widetilde{\Delta}$ has the positive first eigenvalue if there exists a constant $c > 0$ such that for any smooth function ρ with compact support, one has*

$$\int_X (-\widetilde{\Delta}\rho)\rho \geq c\int_X \rho^2. \tag{4.3}$$

And the supremum of c is denoted by $\widetilde{\lambda}_1(X)$.

Definition 4.2 *We say that $\widetilde{\Delta}$ satisfies L^2-Sobolev inequality if there exists a constant $S(X)$ such that for any smooth function ρ with compact support, one has*

$$\int_X (-\widetilde{\Delta}\rho)\rho \geq S(X)\Big(\int_X |\rho|^{\frac{4m}{2m-2}}\Big)^{\frac{2m-2}{2m}}. \tag{4.4}$$

Lemma 4.1 (see [15]) *Let X be an n-dimensional complete Hermitian manifold, and the holomorphic Laplace operator $\widetilde{\Delta}$ has the positive first eigenvalue $\widetilde{\lambda}_1(X)$. Then for a nonnegative continuous function f, the equation*

$$\widetilde{\Delta}u = -f \tag{4.5}$$

has a nonnegative solution $u \in L^{2n}_{2,\mathrm{loc}} \cap C^{1,\alpha}_{\mathrm{loc}}$ $(0 < \alpha < 1)$ if $f \in L_p(X)$ for some $p \geq 2$.

Lemma 4.2 (see [15]) *Let X be an n-dimensional complete Hermitian manifold, and the holomorphic Laplace operator $\widetilde{\Delta}$ satisfies the L^2-Sobolev inequality. Then for a nonnegative continuous function f, the equation*

$$\widetilde{\Delta}u = -f \tag{4.6}$$

has a nonnegative solution $u \in L^{2n}_{2,\mathrm{loc}} \cap C^{1,\alpha}_{\mathrm{loc}}$ $(0 < \alpha < 1)$ if $f \in L_p(X)$ for some $n > p \geq 2$.

Proof of Theorem 1.2 (i) By Lemma 4.1 (Lemma 4.2 for (ii)), (4.2) and the maximum principle, we conclude that

$$\sigma(H_i, K) \leq 2r\exp(u) - 2r \tag{4.7}$$

on Ω_i, and u is the solution of $\widetilde{\Delta}u = -4|\Phi(K)|_K$. After a similar argument with [15], we can show that the C^1-norm of H_i are uniformly bounded on any bounded open subset. Then the standard elliptic theory tells us that, by passing a subsequence, H_i converge uniformly on any compact sub-domain of X to a smooth Hermitian metric H_∞ satisfying

$$\sqrt{-1}\Lambda_\omega G_{H_\infty} = \lambda\mathrm{Id}. \tag{4.8}$$

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References

[1] Collins, T. C., Jacob, A. and Yau, S. T., Poisson metrics on flat vector bundles over non-compact curves, *Comm. Anal. Geom.*, **27**(3), 2019, 529–597.

[2] Corlette, K., Flat G-bundles with canonical metrics, *J. Differential Geom.*, **28**(3), 1988, 361–382.

[3] Ding, W. and Wang, Y., Harmonic maps of complete noncompact Riemannian manifolds, *Internat. J. Math.*, **2**(6), 1991, 617–633.

[4] Donaldson, S. K., Twisted harmonic maps and the self-duality equations, *Proc. London Math. Soc.* (3), **55**(1), 1987, 127–131.

[5] Donaldson, S. K., Boundary value problems for Yang-Mills fields, *J. Geom. Phys.*, **8**(1–4), 1992, 89–122.

[6] Guichard, O., An introduction to the differential geometry of flat bundles and of Higgs bundles, The Geometry, Topology and Physics of Moduli Spaces of Higgs Bundles, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., **36**, 2018, 1–63.

[7] Jost, J. and Yau, S. T., A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry, *Acta Math.*, **170**(2), 1993, 221–254.

[8] Jost, J. and Zuo, K., Harmonic maps and $SL(n, \mathbb{C})$-representations of fundamental groups of quasi-projective manifolds, *J. Algebraic Geom.*, **5**(1), 1996, 77–106.

[9] Jost, J., and Zuo, K., Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasi-projective varieties, *J. Differential Geom.*, **46**(3), 1997, 467–503.

[10] Ni, L., Hermitian harmonic maps from complete Hermitian manifolds to complete Riemannian manifolds, *Math. Z.*, **232**(2), 1999, 331–355.

[11] Ni, L. and Ren, H., Hermitian-Einstein metrics for vector bundles on complete Kähler manifolds, *Trans. Amer. Math. Soc.*, **353**(2), 2001, 441–456.

[12] Simpson, C., Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization, *J. Amer. Math. Soc.*, **1**(4), 1988, 867–918.

[13] Simpson, C., Higgs bundles and local systems, *Inst. Hautes tudes Sci. Publ. Math.*, **75**, 1992, 5–95.

[14] Taylor, M. E., Partial Differential Equations I, Basic Theory, Texts in Applied Mathematics, **115**, Springer-Verlag, New York, 1996.

[15] Zhang, X., Hermitian-Einstein metrics on holomorphic vector bundles over Hermitian manifolds, *J. Geom. Phys.*, **53**(3), 2005, 315–335.

[16] Zhang, X., Hermitian Yang-Mills-Higgs metrics on complete Kähler manifolds, *Canad. J. Math.*, **57**(4), 2005, 871–896.