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Exact Boundary Controllability of Weak Solutions for a Kind of First Order Hyperbolic System — The Constructive Method*

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Abstract In this paper the authors first present the definition and some properties of weak solutions to 1-D first order linear hyperbolic systems. Then they show that the constructive method with modular structure originally given in the framework of classical solutions is still very powerful and effective in the framework of weak solutions to prove the exact boundary (null) controllability and the exact boundary observability for first order hyperbolic systems.

Keywords First order linear hyperbolic system, Exact boundary controllability, Exact boundary observability, Constructive method
2020 MR Subject Classification 93B05, 93C20, 35L50

1 Introduction

1.1 Review on the exact boundary controllability

There are many publications concerning the exact boundary controllability for linear hyperbolic systems. For a kind of 1-D first order linear hyperbolic system, the exact boundary controllability was established by means of the characteristic method by Russell ([17]). Later, the Hilbert Uniqueness Method (HUM for short), a more general and systematic framework was introduced by J.-L. Lions for the study of linear hyperbolic systems, especially, of wave equations ([14–15]), which builds up the relationship between the exact boundary controllability for the system and the exact boundary observability for the adjoint system. Based on the HUM method and Schauder's fixed point theorem, some results were obtained on the exact boundary controllability for semilinear wave equations by Zuazua ([19–20]). An abstract result on the exact boundary controllability for semilinear equations was established by Lasiecka and Triggiani in [7]. Recently, Coron and Nguyen studied the (null) controllability of a general 1-D linear hyperbolic system using boundary controls on one side in optimal time based on the backstepping method (see [5–6]).

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In quasi-linear case there were only few results (see [2–3]) before. Until 2002, Li and Rao proposed a constructive method with modular structure (see [10-11]) and established a complete theory on the exact boundary controllability for the general 1-D first order quasi-linear hyperbolic system without zero eigenvalues

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u) \tag{1.1}$$

with general non-linear boundary conditions based on the semi-global C^1 solution theory (see [8–9]). Under this framework, the local exact boundary controllability was realized by two-sided controls, one-sided controls and two-sided controls with fewer boundary controls, respectively (in the linear case, it gives actually the global controllability). In the case of two-sided controls, the number of boundary controls is equal to the number of unknown variables, namely, the number of all the eigenvalues. In the case of one-sided controls, all the boundary controls act only on the side with more boundary controls, the number of which is reduced to the maximum value of the number of positive eigenvalues and that of negative ones, but the control time must be suitably enlarged. In particular, when the number of positive eigenvalues is equal to that of negative ones, boundary controls can act on either side. In the case of two-sided controls with fewer boundary controls, both the total number of boundary controls and the control time are the same as in the case of one-sided controls, however, the side with fewer boundary controls should be fully controlled, while the rest of boundary controls acts on the other side (see [8]). However, in the last two cases, more strict conditions should be satisfied. For example, for the case of one-sided controls, it requires that the boundary conditions on the side without boundary controls should be reversible. To be explicit, these boundary conditions should be able to be suitably rewritten so that, after adding suitable artificial boundary conditions, the backward problem of this system is well-posed.

The requirement mentioned before doesn't occur for the system of wave equations, since the system of wave equations with usual boundary conditions is always time reversible, thus the exact boundary controllability is always equivalent to the exact boundary null controllability, namely, the system can drive any given initial data U_0 at $t = 0$ exactly to the final data $U \equiv 0$ at $t = T$ by means of boundary controls. Apparently, this equivalence is not true for the first order hyperbolic system in general, therefore the exact boundary controllability is named as the strong exact boundary controllability, and the exact boundary null controllability is named as the weak exact boundary controllability in [12], respectively. The exact boundary controllability implies the exact boundary null controllability, however, the exact boundary null controllability implies the exact boundary controllability only if the system is time reversible (see [12], also see Remark 4.1). On the other hand, in order to realize the one-sided exact boundary null controllability, controls can act on any given side of the boundary, while on the other side the boundary condition should be homogeneous, thus the number of the boundary controls can be further reduced (see [12]).

As a tool, the duality can be used to study the controllability in the linear case, then the study of observability is also of great importance from this point of view. Russell [17] introduced a kind of observability in the 1-D case that for the backward problem of the first order linear hyperbolic system, the boundary observations can uniquely determine the initial data at $t = 0$. It is called as the weak exact boundary observability in [12]. Another kind of observability, which requires that the boundary observations can uniquely determine the final data at $t = T$ for the backward problem, is called as the strong exact boundary observability in [12]. Both of them can be established by a constructive method for 1-D quasi-linear hyperbolic systems in the framework of classical solutions (see [12]), and the strong exact boundary observability implies naturally the weak exact boundary observability, however, in general, one can not obtain the strong exact boundary observability from the weak exact boundary observability unless the system is time reversible (see [12], also see Remark 5.4). Moreover, in the framework of classical solutions, for some special cases in the linear situation, the relationship between the exact boundary (null) controllability for the original system and the strong (weak) exact boundary observability for the adjoint system in the case of one-sided control was discussed in [12]. We will generalize the corresponding conclusions in the framework of weak solutions.

1.2 Purpose of this paper

In this paper, we will consider the exact boundary controllability for the first order linear hyperbolic system in the framework of weak solutions.

First, to study the controllability from coupled systems of wave equations to first order hyperbolic systems is of great significance, since the latter has much wider connotation than the former. For instance, in 1-D case, a wave equation can be always transformed into a first order hyperbolic system, but the number of positive eigenvalues is always equal to that of negative ones (see [8]). If the wave equation explicitly contains the state variable U, there will be also the zero eigenvalue in the corresponding first order hyperbolic system. But for any given first order hyperbolic system, generally speaking, the number of positive eigenvalues is not necessarily equal to that of negative ones, which often makes trouble in the treatment. In this paper we only consider first order hyperbolic systems without zero eigenvalues. Besides, for wave equations with the usual boundary conditions (Dirichlet, Neumann, Robin, even dissipative boundary conditions), the whole system is always time reversible. However, first order hyperbolic systems are not time reversible in general. In fact, when the number of positive eigenvalues is not equal to that of negative ones, or even when they are equal to each other, but the boundary conditions don't satisfy certain reversibility, the corresponding backward problem will not be well-posed. Hence, first order hyperbolic systems have not only much more abundant connotations and practical applications than wave equations, but also they have their own difficulties: the methods to treat wave equations may not remain effective for first order hyperbolic systems. Thus it is necessary to carry out a special research on them.

Second, we will study the controllability for first order hyperbolic systems in the framework of weak solutions. In the present paper, we will show that the constructive method with modular structure is very powerful and effective not only in the framework of classical solutions, but also in the framework of weak solutions.

In what follows, we will first discuss the well-posedness for the following 1-D first order linear hyperbolic system in the framework of weak solutions and establish some related estimates, then we study the corresponding exact boundary controllability and exact boundary observability by the constructive method.

The system under consideration is given by

$$U_t + \Lambda U_x + AU = 0, \quad t \in (0, +\infty),\ x \in (0, L) \tag{1.2}$$

with the boundary conditions

$$U^+(t,0) = G_0 U^-(t,0) + D_0 H^+(t), \quad t \in (0,+\infty), \tag{1.3}$$

$$U^-(t,L) = G_1 U^+(t,L) + D_1 H^-(t), \quad t \in (0,+\infty) \tag{1.4}$$

and the initial data

$$U(0,x) = U_0(x), \quad x \in (0,L), \tag{1.5}$$

where $U = (u_1, \cdots, u_n)^T : (0,+\infty) \times (0,L) \to \mathbb{R}^n$ denotes the state variable, $\Lambda = \mathrm{diag}\{\Lambda^-, \Lambda^+\}$ is a diagonal matrix of order n,

$$\Lambda^- := \mathrm{diag}\{\lambda_1, \cdots, \lambda_m\}, \quad \Lambda^+ := \mathrm{diag}\{\lambda_{m+1}, \cdots, \lambda_n\}$$

with $\lambda_r < 0$ $(r = 1, \cdots, m)$ and $\lambda_s > 0$ $(s = m+1, \cdots, n)$, the coupling matrix $A = (a_{ij})$ is of order n. Let $\overline{m} = n - m$. The boundary coupling matrices G_0 and G_1 are of order $\overline{m} \times m$ and $m \times \overline{m}$, respectively, the boundary control matrices D_0 and D_1 are of order $\overline{m} \times M_0$ and $m \times M_1$ $(M_0 \leq \overline{m}, M_1 \leq m)$, respectively, both of them are full column-rank matrices. All the matrices mentioned above are with constant elements. Moreover, $U = (U^-, U^+)^T$ with $U^- = (u_1, \cdots, u_m)^T$ and $U^+ = (u_{m+1}, \cdots, u_n)^T$, $H = (H^-, H^+)^T$ with $H^- = (h_1, \cdots, h_{M_1})^T$ and $H^+ = (h_{M_1+1}, \cdots, h_M)^T$ $(M = M_0 + M_1 \leq n)$.

All the characteristics $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_s$ $(s = m+1, \cdots, n)$ $\big($resp. $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_r$ $(r = 1, \cdots, m)\big)$ corresponding to the positive (resp. negative) eigenvalues are called to be the coming (resp. departing) characteristics on $x = 0$, since they reach (resp. leave) the boundary $x = 0$ from (resp. to) the interior of the domain $\{(t,x) \mid t \geq 0, 0 \leq x \leq L\}$. Similarly, all the characteristics $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_r$ $(r = 1, \cdots, m)$ $\big($resp. $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_s$ $(s = m+1, \cdots, n)\big)$ corresponding to the negative (resp. positive) eigenvalues are called to be the coming (resp. departing) characteristics on $x = L$.

In order to guarantee the well-posedness of the mixed problem of system (1.2), the boundary conditions should satisfy the following requirements:

1. the number of boundary conditions on each boundary is equal to the number of the coming characteristics on this boundary;

2. the boundary conditions on each boundary can be equivalently rewritten in the form that the state variables corresponding to the coming characteristics are explicitly expressed by the state variables corresponding to the departing characteristics.

Boundary conditions (1.3)–(1.4) are just given according to above requirements.

2 L^2 Estimates of C^1 Solutions

In the framework of classical solutions, by the theory of C^1 solutions for the first order hyperbolic system, we have

Lemma 2.1 (see [13]) *For any given $T > 0$, for any given initial data $U_0 \in (C^1[0,L])^n$ and any given boundary function $H \in (C^1[0,T])^M$, suppose that the conditions of C^1 compatibility are satisfied at the points $(t,x) = (0,0)$ and $(0,L)$, respectively*:

$$U_0^+(0) = G_0 U_0^-(0) + D_0 H^+(0), \quad U_0^-(L) = G_1 U_0^+(L) + D_1 H^-(0) \tag{2.1}$$

and

$$\Lambda^+(U_0^+(0))' + A^+U_0(0) + D_0(H^+(0))' = G_0[\Lambda^-(U_0^-(0))' + A^-U_0(0)],$$
$$\Lambda^-(U_0^-(L))' + A^-U_0(L) + D_1(H^-(L))' = G_1[\Lambda^+(U_0^+(L))' + A^+U_0(L)], \tag{2.2}$$

where A^- *and* A^+ *are the first* m *rows and the last* $\overline{m}$ *rows of* A*, respectively, then the mixed problem* (1.2)–(1.5) *admits a unique* C^1 *solution* $U = U(t,x)$ *on* $R(T) = \{(t,x) \mid 0 \le t \le T, 0 \le x \le L\}$*.*

In order to define a L^2 weak solution and to give its properties, in this section, we first establish some L^2 estimates on the C^1 solution to problem (1.2)–(1.5).

We start with an L^2 estimate of C^1 solutions by the characteristic method.

Theorem 2.1 *For any given* $T > 0$*, the* C^1 *solution* $U = U(t,x)$ *to the mixed problem* (1.2)–(1.5) *satisfies*

$$\|U(T,\cdot)\|_{(L^2(0,L))^n} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^M}), \tag{2.3}$$

here and hereafter, $C(T) > 0$ *denotes a different positive constant, depending only on* T*.*

Proof For simplicity, we denote

$$\widetilde{H}^- = D_1H^-, \quad \widetilde{H}^+ = D_0H^+, \tag{2.4}$$

which are vectors of order m and $\overline{m}$, respectively, and consider the following system:

$$\begin{cases} U_t + \Lambda U_x + AU = 0, & t \in [0,T],\ x \in [0,L], \\ U^+(t,0) = G_0U^-(t,0) + \widetilde{H}^+(t), & t \in [0,T], \\ U^-(t,L) = G_1U^+(t,L) + \widetilde{H}^-(t), & t \in [0,T], \end{cases} \tag{2.5}$$

where $\widetilde{H}^- = (\widetilde{h}_1, \cdots, \widetilde{h}_m)^T$, $\widetilde{H}^+ = (\widetilde{h}_{m+1}, \cdots, \widetilde{h}_n)^T$.

Let

$$T_1 = L \min_{\substack{1 \le r \le m \\ m+1 \le s \le n}} \left\{ \frac{1}{|\lambda_r|}, \frac{1}{\lambda_s} \right\} > 0. \tag{2.6}$$

Let $x = x_s^{(0,0)}(t)$ be the s^{th} characteristic line passing through the point $(t,x) = (0,0)$ ($s = m+1, \cdots, n$). For any given $\overline{t} \in [0,T_1]$, denoting $x_s^{(0,0)}(\overline{t}) = \beta$, the line $t = \overline{t}$ can be separated into two parts:

a. When $\beta \le \overline{x} \le L$, draw the s^{th} ($s = m+1, \cdots, n$) characteristic line $x = x_s(t)$ passing through the point $(\overline{t}, \overline{x})$, it interacts the x-axis at the point $(0, \alpha_s)$, namely, we have

$$\begin{cases} \dfrac{\mathrm{d}x_s}{\mathrm{d}t} = \lambda_s, \\ x_s(0) = \alpha_s, \quad x_s(\overline{t}) = \overline{x}. \end{cases} \tag{2.7}$$

Since λ_s is a constant, $x = x_s(t)$ as a straight line can be rewritten as $x = \lambda_s(t - \overline{t}) + \overline{x}$, then $\alpha_s = \alpha_s(\overline{t}, \overline{x}) = \overline{x} - \lambda_s\overline{t}$. Along this characteristic line, the s^{th} component u_s of U satisfies

$$\begin{cases} \dfrac{\mathrm{d}u_s}{\mathrm{d}_s t} = -A_sU, \\ t = 0: \ u_s = (U_0)_s(\alpha_s), \end{cases} \tag{2.8}$$

where $\frac{\mathrm{d}}{\mathrm{d}_s t} = \frac{\partial}{\partial t} + \lambda_s \frac{\partial}{\partial x}$ is the directional derivative with respect to t along $x = x_s(t)$, A_s is the s^{th} row of A, and $(U_0)_s$ is the s^{th} component of U_0. Integrating (2.8) with respect to t along $x = x_s(t)$ from 0 to $\overline{t}$, we have

$$u_s(\overline{t},\overline{x}) = (U_0)_s(\alpha_s) - \int_0^{\overline{t}} A_s U(\tau, x_s(\tau))\mathrm{d}\tau, \quad \beta \le \overline{x} \le L, \tag{2.9}$$

then

$$(u_s(\overline{t},\overline{x}))^2 \le 2\Big[((U_0)_s(\alpha_s(\overline{t},\overline{x})))^2 + \Big(\int_0^{\overline{t}} A_s U(\tau, x_s(\tau))\mathrm{d}\tau\Big)^2\Big], \quad \beta \le \overline{x} \le L, \tag{2.10}$$

hence

$$\|u_s(\overline{t},\cdot)\|^2_{L^2(\beta,L)} \le 2\Big[\int_\beta^L ((U_0)_s(\alpha_s(\overline{t},\overline{x})))^2\mathrm{d}\overline{x} + \int_\beta^L \Big(\int_0^{\overline{t}} A_s U(\tau, x_s(\tau))\mathrm{d}\tau\Big)^2 \mathrm{d}\overline{x}\Big]. \tag{2.11}$$

Noting $\alpha_s(\overline{t},\overline{x}) = \overline{x} - \lambda_s\overline{t}$, for any given $\overline{t}$, when $\overline{x}$ runs from β to L, α_s runs from 0 to $L - \lambda_s\overline{t}$ $(< L)$, then the first integration on the right-hand side of (2.11) equals to

$$\int_\beta^L ((U_0)_s(\overline{x} - \lambda_s\overline{t}))^2\mathrm{d}\overline{x} \overset{x \triangleq \overline{x} - \lambda_s\overline{t}}{=} \int_0^{L-\lambda_s\overline{t}} ((U_0)_s(x))^2\mathrm{d}x \le \|U_0\|^2_{(L^2(0,L))^n}. \tag{2.12}$$

On the other hand, let $x = x_s^{(\overline{t},L)}(t)$ be the s^{th} characteristic line passing through the point $(\overline{t}, L)$. For the second integration on the right-hand side of (2.11), noting $x_s(\tau) = \lambda_s(\tau - \overline{t}) + \overline{x}$, and changing the order of integration, we have

$$\begin{aligned}
&\int_\beta^L \Big(\int_0^{\overline{t}} A_s U(\tau, x_s(\tau))\mathrm{d}\tau\Big)^2 \mathrm{d}\overline{x} \\
\le\ & \int_\beta^L \int_0^{\overline{t}} |A_s U(\tau, x_s(\tau))|^2 \mathrm{d}\tau\mathrm{d}\overline{x} \\
=\ & \int_0^{\overline{t}} \mathrm{d}\tau \int_\beta^L |A_s U(\tau, \lambda_s(\tau - \overline{t}) + \overline{x})|^2 \mathrm{d}\overline{x} \\
\overset{x \triangleq \lambda_s(\tau-\overline{t})+\overline{x}}{=}\ & \int_0^{\overline{t}} \mathrm{d}\tau \int_{x_s^{(0,0)}(\tau)}^{x_s^{(\overline{t},L)}(\tau)} |A_s U(\tau, x)|^2 \mathrm{d}x \\
\le\ & C\int_0^{\overline{t}} \|U(\tau,\cdot)\|^2_{(L^2(0,L))^n}\mathrm{d}\tau,
\end{aligned} \tag{2.13}$$

here and hereafter, C stands for a different positive constant. By (2.12)–(2.13), it follows from (2.11) that

$$\|u_s(\overline{t},\cdot)\|^2_{L^2(\beta,L)} \le C\Big(\|U_0\|^2_{(L^2(0,L))^n} + \int_0^{\overline{t}} \|U(\tau,\cdot)\|^2_{(L^2(0,L))^n}\mathrm{d}\tau\Big). \tag{2.14}$$

b. When $0 \le \overline{x} \le \beta$, drawing the s^{th} $(s = m+1, \cdots, n)$ characteristic line $x = x_s(t)$ passing through the point $(\overline{t},\overline{x})$, it intersects the t-axis at the point $(t_s, 0)$, where $t_s = t_s(\overline{t},\overline{x}) = \overline{t} - \frac{\overline{x}}{\lambda_s}$. That is to say, we have

$$\begin{cases} \dfrac{\mathrm{d}x_s}{\mathrm{d}t} = \lambda_s, \\ x_s(t_s) = 0, \quad x_s(\overline{t}) = \overline{x}. \end{cases} \tag{2.15}$$

Along this characteristic line, noting the boundary condition in (1.3), the s^{th} component of U satisfies

$$\begin{cases} \dfrac{\mathrm{d}u_s}{\mathrm{d}_s t} = -A_s U, \\ t = t_s: \ u_s(t_s, 0) = (G_0)_s U^-(t_s, 0) + \widetilde{h}_s(t_s), \end{cases} \tag{2.16}$$

where $(G_0)_s$ is the s^{th} row of G_0. Integrating (2.16) with respect to t along $x = x_s(t)$ from t_s to $\overline{t}$, we have

$$u_s(\overline{t}, \overline{x}) = (G_0)_s U^-(t_s, 0) + \widetilde{h}_s(t_s) - \int_{t_s}^{\overline{t}} A_s U(\tau, x_s(\tau))\mathrm{d}\tau, \quad 0 \le \overline{x} \le \beta. \tag{2.17}$$

For any given r $(r = 1, \cdots, m)$, the r^{th} characteristic line $x = x_r(t)$ passing through the point $(t_s, 0)$ intersects the x-axis at the point $(0, \alpha_r) = (0, \alpha_r(\overline{t}, \overline{x}))$, namely, we have

$$\begin{cases} \dfrac{\mathrm{d}x_r}{\mathrm{d}t} = \lambda_r, \\ x_r(0) = \alpha_r, \quad x_r(t_s) = 0. \end{cases} \tag{2.18}$$

Along $x = x_r(t)$, the r^{th} component u_r of U satisfies

$$\begin{cases} \dfrac{\mathrm{d}u_r}{\mathrm{d}_r t} = -A_r U, \\ t = 0: \ u_r = (U_0)_r(\alpha_r), \\ t = t_s: \ u_r = u_r(t_s, 0). \end{cases} \tag{2.19}$$

Integrating (2.19) with respect to t along $x = x_r(t)$ from 0 to t_s, we have

$$u_r(t_s, 0) = (U_0)_r(\alpha_r) - \int_0^{t_s} A_r U(\tau, x_r(\tau))\mathrm{d}\tau.$$

Substituting it into (2.17), we get

$$\begin{aligned} u_s(\overline{t}, \overline{x}) = & \sum_{r=1}^{m} (G_0)_{sr} \Big[(U_0)_r(\alpha_r) - \int_0^{t_s} A_r U(\tau, x_r(\tau))\mathrm{d}\tau \Big] + \widetilde{h}_s(t_s) \\ & - \int_{t_s}^{\overline{t}} A_s U(\tau, x_s(\tau))\mathrm{d}\tau, \ 0 \le \overline{x} \le \beta, \end{aligned} \tag{2.20}$$

where $(G_0)_{sr}$ is the component of G_0 on the s^{th} row and r^{th} column. Similarly, we have

$$\begin{aligned} \|u_s(\overline{t}, \cdot)\|^2_{L^2(0,\beta)} \le C \Big(& \sum_{r=1}^{m} \int_0^{\beta} ((U_0)_r(\alpha_r(\overline{t}, \overline{x})))^2 \mathrm{d}\overline{x} + \int_0^{\beta} \mathrm{d}\overline{x} \Big(\int_{t_s}^{\overline{t}} A_s U(\tau, x_s(\tau))\mathrm{d}\tau \Big)^2 \\ & + \sum_{r=1}^{m} \int_0^{\beta} \mathrm{d}\overline{x} \Big(\int_0^{t_s} A_r U(\tau, x_r(\tau))\mathrm{d}\tau \Big)^2 + \int_0^{\beta} \widetilde{h}_s^2(t_s)\mathrm{d}\overline{x} \Big), \end{aligned} \tag{2.21}$$

in which, noting $t_s = \overline{t} - \frac{\overline{x}}{\lambda_s}$, we have

$$\int_0^{\beta} \widetilde{h}_s^2(t_s)\mathrm{d}\overline{x} = \int_0^{\beta} \widetilde{h}_s^2 \Big(\overline{t} - \frac{\overline{x}}{\lambda_s} \Big) \mathrm{d}\overline{x} \overset{\tau \triangleq \overline{t} - \frac{\overline{x}}{\lambda_s}}{=} \lambda_s \int_0^{\overline{t}} \widetilde{h}_s^2(\tau)\mathrm{d}\tau = \lambda_s \|\widetilde{h}_s\|^2_{L^2(0,\overline{t})}. \tag{2.22}$$

Thus, in a similar way we get

$$\|u_s(\overline{t},\cdot)\|^2_{L^2(0,\beta)} \le C\Big(\|U_0\|^2_{(L^2(0,L))^n} + \|\widetilde{h}_s\|^2_{L^2(0,\overline{t})} + \int_0^{\overline{t}} \|U(\tau,\cdot)\|^2_{(L^2(0,L))^n}\mathrm{d}\tau\Big). \tag{2.23}$$

The combination of (2.14) and (2.23) yields

$$\|u_s(\overline{t},\cdot)\|^2_{L^2(0,L)} \le C\Big(\|U_0\|^2_{(L^2(0,L))^n} + \|\widetilde{h}_s\|^2_{L^2(0,\overline{t})} + \int_0^{\overline{t}} \|U(\tau,\cdot)\|^2_{(L^2(0,L))^n}\mathrm{d}\tau\Big), \quad s = m+1,\cdots,n. \tag{2.24}$$

Similarly, we have

$$\|u_r(\overline{t},\cdot)\|^2_{L^2(0,L)} \le C\Big(\|U_0\|^2_{(L^2(0,L))^n} + \|\widetilde{h}_r\|^2_{L^2(0,\overline{t})} + \int_0^{\overline{t}} \|U(\tau,\cdot)\|^2_{(L^2(0,L))^n}\mathrm{d}\tau\Big), \quad r = 1,\cdots,m. \tag{2.25}$$

Thus, we have

$$\|U(\overline{t},\cdot)\|^2_{(L^2(0,L))^n} \le C\Big(\|U_0\|^2_{(L^2(0,L))^n} + \|\widetilde{H}\|^2_{(L^2(0,\overline{t}))^n} + \int_0^{\overline{t}} \|U(\tau,\cdot)\|^2_{(L^2(0,L))^n}\mathrm{d}\tau\Big). \tag{2.26}$$

Then, by Gronwall's inequality, we get

$$\|U(\overline{t},\cdot)\|_{(L^2(0,L))^n} \le C(\overline{t})(\|U_0\|_{(L^2(0,L))^n} + \|\widetilde{H}\|_{(L^2(0,\overline{t}))^n}), \quad \forall \overline{t} \in [0,T_1]. \tag{2.27}$$

For any given $\overline{\overline{t}} \in [T_1, 2T_1]$, take $U(T_1,x)$ as the initial data, similarly we have

$$\begin{aligned}\|U(\overline{\overline{t}},\cdot)\|_{(L^2(0,L))^n} &\le C(\overline{\overline{t}})(\|U(T_1,\cdot)\|_{(L^2(0,L))^n} + \|\widetilde{H}\|_{(L^2(T_1,\overline{\overline{t}}))^n})\\ &\le C(\overline{\overline{t}})(\|U_0\|_{(L^2(0,L))^n} + \|\widetilde{H}\|_{(L^2(0,\overline{\overline{t}}))^n}), \quad \forall \overline{\overline{t}} \in [T_1,2T_1].\end{aligned} \tag{2.28}$$

Repeat the procedure above for at most $N \le \big[\frac{T}{T_1}\big] + 1$ times, we get

$$\|U(T,\cdot)\|_{(L^2(0,L))^n} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|\widetilde{H}\|_{(L^2(0,T))^n}), \quad \forall T > 0. \tag{2.29}$$

Noting that D_0 and D_1 are full column-rank, (2.29) leads to (2.3).

Next, we estimate the L^2 norm of the trace of the solution on the boundary.

Theorem 2.2 *For any given $T > 0$, the C^1 solution $U = U(t,x)$ to the mixed problem* (1.2)–(1.5) *satisfies*

$$\|U(\cdot,0)\|_{(L^2(0,T))^n} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^M}) \tag{2.30}$$

and

$$\|U(\cdot,L)\|_{(L^2(0,T))^n} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^M}). \tag{2.31}$$

Proof We still consider system (2.5), and prove (2.31) only. Let

$$T_1 = \min_{m+1\le s\le n} \frac{L}{\lambda_s} > 0. \tag{2.32}$$

For any given $\overline{t} \in [0, T_1]$, for each $s = m+1, \cdots, n$, noting (2.32), the s^{th} characteristic line $x = x_s(t)$ passing through the point $(\overline{t}, L)$ must intersect the x-axis at a point $(0, \alpha_s)$, namely, we have

$$\begin{cases} \dfrac{\mathrm{d}x_s}{\mathrm{d}t} = \lambda_s, \\ x_s(\overline{t}) = L, \quad x_s(0) = \alpha_s, \end{cases} \tag{2.33}$$

thus, $x_s(t) = \lambda_s(t - \overline{t}) + L$ and $\alpha_s = \alpha_s(\overline{t}) = L - \lambda_s \overline{t}$. Along this characteristic line, the s^{th} component of U satisfies

$$\begin{cases} \dfrac{\mathrm{d}u_s}{\mathrm{d}_s t} = -A_s U, \\ t = 0: \ u_s = (U_0)_s(\alpha_s). \end{cases} \tag{2.34}$$

Integrating (2.34) with respect to t along this characteristic line from 0 to $\overline{t}$ leads to

$$u_s(\overline{t}, L) = (U_0)_s(\alpha_s) - \int_0^{\overline{t}} A_s U(\tau, x_s(\tau)) \mathrm{d}\tau, \tag{2.35}$$

then

$$(u_s(\overline{t}, L))^2 \le 2\Big[((U_0)_s(\alpha_s(\overline{t})))^2 + \Big(\int_0^{\overline{t}} A_s U(\tau, x_s(\tau)) \mathrm{d}\tau\Big)^2\Big], \tag{2.36}$$

hence

$$\|u_s(\cdot, L)\|^2_{L^2(0,T_1)} \le 2\Big[\int_0^{T_1} ((U_0)_s(\alpha_s(\overline{t})))^2 \mathrm{d}\overline{t} + \int_0^{T_1} \Big(\int_0^{\overline{t}} A_s U(\tau, x_s(\tau)) \mathrm{d}\tau\Big)^2 \mathrm{d}\overline{t}\Big]. \tag{2.37}$$

Noting that $\alpha_s(\overline{t}) = L - \lambda_s \overline{t}$, when $\overline{t}$ runs from 0 to T_1, α_s runs from L to $L - \lambda_s T_1$ $(\le L)$, then the first integration on the right-hand side of (2.37) equals to

$$\int_0^{T_1} ((U_0)_s(L - \lambda_s \overline{t}))^2 \mathrm{d}\overline{t} \overset{\overline{x} \triangleq L - \lambda_s \overline{t}}{=} \frac{1}{\lambda_s} \int_{L - \lambda_s T_1}^{L} ((U_0)_s(\overline{x}))^2 \mathrm{d}\overline{x} \le C \|U_0\|^2_{(L^2(0,L))^n}. \tag{2.38}$$

As to the second integration on the right-hand side of (2.37), let $x = x_s^{(T_1, L)}(t)$ be the s^{th} characteristic line passing through the point (T_1, L). Noting $x_s(\tau) = \lambda_s(\tau - \overline{t}) + L$, and changing the order of integration, we have

$$\begin{aligned} &\int_0^{T_1} \Big(\int_0^{\overline{t}} A_s U(\tau, x_s(\tau)) \mathrm{d}\tau\Big)^2 \mathrm{d}\overline{t} \\ \le &\int_0^{T_1} \int_0^{\overline{t}} |A_s U(\tau, x_s(\tau))|^2 \mathrm{d}\tau \mathrm{d}\overline{t} \\ = &\int_0^{T_1} \mathrm{d}\tau \int_\tau^{T_1} |A_s U(\tau, \lambda_s(\tau - \overline{t}) + L)|^2 \mathrm{d}\overline{t} \\ \overset{\overline{x} \triangleq \lambda_s(\tau - \overline{t}) + L}{=} &\frac{1}{\lambda_s} \int_0^{T_1} \mathrm{d}\tau \int_{x_s^{(T_1, L)}}^{L} |A_s U(\tau, \overline{x})|^2 \mathrm{d}\overline{x} \\ \le &C \int_0^{T_1} \|U(\tau, \cdot)\|^2_{(L^2(0,L))^n} \mathrm{d}\tau. \end{aligned} \tag{2.39}$$

Therefore, it follows from (2.37) that

$$\|u_s(\cdot,L)\|_{L^2(0,T_1)} \le C\Big(\|U_0\|_{(L^2(0,L))^n} + \Big(\int_0^{T_1} \|U(\tau,\cdot)\|^2_{(L^2(0,L))^n}\mathrm{d}\tau\Big)^{\frac12}\Big), \tag{2.40}$$

then by (2.3) we get

$$\|u_s(\cdot,L)\|_{L^2(0,T_1)} \le C(T_1)(\|U_0\|_{(L^2(0,L))^n} + \|\widetilde{H}\|_{(L^2(0,T_1))^n}). \tag{2.41}$$

Taking $U(T_1,x)$ as the initial data, repeating the procedure above on the interval $[T_1, 2T_1]$, we get

$$\begin{aligned}\|u_s(\cdot,L)\|_{L^2(T_1,2T_1)} &\le C(2T_1)(\|U(T_1,\cdot)\|_{(L^2(0,L))^n} + \|\widetilde{H}\|_{(L^2(T_1,2T_1))^n})\\ &\le C(2T_1)(\|U_0\|_{(L^2(0,L))^n} + \|\widetilde{H}\|_{(L^2(0,2T_1))^n}).\end{aligned} \tag{2.42}$$

Repeating the above procedure for at most $N \le [\frac{T}{T_1}]+1$ times, for $s = m+1,\cdots,n$, we get

$$\|u_s(\cdot,L)\|_{L^2(0,T)} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|\widetilde{H}\|_{(L^2(0,T))^n}), \quad \forall T \ge 0, \tag{2.43}$$

namely,

$$\|U^+(\cdot,L)\|_{(L^2(0,T))^{\overline{m}}} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|\widetilde{H}\|_{(L^2(0,T))^n}), \quad \forall T \ge 0. \tag{2.44}$$

Then, by the boundary condition at $x = L$ in (1.4), we have

$$\|U^-(\cdot,L)\|_{(L^2(0,T))^m} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|\widetilde{H}\|_{(L^2(0,T))^n}), \quad \forall T \ge 0. \tag{2.45}$$

Noting that D_0 and D_1 are full column-rank, the combination of (2.44) and (2.45) gives (2.31).

In fact, the result above is true for the trace of the solution at any given $x \in [0,L]$.

Theorem 2.3 *Let $T > 0$. Under the hypotheses of Theorem* 2.1, *for any given $x \in [0,L]$, the C^1 solution $U = U(t,x)$ to the mixed problem* (1.2)–(1.5) *satisfies*

$$\|U(\cdot,x)\|_{(L^2(0,T))^n} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^M}). \tag{2.46}$$

Proof Let $U = U(t,x)$ be the C^1 solution to the mixed problem (1.2)–(1.5) on $[0,T]\times[0,L]$. Changing the role of t and x, $U(t,x)$ is still the unique C^1 solution to the following rightward problem:

$$\begin{cases} U_x + \Lambda^{-1}U_t + \Lambda^{-1}AU = 0, & x \in [0,L],\ t \in [0,T],\\ t = T:\ U^- = U^-(T,x), & x \in [0,L],\\ t = 0:\ U^+ = U_0^+(x), & x \in [0,L]\end{cases} \tag{2.47}$$

and

$$x = 0:\ U = U(t,0), \quad t \in [0,T]. \tag{2.48}$$

For any given $x \in [0,L]$, using Theorem 2.1 for problem (2.47)–(2.48), we have

$$\|U(\cdot,x)\|_{(L^2(0,T))^n} \le C(\|U(\cdot,0)\|_{(L^2(0,T))^n} + \|U_0^+\|_{(L^2(0,L))^{\overline{m}}} + \|U^-(T,\cdot)\|_{(L^2(0,L))^m}). \tag{2.49}$$

Noting (2.30) and (2.3) for the mixed problem (1.2)–(1.5), we have

$$\|U(\cdot,0)\|_{(L^2(0,T))^n} \le C(\|U_0\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^M}) \tag{2.50}$$

and

$$\|U^-(T,\cdot)\|_{(L^2(0,L))^m} \le C(\|U_0\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^M}), \tag{2.51}$$

then we get (2.46).

Besides, for any given C^1 initial data $U_0(x)$ $(0 \le x \le L)$, the Cauchy problem

$$U_t + \Lambda U_x + AU = 0 \tag{2.52}$$

with the initial condition (1.5) admits a unique C^1 solution on the corresponding maximum determinate domain R^c, surrounded by $t = 0$ $(0 \le x \le L)$ and two characteristic lines given by

$$\begin{cases} \dfrac{\mathrm{d}x^{(1)}}{\mathrm{d}t} = \max\limits_{m+1 \le s \le n} \lambda_s, \\ x^{(1)}(0) = 0 \end{cases} \tag{2.53}$$

and

$$\begin{cases} \dfrac{\mathrm{d}x^{(2)}}{\mathrm{d}t} = \min\limits_{1 \le r \le m} \lambda_r, \\ x^{(2)}(0) = L, \end{cases} \tag{2.54}$$

where $x = x^{(1)}(t)$ is the rightmost characteristic line passing through the point $(t,x) = (0,0)$, and $x = x^{(2)}(t)$ is the leftmost characteristic line passing through the point $(t,x) = (0,L)$. Since the solution depends only on the initial data, as a corollary of Theorem 2.1, we have

Corollary 2.1 *Let (t^*,x^*) be the intersection point of $x = x^{(1)}(t)$ and $x = x^{(2)}(t)$. For any given $t \in [0,t^*]$, the C^1 solution $U = U(t,x)$ to Cauchy problem* (2.52) *and* (1.5) *satisfies*

$$\|U(t,\cdot)\|_{(L^2(x^{(1)}(t),x^{(2)}(t)))^n} \le C\|U_0\|_{(L^2(0,L))^n}, \quad 0 \le t \le t^*. \tag{2.55}$$

Similarly, by Theorem 2.3 we have

Corollary 2.2 *Let (t^*,x^*) be the intersection point of $x = x^{(1)}(t)$ and $x = x^{(2)}(t)$. For any given $x \in [0,L]$, the C^1 solution $U = U(t,x)$ to Cauchy problem* (2.52) *and* (1.5) *satisfies*

$$\|U(\cdot,x)\|_{(L^2(0,T^*(x)))^n} \le C\|U_0\|_{(L^2(0,L))^n}, \quad 0 \le x \le L, \tag{2.56}$$

where $T^* = T^*(x)$ is determined by

$$\begin{cases} x = x^{(1)}(T^*), & \text{when } 0 \le x \le x^*, \\ x = x^{(2)}(T^*), & \text{when } x^* \le x \le L. \end{cases} \tag{2.57}$$

Furthermore, let R^0 be the domain surrounded by t-axis, x-axis and characteristic line $x = x^{(2)}(t)$, and let R^L be the domain surrounded by x-axis, $x = L$ and characteristic line $x = x^{(1)}(t)$, where $x = x^{(1)}(t)$ and $x = x^{(2)}(t)$ are given by (2.53) and (2.54), respectively. By the theory of one-sided mixed initial-boundary value problem of first order hyperbolic systems ([8]), the unique C^1 solution $U = U(t,x)$ to problem (1.2)–(1.5) on R^0 is uniquely determined by the initial data U_0 and the boundary function $H^+(t)$ on $x = 0$; while, $U = U(t,x)$ on R^L is uniquely determined by the initial data U_0 and the boundary function $H^-(t)$ on $x = L$. Let $t = t_i(x)$ $(i = 1,2)$ be the inverse function of $x = x^{(i)}(t)$ $(i = 1,2)$ for $x \in [0,L]$. We have

Theorem 2.4 *Assume that $U = U(t,x)$ is the unique C^1 solution to problem* (1.2)–(1.5).

(1) *For any given $t \in [0, t_2^0]$ with $t_2^0 = t_2(0)$, $U = U(t,x)$ satisfies*

$$\|U(t,\cdot)\|_{(L^2(0,x^{(2)}(t)))^n} \leq C(\|U_0\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{M_0}}). \tag{2.58}$$

Moreover, for any given $x \in [0,L]$, $U = U(t,x)$ satisfies

$$\|U(\cdot,x)\|_{(L^2(0,t_2(x)))^n} \leq C(\|U_0\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{M_0}}). \tag{2.59}$$

(2) *For any given $t \in [0, t_1^L]$, in which $t_1^L = t_1(L)$, $U = U(t,x)$ satisfies*

$$\|U(t,\cdot)\|_{(L^2(x^{(1)}(t),L))^n} \leq C(\|U_0\|_{(L^2(0,L))^n} + \|H^-\|_{(L^2(0,T))^{M_1}}). \tag{2.60}$$

Moreover, for any given $x \in [0,L]$, $U = U(t,x)$ satisfies

$$\|U(\cdot,x)\|_{(L^2(0,t_1(x)))^n} \leq C(\|U_0\|_{(L^2(0,L))^n} + \|H^-\|_{(L^2(0,T))^{M_1}}). \tag{2.61}$$

3 Definition and Properties of Weak Solution

3.1 Definition of weak solution

In order to give the definition of weak solution, we first prove

Proposition 3.1 *For any given $T > 0$, $U = U(t,x) \in (C^1([0,T] \times [0,L]))^n$ is a C^1 solution to problem* (1.2)–(1.5) *if and only if $U = U(t,x)$ satisfies*

$$\begin{aligned} &\int_0^L \Phi^T(T,x)U(T,x)\mathrm{d}x - \int_0^L \Phi^T(0,x)U_0(x)\mathrm{d}x \\ =& \int_0^T \int_0^L (\Phi_t^T + \Phi_x^T \Lambda - \Phi^T A)U \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T (\Phi^+)^T(t,0)\Lambda^+ D_0 H^+(t)\mathrm{d}t - \int_0^T (\Phi^-)^T(t,L)\Lambda^- D_1 H^-(t)\mathrm{d}t \end{aligned} \tag{3.1}$$

for any given $\Phi \in (C^1([0,T] \times [0,L]))^n$ satisfying

$$\begin{cases} \Phi^-(t,0) = -(\Lambda^-)^{-1} G_0^T \Lambda^+ \Phi^+(t,0), & t \in [0,T], \\ \Phi^+(t,L) = -(\Lambda^+)^{-1} G_1^T \Lambda^- \Phi^-(t,L), & t \in [0,T]. \end{cases} \tag{3.2}$$

Proof Assume that problem (1.2)–(1.5) admits a C^1 solution $U(t,x)$ on the domain $[0,T] \times [0,L]$ for any given $T > 0$. Let $\Phi \in (C^1([0,T] \times [0,L]))^n$ satisfy (3.2). Multiplying Φ^T on both sides of (2.52) and integrating by parts, we get

$$\begin{aligned} 0 =& \int_0^L \Phi^T(T,x)U(T,x)\mathrm{d}x - \int_0^L \Phi^T(0,x)U_0(x)\mathrm{d}x \\ &+ \int_0^T \Phi^T(t,L)\Lambda U(t,L)\mathrm{d}t - \int_0^T \Phi^T(t,0)\Lambda U(t,0)\mathrm{d}t \\ &- \int_0^T \int_0^L (\Phi_t^T + \Phi_x^T \Lambda - \Phi^T A)U \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.3}$$

Noting that Φ satisfies (3.2), using the boundary conditions in (1.3)–(1.4), we immediately get (3.1).

On the contrary, let $\Phi \in (C^1([0,T]\times[0,L]))^n$ vanish at $x=0,L$ and at $t=0,T$. By (3.1), we get

$$\int_0^T\int_0^L (\Phi_t^T + \Phi_x^T\Lambda - \Phi^T A)U\mathrm{d}x\mathrm{d}t = 0. \tag{3.4}$$

Since $U(t,x)\in (C^1([0,T]\times[0,L]))^n$, integrating by parts in (3.4), we get

$$\int_0^T\int_0^L \Phi^T(U_t + \Lambda U_x + AU)\mathrm{d}x\mathrm{d}t = 0. \tag{3.5}$$

Since the set of functions $\Phi \in (C^1([0,T]\times[0,L]))^n$ mentioned above is dense in $(L^1((0,T)\times(0,L)))^n$, (3.5) implies that

$$\int_0^T\int_0^L \Phi^T(U_t + \Lambda U_x + AU)\mathrm{d}x\mathrm{d}t = 0, \quad \forall \Phi \in (L^1((0,T)\times(0,L)))^n, \tag{3.6}$$

then

$$U_t + \Lambda U_x + AU = 0, \quad t\in[0,T],\ x\in[0,L]. \tag{3.7}$$

Now, let $\Phi \in (C^1([0,T]\times[0,L]))^n$ satisfy (3.2). Integrating by parts and noting (3.7), it follows from (3.1) that

$$\begin{aligned}&\int_0^L \Phi^T(0,x)[U(0,x)-U_0(x)]\mathrm{d}x\\ &= \int_0^T (\Phi^+)^T(t,0)\Lambda^+[G_0U^-(t,0) - U^+(t,0) + D_0H^+(t)]\mathrm{d}t\\ &\quad + \int_0^T (\Phi^-)^T(t,L)\Lambda^-[U^-(t,L) - G_1U^+(t,L) - D_1H^-(t)]\mathrm{d}t.\end{aligned} \tag{3.8}$$

Let $\Phi \in (C^1([0,T]\times[0,L]))^n$ vanish at $x=0,L$, we get

$$\int_0^L \Phi^T(0,x)[U(0,x)-U_0(x)]\mathrm{d}x = 0. \tag{3.9}$$

Let $\mathcal{B} : (C^1([0,T]\times[0,L]))^n \to (L^1(0,L))^n$ be defined by

$$\mathcal{B}(\Phi) := \Phi(0,\cdot).$$

Since $\mathcal{B}(\{\Phi \in (C^1([0,T]\times[0,L]))^n, \text{vanishing at } x=0,L\})$ is dense in $(L^1(0,L))^n$, (3.9) implies

$$\int_0^L \Psi^T(x)[U(0,x)-U_0(x)]\mathrm{d}x = 0, \quad \forall \Psi \in (L^1(0,L))^n, \tag{3.10}$$

then we have

$$U(0,x) = U_0(x). \tag{3.11}$$

Similarly, we can prove

$$U^+(t,0) = G_0U^-(t,0) + D_0H^+(t) \tag{3.12}$$

and

$$U^-(t,L) = G_1U^+(t,L) + D_1H^-(t). \tag{3.13}$$

Then, $U = U(t,x)$ is a C^1 solution to the mixed problem (1.2)–(1.5).

Thus, (3.1) is an equivalent definition of the C^1 solution to problem (1.2)–(1.5), which is still valid when $U = U(t,x) \in (L^2(0,T;L^2(0,L)))^n$. Therefore, we can give the following

Definition 3.1 *For any given $T > 0$, $U = U(t,x) \in (L^2(0,T;L^2(0,L)))^n$ is a weak solution to the forward mixed problem* (1.2)–(1.5)*, if* (3.1) *holds for any given $\Phi \in (C^1([0,T]\times[0,L]))^n$ satisfying* (3.2)*.*

3.2 Existence and uniqueness of weak solution

In this section, we construct a sequence of classical solutions to approximate the weak solution, then we prove the existence and uniqueness of weak solution and establish some related estimates.

For any given initial data $U_0 \in (L^2(0,L))^n$, by a mollifier procedure, there exist $U_0^\varepsilon(x)$ in $(\mathcal{C}_0^\infty[0,L])^n$ with $\varepsilon > 0$ so small that

$$\|U_0^\varepsilon\|_{(L^2(0,L))^n} \le \|U_0\|_{(L^2(0,L))^n} \tag{3.14}$$

and

$$U_0^\varepsilon(\cdot) \to U_0(\cdot) \quad \text{in } (L^2(0,L))^n \quad \text{as } \varepsilon \to 0. \tag{3.15}$$

Similarly, for any given $T > 0$, for any given boundary function $H \in (L^2(0,T))^M$, we can find $H^\varepsilon(t) \in (\mathcal{C}_0^\infty[0,T])^M$ such that

$$\|H^\varepsilon\|_{(L^2(0,T))^M} \le \|H\|_{(L^2(0,T))^M} \tag{3.16}$$

and

$$H^\varepsilon(\cdot) \to H(\cdot) \quad \text{in } (L^2(0,T))^M \text{ as } \varepsilon \to 0. \tag{3.17}$$

Thus, for the mixed problem (1.2)–(1.5) (in which $U_0(x) = U_0^\varepsilon(x)$ and $H(t) = H^\varepsilon(t)$), the conditions of C^1 compatibility are always satisfied at the points $(t,x) = (0,0)$ and $(0,L)$, respectively. By Lemma 2.1, the corresponding mixed problem (1.2)–(1.5) admits a unique C^1 solution $U^\varepsilon(t,x)$. By Theorem 2.1, noting (3.14) and (3.16), we have

$$\|U^\varepsilon(t,\cdot)\|_{(L^2(0,L))^n} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^M}), \quad \forall t \in [0,T], \tag{3.18}$$

then, we have

$$\|U^\varepsilon\|_{(L^2(0,T;L^2(0,L)))^n} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^M}). \tag{3.19}$$

By the linearity, for $\varepsilon_1 \neq \varepsilon_2 > 0$, we have

$$\begin{aligned}
&\|(U^{\varepsilon_1} - U^{\varepsilon_2})\|_{(L^2(0,T;L^2(0,L)))^n} \\
&\leq C(T)(\|U_0^{\varepsilon_1} - U_0^{\varepsilon_2}\|_{(L^2(0,L))^n} + \|H^{\varepsilon_1} - H^{\varepsilon_2}\|_{(L^2(0,T))^M}).
\end{aligned} \tag{3.20}$$

Noting (3.15) and (3.17), it follows from (3.20) that $\{U^\varepsilon\}$ is a Cauchy sequence in $(L^2(0,T; L^2(0,L)))^n$, hence there exists $U(t,x) \in (L^2(0,T;L^2(0,L)))^n$, such that

$$U^\varepsilon(t,x) \to U(t,x) \quad \text{in } (L^2(0,T;L^2(0,L)))^n \text{ as } \varepsilon \to 0. \tag{3.21}$$

Noting (3.18), $U(t,x)$ also satisfies (2.3).

Moreover, we have

$$\|U^\varepsilon(\cdot, L)\|_{(L^2(0,T))^n} \leq C(T)(\|U_0\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^M}), \tag{3.22}$$

then $U(t,x)$ satisfies (2.31). Similarly, $U(t,x)$ satisfies (2.30).

Furthermore, by Theorem 2.3, Corollary 2.1 and Corollary 2.2, $U(t,x)$ satisfies (2.46), (2.55) and (2.56). By Theorem 2.4, $U(t,x)$ satisfies (2.58)–(2.61).

Next, we prove that $U(t,x)$ is a weak solution to problem (1.2)–(1.5). Since $U^\varepsilon(t,x) \in C^1$ is a C^1 solution to the corresponding problem (1.2)–(1.5), by (3.1) we have

$$\begin{aligned}
&\int_0^L \Phi^T(T,x)U^\varepsilon(T,x)\mathrm{d}x - \int_0^L \Phi^T(0,x)U_0^\varepsilon(x)\mathrm{d}x \\
&= \int_0^T \int_0^L (\Phi_t^T + \Phi_x^T\Lambda - \Phi^T A)U^\varepsilon \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T (\Phi^+)^T(t,0)\Lambda^+ D_0(H^\varepsilon)^+(t)\mathrm{d}t - \int_0^T (\Phi^-)^T(t,L)\Lambda^- D_1(H^\varepsilon)^-(t)\mathrm{d}t
\end{aligned} \tag{3.23}$$

for any given $\Phi \in (C^1([0,T]\times[0,L]))^n$ satisfying (3.2). Noting (3.15), (3.17) and (3.21), and taking $\varepsilon \to 0$, we immediately get the existence of weak solution.

Now we prove the uniqueness of weak solution. The idea of the proof was first proposed in [4, pp. 27] for the transport equation, it was also mentioned in [1, pp.231] for first order homogeneous linear hyperbolic systems. Assume that U_1 and U_2 are two weak solutions corresponding to the same initial data $U_0 \in (L^2(0,L))^n$ and the same boundary function $H \in (L^2(0,T))^M$. Let $U = U_1 - U_2$. By (3.1), for any given $\Phi \in (C^1([0,T]\times[0,L]))^n$ satisfying (3.2), we have

$$\int_0^L \Phi^T(T,x)U(T,x)\mathrm{d}x = \int_0^T \int_0^L (\Phi_t^T + \Phi_x^T\Lambda - \Phi^T A)U\mathrm{d}x\mathrm{d}t. \tag{3.24}$$

Let $\{F_n \in (C_0^1[0,L])^n\}$ be a sequence of functions such that

$$F_n(\cdot) \to U(T,\cdot) \quad \text{in } (L^2(0,L))^n \text{ as } n \to +\infty. \tag{3.25}$$

Consider the following problem

$$\begin{cases}
\Psi_{nt} + \Lambda\Psi_{nx} + A^T\Psi_n = 0, & t\in[0,T],\ x\in[0,L], \\
\Psi_n^+(t,0) = -(\Lambda^+)^{-1}G_1^T\Lambda^-\Psi_n^-(t,0), & t\in[0,T], \\
\Psi_n^-(t,L) = -(\Lambda^-)^{-1}G_0^T\Lambda^+\Psi_n^+(t,L), & t\in[0,T], \\
t=0:\ \Psi_n(0,x) = F_n(L-x), & x\in[0,L].
\end{cases} \tag{3.26}$$

Let Ψ_n be the C^1 solution to problem (3.26) and let

$$\Phi_n(t,x) := \Psi_n(T-t, L-x), \quad \forall (t,x) \in [0,T] \times [0,L]. \tag{3.27}$$

We have

$$\Psi_n(t,0) = \Phi_n(T-t, L) \quad \text{and} \quad \Psi_n(t,L) = \Phi_n(T-t,0). \tag{3.28}$$

Moreover, noting (3.25), we have

$$\Phi_n(T,x) = \Psi_n(0, L-x) = F_n(x) \to U(T,x) \quad \text{in } (L^2(0,L))^n, \text{ as } n \to +\infty. \tag{3.29}$$

Substituting (3.27) into (3.26), we have

$$\begin{cases} \Phi_{nt} + \Lambda \Phi_{nx} - A^T \Phi_n = 0, & t \in [0,T],\ x \in [0,L], \\ \Phi_n^-(t,0) = -(\Lambda^-)^{-1} G_0^T \Lambda^+ \Phi_n^+(t,0), & t \in [0,T], \\ \Phi_n^+(t,L) = -(\Lambda^+)^{-1} G_1^T \Lambda^- \Phi_n^-(t,L), & t \in [0,T]. \end{cases} \tag{3.30}$$

Thus, $\Phi_n(t,x)$ satisfies (3.2), then, by (3.24) and noting (3.29), we have

$$\int_0^L F_n^T(x) U(T,x) \mathrm{d}x = \int_0^L \Phi_n^T(T,x) U(T,x) \mathrm{d}x = 0. \tag{3.31}$$

Taking $n \to +\infty$, we get

$$\int_0^L |U(T,x)|^2 \mathrm{d}x = 0. \tag{3.32}$$

Since $T > 0$ is arbitrarily given, we have $U(t,\cdot) = 0$ for any given $t \in [0,T]$. The proof is complete.

Therefore, we have

Theorem 3.1 *For any given $T > 0$, for any given initial data $U_0 \in (L^2(0,L))^n$ and any given boundary function $H \in (L^2(0,T))^M$, the mixed problem* (1.2)–(1.5) *admits a unique weak solution $U = U(t,x) \in (L^2(0,T; L^2(0,L)))^n$, satisfying estimates* (2.3)*,* (2.30)–(2.31)*,* (2.46)*,* (2.55)–(2.56) *and* (2.58)–(2.61)*.*

Remark 3.1 Assume that the conditions (2.1) of C^0 compatibility at the points $(t,x) = (0,0)$ and $(0,L)$ are satisfied, respectively. Proceeding a similar procedure as above in the function space $(C^0([0,T]; L^2(0,L)))^n$, the regularity of the weak solution can be improved to $U = U(t,x) \in (C^0([0,T]; L^2(0,L)))^n$.

Remark 3.2 In Definition 3.1, the trace of the weak solution is not defined on any given $x \in [0,L]$, however, in Theorem 3.1, (2.46) indicates that the weak solution on any given $x \in [0,L]$ is in fact an L^2 function and its L^2 norm can be controlled by the L^2 norm of initial data and boundary function.

Moreover, (2.55) and (2.56) imply that for the corresponding Cauchy problem, the weak solution on the maximum determinate domain R^c depends only on the initial data U_0.

Estimates (2.58)–(2.61) in Theorem 3.1 imply the following theorem.

Theorem 3.2 *The weak solution $U = U(t,x) \in (L^2(0,T;L^2(0,L)))^n$ to problem* (1.2)–(1.5) *on the domain R^0 (resp. R^L), defined in Theorem* 2.4, *is uniquely determined by the initial data U_0 and the boundary condition $H^+(t)$ on $x = 0$ (resp. $H^-(t)$ on $x = L$).*

For any given $T > 0$ and any given $f(t,x) \in (L^2(0,T;L^2(0,L)))^n$, one can consider the inhomogeneous system

$$U_t + \Lambda U_x + AU = f(t,x), \quad t \in (0,T), \quad x \in (0,L) \tag{3.33}$$

with the boundary conditions (1.3)–(1.4) and the initial data (1.5) by the method of operator semigroups (see [16, 18]) to get

Theorem 3.3 *For any given $T > 0$, for any given initial data $U_0 \in (L^2(0,L))^n$, any given boundary function $H \in (L^2(0,T))^M$ and any given $f(t,x) \in (L^2(0,T;L^2(0,L)))^n$, the mixed problem* (3.33) *and* (1.3)–(1.5) *admits a unique weak solution $U = U(t,x) \in (L^2(0,T;L^2(0,L)))^n$, satisfying*

$$\|U(T,\cdot)\|_{(L^2(0,L))^n} \le C(T)(\|U_0\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^M} + \|f\|_{(L^2(0,T;L^2(0,L)))^n}). \tag{3.34}$$

3.3 Backward problem

In studying the controllability and the observability of weak solutions by the constructive method, in order to construct the weak solution to the forward problem under consideration, a backward problem should be solved beforehand. For this purpose, the weak solution to the backward problem will be discussed in this subsection.

Since the coming characteristics and the departing characteristics change their roles for backward problem, for any given $T > 0$, we consider the following backward system:

$$\begin{cases} U_t + \Lambda U_x + AU = 0, & t \in (0,T),\ x \in (0,L), \\ U^-(t,0) = \widehat{G}_0 U^+(t,0) + H^-(t), & t \in (0,T), \\ U^+(t,L) = \widehat{G}_1 U^-(t,L) + H^+(t), & t \in (0,T) \end{cases} \tag{3.35}$$

with the final data

$$t = T:\ U(T,x) = U_T(x), \quad x \in (0,L), \tag{3.36}$$

where the boundary coupling matrices $\widehat{G}_0$ and $\widehat{G}_1$ with constant elements are any given matrices of order $m \times \overline{m}$ and $\overline{m} \times m$, respectively, $H = (H^-, H^+)^T$ with $H^- = (h_1, \cdots, h_m)^T$ and $H^+ = (h_{m+1}, \cdots, h_n)^T$.

Let Ψ be any given function in $(C^1([0,T] \times [0,L]))^n$, satisfying

$$\begin{cases} \Psi^+(t,0) = -(\Lambda^+)^{-1} \widehat{G}_0^T \Lambda^- \Psi^-(t,0), & t \in [0,T], \\ \Psi^-(t,L) = -(\Lambda^-)^{-1} \widehat{G}_1^T \Lambda^+ \Psi^+(t,L), & t \in [0,T]. \end{cases} \tag{3.37}$$

We have

Definition 3.2 *For any given $T > 0$, $U = U(t,x) \in (L^2(0,T;L^2(0,L)))^n$ is a weak solution to the backward mixed problem* (3.35)–(3.36), *if we have*

$$\int_0^L \Psi^T(T,x) U_T(x) \mathrm{d}x - \int_0^L \Psi^T(0,x) U(0,x) \mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^T \int_0^L (\Psi_t^T + \Psi_x^T \Lambda - \Psi^T A) U \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T (\Psi^-)^T(t,0)\Lambda^- H^-(t)\mathrm{d}t - \int_0^T (\Psi^+)^T(t,L)\Lambda^+ H^+(t)\mathrm{d}t
\end{aligned} \tag{3.38}
$$

for any given $\Psi \in (C^1([0,T]\times[0,L]))^n$ *satisfying* (3.37).

Theorem 3.4 *For any given* $T > 0$, *for any given final data* $U_T \in (L^2(0,L))^n$ *and any given boundary function* $H \in (L^2(0,T))^n$, *the backward problem* (3.35)–(3.36) *admits a unique weak solution* $U = U(t,x) \in (L^2(0,T;L^2(0,L)))^n$, *satisfying*

$$
\|U(t,\cdot)\|_{(L^2(0,L))^n} \le C(T)(\|U_T\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^n}), \quad \forall t \in [0,T]. \tag{3.39}
$$

If we take $t = T - \overline{t},\ x = L - \overline{x}$ in (3.35)–(3.36) and $W(\overline{t},\overline{x}) = U(T-\overline{t}, L-\overline{x})$, then, the backward problem (3.35)–(3.36) can be formally transformed into a forward problem previously considered:

$$
\begin{cases}
W_{\overline{t}} + \Lambda W_{\overline{x}} - AW = 0, & \overline{t} \in (0,T),\quad \overline{x} \in (0,L),\\
W^+(\overline{t},0) = \widehat{G}_1 W^-(\overline{t},0) + H^+(T-\overline{t}), & \overline{t} \in (0,T),\\
W^-(\overline{t},L) = \widehat{G}_0 W^+(\overline{t},L) + H^-(T-\overline{t}), & \overline{t} \in (0,T)
\end{cases} \tag{3.40}
$$

with the initial data

$$
\overline{t} = 0:\ W(0,\overline{x}) = U_T(L-\overline{x}), \quad \overline{x} \in (0,L). \tag{3.41}
$$

We now verify the validity of this consideration. In fact, by Theorem 3.1, problem (3.40)–(3.41) admits a unique weak solution $W = W(\overline{t},\overline{x}) \in (L^2(0,T;L^2(0,L)))^n$, satisfying

$$
\|W(\overline{t},\cdot)\|_{(L^2(0,L))^n} \le C(T)(\|U_T\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^n}), \quad \forall \overline{t} \in [0,T], \tag{3.42}
$$

such that

$$
\begin{aligned}
&\int_0^L \Phi^T(T,\overline{x}) W(T,\overline{x})\mathrm{d}\overline{x} - \int_0^L \Phi^T(0,\overline{x}) U_T(L-\overline{x})\mathrm{d}\overline{x}\\
&= \int_0^T \int_0^L (\Phi_{\overline{t}}^T + \Phi_{\overline{x}}^T \Lambda + \Phi^T A) W \mathrm{d}\overline{x}\mathrm{d}\overline{t}\\
&\quad + \int_0^T (\Phi^+)^T(\overline{t},0)\Lambda^+ H^+(T-\overline{t})\mathrm{d}\overline{t} - \int_0^T (\Phi^-)^T(\overline{t},L)\Lambda^- H^-(T-\overline{t})\mathrm{d}\overline{t}
\end{aligned} \tag{3.43}
$$

holds for any given $\Phi \in (C^1([0,T]\times[0,L]))^n$ satisfying

$$
\begin{cases}
\Phi^-(\overline{t},0) = -(\Lambda^-)^{-1}\widehat{G}_1^T \Lambda^+ \Phi^+(\overline{t},0), & \overline{t} \in [0,T],\\
\Phi^+(\overline{t},L) = -(\Lambda^+)^{-1}\widehat{G}_0^T \Lambda^- \Phi^-(\overline{t},L), & \overline{t} \in [0,T].
\end{cases} \tag{3.44}
$$

Let $\overline{t} = T - t$ and $\overline{x} = L - x$ in (3.43)–(3.44), and let

$$
U(t,x) = W(T-t, L-x) \quad \text{and} \quad \Psi(t,x) = \Phi(T-t, L-x). \tag{3.45}
$$

It easily follows from (3.43)–(3.44) that $U = U(t,x)$ satisfies (3.38), and $\Psi \in (C^1([0,T]\times[0,L]))^n$ satisfies (3.37). Therefore, by Definition 3.2, $U = U(t,x)$ is the weak solution to the backward problem (3.35)–(3.36). Noting (3.42) and (3.45), $U = U(t,x)$ satisfies (3.39).

Remark 3.3 By the same manner as estimating the solution to the forward problem, under the assumptions of Theorem 3.4, similar estimates as (2.30)–(2.31), (2.46), (2.55)–(2.56) and (2.58)–(2.61) are valid for the backward problem (3.35)–(3.36), for example, we have

$$\|U(\cdot,0)\|_{(L^2(0,T))^n} \le C(T)(\|U_T\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^n}) \tag{3.46}$$

and

$$\|U(\cdot,L)\|_{(L^2(0,T))^n} \le C(T)(\|U_T\|_{(L^2(0,L))^n} + \|H\|_{(L^2(0,T))^n}). \tag{3.47}$$

Remark 3.4 Assume that the backward problem of system (1.2)–(1.4) is solvable, namely, assume that the number of positive eigenvalues is equal to that of negative ones, and G_i $(i=0,1)$ are reversible, namely,

$$\overline{m} = m \quad \text{and} \quad \text{rank}(G_0) = \text{rank}(G_1) = m. \tag{3.48}$$

Then system (1.2)–(1.4) can be equivalently rewritten as

$$\begin{cases} U_t + \Lambda U_x + AU = 0, & t \in (0,T),\ x \in (0,L), \\ U^-(t,0) = G_0^{-1}U^+(t,0) - G_0^{-1}D_0H^+(t), & t \in (0,T), \\ U^+(t,L) = G_1^{-1}U^-(t,L) - G_1^{-1}D_1H^-(t), & t \in (0,T). \end{cases} \tag{3.49}$$

By Theorem 3.4, the backward problem (3.49) and (3.36) is well-posed. Moreover, by the approximation of classical solutions and the uniqueness of weak solution, it is easy to prove that if $U = \widehat{U}(t,x)$ is the weak solution to problem (1.2)–(1.5), then it is also the weak solution to backward problem (3.49) with the final condition

$$t = T:\ U(T,x) = \widehat{U}(T,x), \quad x \in (0,L),$$

and vice versa. Hence, system (1.2)–(1.4) is time reversible.

We point out that, making an equivalent algebraic transformation by multiplying a full rank matrix from the left on the boundary condition of the system doesn't influence the definition and the properties of the weak solution. In the following discussion, this property was used and will be used from time to time.

3.4 Some related results

In order to prove the controllability for system (1.2)–(1.4) and the observability for the adjoint system (4.1) by the constructive method, we prepare some useful lemmas in this section.

First of all, for any given $T > 0$, assume that $\Phi = \widehat{\Phi}(t,x) \in (L^2(0,T;L^2(0,L)))^n$ is a weak solution to the following homogeneous backward system

$$\begin{cases} \Phi_t + \Lambda\Phi_x - A^T\Phi = 0, & t \in (0,T),\ x \in (0,L), \\ x = 0:\ \Phi^-(t,0) = \widehat{G}_0\Phi^+(t,0), & t \in (0,T), \\ x = L:\ \Phi^+(t,L) = \widehat{G}_1\Phi^-(t,L), & t \in (0,T) \end{cases} \tag{3.50}$$

with

$$t = T:\ \Phi(T,x) = \widehat{\Phi}_T(x), \quad x \in (0,L), \tag{3.51}$$

where the boundary coupling matrices $\widehat{G}_0$ and $\widehat{G}_1$ with constant elements are any given matrices of order $m \times \overline{m}$ and $\overline{m} \times m$, respectively. Assume furthermore that

$$\overline{m} \le m \text{ (i.e, } n \le 2m) \quad \text{and} \quad \operatorname{rank}(\widehat{G}_0) = \overline{m}, \tag{3.52}$$

without loss of generality, we assume that $\widehat{G}_0 = \begin{pmatrix} (\widehat{G}_{01})_{\overline{m}\times\overline{m}} \\ (\widehat{G}_{02})_{(m-\overline{m})\times\overline{m}} \end{pmatrix}$, where $\widehat{G}_{01}$ is an reversible matrix of order $\overline{m}$. Thus the boundary condition on $x = 0$ in (3.50) implies

$$x = 0: \ \Phi^+(t, 0) = \widetilde{G}_0 \Phi^-(t, 0), \quad \forall t \in (0, T), \tag{3.53}$$

where $\widetilde{G}_0 = \begin{pmatrix} \widehat{G}_{01}^{-1} & 0 \end{pmatrix}$. Then we can consider the following forward mixed problem:

$$\begin{cases} \Phi_t + \Lambda\Phi_x - A^T\Phi = 0, & t \in (0, T), \ x \in (0, L), \\ x = 0: \ \Phi^+(t, 0) = \widetilde{G}_0\Phi^-(t, 0), & t \in (0, T), \\ x = L: \ \Phi^-(t, L) = \widehat{\Phi}^-(t, L), & t \in (0, T), \\ t = 0: \ \Phi = \widehat{\Phi}(0, x), & x \in (0, L). \end{cases} \tag{3.54}$$

We have

Theorem 3.5 *Under the assumptions mentioned above, assume furthermore that* (3.52) *holds, then the weak solution* $\Phi = \widehat{\Phi}(t, x) \in (L^2(0, T; L^2(0, L)))^n$ *to the backward problem* (3.50)–(3.51) *is the weak solution to the forward problem* (3.54).

Proof By Definition 3.1, $\Phi = \Phi(t, x) \in (L^2(0, T; L^2(0, L)))^n$ is a weak solution to problem (3.54), if

$$\begin{aligned} &\int_0^L \Psi^T(T, x)\Phi(T, x)\mathrm{d}x - \int_0^L \Psi^T(0, x)\widehat{\Phi}(0, x)\mathrm{d}x \\ &= \int_0^T \int_0^L (\Psi_t^T + \Psi_x^T\Lambda + \Psi^T A^T)\Phi \mathrm{d}x\mathrm{d}t - \int_0^T (\Psi^-)^T(t, L)\Lambda^-\widehat{\Phi}^-(t, L)\mathrm{d}t \end{aligned} \tag{3.55}$$

holds for any given $\Psi \in (C^1([0, T] \times [0, L]))^n$ satisfying

$$\begin{cases} \Psi^-(t, 0) = -(\Lambda^-)^{-1}\widetilde{G}_0^T\Lambda^+\Psi^+(t, 0), & t \in [0, T], \\ \Psi^+(t, L) = 0, & t \in [0, T]. \end{cases} \tag{3.56}$$

Let $\Phi = \widehat{\Phi}(t, x) \in (L^2(0, T; L^2(0, L)))^n$ be the weak solution to the backward problem (3.50)–(3.51). By the uniqueness of weak solution, it suffices to prove that $\Phi = \widehat{\Phi}(t, x)$ satisfies (3.55) for any given $\Psi \in (C^1([0, T] \times [0, L]))^n$ satisfying (3.56). By the procedure of approximation mentioned in Section 3.2, we can find a sequence of C^1 solutions $\Phi^k(t, x)$ of problem (3.50)–(3.51) that converges to $\widehat{\Phi}(t, x)$ in $(L^2(0, T; L^2(0, L)))^n$ as $k \to \infty$, such that

$$(\Phi^k)^-(t, L) \to \widehat{\Phi}^-(t, L) \quad \text{in } (L^2(0, T))^m \text{ as } \ k \to \infty \tag{3.57}$$

and

$$\Phi^k(0, x) \to \widehat{\Phi}(0, x), \quad \Phi^k(T, x) \to \widehat{\Phi}(T, x) \quad \text{in } (L^2(0, L))^n \text{ as } \ k \to \infty. \tag{3.58}$$

It is easy to check that $\Phi = \Phi^k(t, x)$ satisfies

$$\Phi_t + \Lambda\Phi_x - A^T\Phi = 0, \quad t \in (0, T), \quad x \in (0, L). \tag{3.59}$$

Moreover, noting the boundary condition on $x = 0$ in (3.50), by (3.52), $\Phi = \Phi^k(t, x)$ satisfies (3.53). Thus, $\Phi = \Phi^k(t, x)$ is the C^1 solution to system (3.59) and

$$\begin{cases} x = 0: \ \Phi^+(t, 0) = \widetilde{G}_0 \Phi^-(t, 0), & t \in (0, T), \\ x = L: \ \Phi^-(t, L) = (\Phi^k)^-(t, L), & t \in (0, T) \end{cases} \tag{3.60}$$

with the initial data

$$t = 0: \ \Phi = \Phi^k(0, x), \quad x \in (0, L). \tag{3.61}$$

By Proposition 3.1, $\Phi^k(t, x)$ satisfies

$$\begin{aligned} &\int_0^L \Psi^T(T, x)\Phi^k(T, x)\mathrm{d}x - \int_0^L \Psi^T(0, x)\Phi^k(0, x)\mathrm{d}x \\ &= \int_0^T \int_0^L (\Psi_t^T + \Psi_x^T \Lambda + \Psi^T A^T)\Phi^k \mathrm{d}x\mathrm{d}t - \int_0^T (\Psi^-)^T(t, L)\Lambda^-(\Phi^k)^-(t, L)\mathrm{d}t \end{aligned} \tag{3.62}$$

for any given $\Psi \in (C^1([0, T] \times [0, L]))^n$ satisfying (3.56). Taking $k \to \infty$ in (3.62) and noting (3.57) and (3.58), we get (3.55), then $\Phi = \widehat{\Phi}(t, x)$ is the weak solution to problem (3.54).

Now we assume that $U = \widehat{U}(t, x) \in (L^2(0, T; L^2(0, L)))^n$ is a weak solution to problem (1.2)–(1.5). Noticing $\lambda_i \neq 0$ ($i = 1, \cdots, n$), and changing the role of t and x in (1.2), we consider the following rightward problem:

$$U_x + \Lambda^{-1}U_t + \Lambda^{-1}AU = 0, \quad x \in (0, L), \ \ t \in (0, T) \tag{3.63}$$

with the boundary condition

$$\begin{cases} t = 0: \ U^+ = \widehat{U}^+(0, x), & x \in (0, L), \\ t = T: \ U^- = \widehat{U}^-(T, x), & x \in (0, L) \end{cases} \tag{3.64}$$

and with the initial data

$$x = 0: \ U = \widehat{U}(t, 0), \quad t \in (0, T). \tag{3.65}$$

We have

Theorem 3.6 *Under the assumptions mentioned above, the weak solution* $U = \widehat{U}(t, x) \in (L^2(0, T; L^2(0, L)))^n$ *to the forward problem* (1.2)–(1.5) *is also the weak solution to the rightward problem* (3.63)–(3.65).

Proof Multiplying Φ^T on both sides of (3.63), and using the boundary conditions (3.64), it is easy to check that $U = U(t, x) \in (L^2(0, T; L^2(0, L)))^n$ is a weak solution to problem (3.63)–(3.65) if

$$\begin{aligned} 0 = &\int_0^T \Phi^T(t, L)U(t, L)\mathrm{d}t - \int_0^T \Phi^T(t, 0)\widehat{U}(t, 0)\mathrm{d}t \\ &+ \int_0^L (\Phi^-)^T(T, x)\Lambda^{-1}\widehat{U}^-(T, x)\mathrm{d}x - \int_0^L (\Phi^+)^T(0, x)\Lambda^{-1}\widehat{U}^+(0, x)\mathrm{d}x \\ &- \int_0^T \int_0^L (\Phi_x^T + \Phi_t^T \Lambda^{-1} - \Phi^T \Lambda^{-1} A)U \mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.66}$$

holds for any given $\Phi \in (C^1([0,T]\times[0,L]))^n$ satisfying

$$\begin{cases}\Phi^-(0,x)=0, & x\in[0,L],\\ \Phi^+(T,x)=0, & x\in[0,L].\end{cases} \tag{3.67}$$

Let $U=\widehat{U}(t,x)\in(L^2(0,T;L^2(0,L)))^n$ be the weak solution to problem (1.2)–(1.5). By the uniqueness of weak solution, it suffices to prove that $U=\widehat{U}(t,x)$ satisfies (3.66) for any given $\Phi\in(C^1([0,T]\times[0,L]))^n$ satisfying (3.67). By the procedure of approximation mentioned in Section 3.2, we can find a sequence of C^1 solutions $U^k(t,x)$ of problem (1.2)–(1.5) that converges to $\widehat{U}(t,x)$ in $(L^2(0,T;L^2(0,L)))^n$ as $k\to\infty$, such that

$$(U^k)^+(t,0)\to\widehat{U}^+(t,0)\quad\text{in }(L^2(0,T))^{\overline{m}};\quad (U^k)^-(t,L)\to\widehat{U}^-(t,L)\quad\text{in }(L^2(0,T))^m \tag{3.68}$$

and

$$U^k(0,x)\to\widehat{U}(0,x)\quad\text{in }(L^2(0,L))^n;\quad U^k(T,x)\to\widehat{U}(T,x)\quad\text{in }(L^2(0,L))^n \tag{3.69}$$

as $k\to\infty$. It is easy to check that $U=U^k(t,x)$ solves the rightward problem of system (3.63) with the boundary condition

$$\begin{cases}t=0:\ U^+=(U^k)^+(0,x), & x\in(0,L),\\ t=T:\ U^-=(U^k)^-(T,x), & x\in(0,L)\end{cases} \tag{3.70}$$

and the initial data

$$x=0:\ U=U^k(t,0),\quad t\in(0,T). \tag{3.71}$$

By Proposition 3.1, $U^k(t,x)$ satisfies

$$\begin{aligned}0=&\int_0^T\Phi^T(t,L)U^k(t,L)\mathrm{d}t-\int_0^T\Phi^T(t,0)U^k(t,0)\mathrm{d}t\\&+\int_0^L(\Phi^-)^T(T,x)\Lambda^{-1}(U^k)^-(T,x)\mathrm{d}x-\int_0^L(\Phi^+)^T(0,x)\Lambda^{-1}(U^k)^+(0,x)\mathrm{d}x\\&-\int_0^T\int_0^L(\Phi_x^T+\Phi_t^T\Lambda^{-1}-\Phi^T\Lambda^{-1}A)U^k\mathrm{d}x\mathrm{d}t\end{aligned} \tag{3.72}$$

for any given $\Phi\in(C^1([0,T]\times[0,L]))^n$ satisfying (3.67). Taking $k\to\infty$ in (3.72) and noting (3.68)–(3.69), we have

$$\begin{aligned}0=&\int_0^T\Phi^T(t,L)\widehat{U}(t,L)\mathrm{d}t-\int_0^T\Phi^T(t,0)\widehat{U}(t,0)\mathrm{d}t\\&+\int_0^L(\Phi^-)^T(T,x)\Lambda^{-1}\widehat{U}^-(T,x)\mathrm{d}x-\int_0^L(\Phi^+)^T(0,x)\Lambda^{-1}\widehat{U}^+(0,x)\mathrm{d}x\\&-\int_0^T\int_0^L(\Phi_x^T+\Phi_t^T\Lambda^{-1}-\Phi^T\Lambda^{-1}A)\widehat{U}\mathrm{d}x\mathrm{d}t\end{aligned} \tag{3.73}$$

for any given $\Phi\in(C^1([0,T]\times[0,L]))^n$ satisfying (3.67), then $\widehat{U}(t,x)$ is the weak solution to problem (3.63)–(3.65).

Similarly to the proof of Theorem 3.6, we have

Theorem 3.7 *Assume that* $U = \widehat{U}(t,x) \in (L^2(0,T;L^2(0,L)))^n$ *is a weak solution to the backward problem* (3.35)–(3.36)*, then it is also the weak solution to the rightward problem* (3.63)–(3.65)*.*

Similarly, for the leftward problem of system (3.63) with the boundary conditions

$$\begin{cases} t=0: \ U^- = \widehat{U}^-(0,x), & x \in (0,L), \\ t=T: \ U^+ = \widehat{U}^+(T,x), & x \in (0,L) \end{cases} \tag{3.74}$$

and the initial data

$$x = L: \ U = \widehat{U}(t,L), \quad t \in (0,T), \tag{3.75}$$

we have

Theorem 3.8 *Under the assumptions mentioned above, the weak solution* $U = \widehat{U}(t,x) \in (L^2(0,T;L^2(0,L)))^n$ *to problem* (1.2)–(1.5) *is also the weak solution to the leftward problem* (3.63) *and* (3.74)–(3.75)*.*

Theorem 3.9 *Assume that* $U = \widehat{U}(t,x) \in (L^2(0,T;L^2(0,L)))^n$ *is a weak solution to the backward problem* (3.35)–(3.36)*, then it is also the weak solution to the leftward problem* (3.63) *and* (3.74)–(3.75)*.*

4 Exact Boundary (Null) Controllability of Weak Solutions by the Constructive Method

In this paper, we will discuss the exact boundary (resp. null) controllability for system (1.2)–(1.4) and the strong (resp. weak) exact boundary observability for the corresponding adjoint system

$$\begin{cases} \Phi_t + \Lambda\Phi_x - A^T\Phi = 0, & t \in (0,T), \ x \in (0,L), \\ \Phi^-(t,0) = -(\Lambda^-)^{-1}G_0^T\Lambda^+\Phi^+(t,0), & t \in (0,T), \\ \Phi^+(t,L) = -(\Lambda^+)^{-1}G_1^T\Lambda^-\Phi^-(t,L), & t \in (0,T) \end{cases} \tag{4.1}$$

with the final data

$$t = T: \ \Phi(T,x) = \Phi_T(x), \quad x \in (0,L). \tag{4.2}$$

As mentioned in Introduction, first order hyperbolic systems are not always time reversible, then the exact boundary controllability and the exact boundary null controllability for system (1.2)–(1.4) are not equivalent to each other in general (see [12]), hence they should be discussed separately. Correspondingly, for the adjoint system (4.1), the strong exact boundary observability and the weak exact boundary observability should be taken into consideration respectively, too.

Based on Theorem 3.2 and Theorems 3.5–3.9, the constructive method in the framework of classical solutions (see [8, 12]) can also be used in the framework of weak solutions to prove the exact boundary (null) controllability and the strong (weak) exact boundary observability for 1-D first order hyperbolic systems.

In this section and in the next section, we will first list some results on controllability and observability obtained by the constructive method in the framework of weak solutions. The procedure of proofs is similar to that in the framework of classical solutions by the constructive method (see [8, 12]). In what follows, we only prove the one-sided exact boundary controllability of weak solutions for system (1.2)–(1.4) and the one-sided strong exact boundary observability of weak solutions for the adjoint system (4.1) to show how the constructive method works.

Lemma 4.1 (Two-sided exact boundary controllability) *Let $T \geq T_0$, where*

$$T_0 = L \max_{\substack{1\leq r\leq m\\ m+1\leq s\leq n}} \Big\{\frac{1}{|\lambda_r|}, \frac{1}{\lambda_s}\Big\} > 0. \tag{4.3}$$

If $M = n$, namely, $M_0 = \mathrm{rank}(D_0) = \overline{m}$ and $M_1 = \mathrm{rank}(D_1) = m$, then for any given initial data $U_0(x) \in (L^2(0, L))^n$ and final data $U_T(x) \in (L^2(0, L))^n$, there exists a boundary control $H \in (L^2(0, T))^n$, satisfying

$$\|H\|_{(L^2(0,T))^n} \leq C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n}), \tag{4.4}$$

here and hereafter, $C > 0$ denotes a constant depending only on T, such that the mixed problem (1.2)–(1.5) *admits a unique weak solution $U = U(t, x)$, satisfying exactly the final state at the time $t = T$:*

$$U(T, x) = U_T(x), \quad 0 < x < L. \tag{4.5}$$

Lemma 4.2 (One-sided exact boundary controllability) *Assume that the number of positive eigenvalues is not larger than that of negative ones:*

$$\overline{m} \leq m \ (i.e.,\ n \leq 2m). \tag{4.6}$$

Assume furthermore that

$$\mathrm{rank}(G_0) = \overline{m}. \tag{4.7}$$

Let $T \geq \overline{T}_0$, where

$$\overline{T}_0 = L\Big(\max_{1\leq r\leq m}\frac{1}{|\lambda_r|} + \max_{m+1\leq s\leq n}\frac{1}{\lambda_s}\Big) > 0. \tag{4.8}$$

If $M = M_1 = \mathrm{rank}(D_1) = m$, then for any given initial data $U_0(x) \in (L^2(0, L))^n$ and final data $U_T(x) \in (L^2(0, L))^n$, for any given boundary function $H^+(t) \in (L^2(0, T))^{\overline{m}}$, there exists a boundary control $H^-(t) \in (L^2(0, T))^m$, satisfying

$$\|H^-\|_{(L^2(0,T))^m} \leq C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}), \tag{4.9}$$

such that system (1.2)–(1.4) *is exactly controllable at the time $t = T$.*

Proof Let

$$\overline{T}_1 = L \max_{1\leq r\leq m}\frac{1}{|\lambda_r|} \tag{4.10}$$

and

$$\overline{T}_2 = T - L \max_{m+1\leq s\leq n}\frac{1}{\lambda_s}. \tag{4.11}$$

First, for any given initial data $U_0(x) \in (L^2(0,L))^n$ and boundary function $H^+(t) \in (L^2(0,T))^{\overline{m}}$ on $x=0$, we consider the forward problem (1.2)–(1.3) and (1.5) with the following artificial boundary condition on $x=L$:

$$x = L:\ U^-(t,L) = F(t), \quad 0 < t < \overline{T}_1, \tag{4.12}$$

where $F(t) \in (L^2(0,\overline{T}_1))^m$ is any given function of t, satisfying

$$\|F\|_{(L^2(0,\overline{T}_1))^m} \le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}). \tag{4.13}$$

By Theorem 3.1, there exists a unique weak solution $U = U_f(t,x) \in (L^2(0,\overline{T}_1; L^2(0,L)))^n$ on $R_f = \{(t,x) | 0 \le t \le \overline{T}_1, 0 \le x \le L\}$, satisfying

$$\|U_f\|_{(L^2(0,\overline{T}_1;L^2(0,L)))^n} \le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}), \tag{4.14}$$

and we can determine the value of $U = U_f(t,x)$ on $x=0$ as

$$x = 0:\ U_f(t,0) = a(t), \quad 0 < t < \overline{T}_1 \tag{4.15}$$

with $a(t) \in (L^2(0,\overline{T}_1))^n$ satisfying

$$\|a\|_{(L^2(0,\overline{T}_1))^n} \le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}). \tag{4.16}$$

Second, noting (4.7), without loss of generality, we may assume that

$$G_0 = \begin{pmatrix} (G_{01})_{\overline{m}\times\overline{m}} & (G_{02})_{\overline{m}\times(m-\overline{m})} \end{pmatrix},$$

where G_{01} is reversible. Then the boundary condition (1.3) on $x=0$ can be equivalently rewritten as

$$x = 0:\ \begin{pmatrix} u_1 \\ \vdots \\ u_{\overline{m}} \end{pmatrix}(t,0) = G_{01}^{-1}U^+(t,0) - G_{01}^{-1}G_{02}\begin{pmatrix} u_{\overline{m}+1} \\ \vdots \\ u_m \end{pmatrix}(t,0) - G_{01}^{-1}D_0H^+(t). \tag{4.17}$$

For any given final data $U_T(x) \in (L^2(0,L))^n$, we consider the backward problem (2.52) with final data (4.5), boundary condition (4.17) and the following artificial boundary conditions

$$x = 0:\ \begin{pmatrix} u_{\overline{m}+1} \\ \vdots \\ u_m \end{pmatrix}(t,0) = P(t), \quad \overline{T}_2 < t < T \tag{4.18}$$

and

$$x = L:\ U^+(t,L) = Q(t), \quad \overline{T}_2 < t < T, \tag{4.19}$$

where $P(t) \in (L^2(\overline{T}_2,T))^{(m-\overline{m})}$ and $Q(t) \in (L^2(\overline{T}_2,T))^{\overline{m}}$ are any given function of t, satisfying

$$\|P\|_{(L^2(\overline{T}_2,T))^{(m-\overline{m})}} \le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}) \tag{4.20}$$

and

$$\|Q\|_{(L^2(\overline{T}_2,T))^{\overline{m}}} \le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}). \tag{4.21}$$

By Theorem 3.1, there exists a unique weak solution $U = U_b(t,x) \in (L^2(\overline{T}_2, T; L^2(0,L)))^n$ on $R_b = \{(t,x) \mid \overline{T}_2 \le t \le T, 0 \le x \le L\}$, satisfying

$$\|U_b\|_{(L^2(\overline{T}_2,T;L^2(0,L)))^n} \le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}). \tag{4.22}$$

Then, we can determine the value of $U = U_b(t,x)$ on $x = 0$ as

$$x = 0: \quad U_b(t,0) = b(t), \quad \overline{T}_2 < t < T \tag{4.23}$$

with $b(t) \in (L^2(\overline{T}_2, T))^n$ satisfying

$$\|b\|_{(L^2(\overline{T}_2,T))^n} \le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}). \tag{4.24}$$

It is easy to check that both $a(t)$ and $b(t)$ satisfy the boundary condition (1.3) on $x = 0$. Noting (4.8) and (4.10)–(4.11), we may find $c(t) \in (L^2(0,T))^n$ satisfying the boundary condition (1.3) on $x = 0$ and

$$\|c\|_{(L^2(0,T))^n} \le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}), \tag{4.25}$$

such that

$$c(t) = \begin{cases} a(t), & 0 < t < \overline{T}_1; \\ b(t), & \overline{T}_2 < t < T. \end{cases} \tag{4.26}$$

We now change the status of t and x and consider a rightward problem for system (3.63) with the initial data

$$x = 0: \ U(t,0) = c(t), \quad 0 < t < T \tag{4.27}$$

and the boundary conditions

$$t = 0: \ U^+ = U_0^+(x), \quad x \in (0,L) \tag{4.28}$$

and

$$t = T: \ U^- = U_T^-(x), \quad x \in (0,L). \tag{4.29}$$

Again, by Theorem 3.1, there exists a unique weak solution $U = U(t,x) \in (L^2(0,T; L^2(0,L)))^n$ on $R(T) = \{(t,x) \mid 0 \le t \le T, 0 \le x \le L\}$, satisfying

$$\|U\|_{(L^2(0,T;L^2(0,L)))^n} \le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}). \tag{4.30}$$

We first prove that $U = U(t,x)$ satisfies the initial condition (1.5) and the final condition (4.5). In fact, by Theorem 3.6, $U = U_f(t,x)$ is also the solution to the rightward problem (3.63) with (4.28),

$$t = \overline{T}_1: \ U^- = U_f^-(\overline{T}_1, x), \quad x \in (0,L) \tag{4.31}$$

and

$$x = 0: \ U(t,0) = c(t), \quad 0 < t < \overline{T}_1 \tag{4.32}$$

on the domain R_f. However, noting (4.10), by Theorem 3.2, the solution to the rightward problem (3.63) with (4.28), (4.31) and (4.32) is uniquely determined on the domain

$$R^0 = \left\{(t,x)\middle|0 \le t < \frac{\overline{T}_1}{L}(L-x), 0 \le x \le L\right\}$$

by (4.28) and (4.32), which are just the initial data and the boundary condition on $t = 0$ for the rightward problem (3.63) and (4.27)–(4.29). Hence,

$$U(t,x) = U_f(t,x) \tag{4.33}$$

on the domain R^0. In particular, we have (1.5).

On the other hand, by Theorem 3.7, $U = U_b(t,x)$ is also the solution to the rightward problem (3.63) with (4.29),

$$t = \overline{T}_2: \ U^+ = U_b^+(\overline{T}_2, x), \quad x \in (0, L) \tag{4.34}$$

and

$$x = 0: \ U(t,0) = c(t), \quad \overline{T}_2 < t < T \tag{4.35}$$

on the domain R_b. However, noting (4.11), by Theorem 3.2, the solution to the rightward problem (3.63) with (4.35)–(4.34) is uniquely determined on the domain

$$R^T = \left\{(t,x)\middle|T - \frac{T-\overline{T}_2}{L}(L-x) < t \le T, 0 \le x \le L\right\}$$

by (4.29) and (4.35). Similarly, we have

$$U(t,x) = U_b(t,x) \tag{4.36}$$

on the domain R_b. In particular, we have the final condition (4.5).

We now prove that $U = U(t,x)$ satisfies

$$\begin{aligned}
&\int_0^L \Phi^T(T,x)U_T(x)\mathrm{d}x - \int_0^L \Phi^T(0,x)U_0(x)\mathrm{d}x \\
&= \int_0^T\int_0^L (\Phi_t^T + \Phi_x^T\Lambda - \Phi^T A)U\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T (\Phi^+)^T(t,0)\Lambda^+ D_0 H^+(t)\mathrm{d}t - \int_0^T \Phi^T(t,L)\Lambda U(t,L)\mathrm{d}t
\end{aligned} \tag{4.37}$$

for any given $\Phi \in (C^1([0,T]\times[0,L]))^n$ satisfying

$$\Phi^-(t,0) = -(\Lambda^-)^{-1}G_0^T\Lambda^+\Phi^+(t,0), \quad t \in [0,T], \tag{4.38}$$

namely, $U = U(t,x)$ satisfies (1.2)–(1.3) with (1.5) and (4.5) in the sense of weak solutions. (The equivalent integral equation (4.37) is formed by a similar manner of (3.1) in Definition 3.1.)

Regarding the following forward problem

$$\begin{cases}
W_t + \Lambda W_x + AW = 0, & t \in (0,T), \ x \in (0,L), \\
x = 0: \ W^+(t,0) = U^+(t,0), & t \in (0,T), \\
x = L: \ W^-(t,L) = U^-(t,L), & t \in (0,T), \\
t = 0: \ W = U_0(x)
\end{cases} \tag{4.39}$$

as the 'leftward' problem of the rightward problem (3.63) and (4.27)–(4.29), noting (1.5) and by Theorem 3.8, $U(t,x)$ is also the weak solution to the forward problem (4.39), we have $W(t,x) = U(t,x)$, then, by Definition 3.1, we get

$$\begin{aligned} &\int_0^L \Phi^T(T,x)U(T,x)\mathrm{d}x - \int_0^L \Phi^T(0,x)U_0(x)\mathrm{d}x \\ =&\int_0^T \int_0^L (\Phi_t^T + \Phi_x^T\Lambda - \Phi^T A)U\mathrm{d}x\mathrm{d}t \\ &+\int_0^T \Phi^T(t,0)\Lambda U(t,0)\mathrm{d}t - \int_0^T \Phi^T(t,L)\Lambda U(t,L)\mathrm{d}t \end{aligned} \tag{4.40}$$

for any given $\Phi \in (C^1([0,T]\times[0,L]))^n$. Let $\Phi \in (C^1([0,T]\times[0,L]))^n$ satisfy (4.38). Noting $U(t,0) = c(t)$, by its construction, $c(t)$ satisfies the boundary condition (1.3) on $x = 0$, and noticing (4.5), (4.37) follows from (4.40) immediately.

Finally, we determine the boundary control function by

$$H^-(t) = D_1^{-1}(U^-(t,L) - G_1U^+(t,L)), \quad t \in (0,T), \tag{4.41}$$

and, noting (4.25), it follows from Theorem 3.1 that

$$\begin{aligned} \|H^-\|_{(L^2(0,T))^m} &\le C\|U(\cdot,L)\|_{(L^2(0,T))^n} \\ &\le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|c\|_{(L^2(0,T))^n}) \\ &\le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n} + \|H^+\|_{(L^2(0,T))^{\overline{m}}}). \end{aligned} \tag{4.42}$$

As a conclusion, for any given initial data $U_0(x) \in (L^2(0,L))^n$, any given final data $U_T(x) \in (L^2(0,L))^n$ and any given boundary function $H^+(t) \in (L^2(0,T))^{\overline{m}}$, we can construct a function $U = U(t,x) \in (L^2(0,T;L^2(0,L)))^n$, which satisfies exactly (1.2)–(1.3), (1.5) and (4.5) in the sense of weak solutions, and by restricting $U = U(t,x)$ on the boundary $x = L$, we obtain a boundary control function $H^-(t) \in (L^2(0,T))^m$ given by (4.41). Thus, with this boundary control $H^-(t)$, $U = U(t,x)$ is the weak solution to problem (1.2)–(1.5), which satisfies exactly the final data (4.5) at the time $t = T$. Hence, system (1.2)–(1.4) is exactly controllable at the time $t = T$.

Remark 4.1 Assumptions (4.6)–(4.7) guarantee that the constructive method with suitable artificial boundary conditions can be applied to system (1.2)–(1.4) for realizing the exact boundary controllability for weak solutions.

Remark 4.2 The case of two-sided exact boundary controllability with fewer boundary controls can be similarly discussed by the constructive method.

The constructive method can be also applied to prove the exact boundary null controllability for system (1.2)–(1.4). Taking the one-sided controls case as an example, we have

Lemma 4.3 (One-sided exact boundary null controllability) *Let $T \ge \overline{T}_0$, where $\overline{T}_0$ is given by* (4.8). *Assume that $H^-(t) \equiv 0$ (resp. $H^+(t) \equiv 0$), if $M = M_0 = \mathrm{rank}(D_0) = \overline{m}$ (resp. $M = M_1 = \mathrm{rank}(D_1) = m$), then for any given initial data $U_0(x) \in (L^2(0,L))^n$, there exists a boundary control $H^+(t) \in (L^2(0,T))^{\overline{m}}$ (resp. $H^-(t) \in (L^2(0,T))^m$), satisfying*

$$\|H^+\|_{(L^2(0,T))^{\overline{m}}} \le C\|U_0\|_{(L^2(0,L))^n} \quad (\textit{resp.}\ \ \|H^-\|_{(L^2(0,T))^m} \le C\|U_0\|_{(L^2(0,L))^n}), \tag{4.43}$$

such that the corresponding mixed problem (1.2)–(1.5) *admits a unique weak solution* $U = U(t, x)$*, satisfying exactly the null final state at the time* $t = T$*:*

$$U(T, x) \equiv 0, \quad 0 < x < L. \tag{4.44}$$

Remark 4.3 Under the assumptions of Lemma 4.3, since the system has homogeneous boundary conditions on the side without controls, we can set $U \equiv 0$ in R_b, instead of solving a backward problem as in the proof of Lemma 4.2, hence assumptions (4.6)–(4.7) are not required.

Moreover, for system with homogenous boundary conditions on the side without controls, boundary controls can be acted on either side of the boundary, so that the number of boundary controls can be further reduced to $\overline{m}$ $(\leq m)$.

In general, the one-sided exact boundary controllability and the one-sided exact boundary null controllability are not equivalent. If system (1.2)–(1.4) is exactly controllable, then it must be exactly null controllable. However, in order to deduce the one-sided exact boundary controllability from the one-sided exact boundary null controllability for system (1.2)–(1.4), the number of positive eigenvalues of Λ should be equal to that of negative ones, and the boundary coupling matrices G_0 and G_1 should be reversible, namely, we have

Theorem 4.1 *The one-sided exact boundary controllability and the one-sided exact boundary null controllability are equivalent if the following hypotheses are satisfied*:

$$1. \quad \overline{m} = m; \quad 2. \quad \operatorname{rank}(G_0) = \operatorname{rank}(G_1) = m. \tag{4.45}$$

Remark 4.4 Under the assumptions of Theorem 4.1, controls can apply on either side of the boundary, and the number of boundary controls is equal to $M = m = \overline{m}$.

5 Strong (weak) Exact Boundary Observability for the Adjoint System by the Constructive Method

For any given final data $\Phi_T \in (L^2(0, L))^n$, the well-posedness of the backward problem (4.1)–(4.2) in the sense of weak solutions is guaranteed by Theorem 3.4. By the constructive method, we have the following result on the strong observability of the adjoint system (4.1).

Lemma 5.1 (Two-sided strong exact boundary observability) *Assume* $T \geq T_0$*, where* T_0 *is given by* (4.3)*. Assume that* $\Phi = \Phi(t, x)$ *is the weak solution to the adjoint problem* (4.1)–(4.2)*. For any given final data* $\Phi_T(x) \in (L^2(0, L))^n$*, the boundary observations* $\Phi^-(t, L)$ *and* $\Phi^+(t, 0)$ *corresponding to the departing characteristics on* $x = L$ *and on* $x = 0$*, respectively, on the interval* $(0, T)$ *can be used to uniquely determine the final data* $\Phi_T(x)$*, and we have the following strong observability inequality*:

$$\|\Phi_T\|_{(L^2(0,L))^n} \leq C(\|\Phi^-(\cdot, L)\|_{(L^2(0,T))^m} + \|\Phi^+(\cdot, 0)\|_{(L^2(0,T))^{\overline{m}}}). \tag{5.1}$$

In particular, if

$$M = n, \quad \textit{namely,} \quad M_0 = \operatorname{rank}(D_0) = \overline{m} \quad \textit{and} \quad M_1 = \operatorname{rank}(D_1) = m, \tag{5.2}$$

then (5.1) *can be written as*

$$\|\Phi_T\|_{(L^2(0,L))^n} \leq C(\|D_0^T \Lambda^+ \Phi^+(\cdot, 0)\|_{(L^2(0,T))^{\overline{m}}} + \|D_1^T \Lambda^- \Phi^-(\cdot, L)\|_{(L^2(0,T))^m}). \tag{5.3}$$

Lemma 5.2 (One-sided strong exact boundary observability) *Let $T \geq \overline{T}_0$, where $\overline{T}_0$ is given by* (4.8). *Assume that $\Phi = \Phi(t,x)$ is the weak solution to the adjoint problem* (4.1)–(4.2). *Assume furthermore that* (4.6)–(4.7) *hold. For any given final data $\Phi_T(x) \in (L^2(0,L))^n$, the boundary observation $\Phi^-(t,L)$ corresponding to the departing characteristics on $x = L$ (the side with fewer coming characteristics) on the interval $(0,T)$ can be used to uniquely determine the final data $\Phi_T(x)$, and we have the following strong observation inequality*:

$$\|\Phi_T\|_{(L^2(0,L))^n} \leq C\|\Phi^-(\cdot,L)\|_{(L^2(0,T))^m}. \tag{5.4}$$

In particular, if

$$M = M_1 = \text{rank}(D_1) = m, \tag{5.5}$$

then (5.4) *can be written as*

$$\|\Phi_T\|_{(L^2(0,L))^n} \leq C\|D_1^T\Lambda^-\Phi^-(\cdot,L)\|_{(L^2(0,T))^m}. \tag{5.6}$$

Proof Assume that $\Phi = \widehat{\Phi}(t,x) \in (L^2(0,T;L^2(0,L)))^n$ is a weak solution to the backward problem (4.1)–(4.2) with the final data $\widehat{\Phi}_T \in (L^2(0,L))^n$.

Noting $\lambda_i \neq 0$ $(i = 1,\cdots,n)$, the following leftward problem

$$\begin{cases} \Phi_x + \Lambda^{-1}\Phi_t - \Lambda^{-1}A^T\Phi = 0, & x \in (0,L),\ t \in (0,T), \\ t = 0: \ \Phi^- = \widehat{\Phi}^-(0,x), & x \in (0,L), \\ t = T: \ \Phi^+ = \widehat{\Phi}_T^+, & x \in (0,L) \end{cases} \tag{5.7}$$

and

$$x = L: \ \Phi = \widehat{\Phi}(t,L), \quad t \in (0,T) \tag{5.8}$$

admits a unique weak solution. By Theorem 3.9, the weak solution $\Phi = \widehat{\Phi}(t,x)$ to the backward problem (4.1)–(4.2) is also the weak solution to the leftward problem (5.7)–(5.8).

Without loss of generality, assume that

$$\max_{1\leq r\leq m}\frac{1}{|\lambda_r|} = \frac{1}{|\lambda_m|}, \quad \max_{m+1\leq s\leq n}\frac{1}{\lambda_s} = \frac{1}{\lambda_{m+1}}. \tag{5.9}$$

Drawing the m^{th} and the $(m+1)^{th}$ characteristic lines $x = x_m(t)$ and $x = x_{m+1}(t)$ passing through the point $(0,L)$ and (T,L), respectively, we have

$$\begin{cases} \dfrac{\mathrm{d}x_m(t)}{\mathrm{d}t} = \lambda_m, \\ x_m(0) = L \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\mathrm{d}x_{m+1}(t)}{\mathrm{d}t} = \lambda_{m+1}, \\ x_{m+1}(T) = L. \end{cases} \tag{5.10}$$

Noting (4.8), the triangular domain R_l, surrounded by $x = x_m(t)$, $x = x_{m+1}(t)$ and $x = L$, intersects $x = 0$ (see Figure 1).

By (2.56) in Theorem 3.1 for the Cauchy problem, the solution $\Phi = \widehat{\Phi}(t,x)$ on R_l is uniquely determined by (5.8). Then, noting the boundary conditions in (4.1), there exists $t = t^*$ $(0 < t^* < T)$ such that

$$\|\widehat{\Phi}(t^*,\cdot)\|_{(L^2(0,L))^n} \leq C\|\widehat{\Phi}(\cdot,L)\|_{(L^2(0,T))^n} \leq C\|\widehat{\Phi}^-(\cdot,L)\|_{(L^2(0,T))^m}. \tag{5.11}$$

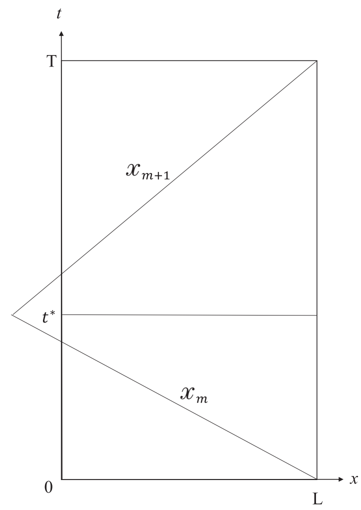

Figure 1 Triangular domain R_l

By (4.6)–(4.7), the boundary condition on $x = 0$ in (4.1) implies that there exists an $\overline{m} \times m$ matrix $\widetilde{G}_0$ such that

$$\Phi^+(t,0) = \widetilde{G}_0 \Phi^-(t,0), \quad \forall t \in (0,T) \tag{5.12}$$

(see (3.53)). Taking $\Phi = \widehat{\Phi}(t^*, x)$ $(x \in (0,L))$ as the initial data, we solve the following forward problem on $R_f = \{(t,x) \mid t^* < t < T,\ 0 < x < L\}$:

$$\begin{cases} \Phi_t + \Lambda \Phi_x - A^T \Phi = 0, & t \in (t^*, T),\ x \in (0,L), \\ x = 0 :\ \Phi^+ = \widetilde{G}_0 \Phi^-(t,0), & t \in (t^*, T), \\ x = L :\ \Phi^- = \widehat{\Phi}^-(t,L), & t \in (t^*, T), \\ t = t^* :\ \Phi = \widehat{\Phi}(t^*, x), & x \in (0,L). \end{cases} \tag{5.13}$$

By Theorem 3.5, $\Phi = \widehat{\Phi}(t,x)|_{R_f}$ is the solution to problem (5.13). By Theorem 3.1, we have

$$\|\widehat{\Phi}_T\|_{(L^2(0,L))^n} = \|\widehat{\Phi}(T,\cdot)\|_{(L^2(0,L))^n} \le C(\|\widehat{\Phi}(t^*,\cdot)\|_{(L^2(0,L))^n} + \|\widehat{\Phi}^-(\cdot,L)\|_{(L^2(0,T))^m}), \tag{5.14}$$

then, noting (5.11), we get immediately

$$\|\widehat{\Phi}_T\|_{(L^2(0,L))^n} \le C\|\widehat{\Phi}^-(\cdot,L)\|_{(L^2(0,T))^m}. \tag{5.15}$$

In particular, under assumption (5.5), D_1 is reversible, and Λ^- is also reversible, hence it follows from (5.15) that

$$\|\widehat{\Phi}_T\|_{(L^2(0,L))^n} \le C\|D_1^T \Lambda^- \widehat{\Phi}^-(\cdot,L)\|_{(L^2(0,T))^m}. \tag{5.16}$$

The proof is complete.

Remark 5.1 In Lemma 5.2, assumptions (4.6) and (4.7) guarantee that we can apply the constructive method with suitable boundary observations to get the observability for the adjoint system.

Remark 5.2 We can similarly discuss the two-sided strong exact boundary observability with fewer boundary observations.

Remark 5.3 The weak exact boundary observability for the adjoint system (4.1) can be similarly studied by means of the constructive method. In this case, the initial data of the solution to the backward adjoint problem (4.1)–(4.2) can be uniquely determined by boundary observations. In one-sided observation case, differently from Lemma 5.2, we don't need assumptions (4.6)–(4.7) for the weak exact boundary observability, boundary observations can be on either side of the boundary, and the number of boundary observations is equal to m or $\overline{m}$, respectively.

Correspondingly, we have

Lemma 5.3 (One-sided weak exact boundary observability) *Let $T \geq \overline{T}_0$, where $\overline{T}_0$ is given by* (4.8). *For any given initial data $\Phi_0(x) \in (L^2(0,L))^n$,*

(1) *the boundary observation $\Phi^-(t,L)$ corresponding to all the departing characteristics on $x = L$ on the interval $(0,T)$ can be used to uniquely determine the initial data $\Phi_0(x)$ at $t = 0$, and we have the following weak observability inequality*:

$$\|\Phi_0\|_{(L^2(0,L))^n} \leq C\|\Phi^-(\cdot,L)\|_{(L^2(0,T))^m}. \tag{5.17}$$

In particular, if $M = M_1 = \operatorname{rank}(D_1) = m$, then (5.17) *can be written as*

$$\|\Phi_0\|_{(L^2(0,L))^n} \leq C\|D_1^T\Lambda^-\Phi^-(\cdot,L)\|_{(L^2(0,T))^m}; \tag{5.18}$$

(2) *the boundary observation $\Phi^+(t,0)$ corresponding to all the departing characteristics on $x = 0$ on the interval $(0,T)$ can be used to uniquely determine the initial data $\Phi_0(x)$ at $t = 0$, and we have the following weak observability inequality*:

$$\|\Phi_0\|_{(L^2(0,L))^n} \leq C\|\Phi^+(\cdot,0)\|_{(L^2(0,T))^{\overline{m}}}. \tag{5.19}$$

In particular, if $M = M_0 = \operatorname{rank}(D_0) = \overline{m}$, then (5.19) *can be written as*

$$\|\Phi_0\|_{(L^2(0,L))^n} \leq C\|D_0^T\Lambda^+\Phi^+(\cdot,0)\|_{(L^2(0,T))^{\overline{m}}}. \tag{5.20}$$

Remark 5.4 Apparently, if system (4.1) possesses the strong exact boundary observability, then it must possess the weak exact boundary observability. Conversely, in order to obtain the one-sided strong exact boundary observability from the one-sided weak exact boundary observability, system (4.1) should satisfy (4.45), namely, it is time reversible. In this case, observations can be on either side of the boundary, and the number of boundary observations $M = m = \overline{m}$.

Lemma 5.4 *If system* (1.2)–(1.4) *is exactly null controllable under boundary controls $H(t) \in (L^2(0,T))^M$ $(M = M_0 + M_1 \leq n)$, then the adjoint system* (4.1) *satisfies the following weak D_0/D_1-observability*:

$$\text{If}\ \ D_0^T\Lambda^+\Phi^+(t,0) = 0 \quad \text{and} \quad D_1^T\Lambda^-\Phi^-(t,L) = 0, \quad \text{then}\ \ \Phi(0,x) \equiv 0. \tag{5.21}$$

Proof By Definition 3.1, and noting the adjoint system (4.1), if system (1.2)–(1.4) is exactly boundary null controllable, then for any given initial data $U_0(x) \in (L^2(0, L))^n$ and null final data $U_T(x) \equiv 0$, there exist boundary controls $H(t) \in (L^2(0, T))^M$, such that

$$\begin{aligned}&\int_0^L \Phi(0, x)^T U_0(x)\mathrm{d}x\\ = -&\int_0^T (\Phi^+)^T(t, 0)\Lambda^+ D_0 H^+(t)\mathrm{d}t + \int_0^T (\Phi^-)^T(t, L)\Lambda^- D_1 H^-(t)\mathrm{d}t.\end{aligned} \tag{5.22}$$

If $D_0^T \Lambda^+\Phi^+(t, 0) = 0$ and $D_1^T \Lambda^-\Phi^-(t, L) = 0$, then the left-hand side of (5.22) is equal to zero for any given initial data $U_0(x)$ in $(L^2(0, L))^n$, thus $\Phi(0, t) \equiv 0$. The proof is complete.

Combining Lemmas 5.3–5.4, we have

Corollary 5.1 *If the number of boundary observations can not be reduced for the one-sided weak exact boundary observability for the adjoint system* (4.1)*, then the number of boundary controls can not be reduced for the one-sided exact boundary null controllability for system* (1.2)–(1.4)*.*

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