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Second Main Theorem for Meromorphic Maps into Algebraic Varieties Intersecting Moving Hypersurfaces Targets*

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Abstract Since the great work on holomorphic curves into algebraic varieties intersecting hypersurfaces in general position established by Ru in 2009, recently there has been some developments on the second main theorem into algebraic varieties intersecting moving hypersurfaces targets. The main purpose of this paper is to give some interesting improvements of Ru's second main theorem for moving hypersurfaces targets located in subgeneral position with index.

Keywords Algebraic varieties, Holomorphic curves, Nevanlinna theory, Moving hypersurfaces

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1 Introduction and Main Results

It is well-known that in 1933, Cartan generalized Nevanlinna theory for meromorphic functions to the case of linearly nondegenerate holomorphic curves into complex projective spaces intersecting hyperplanes in general position, and conjectured that it is still true for moving hyperplanes targets. From then on, higher dimensional Nevanlinna theory has been studied widely (see [7, 11, 16]). In 2009, Ru [13] proposed a great work on second main theorem of algebraically nondegenerate holomorphic curves into smooth complex varieties intersecting hypersurfaces in general position, which is a generalization of the Cartan's second main theorem and his own former result (see [12]) completely solving the Shiffman's conjecture (see [14]) corresponding to the Corvaja-Zannier's theorem (see [1]) in Diophantine approximation (see [4]).

Thus, it is natural and interesting to investigate the Ru's second main theorem into complex projective spaces and even into complex algebraic varieties for the moving hypersurfaces targets. Based on their affirmation of the Shiffman's conjecture for moving hypersurfaces targets (see [2]), recently, Dethloff and Tan [3] continue to prove successfully the following special case where the coefficients of the polynomials Q_j's are constant and the variety V is smooth, namely, the Ru's second main theorem (see [13]).

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Theorem 1.1 (see [3]) *Let $V \subset \mathbb{P}^n(\mathbb{C})$ be an irreducible (possibly singular) variety of dimension ℓ, and let f be a nonconstant holomorphic map of $\mathbb{C}$ into V. Let $\mathcal{D} = \{D_1, \cdots, D_q\}$ be a family of slowly moving hypersurfaces (with respect to f) in general position, and let $\mathcal{Q} = \{Q_1, \cdots, Q_q\}$ be the set of the defined homogeneous polynomials of $\mathcal{D}$ with $\deg Q_j = d_j$ $(j = 1, \cdots, q)$ and $Q_j(f) \not\equiv 0$ for $j = 1, \cdots, q$. Assume that f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$. Then, for any $\varepsilon > 0$,*

$$\sum_{j=1}^{q} \Big(\frac{1}{d_j}\Big) m_f(r, D_j) \le (\ell + 1 + \varepsilon) T_f(r) \tag{1.1}$$

holds for all r outside a set with finite Lebesgue measure.

For the special case $V = \mathbb{P}^n(\mathbb{C})$, Quang [8] recently gave a second main theorem with truncated counting functions for meromorphic mappings into $\mathbb{P}^n(\mathbb{C})$ intersecting a family of moving hypersurfaces in subgeneral position, which any be possibly good at the uniqueness problems of meromorphic mappings.

Theorem 1.2 (see [8]) *Let f be a nonconstant meromorphic map of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $\{Q_i\}_{i=1}^q$ be a collection of slowly moving hypersurfaces in N-subgeneral position with $\deg Q_j = d_j$ $(1 \le j \le q)$. Assume that f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$. Then, for any $\varepsilon > 0$,*

$$(q - (N - n + 1)(n + 1) - \varepsilon) T_f(r) \le \sum_{i=1}^{q} \frac{1}{d_j} N^{[L_0]}(r, f^* Q_i) + o(T_f(r)) \tag{1.2}$$

holds for all r outside a set with finite Lebesgue measure, where

$$L_0 := \binom{L+n}{n} p_0^{\binom{L+n}{n}\left(\binom{L+n}{n}-1\right)\binom{q}{n}-2} - 1$$

with

$$L := (n + 1)d + 2(N - n + 1)(n + 1)^3 I(\varepsilon^{-1})d,$$

where $d := \operatorname{lcm}(d_1, \cdots, d_q)$ is the least common multiple of all $\{d_i\}$, and

$$p_0 := \Big[\frac{\binom{L+n}{n}\big(\binom{L+n}{n} - 1\big)\binom{q}{n} - 1}{\log\big(1 + \frac{\varepsilon}{3(n+1)(N-n+1)}\big)}\Big]^2.$$

In this paper, we mainly combine the methods in [3, 8, 17] together and adopt the new concept of the index of subgeneral position due to Ji-Yan-Yu [6] to obtain some interesting developments of Ru's second main theorem for moving hypersurfaces targets, which are improvements and extensions of Theorems 1.1–1.2.

According to [6], we can give a similar definition for moving hypersurfaces located in m-subgeneral position with index k.

Definition 1.1 *Let V be an algebraic subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $\{D_1, \cdots, D_q\}$ be a family of moving hypersurfaces of $\mathbb{P}^n(\mathbb{C})$. Let N and κ be two positive integers such that $N \ge \dim V \ge \kappa$.*

(a) *The hypersurfaces $\{D_1, \cdots, D_q\}$ are said to be in general position (or say in weakly general position) in V if there exists $z \in \mathbb{C}^m$ (if this condition is satisfied for one $z \in \mathbb{C}^m$,* *it is also satisfied for all* z *except for an analytic set of codimension* ≥ 2) *for any subset* $I \subset \{1, \cdots, q\}$ *with* $\sharp I \leq \dim V + 1$,

$$\operatorname{codim}\Big(\bigcap_{i\in I} D_i(z) \cap V\Big) \geq \sharp I.$$

(b) *The hypersurfaces* $\{D_1, \cdots, D_q\}$ *are said to be in* N*-subgeneral position in* V *if there exists* $z \in \mathbb{C}^m$ (*if this condition is satisfied for one* $z \in \mathbb{C}^m$, *it is also satisfied for all* z *except for an analytic set of codimension* ≥ 2) *for any subset* $I \subset \{1, \cdots, q\}$ *with* $\sharp I \leq N + 1$,

$$\dim\Big(\bigcap_{i\in I} D_i(z) \cap V\Big) \leq N - \sharp I.$$

(c) *The hypersurfaces* $\{D_1, \cdots, D_q\}$ *are said to be in* N*-general position with index* κ *in* V *if* $D_1, \cdots, D_q$ *are in* N*-subgeneral position and if there exists* $z \in \mathbb{C}^m$ (*if this condition is satisfied for one* $z \in \mathbb{C}^m$, *it is also satisfied for all* z *except for an analytic set of codimension* ≥ 2) *for any subset* $I \subset \{1, \cdots, q\}$ *with* $\sharp I \leq \kappa$,

$$\operatorname{codim}\Big(\bigcap_{i\in I} D_i(z) \cap V\Big) \geq \sharp I$$

(*Here we set* $\dim \emptyset = -\infty$).

Now, we state our main result which is an improvement and extension of the above two theorems concerning moving hypersurfaces targets located in subgeneral position with index. Theorem 1.1 is just the following result for the special case whenever $N = \dim V$ and $\kappa = 1$.

Theorem 1.3 *Let* $f : \mathbb{C}^m \to V \subset \mathbb{P}^n(\mathbb{C})$ *be a nonconstant meromorphic map, where* V *is an irreducible algebraic subvariety of dimension* ℓ. *Let* $\mathcal{Q} = \{Q_1, \cdots, Q_q\}$ *be a collection of slowly moving hypersurfaces in* N*-subgeneral position with index* κ *in* V, *and* $\deg Q_j = d_j$ ($j = 1, \cdots, q$). *Assume that* $f : \mathbb{C}^m \to V$ *is algebraically nondegenerate over* $\mathcal{K}_{\mathcal{Q}}$. *Then, for any* $\varepsilon > 0$,

$$\begin{aligned}&\Big(q - \Big(1 + \frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}}\Big)(\ell+1) - \varepsilon\Big)T_f(r)\\ &\leq \sum_{i=1}^{q} \frac{1}{d_j} N(r, f^*Q_i) + o(T_f(r))\end{aligned} \tag{1.3}$$

holds for all r *outside a set with finite Lebesgue measure.*

When $V = P^n(\mathbb{C})$, we can have the following second main theorem with truncation, and thus Theorem 1.2 is just the special case whenever $\kappa = 1$.

Theorem 1.4 *Let* f *be a nonconstant meromorphic map of* $\mathbb{C}^m$ *into* $\mathbb{P}^n(\mathbb{C})$. *Let* $\{Q_i\}_{i=1}^q$ *be a collection of slowly moving hypersurfaces in* N*-subgeneral position with index* κ, *and* $\deg Q_j = d_j$ ($1 \leq j \leq q$). *Assume that* f *is algebraically nondegenerate over* $\mathcal{K}_{\mathcal{Q}}$. *Then, for any* $\varepsilon > 0$,

$$\begin{aligned}&\Big(q - \Big(1 + \frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}}\Big)(n+1) - \varepsilon\Big)T_f(r)\\ &\leq \sum_{i=1}^{q} \frac{1}{d_j} N^{[L_0]}(r, f^*Q_i) + o(T_f(r))\end{aligned} \tag{1.4}$$

holds for all r outside a set with finite Lebesgue measure, where

$$L_0 := \binom{L+n}{n} p_0^{\binom{L+n}{n}\left(\binom{L+n}{n}-1\right)\binom{q}{n}-2} - 1$$

with

$$L := (n+1)d + 2\Big(1 + \frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}}\Big)(n+1)^3 I(\varepsilon^{-1})d,$$

where $d := \mathrm{lcm}(d_1, \cdots, d_q)$ *is the least common multiple of all* $\{d_i\}$, *and*

$$p_0 := \Big[\frac{\binom{L+n}{n}(\binom{L+n}{n}-1)\binom{q}{n}-1}{\log(1+\frac{\varepsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})})}\Big]^2.$$

Remark that very recently, Yan and Yu [17] considered the nonconstant holomorphic curve from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ instead of algebraically nondegenerate and improved Theorem 1.2 without truncation. Thus it is interesting to ask the following question.

Question 1.1 In Theorem 1.3 or Theorem 1.4, is it possible to obtain a second main theorem if the condition "f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$" is omitted?

The remainder is the organization as follows. In the next section, we introduce some basic notions and auxiliary results from Nevanlinna theory. Sections 3–4 are the proofs of Theorems 1.3–1.4, respecitively, in which the methods to deal with moving targets by Dethloff-Tan [3], Yan-Yu [17], and the techniques to deal with hypersurfaces in subgeneral position instead of Nochka's weights owing to Quang [8–9] are used in this paper.

2 Basic Notions and Auxiliary Results from Nevanlinna Theory

2.1 The characteristic function in Nevanlinna theory

We set $\|z\| = (\|z_1\|^2 + \cdots + \|z_m\|^2)^{\frac{1}{2}}$ for $z = (z_1, \cdots, z_m) \in \mathbb{C}^m$ and define

$$B(r) := \{z \in \mathbb{C}^m : \|z\| < r\}, \quad S(r) := \{z \in \mathbb{C}^m : \|z\| = r\}, \quad 0 < r < \infty.$$

Define

$$\begin{aligned} &v_{m-1}(z) := (dd^c\|z\|^2)^{m-1}, \\ &\sigma_m(z) := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1} \quad \text{on } \mathbb{C}^m \backslash \{0\}. \end{aligned}$$

Let F be a nonzero meromorphic function on a domain Ω in $\mathbb{C}^m$. For a set $\alpha = (\alpha_1, \cdots, \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and

$$\mathcal{D}^\alpha F = \frac{\partial^{|\alpha|} F}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_m} z_m}.$$

We denote by ν_F^0, ν_F^∞ and ν_F the zero divisor, the pole divisor, and the divisor of the meromorphic function F, respectively.

For a divisor ν on $\mathbb{C}^m$ and for a positive integer M or $M = \infty$, we set

$$\nu^{[M]}(z) = \min\{M, \nu(z)\},$$

$$n(t) = \begin{cases} \displaystyle\int_{|\nu| \cap B(t)} \nu(z) v_{m-1}, & \text{if } m \geq 2, \\ \displaystyle\sum_{|z| \leq t} \nu(z), & \text{if } m = 1. \end{cases}$$

The counting function of ν is defined by

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2m-1}} \mathrm{d}t, \quad 1 < r < \infty.$$

Similarly, we define $N(r, \nu^{[M]})$ and denote it by $N^{[M]}(r, \nu)$.

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. Define

$$N_\varphi(r) = N(r, \nu_\varphi^0), \quad N_\varphi^{[M]}(r) = N^{[M]}(r, \nu_\varphi^0).$$

For brevity we will omit the character $[M]$ if $M = \infty$.

Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbb{P}^n(\mathbb{C})$, we take a reduced representation $\widetilde{f} = (f_0, \cdots, f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z), \cdots, f_n(z))$ outside the analytic set $I(f) = \{f_0 = \cdots = f_n = 0\}$ of codimension ≥ 2. Set $\|\widetilde{f}\| = (\|f_0\|^2 + \cdots + \|f_n\|^2)^{\frac{1}{2}}$. The characteristic function of f is defined by

$$T_f(r) = \int_{S(r)} \log \|\widetilde{f}\| \sigma_m - \int_{S(1)} \log \|\widetilde{f}\| \sigma_m.$$

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$, which is occasionally regarded as a meromorphic map into $\mathbb{P}^1(\mathbb{C})$. The proximity function of φ is defined by

$$m(r, \varphi) := \int_{S(r)} \log \max(|\varphi|, 1) \sigma_m.$$

The Nevanlinna's characteristic function of φ is defined as follows

$$T(r, \varphi) := N_{\frac{1}{\varphi}}(r) + m(r, \varphi).$$

Then

$$T_\varphi(r) = T(r, \varphi) + O(1).$$

The function φ is said to be small (with respect to f) if $\| T_\varphi(r) = o(T_f(r))$, here the notion $\|$ means that the property holds possibly outside a set with finite Lebesgue measure.

We denote by $\mathcal{M}$ (resp. $\mathcal{K}_f$) the field of all meromorphic functions (resp. small meromorphic functions with respect to f) on $\mathbb{C}^m$.

2.2 Family of moving hypersurfaces and the first main theorem

Denote by $\mathcal{H}_{\mathbb{C}^m}$ the ring of all holomorphic functions on $\mathbb{C}^m$. Let Q be a homogeneous polynomial in $\mathcal{H}_{\mathbb{C}^m}[x_0, \cdots, x_n]$ of degree $d \geq 1$. Denote by $Q(z)$ the homogeneous polynomial over $\mathbb{C}$ obtained by substituting a specific point $z \in \mathbb{C}^m$ into the coefficients of Q. We also call a moving hypersurface in $\mathbb{P}^n(\mathbb{C})$ a homogeneous polynomial $Q \in \mathcal{H}_{\mathbb{C}^m}[x_0, \cdots, x_n]$ such that the common zero set of all coefficients of Q has codimension at least two.

A moving hypersurface Q in $\mathbb{P}^n(\mathbb{C})$ of degree d (≥ 1) is defined by

$$Q(z) = \sum_{I\in\mathcal{I}_d} a_I(z)\omega^I,$$

where $\mathcal{I}_d = \{(i_0,\cdots,i_n) \in \mathbf{N}_0^{n+1}\,;\ i_0+\cdots+i_n = d\}$, $a_I \in \mathcal{H}_{\mathbb{C}^m}$ and $\omega^I = \omega_0^{i_0}\cdots\omega_n^{i_n}$. We consider the meromorphic mapping $Q' : \mathbb{C}^m \to \mathbb{P}^N(\mathbb{C})$, where $N = \binom{n+d}{n}$, given by

$$Q'(z) = (a_{I_0}(z) : \cdots : a_{I_N}(z)) \quad (\mathcal{I}_d = \{I_0,\cdots,I_N\}).$$

Here $I_0 < \cdots < I_N$ in the lexicographic ordering. By changing the homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$ if necessary, we may assume that for each given moving hypersurface as above, $a_{I_0} \neq 0$ (note that $I_0 = (0,\cdots,0,d)$ and a_{I_0} is the coefficient of ω_n^d). We set

$$\widetilde{Q} = \sum_{j=0}^N \frac{a_{I_j}}{a_{I_0}}\omega^{I_j}.$$

The moving hypersurfaces $\mathcal{Q} = \{Q_1,\cdots,Q_q\}$ is said to be "slow" (with respect to f) if $\|T_{Q'}(r) = o(T_f(r))$. This is equivalent to $\|T_{\frac{a_{I_j}}{a_{I_0}}}(r) = o(T_f(r))$ $(\forall 1\leq j\leq N)$, i.e., $\frac{a_{I_j}}{a_{I_0}} \in \mathcal{K}_f$. Let $\{Q_i\}_{i=1}^q$ be a family of moving hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, $\deg Q_i = d_i$. Assume that

$$Q_i = \sum_{I\in\mathcal{I}_{d_i}} a_{iI}\omega^I.$$

We denote by $\mathcal{K}_{\mathcal{Q}}$ the smallest subfield of meromorphic function field $\mathcal{M}$ which contains $\mathbb{C}$ and all $\frac{a_{iI_s}}{a_{iI_t}}$, where $a_{iI_t} \not\equiv 0$, $i \in \{1,\cdots,q\}$, $I_t, I_s \in \mathcal{I}_{d_i}$. We say that f is linearly nondegenerate over $\mathcal{K}_{\mathcal{Q}}$ if there is no nonzero linear form $L \in \mathcal{K}_{\mathcal{Q}}[x_0,\cdots,x_n]$ such that $L(f_0,\cdots,f_n) \equiv 0$, and f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$ if there is no nonzero homogeneous polynomial $Q \in \mathcal{K}_{\mathcal{Q}}[x_0,\cdots,x_n]$ such that $Q(f_0,\cdots,f_n) \equiv 0$.

Let f and Q be as above. The proximity function of f with respect to Q, denoted by $m_f(r,Q)$, is defined by

$$m_f(r,Q) = \int_{S(r)} \lambda_D(\widetilde{f})\sigma_m - \int_{S(1)} \lambda_D(\widetilde{f})\sigma_m,$$

where $Q(\widetilde{f}) = Q(f_0,\cdots,f_n)$, and $\lambda_Q(\widetilde{f}) = \log\frac{\|\widetilde{f}\|^d\|Q\|}{|Q(\widetilde{f})|}$ is the Weil function and $\|Q\| = \max_{I\in\mathcal{I}_d}\{|a_I|\}$, where $\mathcal{I}_d := \{I = (i_0,\cdots,i_n) \in \mathbb{Z}_{\geq 0}^{n+1}\,;\ i_0+\cdots+i_n = d\}$. This definition is independent of the choice of the reduced representation of f.

We denote by f^*Q the pullback of the divisor Q by f. We may see that f^*Q identifies with the zero divisor $\nu^0_{Q(\widetilde{f})}$ of the function $Q(\widetilde{f})$. By Jensen's fomular, we have

$$N_{Q(\widetilde{f})}(r) = \int_{S(r)} \log|Q(\widetilde{f})|\sigma_m - \int_{S(1)} \log|Q(\widetilde{f})|\sigma_m.$$

For convenience, we will denote $N(r, f^*Q) = N_{Q(f)}(r)$.

Theorem 2.1 (First Main Theorem) (see [9]) *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, and let Q be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. If $f(\mathbb{C}^m) \not\subset Q$, then for every real number r with $0 < r < +\infty$,*

$$dT_f(r) = m_f(r,Q) + N_{Q(f)}(r) + O(1),$$

where $O(1)$ is a constant independent of r.

2.3 Some theorems and lemmas

Let f be a nonconstant meromorphic map of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Denote by $\mathcal{C}_f$ the set of all non-negative functions $h : \mathbb{C}^m \backslash A \to [0, +\infty] \subset \overline{\mathbf{R}}$, which are of the form

$$\frac{|g_1| + \cdots + |g_l|}{|g_{l+1}| + \cdots + |g_{l+k}|},$$

where $k, l \in \mathbf{N}, g_1, \cdots, g_{l+k} \in \mathcal{K}_f \backslash \{0\}$ and $A \subset \mathbb{C}^m$, which may depend on $g_1, \cdots, g_{l+k}$, is an analytic subset of codimension at least two. Then, for $h \in \mathcal{C}_f$ we have

$$\int_{S(r)} \log h \sigma_m = o(T_f(r)).$$

Since $\mathcal{Q} = \{Q_1, \cdots, Q_q\}$ are in N-subgeneral position, we have the following lemma.

Lemma 2.1 (see [2, 8, 17]) *For any $Q_{j_1}, \cdots, Q_{j_{N+1}} \in \mathcal{Q}$, there exist functions $h_1, h_2 \in \mathcal{C}_f$ such that*

$$h_2 \|f\|^d \le \max_{k \in \{1, \cdots, N+1\}} |Q_{j_k}(f_0, \cdots, f_n)| \le h_1 \|f\|^d.$$

Lemma 2.2 (Lemma on Logarithmic Derivative) (see [6]) *Let f be a nonzero meromorphic function on $\mathbb{C}^m$.*

$$\Big\| m\Big(r, \frac{\mathcal{D}^\alpha(f)}{f}\Big) = O(\log^+ T(r, f)) \quad (\alpha \in \mathbf{Z}^m_+).$$

Proposition 2.1 (see [5]) *Let $\Phi_1, \cdots, \Phi_k$ be meromorphic functions on $\mathbb{C}^m$ such that $\{\Phi_1, \cdots, \Phi_k\}$ is linearly independent over $\mathbb{C}$. Then there exists an admissible set*

$$\{\alpha_i = (\alpha_{i1}, \cdots, \alpha_{im})\}_{i=1}^k \subset \mathbf{Z}^m_+$$

with $|\alpha_i| = \sum\limits_{j=1}^m |\alpha_{ij}| \le i - 1$ $(1 \le i \le k)$ such that the following are satisfied:

(i) $\{\mathcal{D}^{\alpha_i}\Phi_1, \cdots, \mathcal{D}^{\alpha_i}\Phi_k\}_{i=1}^k$ *is linearly independent over $\mathcal{M}$, i.e.,* $\det(\mathcal{D}^{\alpha_i}\Phi_j) \not\equiv 0$,

(ii) $\det(\mathcal{D}^{\alpha_i}(h\Phi_j)) = h^k \cdot \det(\mathcal{D}^{\alpha_i}\Phi_j)$ *for any nonzero meromorphic function h on $\mathbb{C}^m$.*

Theorem 2.2 (see [10, Theorem 2.31]) *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with a reduced representation $\widetilde{f} = (f_0, \cdots, f_n)$ and let $H_1, \cdots, H_q$ be q arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Then we have*

$$\Big\| \int_{S(r)} \max_K \log\Big(\prod_{j \in K} \frac{\|\widetilde{f}\| \cdot \|H_j\|}{|H_j(\widetilde{f})|}\Big)\sigma_m \le (n+1)T_f(r) - N_{W^\alpha(f_i)}(r) + o(T_f(r)),$$

where α is an admissible set with respect to $\widetilde{f}$ (as in Proposition 2.1) and the maximum is taken over all subsets $K \subset \{1, \cdots, q\}$ such that $\{H_j\,;\ j \in K\}$ is linearly independent.

3 Proof of Theorem 1.3

Firstly, we may assume that $Q_1, \cdots, Q_q$ have the same degree $\deg Q_j = d_j = d$. Set

$$Q_j(z) = \sum_{I \in \mathcal{I}_d} a_{jI}(z)\mathbf{x}^I, \quad j = 1, \cdots, q.$$

For each j, there exists $a_{jI_j}(z)$, one of the coefficients in $Q_j(z)$, such that $a_{jI_j}(z)\not\equiv 0$. We fix this a_{jI_j}, then set $\widetilde{a}_{jI}(z)=\frac{a_{jI}(z)}{a_{jI_j}(z)}$ and

$$\widetilde{Q}_j(z)=\sum_{I\in\mathcal{I}_d}\widetilde{a}_{jI}(z)\mathbf{x}^I,$$

which is a homogeneous polynomial in $\mathcal{K}_{\mathcal{Q}}[x_0,\cdots,x_n]$. By definition of the proximity function and Weil function, we have

$$\lambda_{\widetilde{Q}_j}(f)=\log\frac{\|f\|^d\|Q_j\|}{|Q_j(f)|}=\log\frac{\|f\|^d\|\widetilde{Q}_j\|}{|\widetilde{Q}_j(f)|}$$

for $j=1,\cdots,q$.

For a fixed point $z\in\mathbb{C}^m\setminus\bigcup_{i=1}^{q}\widetilde{Q}_i(\widetilde{f})^{-1}(\{0,\infty\})$. We may assume that there exists a renumbering $\{1,\cdots,q\}$ such that

$$|\widetilde{Q}_{1(z)}(\widetilde{f})(z)|\le|\widetilde{Q}_{2(z)}(\widetilde{f})(z)|\le\cdots\le|\widetilde{Q}_{q(z)}(\widetilde{f})(z)|.$$

By Lemma 2.1, we have $\max_{j\in\{1,\cdots,N+1\}}|\widetilde{Q}_{j(z)}(\widetilde{f})(z)|=|\widetilde{Q}_{N+1(z)}(\widetilde{f})(z)|\ge h\|\widetilde{f}\|^d$ for some $h\in\mathcal{C}_f$, i.e.,

$$\frac{\|\widetilde{f}(z)\|^d}{|\widetilde{Q}_{N+1(z)}(\widetilde{f})(z)|}\le\frac{1}{h(z)}.$$

Hence

$$\prod_{j=1}^{q}\frac{\|\widetilde{f}(z)\|^d}{|\widetilde{Q}_j(\widetilde{f})(z)|}\le\frac{1}{h^{q-N}(z)}\prod_{j=1}^{N}\frac{\|\widetilde{f}(z)\|^d}{|\widetilde{Q}_{j(z)}(\widetilde{f})(z)|}. \tag{3.1}$$

Let $\mathcal{K}_{\mathcal{Q}}$ be an arbitrary field over $\mathbb{C}^m$ generated by a set of meromorphic function on $\mathbb{C}^m$. Let V be a sub-variety in $\mathbb{P}^n(\mathbb{C})$ of dimension ℓ defined by the homogenous ideal $I(V)\subset\mathbb{C}[x_0,\cdots,x_n]$. Denote by $I_{\mathcal{K}_{\mathcal{Q}}}(V)$ the ideal in $\mathcal{K}_{\mathcal{Q}}[x_0,\cdots,x_n]$ generated by $I(V)$. Since $f:\mathbb{C}^m\to V\subset\mathbb{P}^n(\mathbb{C})$ is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$, there is no homogeneous polynomial $P\in\mathcal{K}_{\mathcal{Q}}[x_0,\cdots,x_n]/I_{\mathcal{K}_{\mathcal{Q}}}(V)$ such that $P(f_0,\cdots,f_n)\equiv 0$.

For a positive integer L, let $\mathcal{K}_{\mathcal{Q}}[x_0,\cdots,x_n]_L$ be the vector space of homogeneous polynomials of degree L, and let $I_{\mathcal{K}_{\mathcal{Q}}}(V)_L:=I_{\mathcal{K}_{\mathcal{Q}}}(V)\cap\mathcal{K}_{\mathcal{Q}}[x_0,\cdots,x_n]_L$. Set

$$V_L:=\mathcal{K}_{\mathcal{Q}}[x_0,\cdots,x_n]_L/I_{\mathcal{K}_{\mathcal{Q}}}(V)_L.$$

Denote by $[g]$ the projection of g in V_L. We have the following basic fact from the theory of Hilbert polynomials (see [15]).

Lemma 3.1 *$M:=\dim_{\mathcal{K}_{\mathcal{Q}}}V_L=\frac{\Delta L^\ell}{\ell!}+\rho(L)$, where $\rho(L)$ is an $O(L^{\ell-1})$ function depending on the variety V. Moreover, there exists an integer L_0 such that $\rho(L)$ is a polynomial function of L for $L>L_0$.*

Next, we prove the following lemma concerning on the hypersurfaces located in N-subgeneral position with index κ, which plays the role in this paper. The method of it is originally from Quang [9–10].

Lemma 3.2 *Let $\widetilde{Q}_1, \cdots, \widetilde{Q}_{N+1}$ be homogeneous polynomials in $\mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]$ of the same degree $d \geq 1$, in (weakly) N-subgeneral position with index κ in V. For each point $a \in \mathbb{C}^m$ satisfying the following conditions*:

(i) *The coefficients of $\widetilde{Q}_1, \cdots, \widetilde{Q}_{N+1}$ are holomorphic at a;*

(ii) *$\widetilde{Q}_1(a), \cdots, \widetilde{Q}_{N+1}(a)$ have no non-trivial common zeros in V;*

then there exist homogeneous polynomials $\widetilde{P}_1(a) = \widetilde{Q}_1(a), \cdots, \widetilde{P}_\kappa(a) = \widetilde{Q}_\kappa(a), \widetilde{P}_{\kappa+1}(a), \cdots, \widetilde{P}_{\ell+1}(a) \in \mathbb{C}[x_0, \cdots, x_n]$ with

$$\widetilde{P}_t(a) = \sum_{j=\kappa+1}^{N-\ell+t} c_{tj}\widetilde{Q}_j(a), \quad c_{tj} \in \mathbb{C}, \ t \geq \kappa+1$$

such that

$$\Big(\bigcap_{t=1}^{\ell+1} \widetilde{P}_t(a)\Big) \cap V = \emptyset.$$

Proof We assume that $\widetilde{Q}_i$ $(1 \leq i \leq N+1)$ has the following form

$$\widetilde{Q}_i = \sum_{I \in \tau_d} a_{iI}\omega^I.$$

By the definition of the N-subgeneral position, there exists a point $a \in \mathbb{C}$ such that the following system of equations

$$\widetilde{Q}_i(a)(\omega_0, \cdots, \omega_n) = 0, \quad 1 \leq i \leq N+1$$

has only trivial solution $(0, \cdots, 0)$. We may assume that $\widetilde{Q}_i(a) \not\equiv 0$ for all $1 \leq i \leq N+1$.

For each homogeneous polynomial $\widetilde{Q} \in \mathbb{C}[x_0, \cdots, x_n]$, we denote by D the fixed hypersurface in $\mathbb{P}^n(\mathbb{C})$ defined by $\widetilde{Q}$, i.e.,

$$D = \{(\omega_0, \cdots, \omega_n) \in \mathbb{P}^n(\mathbb{C}) \mid \widetilde{Q}(\omega_0, \cdots, \omega_n) = 0\}.$$

Setting $\widetilde{P}_1(a) = \widetilde{Q}_1(a), \cdots, \widetilde{P}_\kappa(a) = \widetilde{Q}_\kappa(a)$, we see that

$$\dim\Big(\bigcap_{i=1}^{t} D_i(a) \cap V\Big) \leq N - t, \quad t = N - \ell + \kappa + 1, \cdots, N+1,$$

where $\dim \emptyset = -\infty$. According to the definition of N-subgeneral position with index κ, we have $\dim\big(\bigcap_{j=1}^{\kappa} D_j(a) \cap V\big) \leq \ell - \kappa$.

Step 1 We will construct $\widetilde{P}_{\kappa+1}$ as follows. For each irreducible componet Γ of dimension $\ell - \kappa$ of $\big(\bigcap_{i=1}^{\kappa} \widetilde{P}_i(a) \cap V\big)$, we put

$$V_{1\Gamma} = \Big\{c = (c_{\kappa+1}, \cdots, c_{N-\ell+\kappa+1}) \in \mathbb{C}^{N-\ell+1}, \Gamma \subset D_c(a), \\ \text{where } \widetilde{Q}_c = \sum_{j=\kappa+1}^{N-\ell+\kappa+1} c_j\widetilde{Q}_j\Big\}.$$

By definition, $V_{1\Gamma}$ is a subspace of $\mathbb{C}^{N-\ell+1}$. Since

$$\dim\Big(\bigcap_{i=1}^{N-\ell+\kappa+1} D_i(a) \cap V\Big) \leq \ell - \kappa - 1,$$

there exists $i \in \{\kappa+1, \cdots, N-\ell+\kappa+1\}$ such that $\Gamma \not\subset D_i(a)$. This implies that $V_{1\Gamma}$ is a proper subspace of $\mathbb{C}^{N-\ell+1}$. Since the set of irreducible components of dimension $\ell-\kappa$ of $\big(\bigcap\limits_{i=1}^{\kappa}\widetilde{P}_i(a)\cap V\big)$ is at most countable, we have

$$\mathbb{C}^{N-\ell+\kappa}\setminus\bigcup_{\Gamma}V_{1\Gamma}\neq\emptyset.$$

Hence, there exists $(c_{1(\kappa+1)}, \cdots, c_{1(N-\ell+\kappa+1)}) \in \mathbb{C}^{N-\ell+1}$ such that $\Gamma \not\subset \widetilde{P}_{\kappa+1}(a)$, where $\widetilde{P}_{\kappa+1} = \sum\limits_{j=\kappa+1}^{N-\ell+\kappa+1} c_{1j}\widetilde{Q}_j$, for all irreducible components of dimension $\ell-\kappa$ of $\big(\bigcap\limits_{i=1}^{\kappa}\widetilde{P}_i(a)\cap V\big)$. This clearly implies that

$$\dim\Big(\bigcap_{i=1}^{\kappa+1}\widetilde{P}_i(a)\cap V\Big)\leq \ell-(\kappa+1).$$

Step 2 We will construct $\widetilde{P}_{\kappa+2}$ as follows. For each irreducible componet Γ' of dimension $\ell-\kappa-1$ of $\big(\bigcap\limits_{i=1}^{\kappa+1}\widetilde{P}_i(a)\cap V\big)$, we put

$$\begin{aligned}V_{2\Gamma'} = \{c = (c_{\kappa+1}, \cdots, c_{N-\ell+\kappa+2}) \in \mathbb{C}^{N-\ell+2}, \Gamma' \subset D_c(a),\\ \text{where } \widetilde{Q}_c = \sum_{j=\kappa+1}^{N-\ell+\kappa+2} c_j\widetilde{Q}_j\}.\end{aligned}$$

Then $V_{2\Gamma'}$ is a subspace of $\mathbb{C}^{N-\ell+2}$. Since $\dim\big(\bigcap\limits_{i=1}^{N-\ell+\kappa+2} D_i(a)\cap V\big)\leq \ell-\kappa-2$, there exists $i \in \{\kappa+1, \cdots, N-\ell+\kappa+2\}$ such that $\Gamma' \not\subset D_i(a)$. This implies that $V_{2\Gamma'}$ is a proper subspace of $\mathbb{C}^{N-\ell+2}$. Since the set of irreducible components of dimension $\ell-\kappa-1$ of $\big(\bigcap\limits_{i=1}^{\kappa+1}\widetilde{P}_i(a)\cap V\big)$ is at most countable, we also have

$$\mathbb{C}^{N-\ell+\kappa+1}\setminus\bigcup_{\Gamma'}V_{2\Gamma'}\neq\emptyset.$$

Hence, there exists $(c_{2(\kappa+1)}, \cdots, c_{2(N-\ell+\kappa+2)}) \in \mathbb{C}^{N-\ell+2}$ such that $\Gamma' \not\subset \widetilde{P}_{\kappa+2}(a)$, where $\widetilde{P}_{\kappa+2} = \sum\limits_{j=\kappa+1}^{N-\ell+\kappa+2} c_{2j}\widetilde{Q}_j$, for all irreducible components of dimension $\ell-\kappa-1$ of $\big(\bigcap\limits_{i=1}^{\kappa+1}\widetilde{P}_i(a)\cap V\big)$. This clearly implies that

$$\dim\Big(\bigcap_{i=1}^{\kappa+2}\widetilde{P}_i(a)\cap V\Big)\leq \ell-(\kappa+2).$$

Repeat again the above steps, after $(\ell+1-\kappa)$-th step we get the hypersurfaces $\widetilde{P}_{\kappa+1}(a)$, $\cdots$, $\widetilde{P}_{\ell+1}(a)$ satisfying that

$$\dim\Big(\bigcap_{j=1}^{t}\widetilde{P}_j(a)\cap V\Big)\leq \ell-t,$$

where $t = \kappa+1, \cdots, \ell+1$.

In particular, $\big(\bigcap\limits_{j=1}^{\ell+1}\widetilde{P}_j(a)\cap V\big) = \emptyset$. This yields that $\widetilde{P}_1(a), \cdots, \widetilde{P}_{\ell+1}(a)$ are in general position. We complete the proof of the lemma.

Since there are only finitely many choices of $N+1$ polynomials from $\widetilde{Q}_1, \cdots, \widetilde{Q}_q$, the total number of such $\widetilde{P}_j's$ is finite, so there exists a constant $C > 0$, for $t = \kappa+1, \cdots, \ell$ and all $z \in \mathbb{C}^m$ (excluding all zeros and poles of all $\widetilde{Q}_j(f)$), by Lemma 3.2 we can construct $\widetilde{P}_{1(z)} = \widetilde{Q}_{1(z)}, \cdots$, $\widetilde{P}_{\kappa(z)} = \widetilde{Q}_{\kappa(z)}$, $\widetilde{P}_{(\kappa+1)(z)}, \cdots, \widetilde{P}_{(\ell+1)(z)}$ from $\widetilde{Q}_{1(z)}, \cdots, \widetilde{Q}_{(N+1)(z)}$ such that

$$|\widetilde{P}_{t(z)}(\widetilde{f})(z)| \le C \max_{\kappa+1 \le j \le N-\ell+t} |\widetilde{Q}_{j(z)}(\widetilde{f})(z)| = C|\widetilde{Q}_{(N-\ell+t)(z)}(\widetilde{f})(z)|$$

for $\kappa+1 \le t \le \ell$, and thus,

$$\lambda_{\widetilde{Q}_{(N-\ell+t)(z)}(\widetilde{f})(z)} \le \lambda_{\widetilde{P}_{t(z)}(\widetilde{f})(z)} + \log h'', \quad h'' \in \mathcal{C}_f \text{ for } \kappa+1 \le t \le \ell.$$

Combining the above inequality with (3.1), we have

$$\begin{aligned}
&\sum_{j=1}^{q} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} \\
&\le \sum_{j=1}^{\kappa} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} + \sum_{j=\kappa+1}^{N-\ell+\kappa} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} + \sum_{N-\ell+\kappa+1}^{N} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} + \log h' \\
&\le \sum_{j=1}^{\kappa} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} + \sum_{j=\kappa+1}^{N-\ell+\kappa} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} + \sum_{j=\kappa+1}^{\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \log h'', \\
&= \sum_{j=1}^{\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \sum_{j=\kappa+1}^{N-\ell+\kappa} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} + \log h''.
\end{aligned}$$

This gives that if $N-\ell \le \kappa$, we have

$$\begin{aligned}
\sum_{j=1}^{q} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} &\le \sum_{j=1}^{\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \sum_{j=1}^{N-\ell} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} + \log h'' \\
&= \sum_{j=1}^{\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \sum_{j=1}^{N-\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \log h'',
\end{aligned} \tag{3.2}$$

and if $N-\ell \ge \kappa$, we get

$$\begin{aligned}
\sum_{j=1}^{q} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} &\le \sum_{j=1}^{\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \sum_{j=1}^{N-\ell} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} + \log h'' \\
&\le \sum_{j=1}^{\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \frac{N-\ell}{\kappa} \sum_{j=1}^{\kappa} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} + \log h''' \\
&= \sum_{j=1}^{\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \frac{N-\ell}{\kappa} \sum_{j=1}^{\kappa} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \log h''',
\end{aligned} \tag{3.3}$$

where $h''' \in \mathcal{C}_f$. Hence, by (3.2)–(3.3), we get

$$\begin{aligned}
&\sum_{j=1}^{q} \lambda_{\widetilde{Q}_{j(z)}(\widetilde{f})(z)} \\
&\le \sum_{j=1}^{\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}} \sum_{j=1}^{\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \log h^*
\end{aligned}$$

$$= \Big(1 + \frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}}\Big) \sum_{j=1}^{\ell} \lambda_{\widetilde{P}_{j(z)}(\widetilde{f})(z)} + \log h^*, \tag{3.4}$$

where $h^* = \max\{h'', h'''\} \in \mathcal{C}_f$.

Fix a basis $\{[\phi_1], \cdots, [\phi_M]\}$ of V_L with $[\phi_1], \cdots, [\phi_M] \in \mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]$, and let

$$F = [\phi_1(\widetilde{f}), \cdots, \phi_M(\widetilde{f})] : \mathbb{C} \to \mathbb{P}^{M-1}(\mathbb{C}).$$

Since $\widetilde{f}$ satisfies $P(\widetilde{f}) \not\equiv 0$ for all homogeneous polynomials $P \in \mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]/I_{\mathcal{K}_{\mathcal{Q}}}(V)$, F is linearly nondegenerate over $\mathcal{K}_{\mathcal{Q}}$. We have

$$T_F(r) = LT_f(r) + o(T_f(r)). \tag{3.5}$$

For every positive integer L divided by d, we use the following filtration of the vector space V_L with respect to $\widetilde{P}_{1(z)}, \cdots, \widetilde{P}_{\ell(z)}$. This is a generalization of Corvaja-Zannier's filtration [1], see in the three references [3, 12, 17].

We arrange, by the lexicographic order, the ℓ-tuples $\mathbf{i} = (i_1, \cdots, i_\ell)$ of non-negative integers and set $\|\mathbf{i}\| = \sum_j i_j$.

Definition 3.1 (see [3, 12, 17]) (i) *For each $\mathbf{i} \in \mathbf{Z}^{\ell}_{\geq 0}$ and non-negative integer L with $L \geq d\|\mathbf{i}\|$, denote by $I^{\mathbf{i}}_L$ the subspace of $\mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_{L-d\|\mathbf{i}\|}$ consisting of all*

$$\gamma \in \mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_{L-d\|\mathbf{i}\|}$$

such that

$$\widetilde{P}^{i_1}_{1(z)} \cdots \widetilde{P}^{i_\ell}_{\ell(z)} \gamma - \sum_{\mathbf{e}=(e_1,\cdots,e_\ell)>\mathbf{i}} \widetilde{P}^{e_1}_{1(z)} \cdots \widetilde{P}^{e_\ell}_{\ell(z)} \gamma_{\mathbf{e}} \in I_{\mathcal{K}_{\mathcal{Q}}}(V)_L$$
$$(\text{or } [\widetilde{P}^{i_1}_{1(z)} \cdots \widetilde{P}^{i_\ell}_{\ell(z)} \gamma] = \Big[\sum_{\mathbf{e}=(e_1,\cdots,e_\ell)>\mathbf{i}} \widetilde{P}^{e_1}_{1(z)} \cdots \widetilde{P}^{e_\ell}_{\ell(z)} \gamma_{\mathbf{e}} \Big] \quad \text{on } V_L)$$

for some $\gamma_{\mathbf{e}} \in \mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_{L-d\|\mathbf{e}\|}$.

(ii) *Denote by $I^{\mathbf{i}}$ the homogeneous ideal in $\mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]$ generated by $\bigcup_{L \geq d\|\mathbf{i}\|} I^{\mathbf{i}}_L$.*

Remark 3.1 (see [3, 12, 17]) From this definition, we have the following properties.

(i) $(I_{\mathcal{K}_{\mathcal{Q}}}(V), \widetilde{P}_{1(z)} \cdots \widetilde{P}_{\ell(z)})_{L-d\|\mathbf{i}\|} \subset I^{\mathbf{i}}_L \subset \mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_{L-d\|\mathbf{i}\|}$, where $(I_{\mathcal{K}_{\mathcal{Q}}}(V), \widetilde{P}_{1(z)} \cdots \widetilde{P}_{\ell(z)})$ is denoted to be the ideal in $\mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]$ generated by $I_{\mathcal{K}_{\mathcal{Q}}}(V) \cup \{\widetilde{P}_{1(z)} \cdots \widetilde{P}_{\ell(z)}\}$.

(ii) $I^{\mathbf{i}} \cap \mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_{L-d\|\mathbf{i}\|} = I^{\mathbf{i}}_L$.

(iii) $\frac{\mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]}{I^{\mathbf{i}}}$ is a graded module over $\mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]$.

(iv) If $\mathbf{i}_1 - \mathbf{i}_2 := (i_{1,1} - i_{2,1}, \cdots, i_{1,\ell} - i_{2,\ell}) \in \mathbf{Z}^{\ell}_{\geq 0}$, then $I^{\mathbf{i}_2}_L \subset I^{\mathbf{i}_1}_{L+d\|\mathbf{i}_1\|-d\|\mathbf{i}_2\|}$. Hence $I^{\mathbf{i}_2} \subset I^{\mathbf{i}_1}$.

Lemma 3.3 (see [3, 12, 17]) *$\{I^{\mathbf{i}} \mid \mathbf{i} \in \mathbf{Z}^{\ell}_{\geq 0}\}$ is a finite set.*

Denote by

$$\Delta^{\mathbf{i}}_L := \dim_{\mathcal{K}_{\mathcal{Q}}} \frac{\mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_{L-d\|\mathbf{i}\|}}{I^{\mathbf{i}}_L}. \tag{3.6}$$

Lemma 3.4 (see [3, 12, 17]) (i) *There exists a positive integer L_0 such that, for each $\mathbf{i} \in \mathbf{Z}^{\ell}_{\geq 0}$, $\Delta^{\mathbf{i}}_L$ is independent of L for all L satisfying $L - d\|\mathbf{i}\| > L_0$.*

(ii) *There is an integer $\overline{\Delta}$ such that $\Delta^{\mathbf{i}}_L \leq \overline{\Delta}$ for all $\mathbf{i} \in \mathbf{Z}^{\ell}_{\geq 0}$ and L satisfies $L - d\|\mathbf{i}\| > 0$.*

Set $\Delta_0 := \min_{\mathbf{i} \in \mathbf{Z}^{\ell}_{\geq 0}} \Delta^{\mathbf{i}} = \Delta^{\mathbf{i}_0}$ for some $\mathbf{i}_0 \in \mathbf{Z}^{\ell}_{\geq 0}$.

Remark 3.2 (see [3, 12, 17]) By (iv) of Remark 3.1, if $\mathbf{i} - \mathbf{i}_0 \in \mathbf{Z}^{\ell}_{\geq 0}$, then $\Delta^{\mathbf{i}} \leq \Delta^{\mathbf{i}_0}$.

Now, for an integer L big enough, divisible by d, we construct the following filtration of V_L with respect to $\{\widetilde{P}_{1(z)} \cdots \widetilde{P}_{\ell(z)}\}$. Denote by τ_L the set of $\mathbf{i} \in \mathbf{Z}^{\ell}_{\geq 0}$ with $L - d\|\mathbf{i}\| > 0$, arranged by the lexicographic order. Define the spaces $W_{\mathbf{i}} = W_{L,\mathbf{i}}$ by

$$W_{\mathbf{i}} = \sum_{\mathbf{e} > \mathbf{i}} \widetilde{P}^{e_1}_{1(z)} \cdots \widetilde{P}^{e_\ell}_{\ell(z)} \cdot \mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_{L - d\|\mathbf{e}\|}.$$

Clearly, $W_{(0,\cdots,0)} = \mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_L$ and $W_{\mathbf{i}} \supset W_{\mathbf{i}'}$ if $\mathbf{i}' > \mathbf{i}$, so $\{W_{\mathbf{i}}\}$ is a filtration of $\mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_L$. Set $W^*_{\mathbf{i}} = \{[g] \in V_L \mid g \in W_{\mathbf{i}}\}$. Hence,$\{W^*_{\mathbf{i}}\}$ is a filtration of V_L.

Lemma 3.5 (see [3, 12, 17]) *Suppose that $\mathbf{i}'$ follows $\mathbf{i}$ in the lexicographic order, then*

$$\frac{W^*_{\mathbf{i}}}{W^*_{\mathbf{i}'}} \cong \frac{\mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_{L - d\|\mathbf{i}\|}}{I^{\mathbf{i}}_L}.$$

Combining with the notation (3.6), we have

$$\dim \frac{W^*_{\mathbf{i}}}{W^*_{\mathbf{i}'}} = \Delta^{\mathbf{i}}_L.$$

Set

$$\tau^0_L = \{\mathbf{i} \in \tau_L \mid L - d\|\mathbf{i}\| > L_0 \quad \text{and} \quad \mathbf{i} - \mathbf{i}_0 \in \mathbf{Z}^{\ell}_{\geq 0}\}.$$

We have the following properties.

Lemma 3.6 (see [3, 12, 17]) (i) $\Delta_0 = \Delta^{\mathbf{i}}$ *for all* $\mathbf{i} \in \tau^0_L$.

(ii) $\sharp\tau^0_L = \frac{1}{d^\ell}\frac{L^\ell}{\ell!} + O(L^\ell - 1)$.

(iii) $\Delta^{\mathbf{i}}_L = \Delta d^\ell$ *for all* $\mathbf{i} \in \tau^0_L$.

We can choose a basis $\mathcal{B} = \{[\psi_1], \cdots, [\psi_M]\}$ of V_L with respect to the above filtration. Let $[\psi_s]$ be an element of the basis, which lies in $W^*_{\mathbf{i}}/W^*_{\mathbf{i}'}$, we may write $\psi_s = \widetilde{P}^{e_1}_{1(z)} \cdots \widetilde{P}^{e_\ell}_{\ell(z)}\gamma$, where $\gamma \in \mathcal{K}_{\mathcal{Q}}[x_0, \cdots, x_n]_{L-d\|\mathbf{i}\|}$. For every $1 \leq j \leq \ell$, we have

$$\sum_{\mathbf{i} \in \tau_L} \Delta^{\mathbf{i}}_L i_j = \frac{\Delta L^{\ell+1}}{(\ell+1)!d} + O(L^\ell) \tag{3.7}$$

(The proof of (3.7) can be found in [12, (3.6)]). Hence, by (3.7) and the definition of the Weil function, we obtain

$$\sum_{s=1}^{M} \lambda_{\psi_s}(\widetilde{f}(z)) \geq \Big(\frac{\Delta L^{\ell+1}}{(\ell+1)!d} + O(L^\ell)\Big) \cdot \sum_{j=1}^{\ell} \lambda_{\widetilde{P}_{j(z)}}(\widetilde{f}(z)) + \log h^{**}, \tag{3.8}$$

where $h^{**} \in \mathcal{C}_f$. The basis $\{[\psi_1], \cdots, [\psi_M]\}$ can be written as linear forms $L_1, \cdots, L_M$ (over $\mathcal{K}_{\mathcal{Q}}$) in the basis $\{[\phi_1], \cdots, [\phi_M]\}$ and $\psi_s(\widetilde{f}) = L_s(\widetilde{f})$. Since there are only finitely many choices of $\widetilde{Q}_{1(z)}, \cdots, \widetilde{Q}_{(N+1)(z)}$, the collection of all possible linear forms L_s $(1 \leq s \leq M)$ is a finite set, and denote it by $\mathcal{L} := \{L\mu\}^{\Lambda}_{\mu=1}$ $(\Lambda < +\infty)$. It is easy to see that $\mathcal{K}_{\mathcal{L}} \subset \mathcal{K}_{\mathcal{Q}}$.

Lemma 3.7 (Product to the Sum Estimate, see [10]) *Let $H_1, \cdots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Denote by T the set of all injective maps $\mu : \{0, 1, \cdots, n\} \to \{1, \cdots, q\}$. Then*

$$\sum_{j=1}^{q} m_f(r, H_j) \leq \int_0^{2\pi} \max_{\mu\in T} \sum_{i=0}^{n} \lambda_{\widetilde{H}_{\mu(i)}}(f(re^{i\theta}))\frac{d\theta}{2\pi} + O(1)$$

holds for all r outside a set with finite Lebesgue measure.

By (3.4), (3.8) and Lemma 3.7, taking integration on the sphere of radius r, we have

$$\frac{\Delta L^{\ell+1}}{(\ell+1)!d}(1+O(1)) \cdot \sum_{j=1}^{q} m_f(r, \widetilde{Q}_j) \leq \Big(\frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}} + 1\Big) \cdot \int_0^{2\pi} \max_{\mathcal{K}} \sum_{j\in\mathcal{K}} \lambda_{L_j}(\widetilde{f}(re^{i\theta}))\frac{d\theta}{2\pi} + o(T_f(r)) \tag{3.9}$$

for all r outside a set with finite Lebesgue measure, where the set $\mathcal{K}$ ranges over all subsets of $\{1, \cdots, \Lambda\}$ such that the linear forms $\{L_j\}_{j\in\mathcal{K}}$ are linearly independent.

By Theorem 2.2, we have, for any $\varepsilon > 0$,

$$\int_0^{2\pi} \max_{\mathcal{K}} \sum_{j\in\mathcal{K}} \lambda_{L_j}(\widetilde{f}(re^{i\theta}))\frac{d\theta}{2\pi} \leq (M+\varepsilon)T_F(r) - N_W(r, 0) + o(T_f(r)) \tag{3.10}$$

holds for all r outside a set E with finite Lebesgue measure. Taking $\varepsilon = \frac{1}{2}$ in (3.10), and using (3.5) and (3.9), we obtain

$$\begin{aligned}
&\frac{\Delta L^{\ell+1}}{(\ell+1)!d}(1+O(1)) \cdot \sum_{j=1}^{q} m_f(r, \widetilde{Q}_j)\\
&\leq \Big(\frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}} + 1\Big)((M+\varepsilon)T_F(r) - N_W(r, 0) + o(T_f(r)))\\
&\leq \Big(\frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}} + 1\Big)\Big(\frac{\Delta L^{\ell}}{\ell!} + \rho(L) + \varepsilon\Big)LT_f(r) + o(T_f(r))
\end{aligned}$$

holds for all $r \notin E$, where W is the Wronskian of $F_1, \cdots, F_M$.

Take L large enough such that $\varepsilon < \big(\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}} + 1\big)o(1)$, where $\varepsilon > 0$ is any given constant in the theorem. Then we have

$$\sum_{j=1}^{q} \frac{1}{d} m_f(r, \widetilde{Q}_j) \leq \Big(\Big(\frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}} + 1\Big)(\ell+1) + \varepsilon\Big)T_f(r) \tag{3.11}$$

holds for all $r \notin E$.

And by the first main theorem, (3.11) can be written

$$\Big(q - \Big(\frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}} + 1\Big)(\ell+1) - \varepsilon\Big)T_f(r) \leq \sum_{j=1}^{q} \frac{1}{d} N_f(r, \widetilde{Q}_j).$$

Secondly, for the general case whenever all Q_j may not have the same degree, considering $Q_j^{\frac{d}{d_j}}$ instead of Q_j, we have $N_f(r, Q_j) = \frac{d_j}{d} N_f\big(r, Q_j^{\frac{d}{d_j}}\big)$. Then the theorem is proved immediately.

4 Proof of Theorem 1.4

Replacing Q_i by $Q_i^{\frac{d}{d_i}}$ if necessary with the note that

$$\frac{1}{d}N^{[L_0]}\big(r, f * Q_i^{\frac{d}{d_i}}\big) \le \frac{1}{d_i}N^{[L_0]}(r, f * Q_i),$$

we may assume that all hypersurfaces Q_i $(1 \le i \le q)$ are of the same degree d. We may also assume that $q > \big(\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}} + 1\big)(n+1)$.

Consider a reduced representation $\widetilde{f} = (f_0, \cdots, f_n) : \mathbb{C} \to \mathbb{C}^{n+1}$ of f. We also note that

$$N^{[L_0]}_{Q_i(\widetilde{f})}(r) = N^{[L_0]}_{\widetilde{Q}_i(\widetilde{f})}(r) + o(T_f(r)).$$

Then without loss of generality we may assume that $Q_i \in \mathcal{K}_f[x_0, \cdots, x_n]$.

We set

$$\mathcal{I} = \{(i_1, \cdots, i_{N+1});\ 1 \le i_j \le q, i_j \ne i_t,\ \forall j \ne t\}.$$

For each $I = (i_1, \cdots, i_{N+1}) \in \mathcal{I}$, we denote by $P_{I1}, \cdots, P_{I(n+1)}$ the moving hypersurfaces obtained in Lemma 3.2 with respect to the family of moving hypersurfaces $\{Q_{i_1}, \cdots, Q_{i_{N+1}}\}$. It is easy to see that there exists a positive function $h \in \mathcal{C}_f$ such that

$$|P_{It}(\omega)| \le h \max_{\kappa+1\le j\le N+1-n+t} |Q_{i_j}(\omega)|, \quad \kappa + 1 \le t \le n$$

for all $I \in \mathcal{I}$ and $\omega = (\omega_0, \cdots, \omega_n) \in \mathbb{C}^{n+1}$.

For a fixed point $z \in \mathbb{C}^m \setminus \bigcup_{i=1}^{q} Q_i(\widetilde{f})^{-1}(\{0, \infty\})$, we may assume that

$$|Q_{i_1}(\widetilde{f})(z)| \le |Q_{i_2}(\widetilde{f})(z)| \le \cdots \le |Q_{i_q}(\widetilde{f})(z)|.$$

Let $I = (i_1, \cdots, i_{N+1})$. Since $P_{I1}, \cdots, P_{I(n+1)}$ are in weakly general position, there exist functions $g_0, g \in \mathcal{C}_f$, which may be chosen independent of I and z, such that

$$\|\widetilde{f}(z)\|^d \le g_0(z) \max_{1\le j\le n+1} |P_{Ij}(\widetilde{f})(z)| \le g(z)|Q_{i_{N+1}}(\widetilde{f})(z)|.$$

Therefore, we have

$$\begin{aligned}
\prod_{i=1}^{q} \frac{\|\widetilde{f}(z)\|^d}{|Q_i(\widetilde{f})(z)|} &\le g^{q-N}(z) \prod_{j=1}^{N} \frac{\|\widetilde{f}(z)\|^d}{|Q_{i_j}(\widetilde{f})(z)|} \\
&= g^{q-N}(z) \prod_{j=1}^{\kappa} \frac{\|\widetilde{f}(z)\|^d}{|Q_{i_j}(\widetilde{f})(z)|} \cdot \prod_{j=\kappa+1}^{N-n+\kappa} \frac{\|\widetilde{f}(z)\|^d}{|Q_{i_j}(\widetilde{f})(z)|} \cdot \prod_{j=N-n+\kappa+1}^{N} \frac{\|\widetilde{f}(z)\|^d}{|Q_{i_j}(\widetilde{f})(z)|} \\
&\le h_1 \prod_{j=1}^{\kappa} \frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|} \cdot \prod_{j=\kappa+1}^{N-n+\kappa} \frac{\|\widetilde{f}(z)\|^d}{|Q_{i_j}(\widetilde{f})(z)|} \cdot \prod_{j=\kappa+1}^{n} \frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|},
\end{aligned} \tag{4.1}$$

where $h_1 = g^{q-N}(z)h^{n-\kappa}(z)$, $I = (i_1, \cdots, i_{N+1})$. Choose a function ζ in $\mathcal{C}_f$ which is common for all $I \in \mathcal{I}$, such that

$$|P_{Ij}(\omega)| \le \zeta(z)\|\omega\|^d, \quad \forall\omega = (\omega_0, \cdots, \omega_n).$$

We first consider if $N-n\leq\kappa$, and have by the inequality above that,

$$\begin{aligned}\prod_{i=1}^{q}\frac{\|\widetilde{f}(z)\|^d}{|Q_i(\widetilde{f})(z)|}&\leq h_1\prod_{j=1}^{\kappa}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\cdot\Big(\prod_{j=1}^{N-n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)\cdot\prod_{j=\kappa+1}^{n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\\
&\leq\zeta(z)^{\kappa+n-N}h_1\prod_{j=1}^{\kappa}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\cdot\Big(\prod_{j=1}^{N-n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)\Big(\prod_{j=N-n+1}^{\kappa}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)\\
&\quad\cdot\prod_{j=\kappa+1}^{n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\leq h_2\cdot\Big(\prod_{j=1}^{\kappa}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)^2\Big(\prod_{j=\kappa+1}^{n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)^2\\
&=h_2\cdot\Big(\prod_{j=1}^{n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)^2,\end{aligned}\tag{4.2}$$

where $h_2=\zeta(z)^{2n-N}h_1\in\mathcal{C}_f$, however if $N-n\geq\kappa$, we get

$$\begin{aligned}&\prod_{i=1}^{q}\frac{\|\widetilde{f}(z)\|^d}{|Q_i(\widetilde{f})(z)|}\\
&\leq h_1\prod_{j=1}^{\kappa}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big(\prod_{j=1}^{\kappa}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)^{\frac{N-n}{\kappa}}\prod_{j=\kappa+1}^{n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\\
&=h_1\cdot\Big(\prod_{j=1}^{\kappa}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)^{1+\frac{N-n}{\kappa}}\cdot\prod_{j=\kappa+1}^{n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\\
&\leq h_3\cdot\Big(\prod_{j=1}^{\kappa}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)^{1+\frac{N-n}{\kappa}}\Big(\prod_{j=\kappa+1}^{n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)^{1+\frac{N-n}{\kappa}}\\
&=h_3\Big(\prod_{j=1}^{n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)^{1+\frac{N-n}{\kappa}},\end{aligned}\tag{4.3}$$

where $h_3=h_1\zeta^{\frac{(n-\kappa)(N-n)}{\kappa}}(z)\in\mathcal{C}_f$.

Thus by (4.2)–(4.3), we get

$$\prod_{i=1}^{q}\frac{\|\widetilde{f}(z)\|^d}{|Q_i(\widetilde{f})(z)|}\leq h^*\Big(\prod_{j=1}^{n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big)^{1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}},\tag{4.4}$$

where $h^*=\max\{h_2,h_3\}\in\mathcal{C}_f$.

Hence, by taking logarithms in the both sides of (4.4), we can obtain

$$\begin{aligned}&\log\prod_{i=1}^{q}\Big(\frac{\|\widetilde{f}(z)\|^d}{|Q_i(\widetilde{f})(z)|}\Big)\\
&\leq\log h^*+\Big(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}\Big)\log\Big(\prod_{j=1}^{n}\frac{\|\widetilde{f}(z)\|^d}{|P_{Ij}(\widetilde{f})(z)|}\Big).\end{aligned}\tag{4.5}$$

Now, for each non-negative integer L, we denote by V_L the vector space (over $\mathcal{K}_{\mathcal{Q}}$) consisting of all homogeneous polynomials of degree L in $\mathcal{K}_{\mathcal{Q}}[x_0,\cdots,x_n]$ and the zero polynomial. Denote by $(P_{I1},\cdots,P_{In})$ the ideal in $\mathcal{K}_{\mathcal{Q}}[x_0,\cdots,x_n]$ generated by $P_{I1},\cdots,P_{In}$.

Lemma 4.1 (see [2, Proposition 3.3]) *Let $\{P_i\}_{i=1}^q$ $(q \geq n+1)$ be a set of homogeneous polynomials of common degree $d \geq 1$ in $\mathcal{K}_f[x_0, \cdots, x_n]$ in weakly general position. Then for any nonnegative integer L and for any $J := \{j_1, \cdots, j_n\} \subset \{1, \cdots, q\}$, the dimension of the vector space $\frac{V_L}{(P_{j_1}, \cdots, P_{j_n}) \cap V_L}$ is equal to the number of n-tuples $(s_1, \cdots, s_n) \in \mathbf{N}_0^n$ such that $s_1 + \cdots + s_n \leq L$ and $0 \leq s_1, \cdots, s_n \leq d-1$. In particular, for all $L \geq n(d-1)$, we have*

$$\dim \frac{V_L}{(P_{j_1}, \cdots, P_{j_n}) \cap V_L} = d^n.$$

Now, for each positive integer L big enough, divided by d, and $\mathbf{i} = (i_1, \cdots, i_n) \in \mathbf{N}_0^n$ with $\sigma(\mathbf{i}) = \sum_{j=1}^{n} i_j \leq \frac{L}{d}$, we set

$$W_{L,\mathbf{i}} = \sum_{(\mathbf{j})=(j_1,\cdots,j_n)\geq(\mathbf{i})} P_{I1}^{j_1} \cdots P_{In}^{j_n} \cdot V_{L-d\sigma(\mathbf{j})}.$$

It is clear that $W_{L,(0,\cdots,0)} = V_L$ and $W_{L,\mathbf{i}} \supset W_{L,\mathbf{i}'}$ if $\mathbf{i} < \mathbf{i}'$, so $\{W_{L,\mathbf{i}}\}$ is a filtration of V_L. For the proof of the following lemma, we refer to [8].

Lemma 4.2 *Let $\mathbf{i} = (i_1, \cdots, i_n), \mathbf{i}' = (i_1', \cdots, i_n') \in \mathbb{N}_0^n$. Suppose that $\mathbf{i}'$ follows $\mathbf{i}$ in the lexicographic ordering and $d\sigma(\mathbf{i}) < L$. Then*

$$\frac{W_{L,\mathbf{i}}}{W_{L,\mathbf{i}'}} \cong \frac{V_{L-d\sigma(\mathbf{i})}}{(P_{j_1}, \cdots, P_{j_n}) \cap V_{L-d\sigma(\mathbf{i})}}.$$

This lemma yields that

$$\dim \frac{W_{L,\mathbf{i}}}{W_{L,\mathbf{i}'}} = \dim \frac{V_{L-d\sigma(\mathbf{i})}}{(P_{j_1}, \cdots, P_{j_n}) \cap V_{L-d\sigma(\mathbf{i})}}. \tag{4.6}$$

Fix a number L large enough (chosen later). Set $u = u_L := \dim V_L = \binom{L+n}{n}$. We assume that

$$V_L = W_{L,\mathbf{i}_1} \supset W_{L,\mathbf{i}_2} \supset \cdots \supset W_{L,\mathbf{i}_K},$$

where $W_{L,\mathbf{i}_{s+1}}$ follows $W_{L,\mathbf{i}_s}$ in the ordering and $\mathbf{i}_K = \left(\frac{L}{d}, 0, \cdots, 0\right)$. It is easy to see that K is the number of n -tuples $(i_1, \cdots, i_n)$ with $i_j \geq 0$ and $i_1 + \cdots + i_n \leq \frac{L}{d}$. Then we have

$$K = \binom{\frac{L}{d} + n}{n}.$$

For each $k \in \{1, \cdots, K-1\}$, we set $m_k^I = \dim \frac{W_{L,\mathbf{i}_k}}{W_{L,\mathbf{i}_{k+1}}}$ and $m_K^I = 1$. Then by Lemma 4.1, m_k^I does not depend on $\{P_{I1}, \cdots, P_{In}\}$ and k, but only on $\sigma(\mathbf{i}_k)$. Hence, we set $m_k := m_k^I$. We also note that by Lemma 4.1,

$$m_k = d^n$$

for all k with $L - d\sigma(\mathbf{i}_k) \geq n(d-1)$ $\big($it is equivalent to $\sigma(\mathbf{i}_k) \leq \frac{L}{d} - n\big)$.

From the above filtration, we may choose a basis $\{\psi_1^I, \cdots, \psi_u^I\}$ of V_L such that

$$\{\psi_{u-(m_s+\cdots+m_K)+1}^I, \cdots, \psi_u^I\}$$

is a basis of $W_{L,\mathbf{i}_s}$. For each $k \in \{1, \cdots, K\}$ and $l \in \{u-(m_k+\cdots+m_K)+1, \cdots, u-(m_{k+1}+\cdots+m_K)\}$, we may write

$$\psi_l^I = P_{I1}^{i_{1k}} \dots P_{In}^{i_{nk}} h_l, \quad \text{where } (i_{1k}, \cdots, i_{nk}) = (i)_k, h_l \in W^I_{L-d\sigma(\mathbf{i}_k)}.$$

We may choose h_l to be a monomial.

We have the following estimates: Firstly, we see that

$$\sum_{k=1}^{K} m_k i_{sk} = \sum_{l=0}^{\frac{L}{d}} \sum_{k|\sigma(\mathbf{i}_k)=l} m(l) i_{sk} = \sum_{l=0}^{\frac{L}{d}} m(l) \sum_{k|\sigma(\mathbf{i}_k)=l} i_{sk}.$$

Note that, by the symmetry $(i_1, \cdots, i_n) \to (i_{\sigma(1)}, \cdots, i_{\sigma(n)})$ with $\sigma \in S(n)$, $\sum\limits_{k|\sigma(\mathbf{i}_k)=l} i_{sk}$ does not depend on s. We set

$$a := \sum_{k=1}^{K} m_k i_{sk}, \quad \text{which is independent of } s \text{ and } I.$$

Then we have

$$\begin{aligned} |\psi_l^I(\widetilde{f})(z)| &\le |P_{I1}(\widetilde{f})(z)|^{i_{1k}} \cdots |P_{In}(\widetilde{f})(z)|^{i_{nk}} |h_l(\widetilde{f})(z)| \\ &\le c_l |P_{I1}(\widetilde{f})(z)|^{i_{1k}} \cdots |P_{In}(\widetilde{f})(z)|^{i_{nk}} \|\widetilde{f}(z)\|^{L-d\sigma(\mathbf{i})_k} \\ &= c_l \Big(\frac{|P_{I1}(\widetilde{f})(z)|^{i_{1k}}}{\|\widetilde{f}(z)\|^d}\Big)^{i_{1k}} \cdots \Big(\frac{|P_{In}(\widetilde{f})(z)|}{\|\widetilde{f}(z)\|^d}\Big)^{i_{nk}} \|\widetilde{f}(z)\|^L, \end{aligned}$$

where $c_l \in \mathcal{C}_f$, which does not depend on f and z. Taking the product on both sides of the above inequalities over all l and then taking logarithms, we obtain

$$\begin{aligned} \log \prod_{l=1}^{u} |\psi_l^I(\widetilde{f})(z)| \le \sum_{k=1}^{K} m_k \Big(i_{1k} \log \frac{|P_{I1}(\widetilde{f})(z)|}{\|\widetilde{f}(z)\|^d} + \cdots + i_{nk} \log \frac{|P_{In}(\widetilde{f})(z)|}{\|\widetilde{f}(z)\|^d} \Big) \\ + uL \log \|\widetilde{f}(z)\| + \log c_I, \end{aligned} \tag{4.7}$$

where $c_I = \prod\limits_{l=1}^{u} c_l \in \mathcal{C}_f$. By (4.7), it gives

$$\log \prod_{l=1}^{u} |\psi_l^I(\widetilde{f})(z)| \le a\Big(\log \prod_{i=1}^{n} \frac{|P_{Ii}(\widetilde{f})(z)|}{\|\widetilde{f}(z)\|^d} \Big) + uL \log \|\widetilde{f}(z)\| + \log c_I,$$

i.e.,

$$a\Big(\log \prod_{i=1}^{n} \frac{\|\widetilde{f}(z)\|^d}{|P_{Ii}(\widetilde{f})(z)|} \Big) \le \log \prod_{l=1}^{u} \frac{\|\widetilde{f}(z)\|^L}{|\psi_l^I(\widetilde{f})(z)|} + \log c_I. \tag{4.8}$$

Set $c_0 = h^* \prod\limits_{I} \big(1 + c_I^{(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})/a}\big) \in \mathcal{C}_f$.

Combining (4.8) with (4.5), we obtain that

$$\log \prod_{i=1}^{q} \frac{\|\widetilde{f}(z)\|^d}{|Q_i(\widetilde{f})(z)|} \le \frac{1 + \frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}}{a} \log \prod_{l=1}^{u} \frac{\|\widetilde{f}(z)\|^L}{|\psi_l^I(\widetilde{f})(z)|} + \log c_0. \tag{4.9}$$

We now write

$$\psi_l^I = \sum_{J \in \mathcal{I}_L} c_{lJ}^I x_J \in V_L, \quad c_{lJ}^I \in \mathcal{K}_{\mathcal{Q}},$$

where $\mathcal{I}_L$ is the set of all $(n+1)$-tuples $J = (i_0, \cdots, i_n)$ with $\sum_{s=0}^{n} j_s = L, x^J = x_0^{j_0} \cdots x_n^{j_n}$ and $l \in \{1, \cdots, u\}$. For each l, we fix an index $J_l^I \in J$ such that $c_{lJ_l^I}^I \not\equiv 0$. Define

$$\mu_{lJ}^I = \frac{c_{lJ}^I}{c_{lJ_l^I}^I}, \quad J \in \mathcal{I}_L.$$

Set $\Phi = \{\mu_{lJ}^I; I \subset \{1, \cdots, q\}, \sharp I = n, 1 \le l \le u, J \in \mathcal{I}_L\}$. Note that $1 \in \Phi$. Let $B = \sharp\Phi$. We see that $B \le u\binom{q}{n}\left(\binom{L+n}{n} - 1\right) = \binom{q}{n}\left(\binom{L+n}{n} - 1\right)\binom{L+n}{n}$. For each positive integer l, we denote by $\mathcal{L}(\Phi(l))$ the linear span over $\mathbb{C}$ of the set

$$\Phi(l) = \{\gamma_1 \cdots \gamma_l\, ;\ \gamma_i \in \Phi\}.$$

It is easy to see that

$$\dim \mathcal{L}(\Phi(l)) \le \sharp\Phi(l) \le \binom{B+l-1}{B-1}.$$

We may choose a positive integer p such that

$$p \le p_0 := \Big[\frac{B-1}{\log(1 + \frac{\varepsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})})}\Big]^2$$

and

$$\frac{\dim \mathcal{L}(\Phi(p+1))}{\dim \mathcal{L}(\Phi(p))} \le 1 + \frac{\varepsilon}{3(n+1)(1 + \frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}.$$

Indeed, if $\frac{\dim \mathcal{L}(\Phi(p+1))}{\dim \mathcal{L}(\Phi(p))} > 1 + \frac{\varepsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}$ for all $p \le p_0$, we have

$$\dim \mathcal{L}(\Phi(p_0+1)) \ge \Big(1 + \frac{\varepsilon}{3(n+1)(1 + \frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}\Big)^{p_0}.$$

Therefore, we have

$$\begin{aligned}
&\log\Big(1 + \frac{\varepsilon}{3(n+1)(1 + \frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}\Big) \\
&\le \frac{\log \dim \mathcal{L}(\Phi(p_0+1))}{p_0} \le \frac{\log\binom{B+p_0}{B-1}}{p_0} \\
&= \frac{1}{p_0}\log\prod_{i=1}^{B-1}\frac{p_0+i+1}{i} < \frac{(B-1)\log(p_0+2)}{p_0} \\
&\le \frac{B-1}{\sqrt{p_0}} \le \frac{(B-1)\log(1 + \frac{\varepsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})})}{B-1} \\
&= \log\Big(1 + \frac{\varepsilon}{3(n+1)(1 + \frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}\Big).
\end{aligned}$$

This is a contradiction.

We fix a positive integer p satisfying the above condition. Put $s = \dim \mathcal{L}(\Phi(p))$ and $t = \dim \mathcal{L}(\Phi(p+1))$. Let $\{b_1, \cdots, b_t\}$ be an $\mathbb{C}$-basis of $\mathcal{L}(\Phi(p+1))$ such that $\{b_1, \cdots, b_s\}$ be a $\mathbb{C}$-basis of $\mathcal{L}(\Phi(p))$.

For each $l \in 1, \cdots, u$, we set

$$\widetilde{\psi}_l^I = \sum_{J \in \mathcal{I}_L} \mu_{lJ}^I x_I.$$

For each $J \in \mathcal{I}_L$, we consider homogeneous polynomials $\phi_J(x_0, \cdots, x_n) = x^J$. Let F be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^{tu-1}(\mathbb{C})$ with a reduced representation $\widetilde{F} = (hb_i\phi_J(\widetilde{f}))_{1\le i\le t, J\in\mathcal{I}_L}$, where h is a nonzero meromorphic function on $\mathbb{C}^m$. We see that

$$\| \ N_h(r) + N_{1/h}(r) = o(T_f(r)).$$

Since f is assumed to be algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$, F is linearly nondegenerate over $\mathbb{C}$. We see that there exist nonzero functions $c_1, c_2 \in \mathcal{C}_f$ such that

$$c_1|h| \cdot \|\widetilde{f}\|^L \le \|\widetilde{F}\| \le c_2|h| \cdot \|\widetilde{f}\|^L.$$

For each $l \in 1, \cdots, u, 1 \le i \le s$, we consider the linear form L_{il}^I in x^J such that

$$hb_i\widetilde{\psi}_l^I(\widetilde{f}) = L_{il}^I(\widetilde{F}).$$

Since f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$, it is easy to see that $\{b_i\widetilde{\psi}_l^I(\widetilde{f}); 1 \le i \le s, 1 \le l \le M\}$ is linearly independent over $\mathbb{C}$, and so is $\{L_{il}^I(\widetilde{F}); 1 \le i \le s, 1 \le l \le u\}$. This yields that $\{L_{il}^I; 1 \le i \le s, 1 \le l \le u\}$ is linearly independent over $\mathbb{C}$.

For every point z which is neither zero nor pole of any $hb_i\psi_l^I(\widetilde{f})$, we also see that

$$\begin{aligned} s\log\prod_{i=1}^{u}\frac{\|\widetilde{f}(z)\|^L}{|\psi_l^I(\widetilde{f})(z)|} &\le \log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{\|\widetilde{F}(z)\|}{|hb_i\psi_l^I(\widetilde{f})(z)|} + \log c_3 \\ &= \log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{\|\widetilde{F}(z)\| \cdot \|L_{il}^I\|}{|L_{il}^I(\widetilde{F})(z)|} + \log c_4, \end{aligned}$$

where c_3, c_4 are nonzero functions in $\mathcal{C}_f$, not depending on f and I, but on $\{Q_i\}_{i=1}^q$. Combining this inequality and (4.9), we obtain that

$$\begin{aligned} \log\prod_{i=1}^{q}\frac{\|\widetilde{f}(z)\|^d}{|Q_i(\widetilde{f})(z)|} &\le \frac{1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}}{sa} \\ &\quad \cdot \Big(\max_I \log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{\|\widetilde{F}(z)\| \cdot \|L_{il}^I\|}{|L_{il}^I(\widetilde{F})(z)|} + \log c_4\Big) + \log c_0 \end{aligned} \tag{4.10}$$

for all z outside an analytic subset of $\mathbb{C}^m$.

Since $\widetilde{F}$ is linearly nondegenerate over $\mathbb{C}$, according to Proposition 2.1, there exists an admissible set $\alpha = (\alpha_{iJ})_{1\le i\le t, J\in\mathcal{I}_L}$ with $\alpha_{iJ} \in \mathbf{Z}_+^m, \alpha_{iJ} \le tu-1$, such that

$$W^\alpha(hb_i\widetilde{\phi}_J(\widetilde{f})) = \det(\mathcal{D}^{\alpha_{i'j'}}(hb_i\widetilde{\phi}_J(\widetilde{f}))) \not\equiv 0.$$

By Theorem 2.2, we have

$$\begin{aligned}&\Big\| \int_{S(r)} \max_I \Big(\log \prod_{\substack{1\le l\le u\\1\le i\le s}} \frac{\|\widetilde{F}(z)\| \cdot \|L^I_{il}\|}{|L^I_{il}(\widetilde{F})(z)|}\Big)\sigma_m \\ &\le tuT_F(r) - N_{W^\alpha(hb_i\widetilde{\phi}_J(\widetilde{f}))}(r) + o(T_F(r)).\end{aligned} \tag{4.11}$$

Integrating both sides of (4.10) and using (4.11), we obtain that

$$\begin{aligned}qdT_f(r) - \sum_{i=1}^q N(r, f^*Q_i) \le{}& \frac{tu(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}{sa}T_F(r)\\ &- \frac{1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}}{sa}N_{W^\alpha(hb_i\widetilde{\phi}_J(\widetilde{f}))}(r)\\ &+ o(T_F(r)+T_f(r)).\end{aligned} \tag{4.12}$$

We can estimate the following quantity using the method of Quang [8],

$$\sum_{i=1}^q N(r, f^*Q_i) - \frac{1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}}{sa}N_{W^\alpha(hb_i\widetilde{\phi}_J(\widetilde{f}))}(r),$$

thus we can get

$$\sum_{i=1}^q N(r, f^*Q_i) - \frac{1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}}{sa}N_{W^\alpha(hb_i\widetilde{\phi}_J(\widetilde{f}))}(r) \le \sum_{i=1}^q N^{[tu-1]}(r, f^*Q_i).$$

From this inequality and (4.12) with a note that $T_F(r) = LT_f(r) + o(T_f(r))$, we have

$$\begin{aligned}&\Big(q - \frac{tuL(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}{dsa}\Big)T_f(r)\\ &\le \sum_{i=1}^q \frac{1}{d}N^{[tu-1]}(r, f^*Q_i) + o(T_f(r)).\end{aligned} \tag{4.13}$$

Now we give some estimates for A, t and s. For each $I_k = (i_{1k}, \cdots, i_{nk})$ with $\sigma(\mathbf{i}_k) \le \frac{L}{d} - n$, we set

$$i_{(n+1)k} = \frac{L}{d} - n - \sum_{s=1}^n i_s.$$

Since the number of nonnegative integer p-tuples with summation $\le I$ is the same as the number of nonnegative integer $(p+1)$-tuples with summation exactly equal to $I \in \mathbf{Z}$, which is $\binom{I+n}{n}$, and since the sum below is independent of s, we have

$$\begin{aligned}a &= \sum_{\sigma(\mathbf{i}_k)\le\frac{L}{d}} m^I_k i_{sk}\\ &\le \sum_{\sigma(\mathbf{i}_k)\le\frac{L}{d}-n} m^I_k i_{sk}\\ &= \frac{d^n}{n+1}\sum_{\sigma(\mathbf{i}_k)\le\frac{L}{d}-n}\sum_{s=1}^{n+1} i_{sk}\end{aligned}$$

$$
\begin{aligned}
&= \frac{d^n}{n+1}\binom{\frac{L}{d}}{n}\Big(\frac{L}{d}-n\Big)\\
&= d^n\binom{\frac{L}{d}}{n+1}.
\end{aligned}
$$

Now, for every positive number $x \in \big[0, \frac{1}{(n+1)^2}\big]$, we have

$$
\begin{aligned}
(1+x)^n &= 1 + nx + \sum_{i=2}^{n}\binom{n}{i}x^i\\
&\le 1 + nx + \sum_{i=2}^{n}\frac{n^i}{i!(n+1)^{2i-2}}x\\
&\le 1 + nx + \sum_{i=2}^{n}\frac{1}{i!}x\\
&\le 1 + (n+1)x. \qquad (4.14)
\end{aligned}
$$

We chose $L = (n+1)d + 2\big(1 + \frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}\big)(n+1)^3 I(\varepsilon^{-1})d$. Then L is divisible by d and we have

$$
\frac{(n+1)d}{L-(n+1)d} = \frac{(n+1)d}{2(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})(n+1)^3 I(\varepsilon^{-1})d} \le \frac{1}{2(n+1)^2}. \qquad (4.15)
$$

Therefore, using (4.14)–(4.15), we have

$$
\begin{aligned}
\frac{uL}{da} &\le \frac{\binom{L+n}{n}L}{d^{n+1}\binom{\frac{L}{d}}{n+1}} = \frac{L\cdot(L+1)\cdots(L+n)}{1\cdot 2\cdots n}\Big/\frac{(L-nd)\cdot(L-(n-1))\cdots L}{1\cdot 2\cdots(n+1)}\\
&= (n+1)\prod_{i=1}^{n}\frac{L+i}{(L-(n-i+1)d)} < (n+1)\Big(\frac{L}{(L-(n+1)d)}\Big)^n\\
&= (n+1)\Big(1+\frac{(n+1)d}{(L-(n+1)d)}\Big)^n\\
&< (n+1)\Big(1+\frac{(n+1)^2 d}{2(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})(n+1)^3 I(\varepsilon^{-1})d}\Big)\\
&\le (n+1) + \frac{(n+1)^3 d}{2(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})(n+1)^3\varepsilon^{-1}}\\
&\le n+1+\frac{\varepsilon}{2(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\frac{tuL}{das} &\le \Big(1+\frac{\varepsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}\Big)\\
&\quad\cdot\Big(n+1+\frac{\varepsilon}{2(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}\Big)\\
&\le n+1+\frac{\varepsilon}{2(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}\\
&\quad+\frac{\varepsilon}{3(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}+\frac{\varepsilon}{6(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}
\end{aligned}
$$

$$= n + 1 + \frac{\varepsilon}{1 + \frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}}. \tag{4.16}$$

Combining (4.13) and (4.16), we get

$$\Big(q - \Big(1 + \frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}\Big)(n+1) - \varepsilon\Big)T_f(r)$$
$$\leq \sum_{i=1}^{q} \frac{1}{d} N^{[tu-1]}(r, f^*Q_i) + o(T_f(r)). \tag{4.17}$$

Here we note that

$$L := (n+1)d + 2\Big(1 + \frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}\Big)(n+1)^3 I(\varepsilon^{-1})d,$$
$$p_0 := \Big[\frac{B-1}{\log(1 + \frac{\varepsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})})}\Big]^2$$
$$\leq \Big[\frac{\binom{L+n}{n}(\binom{L+n}{n} - 1)\binom{q}{n} - 1}{\log(1 + \frac{\varepsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})})}\Big]^2,$$
$$tu - 1 \leq \binom{L+n}{n}\binom{B+p}{B-1} - 1 \leq \binom{L+n}{n} p^{B-1} - 1$$
$$\leq \binom{L+n}{n} p_0^{\binom{L+n}{n}\big(\binom{L+n}{n}-1\big)\binom{q}{n}-2} - 1 = L_0.$$

By these estimates and by (4.17), we obtain

$$\Big(q - \Big(1 + \frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}\Big)(n+1) - \varepsilon\Big)T_f(r)$$
$$\leq \sum_{i=1}^{q} \frac{1}{d} N^{[L_0]}(r, f^*Q_i) + o(T_f(r)). \tag{4.18}$$

The theorem is proved.

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