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A Künneth Formula for Finite Sets*

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Abstract In this paper, the authors define the homology of sets, which comes from and contains the ideas of path homology and embedded homology. Moreover, A Künneth formula for sets associated to the homology of sets is given.

Keywords Künneth formula, Finite set, Principal ideal domain, Cartesian product, Free R-module
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1 Introduction

Let R be a commutative ring with unit, and let (C, ∂) be a complex of finitely generated free R-modules of rank n. Let $X = \{x_1, \cdots, x_n\}$ be a finite set. Then there is a natural map

$$X \to C, \quad x_i \mapsto e_i,$$

where $e_1, \cdots, e_n$ is a basis of C. For the sake of simplicity, we denote $C = (R[X], \partial)$. Let S be a graded sub R-module of C. Let $\mathrm{Inf}_*(S, C) = (S \cap \partial^{-1} S, \partial)$. Then $\mathrm{Inf}_*(S, C)$ is a subcomplex of C.

Definition 1.1 *Let Y be a subset of X, and let $R[Y]$ be a free R-module generated by Y. The homology of the set Y associated to $C = (R[X], \partial)$ is*

$$H_C(Y; R) = H(\mathrm{Inf}_*(R[Y], C)).$$

If there is no ambiguity, we denote $H(Y) = H_C(Y; R)$.

The idea of the homology of sets is essentially from the path homology of digraphs (see [4]) and multi-graphs (see [5]) and the embedded homology of hypergraphs (see [2]). In this paper, we will always consider free R-modules instead of abelian groups.

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Künneth formulas describe the homology of a product space in terms of the homology of the factors. In [7], Hatcher gave the classical algebraic Künneth formula. In [4, 6], Grigor'yan, Lin, Muranov and Yau studied the Künneth formula for the path homology (with field coefficients) of digraphs. In this paper, we study the Künneth formula for sets which can be applied to digraphs and hypergraphs.

From now on, R is assumed to be a principal ideal domain. For convenience, the tensor product is always over R.

Theorem 1.1 *Let R be a principal ideal domain. Let $C = R[X], C' = R[X']$ be complexes of free R-modules generated by finite sets $X,\ X'$, respectively, and let $Y,\ Y'$ be subsets of $X,\ X'$, respectively. Then there is a natural exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(Y) \otimes H_q(Y') \to H_n(Y \times Y') \to \bigoplus_{p+q=n} \mathrm{Tor}_R(H_p(Y), H_{q-1}(Y')) \to 0,$$

where $Y \times Y'$ is the Cartesian product of sets.

Recently, people are interested in digraphs in topology (see [4, 6]). Let $G = (V, E)$ be a digraph. Let X be the set of regular paths on V. Then we can obtain a chain complex $(C, \partial) = (R[X], \partial)$ (see [6]). Let $A(G)$ be the set of allowed paths on G. We find that the path homology of digraph G coincides with the homology of set $A(G)$, i.e.,

$$H(G) = H_C(A(G)).$$

Grigor'yan et al. studied the Künneth formula for digraphs over a field (see [6]). Let G' be another digraph. In view of Theorem 1.1, in order to get the Künneth formula for digraphs with ring coefficients, we need to show

$$H(A(G) \times A(G')) \cong H(A(G \Box G')),$$

where $\Box$ denotes the Cartesian product of digraphs.

A hypergraph is a potential topic connecting simplicial complex in topology and a graph in combinatorics, which is worth studying both in theory and application (see [1–3, 9]). Let $\mathcal{H}$ be a hypergraph. Let $\mathcal{K}_{\mathcal{H}}$ be the smallest simplicial complex containing $\mathcal{H}$. Note that $\mathcal{H}$ is a set of hyperedges, we observe that

$$H(\mathcal{H}) = H_C(\mathcal{H}),$$

where $(C, \partial) = (C_*(\mathcal{K}_{\mathcal{H}}; R), \partial)$ is the chain complex of simplicial complex $\mathcal{K}_{\mathcal{H}}$. Let $\mathcal{H}'$ be another hypergraph. By Theorem 1.1, we have

$$0 \to \bigoplus_{p+q=n} H_p(\mathcal{H}) \otimes H_q(\mathcal{H}') \to H_n(\mathcal{H} \times \mathcal{H}') \to \bigoplus_{p+q=n} \mathrm{Tor}_R(H_p(\mathcal{H}), H_{q-1}(\mathcal{H}')) \to 0,$$

where $\mathcal{H} \times \mathcal{H}'$ is the Cartesian product of sets. Unfortunately, $\mathcal{H} \times \mathcal{H}'$ is not a hypergraph. In another paper, we give a product of hypergraphs and show the Künneth formula for hypergraphs.

In the next section, we build a basic algebraic language. In Section 3, we prove Theorem 1.1.

2 Preliminaries

In this section, let $(C, \partial) = (R[X], \partial)$ be a complex of free R-modules generated by a finite set X. Let $D = R[Y]$ be a free R-module generated by $Y \subseteq X$.

Proposition 2.1 (see [8]) *Let M be an $m \times n$ matrix over R. Then we have*

$$M = U\Lambda V, \quad U \in R^{m\times m}, V \in R^{n\times n},$$

where $\det(U) = \det(V) = 1$ *and Λ is a matrix of form* $\begin{pmatrix} \Lambda_m & O \end{pmatrix}$ *or* $\begin{pmatrix} \Lambda_n \\ O \end{pmatrix}$. *Here, Λ_m and Λ_n are diagonal matrices.*

Lemma 2.1 *Suppose that $z \in D$ and $\lambda z \in \mathrm{Inf}_*(D, C)$ for some nonzero element $\lambda \in R$. Then we have $z \in \mathrm{Inf}_*(D, C)$.*

Proof Let $X = \{x_1, \cdots, x_n\}$. Then $x_1, \cdots, x_n$ is a basis of $R[X]$. For convenience, we denote $e_X = (x_1, \cdots, x_n)^T$. Let Z be the set of complement of Y in X. Then we have $X = Y \sqcup Z$. Assume that

$$\partial z = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} e_Y \\ e_Z \end{pmatrix},$$

where $\mathbf{a} = (a_1, \cdots, a_{|Y|}) \in R^{1\times|Y|}$, $\mathbf{b} = (b_1, \cdots, b_{|Z|}) \in R^{1\times|Z|}$ and e_Y, e_Z are given by sets Y, Z, respectively. Since $\lambda\partial z \in D$, it follows that

$$\lambda \mathbf{b} e_Z = 0.$$

Since R is an integral domain, we have $\mathbf{b}e_Z = 0$. Thus we obtain

$$\partial z = \mathbf{a} e_Y \in D.$$

The lemma is proved.

Lemma 2.2 *There is a basis $e_1, \cdots, e_{r(D)}$ of D such that $e_1, \cdots, e_\alpha$ is a basis of $\mathrm{Inf}_*(D, C)$ for some α, where $r(D)$ is the rank of D.*

Proof Let $e_1, \cdots, e_n$ be a basis of D, and let $f_1, \cdots, f_\alpha$ be a basis of $\mathrm{Inf}_*(D, C)$. Then we have

$$\mathbf{f} = A\mathbf{e},$$

where $\mathbf{f} = (f_1, \cdots, f_\alpha)^T, \mathbf{e} = (e_1, \cdots, e_n)^T$ and A is an $\alpha \times n$ matrix over R. By Proposition 2.1, we obtain

$$A = U\Lambda V, \quad U \in R^{\alpha\times\alpha}, V \in R^{n\times n},$$

where $\det(U) = \det(V) = 1$ and

$$\Lambda = \begin{pmatrix} d_1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & & \cdots & \\ 0 & & d_\alpha & 0 & \cdots & 0 \end{pmatrix} \in R^{\alpha\times n}.$$

Let $(x_1, \cdots, x_\alpha) = U^{-1}\mathbf{f}$ and $(y_1, \cdots, y_n) = V\mathbf{e}$, then we have

$$x_i = d_i y_i, \quad i = 1, \cdots, \alpha.$$

By Lemma 2.1, we have $y_i \in \mathrm{Inf}_*(D, C), i = 1, \cdots, \alpha$. It follows that $y_1, \cdots, y_\alpha$ is a basis of $\mathrm{Inf}_*(D, C)$. Thus $y_1, \cdots, y_n$ is the desired basis.

Example 2.1 Let $(C, \partial) = (\mathbb{Z}[x, y], \partial), \partial y = x, \partial x = 0, \deg x = 1$, and let $D = \mathbb{Z}[2x, y]$ be a free $\mathbb{Z}$-module generated by $2x, y$. Note that

$$\mathrm{Inf}_*(D, C) = (\mathbb{Z}[2x, 2y], \partial), \quad \partial(2y) = 2x.$$

Thus the condition that D is a free R-module generated by a subset of X is necessary for Lemma 2.2.

Lemma 2.3 *Let $K = \ker \partial \subseteq C$. Then there is a basis $e_1, \cdots, e_{r(C)}$ of C such that $e_1, \cdots, e_\alpha$ is a basis of K for some α, where $r(C)$ is the rank of C.*

Proof By a similar argument with the proof of Lemma 2.2, we have this lemma.

Definition 2.1 *Let M be a finitely generated free R-module, and let $N \subseteq M$ be a free sub R-module of M. We say a family of elements $x_1, \cdots, x_n \in M$ is linearly independent modulo N if the condition*

$$c_1 x_1 + \cdots + c_n x_n \in N, \quad c_1, \cdots, c_n \in R$$

implies $c_1 = \cdots = c_n = 0$.

By Lemma 2.3, we have $C = V \oplus K$, where $K = \ker \partial$ and V is the space of the complement of K in C. Note that a family of elements $x_1, \cdots, x_n \in C$ is linearly independent modulo K if and only if $\partial x_1, \cdots, \partial x_n$ is linearly independent.

3 The Proof of Main Theorem

In this section, let $C = R[X], C' = R[X']$ be complexes of finitely generated free R-modules generated by sets X, X', respectively. Let $D = R[Y], D' = R[Y']$ be finitely generated free R-modules generated by $Y \subseteq X, Y' \subseteq X'$, respectively. For convenience, all the differentials will be denoted by ∂ if there is no ambiguity.

The keypoint of proving Theorem 1.1 is to show

$$\mathrm{Inf}_*(D \otimes D', C \otimes C') = \mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C').$$

We will give some lemmas first.

Lemma 3.1 *Let M, N be finitely generated free R-modules. For each $z \in M \otimes N$, there exists a nonzero element $\lambda \in R$ such that*

$$\lambda z = \sum_{i=1}^{k} x_i \otimes y_i, \quad x_i \in M, y_i \in N, i = 1, \cdots, k,$$

where $\{x_i\}_{1\le i\le k}, \{y_i\}_{1\le i\le k}$ are two families of linearly independent elements in M, N, respectively.

Proof Let $z = \sum\limits_{i=1}^{n} x_i \otimes y_i$, where $x_i \in M, y_i \in N, i = 1, \cdots, n$. If $x_1, \cdots, x_n$ are not linearly independent, we have

$$c_1x_1 + \cdots + c_nx_n = 0, \quad c_1, \cdots, c_n \in R.$$

We may assume $c_n \neq 0$. It follows that

$$c_nz = \sum_{i=1}^{n-1} x_i \otimes (c_ny_i - c_iy_n).$$

Let $z_i = c_ny_i - c_iy_n$. Then we have $c_nz = \sum\limits_{i=1}^{n-1} x_i \otimes z_i$. By finite steps, the above equation can be reduced to

$$\lambda z = \sum_{i=1}^{k} x_i \otimes y_i,$$

where $\lambda \neq 0$ and $\{x_i\}_{1\le i\le k}, \{y\}_{1\le i\le k}$ are two families of linearly independent elements in M and N, respectively.

Remark 3.1 In the above lemma, we can choose $\lambda = 1$. Let $\{e_i\}_{1\le i\le m}, \{f_i\}_{1\le i\le n}$ be the bases of M, N, respectively. Then we have

$$z = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}e_i \otimes f_j, \quad a_{ij} \in R.$$

Let $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ be a matrix over R. By Proposition 2.1, we have

$$A = U\Lambda V, \quad U \in R^{m\times m}, V \in R^{n\times n},$$

where $\det(U) = \det(V) = 1$ and $\Lambda = \begin{pmatrix} \Lambda_k & O_{k\times(n-k)} \\ O_{(m-k)\times k} & O_{(m-k)\times(n-k)} \end{pmatrix}$. Here,

$$\Lambda_k = \mathrm{diag}(\lambda_1, \cdots, \lambda_k), \quad \lambda_i \neq 0, i = 1, \cdots, k.$$

Denote $\mathbf{e} = (e_1, \cdots, e_m)^T$ and $\mathbf{f} = (f_1, \cdots, f_n)^T$. Then we have

$$z = \mathbf{e}^T \otimes A\mathbf{f} = (\mathbf{e}^T U) \otimes \Lambda(V\mathbf{f}),$$

which is the desired result.

The following lemma is a very useful tool in proving our main theorem.

Lemma 3.2 *Let $\{x_i\}_{1\le i\le k}, \{y_i\}_{1\le i\le k}$ be two families of linearly independent elements in C and C', respectively. If $\sum\limits_{i=1}^{k} x_i \otimes y_i \in D \otimes D'$, then we have*

$$x_i \in D, \quad y_i \in D', \quad i = 1, \cdots, k.$$

Proof Let $e_1, \cdots, e_\alpha, e_{\alpha+1}, \cdots, e_m$ be a basis of C such that $e_1, \cdots, e_\alpha$ is a basis of D. Similarly, let $f_1, \cdots, f_\alpha, f_{\alpha+1}, \cdots, f_n$ be a basis of C' such that $f_1, \cdots, f_\beta$ is a basis of D'. Assume that

$$x_i = \sum_{s=1}^{m} a_{is} e_s, \quad y_i = \sum_{t=1}^{n} b_{it} f_t, \quad 1 \leq i \leq k,$$

where $a_{is}, b_{it} \in R$ for $1 \leq s \leq m, 1 \leq t \leq n$. Note that

$$\sum_{i=1}^{k} x_i \otimes y_i = \sum_{s=1}^{m} \sum_{t=1}^{n} \Big(\sum_{i=1}^{k} a_{is} b_{it} \Big) e_s \otimes f_t \in D \otimes D'.$$

We have $\sum\limits_{i=1}^{k} a_{is} b_{it} = 0$ for $s > \alpha$ or $t > \beta$. Let

$$A_0 = (a_{is})_{1 \leq i \leq k, 1 \leq s \leq \alpha}, \quad A_1 = (a_{is})_{1 \leq i \leq k, \alpha+1 \leq s \leq m}$$

and

$$B_0 = (b_{it})_{1 \leq i \leq k, 1 \leq t \leq \beta}, \quad B_1 = (b_{it})_{1 \leq i \leq k, \beta+1 \leq t \leq n}.$$

It follows that

$$\begin{pmatrix} A_0^T \\ A_1^T \end{pmatrix} \begin{pmatrix} B_0 & B_1 \end{pmatrix} = \begin{pmatrix} A_0^T B_0 & O \\ O & O \end{pmatrix}.$$

Since $\operatorname{rank}(A_0^T) \geq \operatorname{rank}(A_0^T B_0) = k$, we have

$$\operatorname{rank}(B_1) \leq k - \operatorname{rank}(A_0^T) + \operatorname{rank}(A_0^T B_1) = 0.$$

Thus we obtain $B_1 = O$. Similarly, we have $A_1 = O$. These imply the lemma.

The following two lemmas are important parts of the proof of Theorem 3.1.

Lemma 3.3 *Let* $z = \sum\limits_{i=1}^{m} x_i \otimes \alpha_i + \sum\limits_{j=1}^{n} \beta_j \otimes y_j \in \operatorname{Inf}_*(D \otimes D', C \otimes C')$ *such that*

$$\alpha_1, \cdots, \alpha_m \in \partial C, \ \beta_1, \cdots, \beta_n \in \partial C'$$

and each of the following sets

$$\{\partial x_1, \cdots, \partial x_m\}, \{\partial y_1, \cdots, \partial y_n\}, \{\alpha_1, \cdots, \alpha_m\}, \{\beta_1, \cdots, \beta_n\}$$

is linearly independent. Then there exists a nonzero element $\lambda \in R$ *such that* $\lambda z \in \operatorname{Inf}_*(D, C) \otimes \operatorname{Inf}_*(D', C')$.

Proof By Lemma 3.2, we have

$$x_i, \beta_j \in D, \quad y_j, \alpha_i \in D', \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

Note that

$$\partial z = \sum_{i=1}^{m} \partial x_i \otimes \alpha_i + \sum_{j=1}^{n} \beta_j \otimes \partial y_j \in D \otimes D'.$$

If $\partial x_k, \beta_1, \cdots, \beta_n$ are not linearly independent, we have

$$c_k \partial x_k = a_{k1}\beta_1 + \cdots + a_{kn}\beta_n, \quad c_k \neq 0, a_{k1}, \cdots, a_{kn} \in R.$$

Then $c_k \partial x_k \in D$. Moreover, we obtain

$$c_k \partial z = \sum_{i \neq k} c_k \partial x_i \otimes \alpha_i + \sum_{j=1}^{n} \beta_j \otimes (c_k \partial y_j + a_{kj}\alpha_k).$$

We may assume that $\partial x_k, \beta_1, \cdots, \beta_n$ are not linearly independent for $m' + 1 \leq k \leq m$. By finite steps, the above equation can be reduced to

$$\lambda \partial z = \lambda \sum_{i=1}^{m'} \partial x_i \otimes \alpha_i + \sum_{j=1}^{n} \beta_j \otimes y'_j$$

for some nonzero element $\lambda \in R$, where $y'_j - \lambda \partial y_j$ $(j = 1, \cdots, n)$ is linearly generated by $\alpha_{m'+1}, \cdots, \alpha_m$. In addition, $\partial x_1, \cdots, \partial x_{m'}, \beta_1, \cdots, \beta_n$ are linearly independent. If $y'_j, \alpha_1, \cdots,$ $\alpha_{m'}$ are not linearly independent, we can change y'_j similarly as above. Then the above equation can be reduced to

$$\lambda_1 \lambda \partial z = \sum_{i=1}^{m'} x'_i \otimes \alpha_i + \lambda_1 \sum_{j=1}^{n'} \beta_j \otimes y'_j$$

for some nonzero elements $\lambda, \lambda_1 \in R$, where $x'_i - \lambda_1 \lambda \partial x_i$ $(i = 1, \cdots, m')$ is linearly generated by $\beta_{n'+1}, \cdots, \beta_n$. In addition, $y'_1, \cdots, y'_{n'}, \alpha_1, \cdots, \alpha_{m'}$ are linearly independent. If $x'_1, \cdots, x'_{m'}, \beta_1, \cdots, \beta_{n'}$ are not linearly independent, then $\partial x_1, \cdots, \partial x_{m'}, \beta_1, \cdots, \beta_n$ are not linearly independent, which contradicts to our construction. Thus $x'_1, \cdots, x'_{m'}, \beta_1, \cdots, \beta_{n'}$ are linearly independent. By Lemma 3.2, we have

$$x'_i \in D, \quad y'_j \in D', \quad 1 \leq i \leq m', 1 \leq j \leq n'.$$

It follows that

$$\lambda_1 \lambda \partial x_i \in D, \quad \lambda \partial y_j \in D', \quad 1 \leq i \leq m', 1 \leq j \leq n'.$$

Recall that we have $c_k \partial x_k \in D, c_k \neq 0$ for $m' + 1 \leq k \leq m$. It follows that $\lambda \partial x_k \in D$ for $m' + 1 \leq k \leq m$. Similarly, we have $\lambda_1 y'_t \in D'$ for $n' + 1 \leq t \leq n$. Hence, we obtain that

$$\lambda_1 \lambda \partial x_i \in D, \quad \lambda_1 \lambda \partial y_j \in D', \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

Thus there exists a nonzero element $\lambda_2 \in R$ such that $\lambda_2 z \in \mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C')$.

Lemma 3.4 *Let $C = V \oplus K$ and $C' = V' \oplus K'$, where K and K' are the spaces of cycles in C and C', respectively. For each element $z \in C \otimes C'$, there exists a nonzero element $\lambda \in R$ such that*

$$\lambda z = \sum_{i=1}^{N_1} x_i \otimes x'_i + \sum_{j=1}^{N_2} u_j \otimes y'_j + \sum_{k=1}^{N_3} y_k \otimes u'_k + \sum_{l=1}^{N_4} v_l \otimes v'_l,$$

where

$$x_i, y_k \in C, \quad u_j, v_l \in K, \quad x'_i, y'_j \in C', \quad u'_k, v'_l \in K'$$

for $1 \le i \le N_1, 1 \le j \le N_2, 1 \le k \le N_3, 1 \le l \le N_4$ *and*

(i) $x_1, \cdots, x_{N_1}, y_1, \cdots, y_{N_3}, u_1, \cdots, u_{N_2}, v_1, \cdots, v_{N_4}$ *are linearly independent*;

(ii) $x_1, \cdots, x_{N_1}, y_1, \cdots, y_{N_3}$ *are linearly independent modulo* K;

(iii) $x'_1, \cdots, x'_{N_1}, y'_1, \cdots, y'_{N_2}, u'_1, \cdots, u'_{N_3}, v'_1, \cdots, v'_{N_4}$ *are linearly independent*;

(iv) $x'_1, \cdots, x'_{N_1}, y'_1, \cdots, y'_{N_2}$ *are linearly independent modulo* K'.

Proof Note that

$$C \otimes C' = (V \otimes V') \oplus (K \otimes V') \oplus (V \otimes K') \oplus (K \otimes K').$$

In view of Lemma 3.1, for each element $z \in C \otimes C'$, we have

$$\lambda_1 z = \sum_{i=1}^{N_1} x_i \otimes x'_i + \sum_{j=1}^{N_2} u_j \otimes y'_j + \sum_{k=1}^{N_3} y_k \otimes u'_k + \sum_{l=1}^{N_4} v_l \otimes v'_l$$

for some $\lambda_1 \in R$, where

$$x_i, y_k \in V, \quad x'_i, y'_j \in V', \quad u_j, v_l \in K, \quad u'_k, v'_l \in K'$$

for $1 \le i \le N_1, 1 \le j \le N_2, 1 \le k \le N_3, 1 \le l \le N_4$ and each of the following sets

$$\{x_i\}_{1\le i\le N_1}, \{x'_i\}_{1\le i\le N_1}, \{u_j\}_{1\le j\le N_2}, \{u'_k\}_{1\le k\le N_3},$$
$$\{y_k\}_{1\le k\le N_3}, \{y'_j\}_{1\le j\le N_2}, \{v_l\}_{1\le l\le N_4}, \{v_l\}_{1\le l\le N_4}$$

is a family of linearly independent elements. If $x_1, \cdots, x_{N_1}, y_{k_0}$ are not linearly independent, we obtain

$$c_{k_0} y_{k_0} = a_{k_0 1} x_1 + \cdots + a_{k_0 N_1} x_{N_1}, \quad a_{k_0 1}, \cdots, a_{k_0 N_1} \in R$$

for some nonzero element $c_{k_0} \in R$. Thus we have

$$c_{k_0} \lambda_1 z = \sum_{i=1}^{N_1} x_i \otimes (a_{k_0 i} x'_i + c_{k_0} u'_k) + c_{k_0} \sum_{j=1}^{N_2} u_j \otimes y'_j + c_{k_0} \sum_{k \ne k_0} y_k \otimes u'_k + c_{k_0} \sum_{l=1}^{N_4} v_l \otimes v'_l.$$

By finite steps, the above equation can be reduced to

$$\lambda_2 \lambda_1 z = \sum_{i=1}^{N_1} x_i \otimes \overline{x}'_i + \lambda_2 \sum_{j=1}^{N_2} u_j \otimes y'_j + \lambda_2 \sum_{k=1}^{N'_3} y_k \otimes u'_k + \lambda_2 \sum_{l=1}^{N_4} v_l \otimes v'_l,$$

where $\overline{x}'_1, \cdots, \overline{x}'_{N_1} \in C'$ are linearly independent modulo K' and $x_1, \cdots, x_{N_1}, y_1, \cdots, y_{N'_3}$ are linearly independent. If $y'_{j_0}, \overline{x}'_1, \cdots, \overline{x}'_{N_1}$ are not linearly independent, by a similar substitution, we can obtain

$$\lambda_3 \lambda_2 \lambda_1 z = \sum_{i=1}^{N_1} \overline{x}_i \otimes \overline{x}'_i + \lambda_3 \lambda_2 \sum_{j=1}^{N'_2} u_j \otimes y'_j + \lambda_3 \lambda_2 \sum_{k=1}^{N'_3} y_k \otimes u'_k + \lambda_3 \lambda_2 \sum_{l=1}^{N_4} v_l \otimes v'_l,$$

such that

(i) $\overline{x}_1, \cdots, \overline{x}_{N_1}, y_1, \cdots, y_{N'_3}, u_1, \cdots, u_{N'_2}$ are linearly independent;

(ii) $\overline{x}_1, \cdots, \overline{x}_{N_1}, y_1, \cdots, y_{N'_3}$ are linearly independent modulo K;

(iii) $\overline{x}'_1, \cdots, \overline{x}'_{N_1}, y'_1, \cdots, y'_{N'_2}, u'_1, \cdots, u'_{N'_3}$ are linearly independent;

(iv) $\overline{x}'_1, \cdots, \overline{x}'_{N_1}, y'_1, \cdots, y'_{N'_2}$ are linearly independent modulo K'.

To complete our proof, it suffices to consider the elements $v_1, \cdots, v_{N_4}$ and $v'_1, \cdots, v'_{N_4}$. If $v_{l_0}, u_1, \cdots, u_{N'_2}$ are linearly independent, by a similar method as above, we can obtain the desired result.

Now, we return to the theorem mentioned before.

Theorem 3.1 $\mathrm{Inf}_*(D \otimes D', C \otimes C') = \mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C')$.

Proof It can be directly verified that

$$(D \otimes D') \cap \partial^{-1}(D \otimes D') \supseteq (D \cap \partial^{-1} D) \otimes (D' \cap \partial^{-1} D').$$

Our main work is to show the inverse.

For each element $z \in \mathrm{Inf}_*(D \otimes D', C \otimes C')$, we have

$$\lambda z = \sum_{i=1}^{N_1} x_i \otimes x'_i + \sum_{j=1}^{N_2} u_j \otimes y'_j + \sum_{k=1}^{N_3} y_k \otimes u'_k + \sum_{l=1}^{N_4} v_l \otimes v'_l,$$

where $\lambda \in R, x_i, y_k \in C, u_j, v_l \in K, x'_i, y'_k \in C', u'_k, v'_l \in K'$ are given in Lemma 3.4. Since $z \in D \otimes D'$, by Lemma 3.2, we have

$$x_i, y_k, u_j, v_l \in D, \quad x'_i, y'_k, u'_k, v'_l \in D'$$

for $1 \le i \le N_1, 1 \le j \le N_2, 1 \le k \le N_3, 1 \le l \le N_4$. Note that

$$\lambda \partial z = \sum_{i=1}^{N_1} \partial x_i \otimes x'_i + \sum_{i=1}^{N_1} (-1)^{\deg x_i} x_i \otimes \partial x'_i + \sum_{j=1}^{N_2} (-1)^{\deg u_j} u_j \otimes \partial y'_j + \sum_{k=1}^{N_3} \partial y_k \otimes u'_k.$$

Since $x_1, \cdots, x_{N_1}, y_1, \cdots, y_{N_3}$ are linearly independent modulo K, we obtain that

$$x_1, \cdots, x_{N_1}, \partial x_1, \cdots, \partial x_{N_1}, \partial y_1, \cdots, \partial y_{N_3}$$

are linearly independent. If $u_{j_0}, \partial x_1, \cdots, \partial x_{N_1}, \partial y_1, \cdots, \partial y_{N_3}$ are not linearly independent, we have

$$c_{j_0} u_{j_0} = \sum_{i=1}^{N_1} a_{j_0 i} \partial x_i + \sum_{k=1}^{N_3} b_{j_0 k} \partial y_k, \quad c_{j_0} \neq 0.$$

It follows that

$$\begin{aligned} c_{j_0} \lambda \partial z = {} & \sum_{i=1}^{N_1} \partial x_i \otimes (c_{j_0} x'_i + (-1)^{\deg u_{j_0}} a_{j_0 i} \partial y'_{j_0}) + c_{j_0} \sum_{i=1}^{N_1} (-1)^{\deg x_i} x_i \otimes \partial x'_i \\ & + c_{j_0} \sum_{j \neq j_0} (-1)^{\deg u_j} u_j \otimes \partial y'_j + \sum_{k=1}^{N_3} \partial y_k \otimes (c_{j_0} u'_k + (-1)^{\deg u_{j_0}} b_{j_0 k} \partial y'_{j_0}). \end{aligned}$$

We may assume that $u_{j_0}, \partial x_1, \cdots, \partial x_{N_1}, \partial y_1, \cdots, \partial y_{N_3}$ are not linearly independent for $N_2'+1 \leq j_0 \leq N_2$. By finite steps, we can reduce the above equation to

$$\lambda_1 \partial z = \sum_{i=1}^{N_1} \partial x_i \otimes \overline{x}_i' + \mu_1 \sum_{i=1}^{N_1} x_i \otimes \partial x_i' + \mu_2 \sum_{j=1}^{N_2'} u_j \otimes \partial y_j' + \sum_{k=1}^{N_3} \partial y_k \otimes \overline{u}_k'$$

for some nonzero elements $\lambda_1, \mu_1, \mu_2 \in R$, where

$$x_1, \cdots, x_{N_1}, \partial x_1, \cdots, \partial x_{N_1}, \partial y_1, \cdots, \partial y_{N_3}, u_1, \cdots, u_{N_2'}$$

are linearly independent. By the above construction, we have that

$$\overline{x}_1', \cdots, \overline{x}_{N_1}', \partial x_1', \cdots, \partial x_{N_1}', \partial y_1', \cdots, \partial y_{N_2'}'$$

are linearly independent. If $\overline{u}_{k_0}', \partial x_1', \cdots, \partial x_{N_1}', \partial y_1', \cdots, \partial y_{N_2'}'$ are not linearly independent, by a similar progress, we can obtain

$$\lambda_2 \partial z = \nu_1 \sum_{i=1}^{N_1} \partial x_i \otimes \overline{x}_i' + \nu_2 \sum_{i=1}^{N_1} \overline{x}_i \otimes \partial x_i' + \nu_3 \sum_{j=1}^{N_2'} \overline{u}_j \otimes \partial y_j' + \nu_4 \sum_{k=1}^{N_3'} \partial y_k \otimes \overline{u}_k'$$

for some nonzero elements $\lambda_2, \nu_1, \nu_2, \nu_3, \nu_4 \in R$, where

$$\overline{x}_1, \cdots, \overline{x}_{N_1}, \partial x_1, \cdots, \partial x_{N_1}, \partial y_1, \cdots, \partial y_{N_3'}, \overline{u}_1, \cdots, \overline{u}_{N_2'}$$

are linearly independent and

$$\overline{x}_1', \cdots, \overline{x}_{N_1}', \partial x_1', \cdots, \partial x_{N_1}', \partial y_1', \cdots, \partial y_{N_2'}', \overline{u}_1', \cdots, \overline{u}_{N_3'}'$$

are linearly independent. Recall that $\partial z \in D \otimes D'$. By Lemma 3.2, we have

$$\overline{x}_1, \cdots, \overline{x}_{N_1}, \partial x_1, \cdots, \partial x_{N_1}, \partial y_1, \cdots, \partial y_{N_3'}, \overline{u}_1, \cdots, \overline{u}_{N_2'} \in D.$$

It follows that

$$x_1, \cdots, x_{N_1}, y_1, \cdots, y_{N_3'} \in D \cap \partial^{-1} D = \mathrm{Inf}_*(D, C).$$

Similarly, we have $x_1', \cdots, x_{N_1}', y_1', \cdots, y_{N_2'}' \in \mathrm{Inf}_*(D', C')$. It implies that

$$\sum_{i=1}^{N_1} x_i \otimes x_i' + \sum_{j=1}^{N_2'} u_j \otimes y_j' + \sum_{k=1}^{N_3'} y_k \otimes u_k' + \sum_{l=1}^{N_4} v_l \otimes v_l' \in \mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C').$$

Let

$$z_1 = \sum_{j=N_2'+1}^{N_2} u_j \otimes y_j' + \sum_{k=N_3'+1}^{N_3} y_k \otimes u_k'.$$

The previous construction implies that $u_{N_2'+1}, \cdots, u_{N_2}$ and $u_{N_3'+1}', \cdots, u_{N_3}'$ are boundaries. By Lemma 3.3, there exists a nonzero element $\lambda' \in R$ such that $\lambda' z_1 \in \mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C')$. Therefore we have

$$\lambda \lambda' z \in \mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C').$$

By Lemma 2.2, there exists a basis $S_1 \sqcup T_1$ of D such that S_1 is a basis of $\mathrm{Inf}_*(D, C)$. Similarly, there is a basis $S_2 \sqcup T_2$ of D' such that S_2 is a basis of $\mathrm{Inf}_*(D', C')$. Let $S = S_1 \otimes S_2$. Thus we can choose a basis $S \sqcup T$ of $D \otimes D'$ such that S is a basis of $\mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C')$. Assume that

$$z = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} e_S \\ e_T \end{pmatrix} \in D \otimes D',$$

where $\mathbf{a} = (a_1, \cdots, a_{|S|}) \in R^{1\times|S|}$, $\mathbf{b} = (b_1, \cdots, b_{|T|}) \in R^{1\times|T|}$. Since $\lambda\lambda' z \in \mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C')$, it follows that

$$\lambda\lambda' \mathbf{b} e_T = 0.$$

Recall that R is a principal ideal domain, we have $\mathbf{b} e_T = 0$. This implies that

$$z \in \mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C'),$$

which gives the desired result.

Example 3.1 Continuing with Example 2.1, let $(C', \partial) = (\mathbb{Z}[x', y'], \partial), \partial y' = x', \partial x' = 0, \deg x' = 1$, and let $D' = \mathbb{Z}[2x', y']$ be a free $\mathbb{Z}$-module generated by $2x', y'$. Then we have

$$\mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C') = \mathbb{Z}[2x, 2y] \otimes \mathbb{Z}[2x', 2y'].$$

A straightforward calculation shows that

$$\mathrm{Inf}_*(D \otimes D', C \otimes C') = \mathbb{Z}[2x \otimes 2x', 2x \otimes 2y', 2y \otimes 2x', 2y \otimes y'].$$

Thus the result in Theorem 3.1 also depends on the condition that D, D' are free R-modules generated by subsets of X, X', respectively.

Theorem 3.2 (see [7, Theorem 3B.5]) *Let R be a principal ideal domain, and let C, C' be chain complexes of free R-modules. Then there is a natural exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \to H_n(C \otimes C') \to \bigoplus_{p+q=n} \mathrm{Tor}_R(H_p(C), H_{q-1}(C')) \to 0.$$

Proof of Theorem 1.1 Note that $R[Y] \otimes R[Y'] \cong R[Y \times Y']$. The theorem follows from Theorems 3.1–3.2.

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