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On the Asymptotic Stability of Wave Equations Coupled by Velocities of Anti-symmetric Type*

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Abstract In this paper, the authors study the asymptotic stability of two wave equations coupled by velocities of anti-symmetric type via only one damping. They adopt the frequency domain method to prove that the system with smooth initial data is logarithmically stable, provided that the coupling domain and the damping domain intersect each other. Moreover, they show, by an example, that this geometric assumption of the intersection is necessary for 1-D case.

Keywords Wave equations, Coupled by velocities, Logarithmic stability
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1 Introduction and Main Results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We are interested in the asymptotic stability of the following system of two wave equations with Dirichlet boundary condition:

$$\begin{cases} y_{tt} - \sum\limits_{j,k=1}^{n} (g^{jk}(x) y_{x_j})_{x_k} + \alpha(x) z_t + \beta(x) y_t = 0 & \text{in } (0,+\infty)\times\Omega, \\ z_{tt} - \sum\limits_{j,k=1}^{n} (g^{jk}(x) z_{x_j})_{x_k} - \alpha(x) y_t = 0 & \text{in } (0,+\infty)\times\Omega, \\ y = z = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ (y, y_t, z, z_t)|_{t=0} = (y^0, y^1, z^0, z^1) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here the coefficients of elliptic operator $g^{jk}(\cdot) \in C^1(\overline{\Omega};\mathbb{R})$ satisfy

$$g^{jk}(x) = g^{kj}(x), \quad \forall x \in \overline{\Omega},\ j,k = 1,2,\cdots,n \tag{1.2}$$

and

$$\sum_{j,k=1}^{n} g^{jk}\xi^j\overline{\xi}^k \geq a|\xi|^2, \quad \forall (x,\xi^1,\cdots,\xi^n) \in \overline{\Omega}\times\mathbb{C}^n \tag{1.3}$$

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for some constant $a > 0$.

We assume that the coupling coefficient $\alpha \in L^\infty(\Omega;\mathbb{R})$ and the damping coefficient $\beta \in L^\infty(\Omega;\mathbb{R})$ are both nonnegative, and furthermore

$$\omega_\alpha \triangleq \{x \in \Omega \mid \alpha(x) \neq 0\} \neq \emptyset \qquad \text{and} \qquad \omega_\beta \triangleq \{x \in \Omega \mid \beta(x) \neq 0\} \neq \emptyset. \tag{1.4}$$

It is classical to consider system (1.1) as the following Cauchy problem in space $\mathcal{H} \triangleq H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$:

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = \mathcal{A}U, \\ U|_{t=0} = U_0 \triangleq (y^0, y^1, z^0, z^1) \in \mathcal{H} \end{cases} \tag{1.5}$$

with $U = (y, u, z, v)$ and the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined as

$$\begin{cases} \mathcal{A}U = \Big(u, \displaystyle\sum_{j,k=1}^n (g^{jk}(x) y_{x_j})_{x_k} - \alpha(x)v - \beta(x)u, v, \sum_{j,k=1}^n (g^{jk}(x) z_{x_j})_{x_k} + \alpha(x)u\Big), \\ \mathcal{D}(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega). \end{cases} \tag{1.6}$$

It is easy to know from the theory of linear operator semigroup (see [21]) that system (1.5) has a unique solution $U(t) = \mathrm{e}^{t\mathcal{A}} U_0$ in $C^0([0, +\infty), \mathcal{H})$. Then, we can define the total energy of system (1.1) by

$$\mathbb{E}(y,z)(t) = \frac{1}{2} \int_\Omega \Big(\sum_{j,k=1}^n g^{jk} y_{x_j} \overline{y}_{x_k} + |y_t|^2 \Big) \mathrm{d}x + \frac{1}{2} \int_\Omega \Big(\sum_{j,k=1}^n g^{jk} z_{x_j} \overline{z}_{x_k} + |z_t|^2 \Big) \mathrm{d}x, \tag{1.7}$$

which implies immediately the equivalence

$$\mathbb{E}(y,z)(t) \sim \|(y, y_t, z, z_t)(t, \cdot)\|_{\mathcal{H}}^2.$$

Obviously, the total energy is non-increasing:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}(y,z)(t) = -\int_\Omega \beta(x) |y_t|^2 \mathrm{d}x \le 0, \quad \forall t \ge 0. \tag{1.8}$$

We are interested in the following questions:

- Under what conditions on α and β, system (1.1) is asymptotically stable?
- If system (1.1) is stable, what is the decay rate of the total energy $\mathbb{E}(y,z)(t)$ as $t \to +\infty$?

More precisely, the main result that we obtain is the following theorem.

Throughout this paper, we use $C = C(\Omega, (g^{jk})_{n\times n}, \alpha, \beta)$ to denote generic positive constants which may vary from line to line unless otherwise stated.

Theorem 1.1 *Assume that* (1.2)–(1.4) *hold. Assume furthermore that there exist a constant* $\delta > 0$ *and a nonempty open subset* $\omega_\delta \subset \omega_\alpha \cap \omega_\beta \subset \Omega$ *such that*

$$\inf_{\omega_\delta} \alpha \ge \delta \qquad \textit{and} \qquad \inf_{\omega_\delta} \beta \ge \delta. \tag{1.9}$$

Then, there exists a constant $C > 0$, *such that for any initial data* $(y^0, y^1, z^0, z^1) \in \mathcal{D}(\mathcal{A})$, *the energy of solution to* (1.1) *satisfies*

$$\mathbb{E}(y,z)(t) \le \frac{C}{\ln(t+2)} \|(y_0, y_1, z_0, z_1)\|_{\mathcal{D}(\mathcal{A})}^2, \quad \forall t \ge 0. \tag{1.10}$$

Moreover, system (1.1) *is strongly stable in* $\mathcal{H}$*, i.e., for any initial data* $(y^0, y^1, z^0, z^1) \in \mathcal{H}$*,*

$$\lim_{t\to+\infty} \mathbb{E}(y,z)(t) = 0. \tag{1.11}$$

Remark 1.1 In Theorem 1.1, if we assume instead that the damping and coupling coefficients α, β are both continuous on $\overline{\Omega}$, then the assumption (1.9) can be simply replaced by the following geometric condition

$$\omega_\alpha \cap \omega_\beta \neq \emptyset. \tag{1.12}$$

Independently, based on frequency domain method and multiplier method, Kassem-Mortada-Toufayli-Wehbe [15] proved strong stability (1.11) when two waves propagate at different speed under the assumption (1.12). One can also refer to [10, Theorem 2.1] for indirect stability results of other coupled wave system by displacements under the same geometric conditions.

Remark 1.2 We provide an example in Section 4 to show that the geometric assumption $\omega_\alpha \cap \omega_\beta \neq \emptyset$ is necessary in general, which is different from the situation with coupling by displacements. One can refer to the open problem raised in [10, Remark 2.2]. As a supplement, we also refer readers to [13, section 5.2.1.3], some numerical examples have been provided to show that for some initial data, system (1.1) seems also strongly stable when $\omega_\alpha \cap \omega_\beta = \emptyset$.

Remark 1.3 The result on logarithmical stability in Theorem 1.1 is sharp. Indeed, if $\alpha \equiv 0$, system (1.1) is decoupled into a dissipative system for (y, y_t) which is only logarithmically stable (see [18]) and a conservative one for (z, z_t). Hence one can not expect a faster decay rate than the logarithmical one for the coupled system (1.1) no matter what the coupling α is.

Remark 1.4 In the setting of Theorem 1.1, similar stability results still hold for system (1.1) with other types of boundary conditions, for instance, Robin conditions or mixed Dirichlet-Neumann conditions (see [10]). However, there are no such stability results for the system with Neumann conditions, since all the constant states are equilibrium of the system and will stay at the equilibrium all the time.

In order to prove the logarithmic stability of system (1.1) with regular initial data in Theorem 1.1, we adopt the frequency domain approach to prove certain spectral estimates of the infinitesimal generator $\mathcal{A}$ of the solution semigroup. One can refer to [6, 18] for the case of single wave equation and [10] for the case of wave systems.

Let us denote the real part and the imaginary part of $\gamma \in \mathbb{C}$ by $\Re\gamma$ and $\Im\gamma$, respectively. We denote also the resolvent set and spectrum of the operator $\mathcal{A}$ by $\rho(\mathcal{A})$ and $Sp(\mathcal{A})$, respectively.

Theorem 1.2 *Suppose that the assumptions of Theorem 1.1 hold. Then there exists a constant* $C > 0$ *such that*

$$\mathcal{O}_C \triangleq \Big\{\gamma \in \mathbb{C} \mid -\frac{\mathrm{e}^{-C|\Im\gamma|}}{C} \le \Re\gamma \le 0\Big\} \cap \Big\{\gamma \in \mathbb{C} \mid |\gamma| \ge \frac{2}{C}\Big\} \subset \rho(\mathcal{A}), \tag{1.13}$$

and the following estimate holds

$$\|(\mathcal{A} - \gamma I)^{-1}\|_{\mathcal{L}(\mathcal{H})} \le C\mathrm{e}^{C|\Im\gamma|}, \quad \forall \gamma \in \mathcal{O}_C. \tag{1.14}$$

Obviously, the energy decay given by (1.11) implies directly the fact $\rho(\mathcal{A}) \subset \{\gamma \in \mathbb{C} \mid \Re\gamma < 0\}$ and in particular, the origin $O \in \rho(\mathcal{A})$. Since $\rho(\mathcal{A})$ is an open set, then the corollary follows from (1.13), upon choosing C large enough, as a byproduct of Theorem 1.2.

Corollary 1.1 *Suppose that the assumptions of Theorem 1.1 hold. Then there exists a constant $C > 0$ such that*

$$Sp(\mathcal{A}) \subset \Big\{\gamma \in \mathbb{C} \mid \Re\gamma < -\frac{\mathrm{e}^{-C|\Im\gamma|}}{C}\Big\}. \tag{1.15}$$

Remark 1.5 The proof of Theorem 1.2 is based on global Carleman estimates (see [11]), which is quite elementary and applied to address many stabilization problems for the system with lower order terms. Moreover, it can be used to obtain explicit bounds on some estimates of decay rate or constant costs in terms of the coefficients. Roughly speaking, (1.14) is equivalent to an observable estimate with constant cost like $\mathrm{e}^{C|\gamma|}$ for coupled elliptic system, which seems quite natural to adopt global Carleman estimates to obtain these types of estimates (see Lemma 3.1 in section 3.1 for more details).

Remark 1.6 We should point out that we can not directly adopt the approach in this paper to obtain the logarithmic stability of system (1.1) when two waves have different propagating speed. Roughly speaking, one key step in the proof of important Lemma 3.1 is using an easy fact that $\partial_s p \cdot [\partial_{ss} q + \partial_j(g^{jk}\partial_k q)] + \partial_s q \cdot [\partial_{ss} p + \partial_j(g^{jk}\partial_k p)] = \partial_s[\partial_s p \partial_s q + p\partial_j(g^{jk}\partial_k \partial_s q)] + \partial_j(g^{jk}\partial_s q \partial_k p) - \partial_j(g^{jk} p \partial_k \partial_s q)$, which can be used to give an estimate that L^2 norm of the coupling term with force terms can control the H^1 energy. However, this fact is invalid for the case of two waves with different propagating speed.

1.1 Previous results

There are a lot of results about asymptotic stability or stabilization of wave equations. Among them, Rauch-Taylor [24] and Bardos-Lebeau-Rauch [5] pointed out that, the single damped wave equation is exponentially stable if and (almost) only if the geometric control condition (GCC for short) is satisfied: There exists $T > 0$ such that every geodesic flow touches the support set of damping term before T. If the damping acts on a small open set but the GCC is not satisfied, Lebeau [18] and Burq [6] proved that the wave equation is logarithmically stable for regular initial data. There are also many results about polynomial stability of a single wave equation with special condition on the damping domain (see [4, 7, 17, 23]). Recently, Jin proved in [14] that the damped wave equation on hyperbolic surface with constant curvature is exponentially stable even if the damping domain is arbitrarily small.

As for the case of coupled wave equations or other reversible equations, indirect stability is an important issue both in mathematical theory and in engineering application. Indeed, it arises whenever it is impossible or too expensive to damp all the components of the state, and it is hence important to study stabilization properties of coupled systems with a reduced number of feedbacks. For finite dimensional systems, it is fully understood thanks to the Kalman rank condition. While in the case of coupled partial differential equations, the situation is much more complicated. It depends not only on the algebraic structure of coupling but also the geometric properties of the damping and coupling domain.

Alabau-Boussouira first studied indirect stability of a weekly coupled wave system where the coupling is through the displacements. In [1], the author adopted multiplier method to obtain polynomial stability for wave system with anti-symmetric type coupling under stronger geometric conditions for both the coupling and damping terms. Moreover, she proved that this result was sharp for coupling with displacement. In [2], the polynomial stability results for coupled systems under an abstract framework (including wave-wave system, wave-plate system etc.) were obtained under the conditions that both coupling and damping are localized and satisfy the piecewise multipliers geometric conditions (PMGC for short, see [19]). For 1-D case, a sharp decay rate of polynomial stability was obtained by Riesz basis method in [20]. In [10], Fu adopted global Carleman estimates and frequency domain method to prove that the system with coupling by displacement of symmetric type is logarithmically stable with the assumption that coupling domain intersects the damping domain.

The above results concern only the weakly coupled system. In [3], Alabau-Wang-Yu studied the indirect stability for wave equations coupled by velocities with a general nonlinear damping. By multiplier method, they obtained various types of stability results, including exponential stability, under strong geometric condition on the coupling and damping domains. They also pointed out, for the first time, that it is more efficient to transfer the energy in case the of coupling by velocities compared to the case with coupling by displacements. For 1-D case with constant coefficients, the sharp decay rate was explicitly given in [8]. In [16], Klein computed the best exponent for the stabilization of wave equations on compact manifolds. The coefficient he obtained is therefore the solution of some ODE system of matrices. Kassem-Mortada-Toufayli-Wehbe [15] studied a system of two wave equations coupled by velocities with only one localized damping, the waves propagate at different speed and the positivity and smallness assumptions of the continuous coupling coefficient can be removed. They obtained a strong stability result with the assumption that coupling domain intersects the damping domain. Moreover, assuming that coupling and damping coefficient belong to $W^{1,\infty}(\Omega)$ and the intersection of coupling domain and damping domain holds PMGC, based on frequency domain method and a multiplier method, they established an exponential energy decay when the waves propagate at the same speed and a polynomial energy decay when the waves propagate at the different speed. Recently, the exponential energy decay result has been generalized by Gerbi-Kassem-Mortada-Wehbe in [13] to the case that the intersection of the coupling and the damping domain satisfies GCC.

To the authors' knowledge, most known indirect stability results are obtained under the geometric conditions that the damping domain intersects the coupling domain. Indeed, this guarantees effectively the energy transmission in higher space dimension. It is remarkable that Alabau-Boussouira and Léauteau [2] proved an indirect stability result in 1-D case where the damping domain and the coupling domain are two intervals which do not intersect.

1.2 Main contribution and ideas

As already pointed out in [3], the energy transition is more efficient through the first order coupling (by velocities) compared to zero order coupling (by displacements). This is natural since the first order coupling effect can be seen as a bounded perturbation to the system while the zero order coupling is a compact one. Nevertheless, one can not expect a faster decay (than logarithmical one) of the whole system (1.1) even if a first order coupling appears, because the energy of the single wave equation with damping localized in small domain only decays logarithmically. In this sense, the stability results are sharp.

Not surprisingly, the indirect stability result is obtained by assuming essentially the damping and coupling domain intersect. However, we give an example to show that this geometric condition is necessary in general for the wave system coupled by velocities. This is quite different from the system coupled by displacements (see [2]).

As for the proof of the main theorems, we adopt the frequency domain approach to reduce the stability problem to an estimate on resolvent which can be obtained by global Carleman estimates of an elliptic equation as in [10]. Different from the system in [10], there are no zero order terms explicitly in system (1.1). In order to derive the L^2 energy of the solution, we then need to make fully use of the coupling structure together with Poincaré inequality under homogeneous Dirichlet boundary condition.

1.3 Organization of the paper

This paper is organized as follows. In Section 2, we recall some basic facts about frequency domain method and global Carlemann estimates for an elliptic equation. In Section 3, we give the proofs of Theorem 1.1 and Theorem 1.2 as well as the technical Lemma 3.1, which is crucial to the proof of Theorem 1.2. Finally in Section 4, we give an example of system (1.1) with $\omega_\alpha \cap \omega_\beta = \emptyset$, which is indeed unstable.

2 Preliminaries

In this section, we briefly recall the frequency domain method and global Carleman estimates for an elliptic equation.

2.1 Frequency domain method

Thanks to classical semigroup theory, $\mathcal{A}$ generates a C_0-semigroup operator $\{\mathrm{e}^{t\mathcal{A}}\}_{t\geq 0}$ on $\mathcal{H}$. It is well-known that the logarithmic stability of system (1.1) can be obtained by a resolvent estimate (see [6, 18]). More precisely, we have the following lemma.

Lemma 2.1 (see [6, Theorem 3]) *Let $\mathcal{A}$ be defined by* (1.6). *If*

$$\|(\mathcal{A} - \mathrm{i}\sigma I)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C\mathrm{e}^{C|\sigma|}, \quad \forall \sigma \in \mathbb{R}, |\sigma| > 1, \tag{2.1}$$

then, there exists $C > 0$ such that for any $U_0 \in \mathcal{D}(\mathcal{A}^2) \triangleq \{U \in \mathcal{H} \mid \mathcal{A}U \in \mathcal{D}(\mathcal{A})\}$,

$$\|\mathrm{e}^{t\mathcal{A}}U_0\|_{\mathcal{H}} \leq \Big(\frac{C}{\ln(t+2)}\Big)^2 \|U_0\|_{\mathcal{D}(\mathcal{A}^2)}, \quad \forall t \geq 0. \tag{2.2}$$

Obviously, (1.14) in Theorem 1.2 implies the assumption (2.1) in Lemma 2.1. Once Theorem 1.2 is proved, the logarithmical decay estimate (1.10) in Theorem 1.1 can be easily obtained by Calderon-Lions interpolation theorem and (2.2).

2.2 Global Carleman estimates

To obtain resolvent estimate (1.14), we need to introduce the global Carlemann estimates for elliptic equations (see [9, 10, 12]).

Let ω_0 be an open set such that $\omega_0 \subset\subset \omega_\delta \subset \omega_\alpha \cap \omega_\beta$. There exists $\widehat{\psi} \in C^2(\overline{\Omega}; \mathbb{R})$ such that

$$\widehat{\psi} > 0 \ \text{ in } \Omega, \qquad \widehat{\psi} = 0 \ \text{ on } \partial\Omega \qquad \text{and} \qquad |\nabla \widehat{\psi}| > 0 \ \text{ in } \overline{\Omega \backslash \omega_0}. \tag{2.3}$$

Next, we introduce some weight functions

$$\theta = \mathrm{e}^{l}, \quad l = \lambda\phi, \quad \phi = \mathrm{e}^{\mu\psi} \tag{2.4}$$

with

$$\psi = \psi(s,x) \triangleq \frac{\widehat{\psi}(x)}{\|\widehat{\psi}\|_{L^\infty}} + b^2 - s^2, \quad s \in [-b,b], x \in \overline{\Omega}, \tag{2.5}$$

where $b > 0$ and $\lambda, \mu, s \in \mathbb{R}$ are all constants.

Let us consider a single elliptic equation

$$\begin{cases} w_{ss} + \sum\limits_{j,k=1}^{n} (g^{jk} w_{x_j})_{x_k} = f & \text{in } (-b,b)\times\Omega, \\ w = 0 & \text{on } (-b,b)\times\partial\Omega, \\ w(\pm b, \cdot) = 0 & \text{in } \Omega, \end{cases} \tag{2.6}$$

where $g^{jk}(\cdot) \in C^1(\Omega;\mathbb{R})$ satisfy (1.2)–(1.3). Therefore for every $f \in L^2((-b,b)\times\Omega)$, the elliptic system (2.6) has a unique solution $w \in H_0^1((-b,b)\times\Omega)$. Therefore we have the following global Carleman estimates of solution.

Theorem 2.1 (see [10]) *Let $b \in (1,2]$ and $\theta, \phi \in C^2([-b,b]\times\overline{\Omega};\mathbb{R})$ be defined by (2.4)–(2.5). Then there exists $\mu_0 > 0$, such that for any $\mu \geq \mu_0$, there exist $C = C(\mu) > 0$ and $\lambda_0 = \lambda_0(\mu)$ such that for any $f \in L^2((-b,b)\times\Omega)$, the solution w to system (2.6) satisfies*

$$\begin{aligned} &\lambda\mu^2 \int_{-b}^{b}\int_\Omega \theta^2\phi(a|\nabla w|^2 + |w_s|^2 + \lambda^2\mu^2\phi^2|w|^2)\mathrm{d}x\mathrm{d}s \\ &\leq C\int_{-b}^{b}\int_\Omega \theta^2|f|^2\mathrm{d}x\mathrm{d}s + C\lambda\mu^2\int_{-b}^{b}\int_{\omega_0} \theta^2\phi(|\nabla w|^2 + |w_s|^2 + \lambda^2\mu^2\phi^2|w|^2)\mathrm{d}x\mathrm{d}s \end{aligned} \tag{2.7}$$

for all $\lambda \geq \lambda_0(\mu)$.

3 Proofs of Main Theorems

In this section, we give the proofs of the main theorems, i.e., Theorem 1.1 and Theorem 1.2. First in Subsection 3.1, we prove Theorem 1.2, particularly the resolvent estimate (3.16), based on some interpolation estimates on elliptic equations. Then by Theorem 1.2 and Calderon-Lions interpolation inequality, we conclude Theorem 1.1 in Subsection 3.2. Finally in Subsection 3.3, we prove Lemma 3.1 concerning an interpolation inequality of coupled elliptic equations, which is crucial to the proof of Theorem 1.2.

3.1 Proof of Theorem 1.2

Let $F = (f^0, f^1, g^0, g^1) \in \mathcal{H}$ and $U_0 = (y^0, y^1, z^0, z^1) \in \mathcal{D}(\mathcal{A})$ be such that

$$(\mathcal{A} - \gamma I)U_0 = F, \tag{3.1}$$

where $\gamma \in \mathbb{C}$ and $\mathcal{A}$ is given by (1.6). Then (3.1) is equivalent to

$$\begin{cases}-\gamma y^0 + y^1 = f^0 & \text{in } \Omega,\\ \sum_{j,k=1}^{n} (g^{jk} y^0_{x_j})_{x_k} - \alpha(x) z^1 - (\beta(x)+\gamma) y^1 = f^1 & \text{in } \Omega,\\ -\gamma y^0 + y^1 = g^0 & \text{in } \Omega,\\ \sum_{j,k=1}^{n} (g^{jk} z^0_{x_j})_{x_k} - \gamma y^1 + \alpha(x) y^1 = g^1 & \text{in } \Omega,\\ y^0 = z^0 = 0 & \text{on } \partial\Omega,\end{cases}$$

or furthermore

$$\begin{cases}\sum_{j,k=1}^{n} (g^{jk} y^0_{x_j})_{x_k} - \gamma^2 y^0 - \gamma\alpha(x) z^0 - \gamma\beta(x) y^0 = F^0 \triangleq (\beta(x)+\gamma) f^0 + f^1 & \text{in } \Omega,\\ \sum_{j,k=1}^{n} (g^{jk} z^0_{x_j})_{x_k} - \gamma^2 z^0 + \gamma\alpha(x) z^0 = F^1 \triangleq \gamma g^0 + g^1 & \text{in } \Omega,\\ y^0 = z^0 = 0 & \text{on } \partial\Omega,\\ y^1 = f^0 + \gamma y^0,\ z^1 = g^0 + \gamma z^0 & \text{in } \Omega.\end{cases} \tag{3.2}$$

In order to prove (1.14), it suffices to prove that there exists a constant $C > 0$, such that

$$\|(y^0, y^1, z^0, z^1)\|_{\mathcal{H}} \le C\mathrm{e}^{C|\Im\gamma|} \|(f^0, f^1, g^0, g^1)\|_{\mathcal{H}}, \quad \forall \gamma \in \mathcal{O}_C. \tag{3.3}$$

For this purpose, we set

$$p(s,x) = \mathrm{e}^{\mathrm{i}\gamma s} y^0(x), \quad q(s,x) = \mathrm{e}^{\mathrm{i}\gamma s} z^0(x), \quad (s,x) \in (-2,2)\times\Omega. \tag{3.4}$$

Then p and q satisfy the following coupled elliptic align

$$\begin{cases} p_{ss} + \sum_{j,k=1}^{n} (g^{jk} p_{x_j})_{x_k} + \mathrm{i}\alpha(x) q_s + \mathrm{i}\beta(x) p_s = G^0 \triangleq F^0 \mathrm{e}^{\mathrm{i}\gamma s} & \text{in } (-2,2)\times\Omega,\\ q_{ss} + \sum_{j,k=1}^{n} (g^{jk} q_{x_j})_{x_k} - \mathrm{i}\alpha(x) p_s = G^1 \triangleq F^1 \mathrm{e}^{\mathrm{i}\gamma s} & \text{in } (-2,2)\times\Omega,\\ p = q = 0 & \text{on } (-2,2)\times\partial\Omega.\end{cases} \tag{3.5}$$

Note that there are no boundary conditions on $s = \pm 2$ in the above system (3.5). We have the following lemma on interpolation estimate, while its proof is left in Subsection 3.3.

Lemma 3.1 *Under the assumption of Theorem 1.1, there exists a constant $C > 0$ such that for any $\lambda > 0$ big enough, the solution (p,q) to (3.5) with form (3.4) satisfies*

$$\begin{aligned}\|p\|_{H^1(Y)} + \|q\|_{H^1(Y)} \le{}& C\mathrm{e}^{C\lambda}(\|G^0\|_{L^2(X)} + \|G^1\|_{L^2(X)} + \|p\|_{H^1(-2,2;L^2(\omega_\delta))})\\ &+ C\mathrm{e}^{-2\lambda}(\|p\|_{H^1(X)} + \|q\|_{H^1(X)}),\end{aligned} \tag{3.6}$$

where

$$X \triangleq (-2,2)\times\Omega, \quad Y \triangleq (-1,1)\times\Omega, \quad Z \triangleq (-2,2)\times\omega_\delta. \tag{3.7}$$

On the other hand, by (3.4), we have

$$\begin{cases}\|y^0\|_{H_0^1(\Omega)}+\|z^0\|_{H_0^1(\Omega)}\le C\mathrm{e}^{C|\Im\gamma|}(\|p\|_{H^1(-1,1;H_0^1(\Omega))}+\|q\|_{H^1(-1,1;H_0^1(\Omega))}),\\ \|p\|_{H^1(-2,2;H_0^1(\Omega))}+\|q\|_{H^1(-2,2;H_0^1(\Omega))}\le C\mathrm{e}^{C|\Im\gamma|}(|\gamma|+1)(\|y^0\|_{H_0^1(\Omega)}+\|z^0\|_{H_0^1(\Omega)}),\\ \|p\|_{H^1(-2,2;L^2(\omega_\delta))}\le C\mathrm{e}^{C|\Im\gamma|}|\gamma|\|y^0\|_{L^2(\omega_\delta)}\end{cases}\tag{3.8}$$

for some constant $C>0$. Combining (3.6) and (3.8), we get

$$\|y^0\|_{H_0^1(\Omega)}+\|z^0\|_{H_0^1(\Omega)}\le C\mathrm{e}^{C|\Im\gamma|}(\|(f^0,f^1,g^0,g^1)\|_{\mathcal{H}}+\|y^0\|_{L^2(\omega_\delta)}).\tag{3.9}$$

Next, we turn to estimate $\|y^0\|_{L^2(\omega_\delta)}$. Let $\zeta\in C_0^2(\Omega;\mathbb{R})$ be a cutoff function such that

$$0\le\zeta(x)\le1\quad\text{in }\Omega\qquad\text{and}\qquad\zeta(x)\equiv1\text{ in }\omega_\delta\subset\omega_\alpha\cap\omega_\beta.\tag{3.10}$$

Multiplying y-equation in (3.2) by $2\zeta\overline{y}^0$ and integrating by parts on Ω yield that

$$\begin{aligned}&\int_\Omega\Big(-\sum_{j,k=1}^n(g^{jk}y^0_{x_j})_{x_k}+\gamma^2y^0+\gamma\alpha(x)z^0+\gamma\beta(x)y^0\Big)\cdot2\zeta\overline{y}^0\mathrm{d}x\\ =\;&2\gamma^2\int_\Omega\zeta|y^0|^2\mathrm{d}x+2\int_\Omega\zeta\sum_{j,k=1}^n g^{jk}y^0_{x_j}\overline{y}^0_{x_k}\mathrm{d}x-\int_\Omega\sum_{j,k=1}^n(g^{jk}\zeta_{x_j})_{x_k}|y^0|^2\mathrm{d}x\\ &+2\gamma\int_\Omega\alpha(x)\zeta\overline{y}^0z^0\mathrm{d}x+2\gamma\int_\Omega\beta(x)\zeta|y^0|^2\mathrm{d}x\\ =\;&\int_\Omega F^0\cdot2\zeta\overline{y}^0\mathrm{d}x.\end{aligned}\tag{3.11}$$

Similarly, multiplying z-equation in (3.2) by $2\zeta\overline{z}^0$ and integrating by parts on Ω yield that

$$\begin{aligned}&\int_\Omega\Big(-\sum_{j,k=1}^n(g^{jk}z^0_{x_j})_{x_k}+\gamma^2z^0-\gamma\alpha(x)y^0\Big)\cdot2\zeta\overline{z}^0\mathrm{d}x\\ =\;&2\gamma^2\int_\Omega\zeta|z^0|^2\mathrm{d}x+2\int_\Omega\zeta\sum_{j,k=1}^n g^{jk}z^0_{x_j}\overline{z}^0_{x_k}\mathrm{d}x-\int_\Omega\sum_{j,k=1}^n(g^{jk}\zeta_{x_j})_{x_k}|z^0|^2\mathrm{d}x\\ &-2\gamma\int_\Omega\alpha(x)\zeta y^0\overline{z}^0\mathrm{d}x\\ =\;&\int_\Omega F^1\cdot2\zeta\overline{z}^0\mathrm{d}x.\end{aligned}\tag{3.12}$$

Note that

$$\Im(\gamma\overline{y}^0z^0-\gamma y^0\overline{z}^0)=\Im[\gamma(\overline{y}^0z^0-y^0\overline{z}^0)]=\Re\gamma\cdot2\Im(\overline{y}^0z^0).$$

Adding (3.11) to (3.12) and taking the imaginary part result in

$$\begin{aligned}&4\Re\gamma\Im\gamma\int_\Omega\zeta(|z^0|^2+|y^0|^2)\mathrm{d}x+2\Im\gamma\int_\Omega\zeta\beta(x)|y^0|^2\mathrm{d}x+2\Re\gamma\int_\Omega\alpha\zeta\Im(\overline{y}^0z^0)\mathrm{d}x\\ =\;&\int_\Omega2\zeta\cdot\Im(F^0\overline{y}^0+F^1\overline{z}^0)\mathrm{d}x.\end{aligned}$$

Then it follows by Cauchy inequality and the definition of β,F^0,F^1 that

$$|\Im\gamma|\int_{\omega_\delta}\beta(x)|y^0|^2\mathrm{d}x\le|\Im\gamma|\int_\Omega\zeta\beta(x)|y^0|^2\mathrm{d}x$$

$$\begin{aligned}&\le C|\Re\gamma|(|\Im\gamma|+1)(\|y^0\|_{L^2(\Omega)}^2+\|z^0\|_{L^2(\Omega)}^2)\\&\quad+C(\|F^0\|_{L^2(\Omega)}^2+\|F^1\|_{L^2(\Omega)}^2)\\&\le C|\Re\gamma|(|\Im\gamma|+1)(\|y^0\|_{L^2(\Omega)}^2+\|z^0\|_{L^2(\Omega)}^2)\\&\quad+C(|\gamma|+1)\|(f^0,f^1,g^0,g^1)\|_{\mathcal{H}}^2.\end{aligned}$$

Thus by (3.9), we get

$$\begin{aligned}|\Im\gamma|\int_{\omega_\delta}\beta(x)|y^0|^2\mathrm{d}x&\le C|\Re\gamma|(|\Im\gamma|+1)\mathrm{e}^{C|\Im\gamma|}\|y^0\|_{L^2(\omega_\delta)}^2\\&\quad+C(|\Re\gamma||\Im\gamma|+|\gamma|+1)\mathrm{e}^{C|\Im\gamma|}\|(f^0,f^1,g^0,g^1)\|_{\mathcal{H}}^2.\end{aligned}\tag{3.13}$$

By the definition of $\mathcal{O}_C$, we take $C>0$ large enough such that

$$C|\Re\gamma|(|\Im\gamma|+1)\mathrm{e}^{C|\Im\gamma|}\le\frac{\delta|\Im\gamma|}{2}\quad\text{and}\quad\Im\gamma>\frac{1}{C}\tag{3.14}$$

for all $\gamma\in\mathcal{O}_C$. Note also the fact that $\beta(x)\ge\delta$ a.e. in ω_δ. Then it follows from (3.13)–(3.14) that

$$\|y^0\|_{L^2(\omega_\delta)}\le C\mathrm{e}^{C|\Im\gamma|}\|(f^0,f^1,g^0,g^1)\|_{\mathcal{H}}\tag{3.15}$$

for some $C>0$ large enough. Combining (3.9) and (3.15) gives

$$\|y^0\|_{H_0^1(\Omega)}+\|z^0\|_{H_0^1(\Omega)}\le C\mathrm{e}^{C|\Im\gamma|}\|(f^0,f^1,g^0,g^1)\|_{\mathcal{H}}.$$

Since $y^1=f^0+\gamma y^0, z^1=g^0+\gamma z^0$, we have also

$$\begin{aligned}\|y^1\|_{L^2(\Omega)}+\|z^1\|_{L^2(\Omega)}&\le\|f^0\|_{H_0^1(\Omega)}+\|g^0\|_{H_0^1(\Omega)}+|\gamma|(\|y^0\|_{L^2(\Omega)}+\|z^0\|_{L^2(\Omega)})\\&\le C\mathrm{e}^{C|\Im\gamma|}\|(f^0,f^1,g^0,g^1)\|_{\mathcal{H}}.\end{aligned}$$

Hence the desired estimate (3.3) indeed holds for all $\gamma\in\mathcal{O}_C$.

Consequently, $\mathcal{A}-\gamma I$ is a bijection from $\mathcal{D}(\mathcal{A})$ to $\mathcal{H}$, which satisfies the resolvent estimate (1.14). We conclude the proof of Theorem 1.2.

3.2 Proof of Theorem 1.1

As a corollary of Theorem 1.2, there exists $C>0$ such that

$$\|(\mathcal{A}-\mathrm{i}\sigma I)^{-1}\|_{L^2(\mathcal{H})}\le C\mathrm{e}^{C|\sigma|},\quad\forall\sigma\in\mathbb{R},|\sigma|>1.\tag{3.16}$$

Then by Lemma 2.1, we have for $U_0\in\mathcal{D}(\mathcal{A}^2)$ that

$$\|\mathrm{e}^{t\mathcal{A}}U_0\|_{\mathcal{H}}\le\Big(\frac{C}{\ln(t+2)}\Big)^2\|U_0\|_{\mathcal{D}(\mathcal{A}^2)},\quad\forall t\ge0.\tag{3.17}$$

On the other hand, the contraction of the semigroup $\mathrm{e}^{t\mathcal{A}}$ implies that

$$\|\mathrm{e}^{t\mathcal{A}}U_0\|_{\mathcal{H}}\le\|U_0\|_{\mathcal{H}},\quad\forall t\ge0.\tag{3.18}$$

Note that $\mathcal{D}(\mathcal{A})$ is an interpolate space between $\mathcal{D}(\mathcal{A}^2)$ and $\mathcal{H}$. Combining (3.17)–(3.18) and using Calderon-Lions interpolation theorem (see [25, p. 38, Example 1 and p. 44, Proposition 8]), we conclude for all $U_0 \in \mathcal{D}(\mathcal{A})$ that

$$\|\mathrm{e}^{t\mathcal{A}}U_0\|_{\mathcal{H}} \le \frac{C}{\ln(t+2)}\|U_0\|_{\mathcal{D}(\mathcal{A})}, \quad \forall t \ge 0, \tag{3.19}$$

which is equivalent to the logarithmical decay estimate (1.10).

Finally we conclude by (3.19) and density argument that system (1.1) is strongly stable, i.e., for all $U_0 \in \mathcal{H}$,

$$\lim_{t\to+\infty} \|\mathrm{e}^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

The proof of Theorem 1.2 is complete.

3.3 Proof of Lemma 3.1

In this subsection, we give the proof of Lemma 3.1 which plays a key role in proving Theorem 1.2. The proof is divided into 6 steps. In this subsection, we denote $C > 0$ various constants independent of λ which can be different from one line to another.

Step 1 We derive a weighted estimate (3.28) for (p, q), the solution of (3.5).

Note that there are no boundary conditions on $p(\pm 2, \cdot), q(\pm 2, \cdot)$ in (3.5). Let us introduce a cutoff function $\varphi = \varphi(s) \in C_0^3((-b, b); \mathbb{R})$ (see for instance [22]) such that

$$0 \le \varphi(s) \le 1 \ \text{ in } [-b, b] \qquad \text{and} \qquad \varphi(s) \equiv 1 \ \text{ in } [-b_0, b_0], \tag{3.20}$$

where the constants b_0, b are given by

$$b \triangleq \sqrt{1 + \frac{1}{\mu}\ln(2 + \mathrm{e}^{\mu})}, \quad b_0 \triangleq \sqrt{b^2 - \frac{1}{\mu}\ln\Big(\frac{1+\mathrm{e}^{\mu}}{\mathrm{e}^{\mu}}\Big)}, \tag{3.21}$$

respectively, for μ large enough which enables to apply Theorem 2.1. Obviously, if $\mu > \ln 2$, then

$$1 < b_0 < b < 2. \tag{3.22}$$

Let

$$\widehat{p}(s, x) = \varphi(s)p(s, x), \quad \widehat{q}(s, x) = \varphi(s)q(s, x), \quad (s, x) \in (-b, b) \times \Omega. \tag{3.23}$$

Then, we consider the elliptic equations that $\widehat{p}, \widehat{q}$ satisfy in $(-b, b) \times \Omega$

$$\begin{cases} \widehat{p}_{ss} + \sum\limits_{j,k=1}^{n} (g^{jk}\widehat{p}_{x_j})_{x_k} = \widehat{G}^0 & \text{in } (-b, b) \times \Omega, \\ \widehat{q}_{ss} + \sum\limits_{j,k=1}^{n} (g^{jk}\widehat{q}_{x_j})_{x_k} = \widehat{G}^1 & \text{in } (-b, b) \times \Omega, \\ \widehat{p} = \widehat{q} = 0 & \text{on } (-b, b) \times \partial\Omega, \\ \widehat{p}(\pm b, \cdot) = \widehat{q}(\pm b, \cdot) = 0 & \text{in } \Omega, \end{cases} \tag{3.24}$$

where

$$\begin{aligned}\widehat{G}^0 &\triangleq \varphi_{ss}p + 2\varphi_s p_s + \varphi G^0 + \mathrm{i}\alpha(x)(\varphi_s q - \widehat{q}_s) + \mathrm{i}\beta(x)(\varphi_s p - \widehat{p}_s),\\ \widehat{G}^1 &\triangleq \varphi_{ss}q + 2\varphi_s q_s + \varphi G^1 - \mathrm{i}\alpha(x)(\varphi_s p - \widehat{p}_s).\end{aligned} \tag{3.25}$$

By applying Theorem 2.1 to both $\widehat{p}$ and $\widehat{q}$, there exists $\mu_0 > \ln 2$ such that for any $\mu \geq \mu_0$, there exist $C = C(\mu) > 0$ and $\lambda_0 = \lambda_0(\mu)$ such that for any $\lambda \geq \lambda_0(\mu)$, we have

$$\begin{aligned}&\lambda\mu^2 \int_{-b}^{b}\int_\Omega \theta^2\phi(a|\nabla\widehat{p}|^2 + |\widehat{p}_s|^2 + \lambda^2\mu^2\phi^2|\widehat{p}|^2)\mathrm{d}x\mathrm{d}s\\ &\leq C\int_{-b}^{b}\int_\Omega \theta^2|\widehat{G}^0|^2\mathrm{d}x\mathrm{d}s + C\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\phi(|\nabla\widehat{p}|^2 + |\widehat{p}_s|^2 + \lambda^2\mu^2\phi^2|\widehat{p}|^2)\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{3.26}$$

$$\begin{aligned}&\lambda\mu^2 \int_{-b}^{b}\int_\Omega \theta^2\phi(a|\nabla\widehat{q}|^2 + |\widehat{q}_s|^2 + \lambda^2\mu^2\phi^2|\widehat{q}|^2)\mathrm{d}x\mathrm{d}s\\ &\leq C\int_{-b}^{b}\int_\Omega \theta^2|\widehat{G}^1|^2\mathrm{d}x\mathrm{d}s + C\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\phi(|\nabla\widehat{q}|^2 + |\widehat{q}_s|^2 + \lambda^2\mu^2\phi^2|\widehat{q}|^2)\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{3.27}$$

where $\widehat{G}^0, \widehat{G}^1$ are given by (3.25) and $\omega_0 \subset\subset \omega_\delta \subset \omega_\alpha \cap \omega_\beta$. Adding (3.26) to (3.27) gives

$$\mathbf{I_0} \leq \mathbf{I_1} + \mathbf{I_2}, \tag{3.28}$$

where

$$\begin{aligned}\mathbf{I_0} &\triangleq \lambda\mu^2\int_{-b}^{b}\int_\Omega \theta^2\phi(a|\nabla\widehat{p}|^2 + |\widehat{p}_s|^2 + \lambda^2\mu^2\phi^2|\widehat{p}|^2 + a|\nabla\widehat{q}|^2 + |\widehat{q}_s|^2 + \lambda^2\mu^2\phi^2|\widehat{q}|^2)\mathrm{d}x\mathrm{d}s,\\ \mathbf{I_1} &\triangleq C\int_{-b}^{b}\int_\Omega \theta^2(|\widehat{G}^0|^2 + |\widehat{G}^1|^2)\mathrm{d}x\mathrm{d}s,\\ \mathbf{I_2} &\triangleq C\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\phi(|\nabla\widehat{p}|^2 + |\widehat{p}_s|^2 + \lambda^2\mu^2\phi^2|\widehat{p}|^2 + |\nabla\widehat{q}|^2 + |\widehat{q}_s|^2 + \lambda^2\mu^2\phi^2|\widehat{q}|^2)\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.29}$$

Step 2 We estimate $\mathbf{I_0}$ in (3.29) from below.

By the choices of θ, l, ϕ in (2.4) and b, b_0 in (3.21), we have

$$\theta \geq \mathrm{e}^{\lambda(2+\mathrm{e}^\mu)}, \quad \forall |s| \leq 1, x \in \Omega. \tag{3.30}$$

Then

$$\begin{aligned}\mathbf{I_0} &\geq \lambda\mu^2\int_{-b_0}^{b_0}\int_\Omega \theta^2\phi(a|\nabla p|^2 + |p_s|^2 + \lambda^2\mu^2\phi^2|p|^2 + a|\nabla q|^2 + |q_s|^2 + \lambda^2\mu^2\phi^2|q|^2)\mathrm{d}x\mathrm{d}s\\ &\geq \lambda\mathrm{e}^{2\lambda(2+\mathrm{e}^\mu)}C(\mu)\int_{-1}^{1}\int_\Omega (|\nabla p|^2 + |p_s|^2 + |p|^2 + |\nabla q|^2 + |q_s|^2 + |q|^2)\mathrm{d}x\mathrm{d}s\\ &= \lambda\mathrm{e}^{2\lambda(2+\mathrm{e}^\mu)}C(\mu)(\|p\|^2_{H^1(Y)} + \|q\|^2_{H^1(Y)}).\end{aligned} \tag{3.31}$$

Step 3 We estimate $\mathbf{I_1}$ in (3.29) from above.

Using Cauchy inequality, we get easily

$$\mathbf{I_1} \leq \mathbf{I_{11}} + \mathbf{I_{12}} + \mathbf{I_{13}}, \tag{3.32}$$

where

$$\begin{aligned}
\mathbf{I_{11}} &\triangleq C\int_{-b}^{b}\int_{\Omega}\theta^2(|\widehat{G}_1^0|^2+|\widehat{G}_1^1|^2)\mathrm{d}x\mathrm{d}s,\\
\mathbf{I_{12}} &\triangleq C\int_{-b}^{b}\int_{\Omega}\theta^2(|\mathrm{i}\alpha(x)\widehat{q}_s+\mathrm{i}\beta(x)\widehat{p}_s|^2+|\mathrm{i}\alpha(x)\widehat{p}_s|^2)\mathrm{d}x\mathrm{d}s,\\
\mathbf{I_{13}} &\triangleq C\int_{-b}^{b}\int_{\Omega}\theta^2\varphi^2(|G^0|^2+|G^1|^2)\mathrm{d}x\mathrm{d}s,
\end{aligned}\tag{3.33}$$

where $\widehat{G}_1^0=\varphi_{ss}p+2\varphi_sp_s+\mathrm{i}\alpha(x)\varphi_sq+\mathrm{i}\beta(x)\varphi_sp$, $\widehat{G}_1^1=\varphi_{ss}q+2\varphi_sq_s-\mathrm{i}\alpha(x)\varphi_sp$, and we denote $\widehat{G}_1^0,\widehat{G}_1^1$ some terms concerning the derivatives of φ in $\widehat{G}^0,\widehat{G}^1$, which are useful for the estimation below.

By the choices of θ,l,ϕ in (2.4) and b,b_0 in (3.21), we have

$$\theta\le \mathrm{e}^{\lambda(1+\mathrm{e}^{\mu})},\quad \forall b_0\le|s|\le b,x\in\Omega.\tag{3.34}$$

Then

$$\begin{aligned}
\mathbf{I_{11}} &= C\int_{(-b,-b_0)\bigcup(b_0,b)}\int_{\Omega}\theta^2(|\varphi_{ss}p+2\varphi_sp_s+\mathrm{i}\alpha(x)\varphi_sq+\mathrm{i}\beta(x)\varphi_sp|^2\\
&\quad+|\varphi_{ss}q+2\varphi_sq_s-\mathrm{i}\alpha(x)\varphi_sp|^2)\mathrm{d}x\mathrm{d}s\\
&\le \mathrm{e}^{2\lambda(1+\mathrm{e}^{\mu})}C(\mu)\int_{(-b,-b_0)\bigcup(b_0,b)}\int_{\Omega}(|p|^2+|p_s|^2+|q|^2+|q_s|^2)\mathrm{d}x\mathrm{d}s\\
&\le \mathrm{e}^{2\lambda(1+\mathrm{e}^{\mu})}C(\mu)(\|p\|^2_{H^1(X)}+\|q\|^2_{H^1(X)}).
\end{aligned}\tag{3.35}$$

Obviously,

$$\mathbf{I_{12}}\le C\int_{-b}^{b}\int_{\Omega}\theta^2(|\widehat{p}_s|^2+|\widehat{q}_s|^2)\mathrm{d}x\mathrm{d}s\le\frac{C(\mu)}{\lambda}\mathbf{I_0}\le\lambda^{-\frac12}\mathbf{I_0},\tag{3.36}$$

which can be absorbed by $\mathbf{I_0}$ if λ is large enough.

Therefore,

$$\begin{aligned}
\mathbf{I_1} &\le \mathrm{e}^{2\lambda(1+\mathrm{e}^{\mu})}C(\mu)(\|p\|^2_{H^1(X)}+\|q\|^2_{H^1(X)})+\lambda^{-\frac12}\mathbf{I_0}\\
&\quad+C\int_{-b}^{b}\int_{\Omega}\theta^2\varphi^2(|G^0|^2+|G^1|^2)\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{3.37}$$

Step 4 We estimate the localized term $\mathbf{I_2}$ in (3.29) from above.

We write it as

$$\mathbf{I_2}=\mathbf{I_{21}}+\mathbf{I_{22}}+\mathbf{I_{23}}+\mathbf{I_{24}},\tag{3.38}$$

where

$$\begin{aligned}
\mathbf{I_{21}} &\triangleq C\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\phi(|\nabla\widehat{p}|^2+|\widehat{p}_s|^2)\mathrm{d}x\mathrm{d}s,\\
\mathbf{I_{22}} &\triangleq C\lambda^3\mu^4\int_{-b}^{b}\int_{\omega_0}\theta^2\phi^3|\widehat{p}|^2\mathrm{d}x\mathrm{d}s,\\
\mathbf{I_{23}} &\triangleq C\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\phi(|\nabla\widehat{q}|^2+|\widehat{q}_s|^2)\mathrm{d}x\mathrm{d}s,\\
\mathbf{I_{24}} &\triangleq C\lambda^3\mu^4\int_{-b}^{b}\int_{\omega_0}\theta^2\phi^3|\widehat{q}|^2\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{3.39}$$

Recall the definitions $\widehat{p}, \widehat{q}$ in (3.23), φ in (3.20) and θ, l, ϕ in (2.3)–(2.5). It is easy to check that

$$\begin{aligned} &l_s = -2\lambda\mu s\phi, \quad l_{x_j} = \lambda\mu\phi\psi_{x_j}, \quad l_{x_j s} = -2\lambda\mu^2 s\phi\psi_{x_j}, \\ &l_{ss} = 4\lambda\mu^2 s^2\phi - 2\lambda\mu\phi, \quad l_{x_j x_k} = \lambda\mu^2\phi\psi_{x_j}\psi_{x_k} + \lambda\mu\phi\psi_{x_j x_k}. \end{aligned} \tag{3.40}$$

Denote ω_k $(k = 1, 2, 3)$ some open subsets in Ω such that $\omega_0 \subset\subset \omega_1 \subset\subset \omega_2 \subset\subset \omega_3 \subset\subset \omega_\delta$. Let $\eta_j \in C_0^3(\omega_j; \mathbb{R})$ $(j = 1, 2, 3)$ be suitable cut-off functions such that

$$\eta_j(x) \equiv 1 \quad \text{in } \omega_{j-1}, \qquad 0 \le \eta_j(x) \le 1 \quad \text{in } \omega_j, \qquad \eta_j(x) \equiv 0 \quad \text{in } \Omega \backslash \omega_j. \tag{3.41}$$

Moreover, we choose further η_2 such that

$$|(\eta_2)_{x_j}(\eta_2)_{x_k}| \le C\eta_2, \quad \forall x \in \omega_2 \tag{3.42}$$

for some constant $C > 0$. The existence of η_2 is shown at the end of the proof.

Step 4.1 We estimate a weighted energy for $(\nabla\widehat{q}, \widehat{q}_s)$, i.e., $\int_{-b}^{b}\int_{\omega_0} \theta^2\phi(|\nabla\widehat{q}|^2 + |\widehat{q}_s|^2)\mathrm{d}x\mathrm{d}s$.

By the definition of η_1, it suffices to estimate $\int_{-b}^{b}\int_{\Omega} \theta^2\phi\eta_1^2(a|\nabla\widehat{q}|^2 + |\widehat{q}_s|^2)\mathrm{d}x\mathrm{d}s$. To do this, we multiply $\widehat{q}$-equation in (3.24) by $\theta^2\phi\eta_1^2\overline{\widehat{q}}$,

$$\begin{aligned} \theta^2\phi\eta_1^2\overline{\widehat{q}}\widehat{G}^1 &= \theta^2\phi\eta_1^2\overline{\widehat{q}} \cdot \Big[\widehat{q}_{ss} + \sum_{j,k=1}^{n}(g^{jk}\widehat{q}_{x_j})_{x_k}\Big] \\ &= (\theta^2\phi\eta_1^2\overline{\widehat{q}}\widehat{q}_s)_s - \theta^2\phi\eta_1^2|\widehat{q}_s|^2 - (\theta^2\phi\eta_1^2)_s\overline{\widehat{q}}\widehat{q}_s + \sum_{k=1}^{n}\Big[\theta^2\phi\eta_1^2\sum_{j=1}^{n} g^{jk}\overline{\widehat{q}}\widehat{q}_{x_j}\Big]_{x_k} \\ &\quad - \theta^2\phi\eta_1^2\sum_{j,k=1}^{n} g^{jk}\widehat{q}_{x_j}\overline{\widehat{q}}_{x_k} - \sum_{j,k=1}^{n} g^{jk}(\theta^2\phi\eta_1^2)_{x_j}\overline{\widehat{q}}\widehat{q}_{x_j}. \end{aligned}$$

It reduces equivalently to

$$\begin{aligned} &\theta^2\phi\eta_1^2 \cdot \Big[|\widehat{q}_s|^2 + \sum_{j,k=1}^{n} g^{jk}\widehat{q}_{x_j}\overline{\widehat{q}}_{x_k}\Big] \\ &= \theta^2\phi\eta_1^2\overline{\widehat{q}}\widehat{G}^1 + (\theta^2\phi\eta_1^2\overline{\widehat{q}}\widehat{q}_s)_s - (\theta^2\phi\eta_1^2)_s\overline{\widehat{q}}\widehat{q}_s + \sum_{k=1}^{n}\Big[\theta^2\phi\eta_1^2\sum_{j=1}^{n} g^{jk}\overline{\widehat{q}}\widehat{q}_{x_j}\Big]_{x_k} \\ &\quad - \sum_{j,k=1}^{n} g^{jk}(\theta^2\phi\eta_1^2)_{x_j}\overline{\widehat{q}}\widehat{q}_{x_j}. \end{aligned}$$

Integrating (3.3) over $(-b, b) \times \Omega$, upon using (3.24), (1.2)–(1.3) and Cauchy inequality, we obtain

$$\begin{aligned} \int_{-b}^{b}\int_{\omega_0} \theta^2\phi(a|\nabla\widehat{q}|^2 + |\widehat{q}_s|^2)\mathrm{d}x\mathrm{d}s &\le \int_{-b}^{b}\int_{\Omega} \theta^2\phi\eta_1^2(a|\nabla\widehat{q}|^2 + |\widehat{q}_s|^2)\mathrm{d}x\mathrm{d}s \\ &\le \frac{C}{\lambda\mu^2}\int_{-b}^{b}\int_{\Omega} \theta^2|\widehat{G}^1|^2\mathrm{d}x\mathrm{d}s + C\lambda^2\mu^2\int_{-b}^{b}\int_{\omega_1} \theta^2\phi^3|\widehat{q}|^2\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{3.43}$$

The factor $\frac{C}{\lambda\mu^2}$ is important for the estimate of $\mathbf{I_2}$ (see (3.58)).

Step 4.2 We estimate a weighted energy for $\widehat{q}$, i.e., $\int_{-b}^{b}\int_{\omega_1} \theta^2\phi^3|\widehat{q}|^2\mathrm{d}x\mathrm{d}s$.

By definition of η_2, it suffices to find an up bound for $\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}|^2\mathrm{d}x\mathrm{d}s$. We claim that there exists a constant $C>0$, such that

$$\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}|^2\mathrm{d}x\mathrm{d}s \le C\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}_s|^2\mathrm{d}x\mathrm{d}s. \tag{3.44}$$

Indeed, in view of (3.4), we have

$$|\widehat{q}| = |\varphi z^0|, \quad |\widehat{q}_s| = |\varphi_s + \mathrm{i}\gamma\varphi||z^0|, \quad \forall (s,x)\in[-b,b]\times\Omega.$$

Notice that

$$|\varphi_s + \mathrm{i}\gamma\varphi| = |\gamma||\varphi| \ge C|\varphi|, \qquad \forall s\in(-b_0,b_0), \gamma\in\mathcal{O}_C,$$

then

$$\int_{-b_0}^{b_0}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}|^2\mathrm{d}x\mathrm{d}s \le C\int_{-b_0}^{b_0}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}_s|^2\mathrm{d}x\mathrm{d}s \le C\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}_s|^2\mathrm{d}x\mathrm{d}s. \tag{3.45}$$

Recalling (3.30) and (3.34), we can obtain from the choices of φ, b, b_0 in (3.20) and (3.22), that

$$\begin{aligned}
&\int_{(-b,-b_0)\bigcup(b_0,b)}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}|^2\mathrm{d}x\mathrm{d}s\\
={}&\int_{(-b,-b_0)\bigcup(b_0,b)}\int_\Omega \theta^2\phi^3\varphi^2\eta_2^2|z^0|^2\mathrm{d}x\mathrm{d}s\\
={}&\int_\Omega \eta_2^2|z^0|^2\int_{(-b,-b_0)\bigcup(b_0,b)}\theta^2\phi^3\varphi^2\mathrm{d}s\mathrm{d}x\\
\le{}&\int_\Omega \eta_2^2|z^0|^2\int_{(-b,-b_0)\bigcup(b_0,b)}\mathrm{e}^{2\lambda(1+\mathrm{e}^\mu)}\mathrm{e}^{3\mu\psi(\pm1,x)}\mathrm{d}s\mathrm{d}x\\
\le{}&\int_\Omega \eta_2^2|z^0|^2\int_{-1}^{1}\mathrm{e}^{2\lambda(2+\mathrm{e}^\mu)}\mathrm{e}^{3\mu\psi(\pm1,x)}\mathrm{d}s\mathrm{d}x\\
\le{}&\int_\Omega \eta_2^2|z^0|^2\int_{-1}^{1}\theta^2\phi^3\varphi^2\mathrm{d}s\mathrm{d}x\\
\le{}&\int_{-b_0}^{b_0}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}|^2\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{3.46}$$

Combining (3.45) and (3.46) yields (3.44) immediately.

Step 4.3 We estimate $\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}_s|^2\mathrm{d}x\mathrm{d}s$ by using the coupling relation in (3.24). Multiplying $\widehat{p}$-equation in (3.24) by $\mathrm{i}\theta^2\phi^3\eta_2^2\overline{\widehat{q}}_s$, and arranging the terms, we get

$$\begin{aligned}
\alpha(x)\theta^2\phi^3\eta_2^2|\widehat{q}_s|^2 ={}& \mathrm{i}\theta^2\phi^3\eta_2^2\overline{\widehat{q}}_s\Big[\widehat{p}_{ss} + \sum_{j,k=1}^n (g^{jk}\widehat{p}_{x_j})_{x_k}\Big]\\
&- \mathrm{i}\theta^2\phi^3\eta_2^2\overline{\widehat{q}}_s\,[\widehat{G}_1^0 + \varphi G^0 - \mathrm{i}\beta(x)\widehat{p}_s],
\end{aligned} \tag{3.47}$$

where $\widehat{G}_1^0$ is given in (3.33) and it only concerns the terms with derivates of φ.

Direct computations yield that

$$\begin{aligned}
&\theta^2\phi^3\eta_2^2\overline{\widehat{q}}_s\Big[\widehat{p}_{ss}+\sum_{j,k=1}^{n}(g^{jk}\widehat{p}_{x_j})_{x_k}\Big]\\
&=(\theta^2\phi^3\eta_2^2\overline{\widehat{q}}_s\widehat{p}_s)_s-(\theta^2\phi^3\eta_2^2)_s\overline{\widehat{q}}_s\widehat{p}_s-\theta^2\phi^3\eta_2^2\widehat{p}_s\Big[\overline{\widehat{q}}_{ss}+\sum_{j,k}(g^{jk}\overline{\widehat{q}}_{x_k})_{x_j}\Big]\\
&\quad+\sum_{j,k}(\theta^2\phi^3\eta_2^2\overline{\widehat{q}}_s g^{jk}\widehat{p}_{x_j})_{x_k}-\sum_{j,k}(\theta^2\phi^3\eta_2^2)_{x_k}\overline{\widehat{q}}_s g^{jk}\widehat{p}_{x_j}-\sum_{j,k}(\theta^2\phi^3\eta_2^2\overline{\widehat{q}}_{x_k}g^{jk}\widehat{p}_{x_j})_s\\
&\quad+\sum_{j,k}(\theta^2\phi^3\eta_2^2)_s\overline{\widehat{q}}_{x_k}g^{jk}\widehat{p}_{x_j}+\sum_{j,k}(\theta^2\phi^3\eta_2^2 g^{jk}\overline{\widehat{q}}_{x_k}\widehat{p}_s)_{x_j}-\sum_{j,k}(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}\overline{\widehat{q}}_{x_k}\widehat{p}_s
\end{aligned}\tag{3.48}$$

and

$$\begin{aligned}
\sum_{j,k}(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}\overline{\widehat{q}}_{x_k}\widehat{p}_s&=\sum_{j,k}[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}\overline{\widehat{q}}\widehat{p}_s]_{x_k}-\sum_{j,k}[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}\overline{\widehat{q}}\widehat{p}_{x_k}]_s\\
&\quad-\sum_{j,k}[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}]_{x_k}\overline{\widehat{q}}\widehat{p}_s+\sum_{j,k}[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}]_s\overline{\widehat{q}}\widehat{p}_{x_k}\\
&\quad+\sum_{j,k}(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}\overline{\widehat{q}}_s\widehat{p}_{x_k}.
\end{aligned}\tag{3.49}$$

We integrate (3.47) on $(-b,b)\times\Omega$. Then from (3.48)–(3.49) and particularly the facts that $\widehat{p}_s=\widehat{q}_s=0$ on the boundary of $(-b,b)\times\Omega$ and $\nabla\widehat{p}(\pm b,\cdot)=\nabla\widehat{q}_s(\pm b,\cdot)=0$, we derive

$$\begin{aligned}
&\int_{-b}^{b}\int_\Omega\alpha(x)\theta^2\phi^3\eta_2^2|\widehat{q}_s|^2\mathrm{d}x\mathrm{d}s\\
&\le\frac{\delta}{2}\int_{-b}^{b}\int_\Omega\theta^2\phi^3\eta_2^2|\widehat{q}_s|^2\mathrm{d}x\mathrm{d}s\\
&\quad+C\int_{-b}^{b}\int_{\omega_2}\theta^2\phi^3\eta_2^2(|\widehat{G}_1^0|^2+|G^0|^2+|\widehat{G}_1^1|^2+|G^1|^2)\mathrm{d}x\mathrm{d}s\\
&\quad+C\lambda^2\mu^2\int_{-b}^{b}\int_{\omega_3}\theta^2\phi^5(|\widehat{p}|^2+|\widehat{p}_s|^2+|\nabla\widehat{p}|^2)\mathrm{d}x\mathrm{d}s\\
&\quad+\Big|\int_{-b}^{b}\int_\Omega\sum_{j,k}[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}]_s\overline{\widehat{q}}\widehat{p}_{x_k}\mathrm{d}x\mathrm{d}s\Big|\\
&\quad+\Big|\int_{-b}^{b}\int_\Omega\sum_{j,k}[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}]_{x_k}\overline{\widehat{q}}\widehat{p}_s\mathrm{d}x\mathrm{d}s\Big|.
\end{aligned}\tag{3.50}$$

Since $\alpha(x)\ge\delta>0$ on ω_δ, the first term on the right hand side of (3.50) can be absorbed by the left hand side. Moreover, by direct computations, we have the following two facts

$$\begin{aligned}
[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}]_{x_k}&=(\theta^2\phi^3\eta_2^2)_{x_j,x_k}g^{jk}+(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}_{x_k}\\
&=g^{jk}[2\theta_{x_j,x_k}\theta\phi^3\eta_2^2+2\theta_{x_j}\theta_{x_k}\phi^3\eta_2^2+6\theta_{x_j}\theta\phi^2\phi_{x_k}\eta_2^2\\
&\quad+4\theta_{x_j,x_k}\theta\phi^3(\eta_2)_{x_k}\eta_2+6\theta_{x_k}\theta\phi_{x_j}\phi^2\eta_2^2+3\theta^2\phi_{x_j,x_k}\phi^2\eta_2^2\\
&\quad+6\theta^2\phi_{x_j}\phi\phi_{x_k}\eta_2^2+6\theta^2\phi_{x_j}\phi^2\eta_2(\eta_2)_{x_k}+2(\eta_2)_{x_j,x_k}\eta_2\theta^2\phi^3\\
&\quad+2(\eta_2)_{x_j}(\eta_2)_{x_k}\theta^2\phi^3+4(\eta_2)_{x_j}\eta_2\theta_{x_k}\theta\phi^3+6(\eta_2)_{x_j}\eta_2\theta^2\phi^2\phi_{x_k}]\\
&\quad+[2\theta_{x_j}\theta\phi^3\eta_2^2+3\theta^2\phi_{x_j}\phi^2\eta_2^2+2(\eta_2)_{x_j}\eta_2\theta^2\phi^3]g^{jk}_{x_k}
\end{aligned}\tag{3.51}$$

and

$$[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}]_s = g^{jk}[2\theta_{x_j,s}\theta\phi^3\eta_2^2 + 2\theta_{x_j}\theta_s\phi^3\eta_2^2 + 6\theta_{x_j}\theta\phi_s\phi^2\eta_2^2 + 6\theta_s\theta\phi_{x_j}\phi^2\eta_2^2 + 3\theta^2\phi_{x_j,s}\phi^2\eta_2^2 + 6\theta^2\phi_{x_j}\phi_s\phi\eta_2^2 + 2(\eta_2)_{x_j}\eta_2\theta_s\theta\phi^3 + 6(\eta_2)_{x_j}\eta_2\theta^2\phi^2\phi_s]. \tag{3.52}$$

By definitions of η_2, θ, ϕ, in particular (3.40), (3.42) and $g^{jk}(\cdot) \in C^1(\overline{\Omega};\mathbb{R})$, one can obtain the following estimate

$$|[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}]_s| + |[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}]_{x_k}| \le C\lambda^2\mu^2\eta_2\theta^2\phi^5. \tag{3.53}$$

Thus by above estimate (3.53) and Cauchy inequality, we get for all $\varepsilon > 0$ that

$$\Big|\int_{-b}^{b}\int_\Omega \sum_{j,k}[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}]_s\overline{\widehat{q}}\widehat{p}_{x_k}\mathrm{d}x\mathrm{d}s\Big| + \Big|\int_{-b}^{b}\int_\Omega \sum_{j,k}[(\theta^2\phi^3\eta_2^2)_{x_j}g^{jk}]_{x_k}\overline{\widehat{q}}\widehat{p}_s\mathrm{d}x\mathrm{d}s\Big|$$
$$\le \varepsilon\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}|^2\mathrm{d}x\mathrm{d}s + \frac{C\lambda^4\mu^4}{\varepsilon}\int_{-b}^{b}\int_{\omega_3}\theta^2\phi^7(|\widehat{p}_s|^2 + |\nabla\widehat{p}|^2)\mathrm{d}x\mathrm{d}s,$$

Therefore we obtain from (3.44) and by choosing $\varepsilon > 0$ small that

$$\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}_s|^2\mathrm{d}x\mathrm{d}s \le C\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2(|\widehat{G}_1^0|^2 + |G^0|^2 + |\widehat{G}_1^1|^2 + |G^1|^2)\mathrm{d}x\mathrm{d}s + C\lambda^4\mu^4\int_{-b}^{b}\int_{\omega_3}\theta^2\phi^7(|\widehat{p}|^2 + |\widehat{p}_s|^2 + |\nabla\widehat{p}|^2)\mathrm{d}x\mathrm{d}s, \tag{3.54}$$

$$\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2|\widehat{q}|^2\mathrm{d}x\mathrm{d}s \le C\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2(|\widehat{G}_1^0|^2 + |G^0|^2 + |\widehat{G}_1^1|^2 + |G^1|^2)\mathrm{d}x\mathrm{d}s + C\lambda^4\mu^4\int_{-b}^{b}\int_{\omega_3}\theta^2\phi^7(|\widehat{p}|^2 + |\widehat{p}_s|^2 + |\nabla\widehat{p}|^2)\mathrm{d}x\mathrm{d}s. \tag{3.55}$$

Step 4.4 We estimate $\int_{-b}^{b}\int_{\omega_3}\theta^2\phi^7(|\nabla\widehat{p}|^2 + |\widehat{p}_s|^2)\mathrm{d}x\mathrm{d}s$.
Similarly to (3.43), we can derive

$$\int_{-b}^{b}\int_{\omega_3}\theta^2\phi^7(|\nabla\widehat{p}|^2 + |\widehat{p}_s|^2)\mathrm{d}x\mathrm{d}s \le \frac{C}{\lambda^7\mu^8}\int_{-b}^{b}\int_\Omega \theta^2|\widehat{G}^0|^2\mathrm{d}x\mathrm{d}s + C\lambda^7\mu^8\int_{-b}^{b}\int_{\omega_\delta}\theta^2\phi^{14}|\widehat{p}|^2\mathrm{d}x\mathrm{d}s. \tag{3.56}$$

The factor $\frac{C}{\lambda^7\mu^8}$ is important for the estimate of $\mathbf{I_2}$ (see (3.58)).

Step 4.5 We summarize the estimate of $\mathbf{I_2}$ from above.
Applying (3.43), (3.54)–(3.56) for $\mathbf{I_{21}}, \mathbf{I_{23}}, \mathbf{I_{24}}$ in (3.38), we end up with

$$\mathbf{I_2} = \mathbf{I_{21}} + \mathbf{I_{22}} + \mathbf{I_{23}} + \mathbf{I_{24}} \le C\lambda^3\mu^4\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2(|\widehat{G}_1^0|^2 + |\widehat{G}_1^1|^2)\mathrm{d}x\mathrm{d}s + C\int_{-b}^{b}\int_\Omega \theta^2(|\widehat{G}^0|^2 + |\widehat{G}^1|^2)\mathrm{d}x\mathrm{d}s + C\lambda^3\mu^4\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2(|G^0|^2 + |G^1|^2)\mathrm{d}x\mathrm{d}s + C\lambda^{14}\mu^{16}\int_{-b}^{b}\int_{\omega_\delta}\theta^2\phi^{14}|\widehat{p}|^2\mathrm{d}x\mathrm{d}s. \tag{3.57}$$

Similarly to the estimate of $\mathbf{I_{11}}$ in (3.35), we have

$$C\lambda^3\mu^4\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2(|\widehat{G}_1^0|^2+|\widehat{G}_1^1|^2)\mathrm{d}x\mathrm{d}s \le \lambda^3\mathrm{e}^{2\lambda(1+\mathrm{e}^\mu)}C(\mu)(\|p\|^2_{H^1(X)}+\|q\|^2_{H^1(X)}).$$

Note that $C\int_{-b}^{b}\int_\Omega \theta^2(|\widehat{G}^0|^2+|\widehat{G}^1|^2)\mathrm{d}x\mathrm{d}s$ is the same as $\mathbf{I_1}$ and the estimate has been given by (3.37). Therefore, we have

$$\begin{aligned}\mathbf{I_2} &\le C\mathbf{I_1}+\lambda^3\mathrm{e}^{2\lambda(1+\mathrm{e}^\mu)}C(\mu)(\|p\|^2_{H^1(X)}+\|q\|^2_{H^1(X)})\\ &\quad +C\lambda^3\mu^4\int_{-b}^{b}\int_\Omega \theta^2\phi^3\eta_2^2(|G^0|^2+|G^1|^2)\mathrm{d}x\mathrm{d}s+C\lambda^{14}\mu^{16}\int_{-b}^{b}\int_{\omega_\delta}\theta^2\phi^{14}|\widehat{p}|^2\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.58}$$

Step 5 We derive the estimate (3.6) based on Steps 1–4.

Letting λ be large enough, from (3.28), (3.31), (3.37) and (3.58), we finally obtain the desired interpolation estimate (3.6).

Step 6 We provide an example of the cut-off function $\eta_2 \in C_0^3(\omega_2)$ such that (3.41)–(3.42) indeed hold.

Without loss of generality, we may assume $\omega_1 \subset B(0,\frac{r}{2}) \subset B(0,r) \subset \omega_2$. Set

$$\eta_2(x) \triangleq \begin{cases} 1, & \text{if } |x|<\dfrac{r}{2},\\ \dfrac{128}{81r^{10}}[r^2-|x|^2]^3[16|x|^4-2r^2|x|^2+r^4], & \text{if } \frac{r}{2}\le|x|\le r,\\ 0, & \text{if } |x|>r.\end{cases}$$

It is easy to check that $\eta_2 \in C_0^3(\omega_2)$ and

$$\lim_{|x|\to r}\frac{|(\eta_2)_{x_k}(\eta_2)_{x_j}|}{\eta_2}=0,\qquad \lim_{|x|\to \frac{r}{2}}\frac{|(\eta_2)_{x_k}(\eta_2)_{x_j}|}{\eta_2}<\infty.$$

Then it follows the property (3.42).

This finally concludes the proof of Lemma 3.1.

4 An Example with $\omega_\alpha \cap \omega_\beta = \emptyset$

In this section, we show, by an example, that the geometric assumption $\omega_\alpha \cap \omega_\beta \ne \emptyset$ is necessary in general for asymptotic stability of the coupled system (1.1).

More precisely, we consider the following 1-D coupled wave equations

$$\begin{cases} y_{tt}-y_{xx}+\alpha(x)z_t+\beta(x)y_t=0 & \text{in } (0,+\infty)\times(0,2\pi),\\ z_{tt}-z_{xx}-\alpha(x)y_t=0 & \text{in } (0,+\infty)\times(0,2\pi),\\ y(0)=y(2\pi)=z(0)=z(2\pi)=0 & \text{in } (0,+\infty),\\ (y,y_t,z,z_t)|_{t=0}=(y^0,y^1,z^0,z^1) & \text{in } (0,2\pi),\end{cases} \tag{4.1}$$

where the coefficients α and β are given by

$$\alpha(x)=\begin{cases}\dfrac{24}{5}, & \text{if } x\in[0,\pi),\\ 0, & \text{if } x\in(\pi,2\pi),\end{cases}\qquad \beta(x)=\begin{cases}0, & \text{if } x\in[0,\pi),\\ 1, & \text{if } x\in(\pi,2\pi)\end{cases}$$

such that $\omega_\alpha \cap \omega_\beta = \emptyset$.

Let the initial data be the following

$$y^0 = 0 \quad \text{in } (0, 2\pi), \quad y^1(x) = \begin{cases} 5(7\sin x - \sin(7x)), & \text{if } x \in (0, \pi), \\ 0, & \text{if } x \in [\pi, 2\pi) \end{cases} \tag{4.2}$$

and

$$z^0(x) = \begin{cases} -7\sin x - \sin(7x), & \text{if } x \in (0, \pi), \\ -\dfrac{14}{5}\sin(5x), & \text{if } x \in [\pi, 2\pi), \end{cases} \qquad z^1 = 0 \quad \text{in } (0, 2\pi). \tag{4.3}$$

Clearly, (y^0, y^1, z^0, z^1) belongs to $(H^2(0, 2\pi) \cap H_0^1(0, 2\pi)) \times H_0^1(0, 2\pi) \times H^2(0, 2\pi) \cap (H_0^1(0, 2\pi) \times H_0^1(0, 2\pi))$. It is not hard to check that the unique solution of system (4.1)–(4.3) can be explicitly given by

$$y(t, x) = \begin{cases} \sin(5t)(7\sin x - \sin(7x)), & \text{if } t \in (0, +\infty), x \in (0, \pi), \\ 0, & \text{if } t \in (0, +\infty), x \in [\pi, 2\pi) \end{cases} \tag{4.4}$$

and

$$z(t, x) = \begin{cases} -\cos(5t)(7\sin x + \sin(7x)), & \text{if } t \in (0, +\infty), x \in (0, \pi), \\ -\dfrac{14}{5}\cos(5t)\sin(5x), & \text{if } t \in (0, +\infty), x \in [\pi, 2\pi). \end{cases} \tag{4.5}$$

In contrast to (1.11) in Theorem 1.1, the energy of system (4.1) is conserved

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(y, z)(t) = -\int_0^{2\pi} \beta(x)|y_t|^2 \mathrm{d}x = 0, \quad \forall t \geq 0,$$

therefore the system is not asymptotically stable. According to the above example, we conclude that the decay estimate (1.10) may not hold if $\omega_\alpha \cap \omega_\beta = \emptyset$.

References

[1] Alabau-Boussouira, F., Indirect boundary stabilization of weakly coupled hyperbolic systems, *SIAM J. Control Optim.*, **41**(2),2002, 511–541.

[2] Alabau-Boussouira, F. and Léautaud, M., Indirect stabilization of locally coupled wave-type systems, *ESAIM Control Optim. Calc. Var.*, **18**(2), 2012, 548–582.

[3] Alabau-Boussouira, F., Wang, Z. Q., and Yu, L. X., A one-step optimal energy decay formula for indirectly nonlinearly damped hyperbolic systems coupled by velocities, *ESAIM Control Optim. Calc. Var.*, **23**(2), 2017, 721–749.

[4] Anantharaman, N. and Léautaud, M., Sharp polynomial decay rates for the damped wave equation on the torus, *Anal. PDE*, **7**(1), 2014, 159–214. With an appendix by Stéphane Nonnenmacher.

[5] Bardos, C., Lebeau, G., and Rauch, J., Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, *SIAM J. Control Optim.*, **30**(5), 1992, 1024–1065.

[6] Burq, N., Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel, *Acta Math.*, **180**(1), 1998, 1–29.

[7] Burq, N. and Hitrik, M., Energy decay for damped wave equations on partially rectangular domains, *Math. Res. Lett.*, **14**(1), 2007, 35–47.

[8] Cui, Y. and Wang, Z. Q., Asymptotic stability of wave equations coupled by velocities, *Math. Control Relat. Fields*, **6**(3), 2016, 429–446.

[9] Fu, X. Y., Logarithmic decay of hyperbolic equations with arbitrary small boundary damping, *Communications in Partial Differential Equations*, **34**(9), 2009, 957–975.

[10] Fu, X. Y., Sharp decay rates for the weakly coupled hyperbolic system with one internal damping, *SIAM J. Control Optim.*, **50**(3), 2012, 1643–1660.

[11] Fu, X. Y., Lü, Q., and Zhang, X., Carleman Estimates for Second Order Partial Differential Operators and Applications, a Unified Approach, Springer Briefs in Mathematics, Springer, Cham, 2019.

[12] Fursikov, A. V. and Imanuvilov, O. Yu., Controllability of evolution equations, *Lecture Notes Series*, **34**, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[13] Gerbi, S., Kassem, C., Mortada, A. and Wehbe, A., Exact controllability and stabilization of locally coupled wave equations, arXiv:2003.14001, 2020.

[14] Jin, L., Damped wave equations on compact hyperbolic surfaces, *Commun. Math. Phys*, **373**(3), 2020, 771–794.

[15] Kassem, C., Mortada, A., Toufayli, L., and Wehbe, A., Local indirect stabilization of n-d system of two coupled wave equations under geometric conditions, *Comptes Rendus Mathematique*, **357**(6), 2019, 494–572.

[16] Klein, G., Best exponential decay rate of energy for the vectorial damped wave equation, *SIAM J. Control Optim.*, **56**, 2017, 3432–3453.

[17] Léautaud, M. and Lerner, N., Sharp polynomial energy decay for locally undamped waves, in Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2014–2015, pages Exp. No. XXI, 13. Ed. Éc. Polytech., Palaiseau, 2016.

[18] Lebeau, G., Équation des ondes amorties, in Algebraic and geometric methods in mathematical physics (Kaciveli, 1993), volume 19 of *Math. Phys. Stud.*, pages 73–109. Kluwer Acad. Publ., Dordrecht, 1996.

[19] Liu, K. S., Locally distributed control and damping for the conservative systems, *SIAM J. Control Optim.*, **35**(5), 1997, 1574–1590.

[20] Liu, Z. Y. and Rao, B. P., Frequency domain approach for the polynomial stability of a system of partially damped wave equations, *J. Math. Anal. Appl.*, **335**(2), 2007, 860–881.

[21] Pazy, A., Semigroups of linear operators and applications to partial differential equations, *Applied Mathematical Sciences*, Vol. 44, Springer-Verlag, New York, 1983.

[22] Phung, K.-D. and Zhang, X., Time reversal focusing of the initial state for kirchoff plate, *SIAM J. Appl. Math.*, **68**(6), 2008, 1535–1556.

[23] Phung, K.-D., Polynomial decay rate for the dissipative wave equation, *J. Differential Equations*, **240**(1), 2007, 92–124.

[24] Rauch, J. and Taylor, M., Exponential decay of solutions to hyperbolic equations in bounded domains, *Indiana Univ. Math. J.*, **24**, 1974, 79–86.

[25] Reed, M. and Simon, B., Methods of Modern Mathematical Physics. I, Functional Analysis, Academic Press, New York-London, 1972.