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On a Lotka-Volterra Competition Diffusion Model with Advection

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Abstract In this paper, the author focuses on the joint effects of diffusion and advection on the dynamics of a classical two species Lotka-Volterra competition-diffusion-advection system, where the ratio of diffusion and advection rates are supposed to be a positive constant. For comparison purposes, the two species are assumed to have identical competition abilities throughout this paper. The results explore the condition on the diffusion and advection rates for the stability of former species. Meanwhile, an asymptotic behavior of the stable coexistence steady states is obtained.

Keywords Competition-diffusion-advection, Stability, Dynamics
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1 Introduction and Statement of Main Results

The Lotka-Volterra competition-diffusion system

$$\begin{cases} u_t = d_1\Delta u + u(m(x) - u - bv) & \text{in } \Omega \times (0, +\infty), \\ v_t = d_2\Delta v + v(m(x) - cu - v) & \text{in } \Omega \times (0, +\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) \gneqq 0 & \text{in } \Omega, \\ v(x, 0) = v_0(x) \gneqq 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

models two competing species. Here $u(x, t)$ and $v(x, t)$ denote respectively the population densities of two competing species at location $x \in \Omega$ and time $t > 0$, and $d_1, d_2 > 0$ are random diffusion rates of species u and v respectively. The habitat Ω is a bounded region in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, n denotes the unit outer normal vector on $\partial\Omega$, and the no flux boundary condition means that no individuals cross the boundary. The function $m(x)$ represents their common intrinsic growth rate or local carrying capacity, which is non-constant. $1 \geq b > 0$ and $1 \geq c > 0$ are interspecific competition coefficients. Then the maximum principle yields that $u(x, t) > 0$, $v(x, t) > 0$ for every $x \in \overline{\Omega}$ and every $t > 0$ (see [27]). By both mathematicians and ecologists, particular interests in two-species Lotka-Volterra competition models with spatially homogeneous or heterogeneous interactions are the dynamics of (1.1) (see [2–3, 8–10, 12, 14–16, 18–22, 26] and the references therein). We say that a steady state (U, V) of (1.1) is a coexistence state if both components are positive, and it is a semi-trivial state if one component is positive and the other is identically zero.

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If $m(x) \in C^{\alpha}(\overline{\Omega})$ ($\alpha \in (0,1)$) with $\int_\Omega m(x)\mathrm{d}x \geq 0$ and $m \not\equiv 0$, then we denote by Θ_d the unique positive solution of

$$\begin{cases} d\Delta\Theta + \Theta(m(x) - \Theta) = 0 & \text{in } \Omega, \\ \dfrac{\partial \Theta}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

One can refer to [2, 25] for the proof of existence and uniqueness results of (1.2). (1.2) indicates that (1.1) has two semi-trivial steady states, denoted by $(\Theta_{d_1}, 0)$ and $(0, \Theta_{d_2})$, for every $d_1 > 0$ and $d_2 > 0$.

Under above conditions, He and Ni [14] provided a complete classification on the global dynamics of system (1.1), which says that either one of the two semi-trivial steady states is globally asymptotically stable, or there is a unique coexistence steady state which is globally asymptotically stable, or the system is degenerate in the sense that there is a compact global attractor consisting of a continuum of steady states which connect the two semi-trivial steady states (see [14, Theorems 1.3 and 3.4]). We refer the interested readers to [14–16] for more investigations on system (1.1).

Besides random dispersal, it seems reasonable to argue that it is also plausible that species could move upward along the resource gradient (see e.g. [2, 7] and the references therein). A more general problem as follows was considered in [31],

$$\begin{cases} u_t = d_1\Delta u - \alpha_1 \nabla \cdot (u\nabla P(x)) + u(m_1(x) - u - bv) & \text{in } \Omega \times (0, +\infty), \\ v_t = d_2\Delta v - \alpha_2 \nabla \cdot (v\nabla P(x)) + v(m_2(x) - cu - v) & \text{in } \Omega \times (0, +\infty), \\ d_1 \dfrac{\partial u}{\partial n} - \alpha_1 u \dfrac{\partial P}{\partial n} = d_2 \dfrac{\partial v}{\partial n} - \alpha_2 v \dfrac{\partial P}{\partial n} = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x) \gneqq 0 & \text{in } \Omega, \\ v(x,0) = v_0(x) \gneqq 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where the non-constant function $P(x) \in C^2(\overline{\Omega})$ is used to specify the advective direction, and the advection rates of two species are denoted by $\alpha_1, \alpha_2 > 0$, respectively. Here the movement strategies, growth rates and competition abilities of two species are taken into account and allowed to be different. Throughout this paper, we make the following basic hypotheses.

Assumption 1.1

$$\frac{\alpha_1}{d_1} = \frac{\alpha_2}{d_2} =: \eta > 0. \tag{1.4}$$

Assumption 1.2

$$m(x) \text{ is Hölder continuous, } m(x)\mathrm{e}^{-\eta P(x)} \text{ is non-constant, and } m \geq 0, m \not\equiv 0. \tag{1.5}$$

This paper is devoted to some dynamics of the following problem for all (d_1, d_2, η) in the special case $b = c = 1$,

$$\begin{cases} u_t = d_1\Delta u - \alpha_1 \nabla \cdot (u\nabla P(x)) + u(m(x) - u - v) & \text{in } \Omega \times (0, +\infty), \\ v_t = d_2\Delta v - \alpha_2 \nabla \cdot (v\nabla P(x)) + v(\widehat{m}(x) - u - v) & \text{in } \Omega \times (0, +\infty), \\ d_1 \dfrac{\partial u}{\partial n} - \alpha_1 u \dfrac{\partial P}{\partial n} = d_2 \dfrac{\partial v}{\partial n} - \alpha_2 v \dfrac{\partial P}{\partial n} = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x) \gneqq 0 & \text{in } \Omega, \\ v(x,0) = v_0(x) \gneqq 0 & \text{in } \Omega, \end{cases} \tag{1.6}$$

where

$$\widehat{m}(x) = \mathrm{e}^{\eta P} \frac{\int_\Omega m\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}. \tag{1.7}$$

Here all intra- and inter-specific competition coefficients are normalized to 1, which means that the two species have identical competition abilities.

1.1 Motivation and related work

System (1.3) has important applications in biological scenarios. For example, by letting $m_1(x) = m_2(x) = m(x) = P(x)$, $b = c = 1$, one obtains the following model

$$\begin{cases} u_t = d_1\Delta u - \alpha_1\nabla\cdot(u\nabla m(x)) + u(m(x) - u - v) & \text{in } \Omega\times(0,+\infty),\\ v_t = d_2\Delta v - \alpha_2\nabla\cdot(v\nabla m(x)) + v(m(x) - u - v) & \text{in } \Omega\times(0,+\infty),\\ d_1\dfrac{\partial u}{\partial n} - \alpha_1 u\dfrac{\partial m}{\partial n} = d_2\dfrac{\partial v}{\partial n} - \alpha_2 v\dfrac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega\times(0,+\infty),\\ u(x,0) = u_0(x) \gneqq 0 & \text{in } \Omega,\\ v(x,0) = v_0(x) \gneqq 0 & \text{in } \Omega. \end{cases} \tag{1.8}$$

Recently (1.8) has been frequently used as a standard model to study the evolution of conditional dispersal (see, e.g., [1, 6, 13] for $\alpha_1, \alpha_2 > 0$, [4–5, 23–24] for $\alpha_1 > 0 = \alpha_2$, and the textbook [11]). Basically speaking, system (1.8) models the competition between two species with the same population dynamics but different movement strategies as reflected by their diffusion and/or advection rates.

For system (1.3), it is known that Xiao and Zhou gave a complete classification on the global dynamics (see [31, Theorems 1.1–1.3]). We also know that when $\alpha_1 = \alpha_2 = 0$, system (1.3) with $m_2 = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega m_1(x)\mathrm{d}x$ was considered in [15], where the effect of homogeneous versus heterogeneous distribution of resource was compared. Motivated by the above works, we hope to extend some arguments above to system (1.3). That is, we also look forward to comparing the effect of homogeneous versus heterogeneous distribution of resource. Moreover, compared with [15] and [31], we will show the influence of advection and do some further studies. For technical reasons, in this paper, we assume that $m_2 = \mathrm{e}^{\eta P}\frac{\int_\Omega m\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}$. Since $\lim\limits_{\eta\to 0} \mathrm{e}^{\eta P}\frac{\int_\Omega m\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega m_1(x)\mathrm{d}x$, for sufficiently small η, it would be interesting to extend some of the results in this paper to the case where $m_2 = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega m_1(x)\mathrm{d}x$ further in another paper.

The purpose of this paper is to consider some more dynamics of (1.6) by regarding the movement rates $d_1, d_2, \alpha_1, \alpha_2$ as variable parameters with others fixed.

The rest of this paper is organized as follows. In Subsection 1.2, we present some preliminary results, which may be helpful to verify our results. In Subsection 1.3, we establish our main results (Theorems 1.1–1.4). The proofs will be given in Section 2.

1.2 Preliminaries

Before describing our results, we first introduce some notations and do some preparations. Under (1.5), (1.6) has two semi-trivial steady states for all $d_1, d_2 > 0$ and $\alpha_1, \alpha_2 \geq 0$ (see [2]), denoted by $(\theta_{d_1}, 0)$, $(0, \theta_{d_2})$ respectively, where $\theta_{d_1} \in C^2(\overline{\Omega})$ is the unique positive solution of

$$\begin{cases} d_1 \nabla \cdot (\nabla \theta_{d_1} - \eta \theta_{d_1} \nabla P) + \theta_{d_1}(m(x) - \theta_{d_1}) = 0 & \text{in } \Omega, \\ \dfrac{\partial \theta_{d_1}}{\partial n} - \eta \theta_{d_1} \dfrac{\partial P}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

and $\theta_{d_2} \in C^2(\overline{\Omega})$ is the unique positive solution of

$$\begin{cases} d_2 \nabla \cdot (\nabla \theta_{d_2} - \eta \theta_{d_2} \nabla P) + \theta_{d_2}(\widehat{m}(x) - \theta_{d_2}) = 0 & \text{in } \Omega, \\ \dfrac{\partial \theta_{d_2}}{\partial n} - \eta \theta_{d_2} \dfrac{\partial P}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

Defining $\widetilde{\theta}_{d_1} = \theta_{d_1} \mathrm{e}^{-\eta P}$ and $\widetilde{\theta}_{d_2} = \theta_{d_2} \mathrm{e}^{-\eta P}$, we then have the equivalent equations

$$\begin{cases} d_1 \Delta \widetilde{\theta}_{d_1} + \alpha_1 \nabla P \cdot \nabla \widetilde{\theta}_{d_1} + \widetilde{\theta}_{d_1}(m(x) - \theta_{d_1}) = 0 & \text{in } \Omega, \\ \dfrac{\partial \widetilde{\theta}_{d_1}}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

$$\begin{cases} d_2 \Delta \widetilde{\theta}_{d_2} + \alpha_2 \nabla P \cdot \nabla \widetilde{\theta}_{d_2} + \widetilde{\theta}_{d_2}(\widehat{m}(x) - \theta_{d_2}) = 0 & \text{in } \Omega, \\ \dfrac{\partial \widetilde{\theta}_{d_2}}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

which immediately implies that $\widetilde{\theta}_{d_2} \equiv \frac{\int_\Omega m \mathrm{e}^{\eta P} \mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x}$.

Following the approach in [14], we now define

$$\begin{cases} \Sigma_u := \{(d_1, d_2, \eta) \in \Gamma : (\theta_{d_1}, 0) \text{ is linearly stable}\}; \\ \Sigma_v := \{(d_1, d_2, \eta) \in \Gamma : (0, \theta_{d_2}) \text{ is linearly stable}\}; \\ \Sigma_- := \{(d_1, d_2, \eta) \in \Gamma : \text{both } (\theta_{d_1}, 0) \text{ and } (0, \theta_{d_2}) \text{ are linearly unstable}\}, \end{cases} \tag{1.13}$$

where

$$\Gamma := \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+. \tag{1.14}$$

Thus to study the dynamics of system (1.6), we should study the stability of semi-trivial steady states $(\theta_{d_1}, 0)$, $(0, \theta_{d_2})$. Mathematically, the stability of $(\theta_{d_1}, 0)$ is determined by the following linear eigenvalue problem

$$\begin{cases} d_2 \nabla \cdot (\nabla \psi - \eta \psi \nabla P(x)) + (\widehat{m}(x) - \theta_{d_1})\psi = \sigma \psi & \text{in } \Omega, \\ \dfrac{\partial \psi}{\partial n} - \eta \psi \dfrac{\partial P}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

Similarly, the stability of $(0, \theta_{d_2})$ is determined by the linear eigenvalue problem as follows:

$$\begin{cases} d_1 \nabla \cdot (\nabla \varphi - \eta \varphi \nabla P(x)) + (m(x) - \theta_{d_2})\varphi = \sigma \varphi & \text{in } \Omega, \\ \dfrac{\partial \varphi}{\partial n} - \eta \varphi \dfrac{\partial P}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.16}$$

The Krein-Rutman theorem (see [17, p20, Theorems 7.1–7.2]) reads that problem (1.15) and (1.16) admit a principal eigenvalue, denoted by $\sigma_1(d_2,\eta,\widehat{m}-\theta_{d_1})$, $\sigma_1(d_1,\eta,m-\theta_{d_2})$, and their corresponding eigenfunction can be chosen to be strictly positive in Ω. The following lemma characterizes the linear stability of the two semitrivial steady states of (1.6).

Lemma 1.1 (see [2]) *$(\theta_{d_1},0)$ is linearly stable if $\sigma_1(d_2,\eta,\widehat{m}-\theta_{d_1})<0$ and is linearly unstable if $\sigma_1(d_2,\eta,\widehat{m}-\theta_{d_1})>0$. Similarly, $(0,\theta_{d_2})$ is linearly stable if $\sigma_1(d_1,\eta,m-\theta_{d_2})<0$ and is linearly unstable if $\sigma_1(d_1,\eta,m-\theta_{d_2})>0$.*

Hence, we obtain the following equivalent descriptions:

$$\begin{cases}\Sigma_u:=\{(d_1,d_2,\eta)\in\Gamma:\sigma_1(d_2,\eta,\widehat{m}-\theta_{d_1})<0\};\\ \Sigma_v:=\{(d_1,d_2,\eta)\in\Gamma:\sigma_1(d_1,\eta,m-\theta_{d_2})<0\};\\ \Sigma_-:=\{(d_1,d_2,\eta)\in\Gamma:\sigma_1(d_2,\eta,\widehat{m}-\theta_{d_1})>0\text{ and }\sigma_1(d_1,\eta,m-\theta_{d_2})>0\}.\end{cases}\tag{1.17}$$

To understand the dynamics of system (1.6), we also need to consider the neutrally stable case, which leads us to further define

$$\begin{cases}\Sigma_{0,0}:=\{(d_1,d_2,\eta)\in\Gamma:\sigma_1(d_2,\eta,\widehat{m}-\theta_{d_1})=\sigma_1(d_1,\eta,m-\theta_{d_2})=0\};\\ \Sigma_{u,0}:=\{(d_1,d_2,\eta)\in\Gamma:\sigma_1(d_2,\eta,\widehat{m}-\theta_{d_1})=0\};\\ \Sigma_{v,0}:=\{(d_1,d_2,\eta)\in\Gamma:\sigma_1(d_1,\eta,m-\theta_{d_2})=0\}.\end{cases}\tag{1.18}$$

By definition, it is easy to see $\Sigma_{0,0}=\Sigma_{u,0}\bigcap\Sigma_{v,0}$.

Let $\lambda_1(\eta,h)$ denote the unique nonzero principal eigenvalue of

$$\begin{cases}\nabla\cdot(\nabla\phi-\eta\phi\nabla P(x))+\lambda h(x)\phi=0 & \text{in }\Omega,\\ \dfrac{\partial\phi}{\partial n}-\eta\phi\dfrac{\partial P(x)}{\partial n}=0 & \text{on }\partial\Omega.\end{cases}\tag{1.19}$$

In fact, $\lambda_1(\eta,h)$ is also the nonzero principal eigenvalue of

$$\begin{cases}\Delta\zeta+\eta\nabla P(x)\cdot\nabla\zeta+\lambda h(x)\zeta=0 & \text{in }\Omega,\\ \dfrac{\partial\zeta}{\partial n}=0 & \text{on }\partial\Omega.\end{cases}\tag{1.20}$$

We now collect some properties about $\lambda_1(\eta,h)$, which can be derived in [2].

Lemma 1.2 *The problem (1.19) has a nonzero principal eigenvalue $\lambda_1=\lambda_1(\eta,h)$ if and only if h changes sign and $\int_\Omega h(x)e^{\eta P}dx\neq 0$. More precisely, if h changes sign, then*

(i) $\int_\Omega he^{\eta P}dx=0\Leftrightarrow 0$ *is the only principal eigenvalue.*

(ii) $\int_\Omega he^{\eta P}dx<0\Leftrightarrow\lambda_1(\eta,h)>0$.

(iii) $\int_\Omega he^{\eta P}dx>0\Leftrightarrow\lambda_1(\eta,h)<0$.

(iv) $\lambda_1(\eta,h_1)>\lambda_1(\eta,h_2)$ *if $h_1\le h_2$, $h_1\not\equiv h_2$, and both h_1,h_2 change sign.*

(v) $\lambda_1(\eta,h)$ *is continuous in h.*

In order to analyze the principal eigenvalue of problems (1.15) and (1.16), it is more convenient to consider the following more general form of eigenvalue problem:

$$\begin{cases}d_1\nabla\cdot(\nabla\phi-\eta\phi\nabla P)+h(x)\phi=\sigma\phi & \text{in }\Omega,\\ \dfrac{\partial\phi}{\partial n}-\eta\phi\dfrac{\partial P}{\partial n}=0 & \text{on }\partial\Omega,\end{cases}\tag{1.21}$$

which is equivalent to

$$\begin{cases} d_1\nabla\cdot(\mathrm{e}^{\eta P}\nabla\psi)+\mathrm{e}^{\eta P}h(x)\psi=\sigma\mathrm{e}^{\eta P}\psi & \text{in } \Omega,\\ \dfrac{\partial\psi}{\partial n}=0 & \text{on } \partial\Omega. \end{cases} \tag{1.22}$$

The principal eigenvalue of problem (1.21), denoted by $\sigma_1(d_1,\eta,h)$, is expressed by the following variational formula (see, e.g. [2])

$$\sigma_1(d_1,\eta,h)=\max_{\psi\in W^{1,2}(\Omega),\psi\not\equiv 0}\frac{-d_1\int_\Omega \mathrm{e}^{\eta P}|\nabla\psi|^2\mathrm{d}x+\int_\Omega \mathrm{e}^{2\eta P}h(x)\psi^2\mathrm{d}x}{\int_\Omega \mathrm{e}^{\eta P}\psi^2\mathrm{d}x}. \tag{1.23}$$

The following lemma collects a useful property of $\sigma_1(d_1,\eta,h)$ (see e.g. [2]).

Lemma 1.3 *The first eigenvalue $\sigma_1(d_1,\eta,h)$ of (1.21) has the following property: If $\lambda_1(\eta,h)>0$, then $\sigma_1(d_1,\eta,h)<0\Leftrightarrow d_1>\dfrac{1}{\lambda_1(\eta,h)}$.*

1.3 Main results

Based on the above preparations, we are now ready to state our main results concerning the steady states of (1.6). Before giving the first theorem, motivated by [14], we simply need to define

$$L_u:=\inf_{d_1>0}\frac{\int_\Omega \widehat{m}\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \theta_{d_1}\mathrm{e}^{\eta P}\mathrm{d}x}\geq 0,\quad S_u:=\sup_{d_1>0}\sup_{\Omega}\frac{\widehat{m}}{\theta_{d_1}}\leq+\infty, \tag{1.24}$$

$$L_v:=\inf_{d_2>0}\frac{\int_\Omega m\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \theta_{d_2}\mathrm{e}^{\eta P}\mathrm{d}x}\geq 0,\quad S_v:=\sup_{d_2>0}\sup_{\Omega}\frac{m}{\theta_{d_2}}\leq+\infty, \tag{1.25}$$

$$\begin{cases} I:=\left\{(d_1,\eta)\in\mathbb{R}^+\times\mathbb{R}^+:\displaystyle\int_\Omega(\widehat{m}-\theta_{d_1})\mathrm{e}^{\eta P}\mathrm{d}x<0\right\}=I^0\cup I^1,\\ I^0:=\{(d_1,\eta)\in\mathbb{R}^+\times\mathbb{R}^+:\widehat{m}-\theta_{d_1}\leq\not\equiv 0\},\\ I^1:=\left\{(d_1,\eta)\in I:\sup\limits_{\overline{\Omega}}(\widehat{m}-\theta_{d_1})>0\right\}. \end{cases} \tag{1.26}$$

Since by (1.11)–(1.12), it is easy to verify that

$$\int_\Omega m\mathrm{e}^{\eta P}\mathrm{d}x<\int_\Omega \theta_{d_1}\mathrm{e}^{\eta P}\mathrm{d}x,\quad \int_\Omega \widehat{m}\mathrm{e}^{\eta P}\mathrm{d}x<\int_\Omega \theta_{d_2}\mathrm{e}^{\eta P}\mathrm{d}x, \tag{1.27}$$

and

$$\lim_{d_1\to 0}\widetilde{\theta}_{d_1}=m\mathrm{e}^{-\eta P},\quad \lim_{d_1\to+\infty}\widetilde{\theta}_{d_1}=\frac{\int_\Omega m\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}. \tag{1.28}$$

Indeed, (1.11) is equivalent to

$$\begin{cases} d_1\nabla\cdot(\mathrm{e}^{\eta P}\nabla\widetilde{\theta}_{d_1})+\mathrm{e}^{2\eta P}\widetilde{\theta}_{d_1}(m(x)-\widetilde{\theta}_{d_1})=0 & \text{in } \Omega,\\ \dfrac{\partial\widetilde{\theta}_{d_1}}{\partial n}=0 & \text{on } \partial\Omega. \end{cases} \tag{1.29}$$

After dividing by $\mathrm{e}^{\eta P}\widetilde{\theta}_{d_1}$ on both side in the first equation above and integrating over Ω, we obtain the first inequality in (1.27). The second inequality holds similarly. Meanwhile (1.28) can be deduced in [30]. Then we get that

$$\begin{cases} 0 < L_u < 1, L_v = 1, \\ L_u S_v > 1, \text{if } L_u > 0 \text{ and } S_v < +\infty, \\ L_v S_u > 1, \text{if } L_v > 0 \text{ and } S_u < +\infty. \end{cases} \tag{1.30}$$

Theorem 1.1 *Assume that* (1.4)–(1.5) *hold. Let* L_u, S_u, L_v *and* S_v *be defined as in* (1.24) *and* (1.25)*. Then the following statements hold for* (1.6):

(i) *For* Σ_u*, we have that* $\Sigma_u = \{(d_1, d_2, \eta) : d_1 \in I, d_2 > d_2^*(d_1, \eta)\}$*, where* I *is defined as in* (1.26) *and* $d_2^*(d_1, \eta)$ *is defined as in* (2.4).

(ii) *For* Σ_v*, we have that* $\Sigma_v = \emptyset$.

Combined with [31, Theorems 1.1–1.3]) and Theorem 1.1, we can characterize the sets Σ_- and $\Sigma_{0,0}$ directly, i.e., Theorem 1.2. Thus the proof is omitted here.

Theorem 1.2 *Assume that* (1.4)–(1.5) *hold. Let* L_u, S_u, L_v *and* S_v *be defined as in* (1.24) *and* (1.25)*. Then the following statements hold for* (1.6):

(i) *For* Σ_-*, we have that* $\Sigma_- = \Gamma \backslash (\Sigma_{u,0} \bigcup \Sigma_{v,0} \bigcup \Sigma_u)$.

(ii) *For* $\Sigma_{0,0}$*, we have the following characterization*:

$$\Sigma_{0,0} = \{(d_1, d_2, \eta) \in \Gamma : \theta_{d_1} = \theta_{d_2} \text{ in } \overline{\Omega}\}. \tag{1.31}$$

Hence, $\Sigma_{0,0} \neq \emptyset$ *if and only if there exists* $(d_1, d_2, \eta) \in \Gamma$ *such that* $\theta_{d_1} = \theta_{d_2}$.

Based on Theorem 1.2, we will consider whether the set Σ_- is empty for large d_1. Furthermore, if Σ_- is nonempty, we study what the asymptotic behavior of the unique coexistence steady state of (1.6) is as $d_1 \to +\infty$ when $(d_1, d_2) \in \Sigma_-$. To deal with these problems, we shall analyze the asymptotic behavior of $d_2^*(d_1, \eta)$ as $d_1 \to +\infty$ more carefully.

For each $D > 0$, we set $\Gamma_D := \{(d_1, d_2, \eta) \in \Gamma : d_1 > D\}$. Denote by $\rho_{m,\eta,P}$ the unique solution satisfying:

$$\begin{cases} \Delta \rho_{m,\eta,P} + \eta \nabla P \cdot \nabla \rho_{m,\eta,P} + \widehat{m} \mathrm{e}^{-\eta P}(m - \widehat{m}) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial \rho_{m,\eta,P}}{\partial n} = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \rho_{m,\eta,P} \mathrm{e}^{2\eta P} \mathrm{d}x = 0, \end{cases} \tag{1.32}$$

and

$$C(m, \eta, P) = \frac{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x \int_\Omega \mathrm{e}^{\eta P} |\nabla \rho_{m,\eta,P}|^2 \mathrm{d}x}{\left(\int_\Omega m \mathrm{e}^{\eta P} \mathrm{d}x \right)^2}. \tag{1.33}$$

Theorem 1.3 *Assume that* (1.4)–(1.5) *hold. Then there exists a* $D_{m,\eta,P} > 0$ *depending only on* m, η, P *such that the followings hold for* (1.6):

(i) *If* $\inf\limits_\Omega \rho_{m,\eta,P} + C(m, \eta, P) > 0$*, then for all* $d_1 > D_{m,\eta,P}$*,* $(\theta_{d_1}, 0)$ *is linearly stable, i.e.,* $\Sigma_u \bigcap \Gamma_{D_{m,\eta,P}} = \Gamma_{D_{m,\eta,P}}$.

(ii) *If* $\inf\limits_{\Omega}\rho_{m,\eta,P}+C(m,\eta,P)=0$, *then* $d_2^*(d_1,\eta)=O\big(\frac{1}{d_1^2}\big)$ *for all* $d_1>D_{m,\eta,P}$.

(iii) *If* $\inf\limits_{\Omega}\rho_{m,\eta,P}+C(m,\eta,P)<0$, *then for all* $d_1>D_{m,\eta,P}$, *there exist two numbers*

$$\Lambda_{m,\eta,P}:=\frac{1}{\lambda_1(\eta,(-\rho_{m,\eta,P}-C(m,\eta,P))\mathrm{e}^{\eta P})} \tag{1.34}$$

and $\Pi_{m,\eta,P}\in\mathbb{R}$ *depending only on* $m,\eta,P(x)$ *such that*

$$d_2^*(d_1,\eta)=\frac{\Lambda_{m,\eta,P}}{d_1}+\frac{\Pi_{m,\eta,P}}{d_1^2}+O\Big(\frac{1}{d_1^3}\Big), \tag{1.35}$$

which implies that $\Gamma_{D_{m,\eta,P}}\bigcup\Sigma_-=\Gamma_{D_{m,\eta,P}}\backslash\overline{\Sigma_u}$ *is nonempty.*

Finally, we state a result which characterizes the asymptotical behavior of the coexistence steady state in details for the case $\inf\limits_{\Omega}\rho_{m,\eta,P}+C(m,\eta,P)\le 0$ as $d_1\to+\infty$ and $d_2\to 0$.

Theorem 1.4 *Assume that* (1.4)–(1.5) *hold and* $\inf\limits_{\Omega}\rho_{m,\eta,P}+C(m,\eta,P)\le 0$. *Let* $(d_1,d_2,\eta)\in\Sigma_-$ *and* (U,V) *be the corresponding unique coexistence steady state of* (1.6). *Then there exists a constant* $D_{m,\eta,P}>0$ *depending only on* m,η,P *such that the following holds*:

$$\begin{cases}\widetilde{U}=\dfrac{\int_\Omega m\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}+O\Big(\dfrac{1}{d_1}\Big),\\ \|V\|_\infty=O\Big(\dfrac{1}{d_1}\Big),\end{cases}\quad \textit{uniformly in } d_2,\ \textit{for all } d_1>D_{m,\eta,P}. \tag{1.36}$$

If we assume further that $\inf\limits_{\Omega}\rho_{m,\eta,P}+C(m,\eta,P)<0$, *then*

$$0<C_1=\liminf_{\substack{d_1\to+\infty\\ d_1d_2\to p}} d_1\|V\|_\infty\le\limsup_{\substack{d_1\to+\infty\\ d_1d_2\to p}} d_1\|V\|_\infty=C_2, \tag{1.37}$$

where $p\in[0,\Lambda_{m,\eta,P})$, *and* C_1 *and* C_2 *are two positive constants depending only on* η,P *and* m.

In fact, in [31], the existence and the globally asymptotic stability of co-existence steady state of (1.3) has been considered. Hence, combined with the results in this paper, some properties of steady states of (1.6) are clear.

2 Proofs of the Main Results

Proof of Theorem 1.1 The proof is divided into three steps.

Step 1 $(d_1,d_2,\eta)\in\Sigma_u$ indicates that $(d_1,\eta)\in I$.

Suppose that $(d_1,\eta)\notin I$, where I is defined as in (1.26). Then

$$\int_\Omega(\widehat{m}-\theta_{d_1})\mathrm{e}^{\eta P}\mathrm{d}x\ge 0, \tag{2.1}$$

which implies that $\sigma_1(d_2,\eta,\widehat{m}-\theta_{d_1})\ge 0$, i.e., $(d_1,d_2,\eta)\notin\Sigma_u$. Hence $(d_1,d_2,\eta)\in\Sigma_u$ implies that $(d_1,\eta)\in I$.

We next characterize the set I in detail.

Step 2 $I^1\ne\emptyset$ if and only if $L_u<1<S_u$.

Indeed, it suffices to show that $L_u < 1 < S_u$ can lead to $I^1 \neq \emptyset$. Since by (1.30), $L_u < 1 < S_u$ always holds, and by which, there exists some $d_1' > 0$, $y_0 \in \overline{\Omega}$, such that

$$\int_\Omega (\widehat{m} - \theta_{d_1'}) \mathrm{e}^{\eta P} \mathrm{d}x < 0 \quad \text{and} \quad (\widehat{m} - \theta_{d_1'})(y_0) > 0. \tag{2.2}$$

That is, $(d_1', \eta) \in I^1 \neq \emptyset$, which finishes the proof of Step 2.

Step 3 Since $L_u < 1 < S_u$, it immediately follows that $I = I^0 \bigcup I^1 \subset \mathbb{R}^+ \times \mathbb{R}^+$. If $(d_1, \eta) \in I^0$, then $\sigma_1(d_2, \eta, \widehat{m} - \theta_{d_1}) < 0$ by (1.23). If $(d_1, \eta) \in I^1$, then

$$\sigma_1(d_2, \eta, \widehat{m} - \theta_{d_1}) < 0 \Leftrightarrow d_2 > \frac{1}{\lambda_1(\eta, \widehat{m} - \theta_{d_1})} > 0 \tag{2.3}$$

by Lemmas 1.2–1.3. Hence after defining

$$d_2^*(d_1, \eta) = \begin{cases} 0 & (d_1, \eta) \in I^0, \\ \dfrac{1}{\lambda_1(\eta, \widehat{m} - \theta_{d_1})} & (d_1, \eta) \in I^1, \end{cases} \tag{2.4}$$

we obtain that $(d_1, d_2, \eta) \in \Sigma_u$ if and only if $(d_1, \eta) \in I$ and $d_2 > d_2^*$. This finishes the proof of Theorem 1.1(i). The proof of Theorem 1.1(ii) is in fact the same as (i) and is thus omitted.

Next in order to establish Theorem 1.3, motivated by [15], we need to verify the following asymptotic expansion of θ_d as $d \to +\infty$, which will be used later.

Proposition 2.1 *Assume that* (1.4)–(1.5) *hold. Let θ_d be the unique solution of*

$$\begin{cases} d\nabla \cdot (\nabla \theta_d - \eta \theta_d \nabla P) + \theta_d (m(x) - \theta_d) = 0 & \text{in } \Omega, \\ \dfrac{\partial \theta_d}{\partial n} - \eta \dfrac{\partial P}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Then there exists a constant $D_{m,\eta,P} > 0$ depending only on m, η, P such that

$$\theta_d = \widehat{m} + \mathrm{e}^{\eta P} \Big(\frac{\rho_{m,\eta,P} + C(m,\eta,P)}{d} + \frac{\gamma_{m,\eta,P} + K(m,\eta,P)}{d^2} \Big) + O\Big(\frac{1}{d^3}\Big) \tag{2.6}$$

for all $d > D_{m,\eta,P}$, where $\rho_{m,\eta,P}$, $C(m,\eta,P)$ are defined as in (1.32)–(1.33), *and $\gamma_{m,\eta,P}$, $K(m,\eta,P)$ are defined below:*

$$\begin{cases} \Delta \gamma_{m,\eta,P} + \eta \nabla P \cdot \nabla \gamma_{m,\eta,P} + (m - 2\mathrm{e}^{\eta P} \widehat{m})(\rho_{m,\eta,P} + C(m,\eta,P)) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial \gamma_{m,\eta,P}}{\partial n} = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \gamma_{m,\eta,P} \mathrm{e}^{2\eta P} \mathrm{d}x = 0, \end{cases} \tag{2.7}$$

$$K(m,\eta,P) = \frac{\displaystyle\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x \int_\Omega \rho_{m,\eta,P}^2 \mathrm{e}^{2\eta P} (m - 3\widehat{m} \mathrm{e}^{-\eta P}) \mathrm{d}x}{\Big(\displaystyle\int_\Omega m \mathrm{e}^{\eta P} \mathrm{d}x \Big)^2}. \tag{2.8}$$

Proof Multiplying the equation of $\rho_{m,\eta,P}$ by $\gamma_{m,\eta,P} \mathrm{e}^{\eta P}$ and the equation of $\gamma_{m,\eta,P}$ by $\rho_{m,\eta,P} \mathrm{e}^{\eta P}$, and then we see by (1.33), from integrating by parts that

$$\widehat{m} \mathrm{e}^{-\eta P} \int_\Omega \gamma_{m,\eta,P} (m \mathrm{e}^{\eta P} - 2\mathrm{e}^{\eta P} \widehat{m}) \mathrm{d}x$$

$$
\begin{aligned}
&= \int_\Omega \mathrm{e}^{\eta P} \nabla \rho_{m,\eta,P} \cdot \nabla \gamma_{m,\eta,P} \mathrm{d}x \\
&= \int_\Omega \rho_{m,\eta,P} (m\mathrm{e}^{\eta P} - 2\mathrm{e}^{\eta P} \widehat{m})(\rho_{m,\eta,P} + C(m,\eta,P))\mathrm{d}x \\
&= \int_\Omega \rho^2_{m,\eta,P} (m\mathrm{e}^{\eta P} - 2\mathrm{e}^{\eta P} \widehat{m})\mathrm{d}x + 2C^2(m,\eta,P) \int_\Omega m\mathrm{e}^{\eta P} \mathrm{d}x. \qquad (2.9)
\end{aligned}
$$

This, together with (2.8), implies that

$$
\int_\Omega (m\mathrm{e}^{\eta P} - 2\mathrm{e}^{\eta P}\widehat{m})(\gamma_{m,\eta,P} + K(m,\eta,P))\mathrm{d}x = \int_\Omega \mathrm{e}^{2\eta P}(\rho_{m,\eta,P} + C(m,\eta,P))^2 \mathrm{d}x. \qquad (2.10)
$$

Hence there exists a unique θ_3 satisfying

$$
\begin{cases}
\Delta\theta_3 + \eta\nabla P \cdot \nabla\theta_3 + (m - 2\mathrm{e}^{\eta P}\widehat{m})(\gamma_{m,\eta,P} + K(m,\eta,P)) \\
-\mathrm{e}^{\eta P}(\rho_{m,\eta,P} + C(m,\eta,P))^2 = 0 \quad \text{in } \Omega, \\
\dfrac{\partial\theta_3}{\partial n} = 0 \quad \text{on } \partial\Omega, \\
\displaystyle\int_\Omega \theta_3 \mathrm{e}^{2\eta P}\mathrm{d}x = 0.
\end{cases} \qquad (2.11)
$$

Let now θ_4 be the unique solution to

$$
\begin{cases}
\Delta\theta_4 + \eta\nabla P \cdot \nabla\theta_4 + (m - 2\mathrm{e}^{\eta P}\widehat{m})(\theta_3 + \widetilde{C}) \\
-2\mathrm{e}^{\eta P}(\rho_{m,\eta,P} + C(m,\eta,P))(\gamma_{m,\eta,P} + K(m,\eta,P)) = 0 \quad \text{in } \Omega, \\
\dfrac{\partial\theta_4}{\partial n} = 0 \quad \text{on } \partial\Omega, \\
\displaystyle\int_\Omega \theta_4 \mathrm{e}^{2\eta P}\mathrm{d}x = 0,
\end{cases} \qquad (2.12)
$$

where $\widetilde{C}$ is the unique number such that (2.12) has a solution. Define

$$
\begin{aligned}
\theta^\pm := \widehat{m} &+ \frac{(\rho_{m,\eta,P} + C(m,\eta,P))\mathrm{e}^{\eta P}}{d} + \frac{(\gamma_{m,\eta,P} + K(m,\eta,P))\mathrm{e}^{\eta P}}{d^2} \\
&+ \frac{(\theta_3 + \widetilde{C} \pm 1)\mathrm{e}^{\eta P}}{d^3} + \frac{\theta_4 \mathrm{e}^{\eta P}}{d^4} \pm \frac{\rho_{m,\eta,P}\mathrm{e}^{\eta P}\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}{d^4 \int_\Omega m\mathrm{e}^{\eta P}\mathrm{d}x}. \qquad (2.13)
\end{aligned}
$$

By some straightforward computations, we have

$$
\begin{aligned}
& d\nabla \cdot (\mathrm{e}^{\eta P}\nabla\widetilde{\theta}^\pm) + \theta^\pm(m - \theta^\pm) \\
&= \nabla \cdot (\mathrm{e}^{\eta P}\nabla\rho_{m,\eta,P}) + d^{-1}\nabla \cdot (\mathrm{e}^{\eta P}\nabla\gamma_{m,\eta,P}) \\
&\quad + d^{-2}\nabla \cdot (\mathrm{e}^{\eta P}\nabla\theta_3) + d^{-3}\nabla \cdot (\mathrm{e}^{\eta P}\nabla\theta_4) \pm d^{-3}\frac{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}{\int_\Omega m\mathrm{e}^{\eta P}\mathrm{d}x}\nabla \cdot (\mathrm{e}^{\eta P}\nabla\rho_{m,\eta,P}) \\
&\quad + \widehat{m}(m - \widehat{m}) + \frac{\mathrm{e}^{\eta P}}{d}(m - 2\widehat{m})(\rho_{m,\eta,P} + C(m,\eta,P)) + \frac{\mathrm{e}^{\eta P}}{d^2}((\gamma_{m,\eta,P} + K(m,\eta,P))(m - 2\widehat{m}) \\
&\quad - \mathrm{e}^{\eta P}(\rho_{m,\eta,P} + C(m,\eta,P))^2) + \frac{\mathrm{e}^{\eta P}}{d^3}((m - 2\widehat{m})(\theta_3 + \widetilde{C} \pm 1) \\
&\quad - 2\mathrm{e}^{\eta P}(\rho_{m,\eta,P} + C(m,\eta,P))(\gamma_{m,\eta,P} + K(m,\eta,P))) + O\Big(\frac{1}{d^4}\Big) = \mp\frac{1}{d^3}\widehat{m} + O\Big(\frac{1}{d^4}\Big). \qquad (2.14)
\end{aligned}
$$

Thus $\theta^{\pm}$ is a pair of upper and lower solutions to (2.5) for all d sufficiently large. Note that for all d sufficiently large, $0<\theta^-<\theta^+$, by the upper/lower solution method (see [29]) and the uniqueness of θ_d, we have that $\theta^-\le\theta_d\le\theta^+$. This thus finishes the proof of (2.6).

Now we are going to prove Theorem 1.3.

Proof of Theorem 1.3 We divide this proof into several cases.

Case 1 $\inf_{\Omega}\rho_{m,\eta,P}+C(m,\eta,P)>0$.

It follows from (2.6) that there exists a constant $D_{m,\eta,P}>0$ such that $\widehat{m}<\theta_{d_1}$ on $\overline{\Omega}$ for all $d_1>D_{m,\eta,P}$. Hence, $\sigma_1(d_2,\widehat{m}-\theta_{d_1})<\sigma_1(d_2,0)=0$ by (1.23), which implies that $(\theta_{d_1},0)$ is linearly stable.

Case 2 $\inf_{\Omega}\rho_{m,\eta,P}+C(m,\eta,P)=0$. Thus $\rho_{m,\eta,P}+C(m,\eta,P)\ge 0$.

It is obvious that $\widehat{m}-\theta_{d_1}\le\not\equiv 0$ leads to $\sigma_1(d_2,\widehat{m}-\theta_{d_1})<\sigma_1(d_2,0)=0$, which implies $(\theta_{d_1},0)$ is linearly stable.

Now without loss of generality, we may assume that $\widehat{m}-\theta_{d_1}$ changes sign in Ω for all d_1 large enough. By (1.27) and Lemma 1.2, $\lambda_1(\eta,\widehat{m}-\theta_{d_1})>0$. Hence in order to prove (ii), it suffices to show that there exist two constants $C_{m,\eta,P}>0$ and $D_{m,\eta,P}>0$ depending only on m,η,P such that

$$\lambda_1(\eta,\widehat{m}-\theta_{d_1})=d_1^2\lambda_1(\eta,d_1^2\widehat{m}-d_1^2\theta_{d_1})>C_{m,\eta,P}d_1^2\quad\text{for all } d_1>D_{m,\eta,P},\tag{2.15}$$

i.e., $\lambda_1(\eta,d_1^2\widehat{m}-d_1^2\theta_{d_1})>C_{m,\eta,P}$ for all $d_1>D_{m,\eta,P}$. By (2.6), there exists a constant $D_{m,\eta,P}>0$ such that

$$\begin{aligned}d_1^2\widehat{m}-d_1^2\theta_{d_1}&=-d_1(\rho_{m,\eta,P}+C(m,\eta,P))\mathrm{e}^{\eta P}-(\gamma_{m,\eta,P}+K(m,\eta,P))\mathrm{e}^{\eta P}\\&\quad+O\Big(\frac{1}{d_1}\Big)\quad\text{for all } d_1>D_{m,\eta,P}.\end{aligned}\tag{2.16}$$

For each $L>0$,

$$\Theta_{d_1}^L:=\begin{cases}d_1^2\widehat{m}-d_1^2\theta_{d_1} & \text{if } d_1^2\widehat{m}-d_1^2\theta_{d_1}>-L,\\ -L & \text{if } d_1^2\widehat{m}-d_1^2\theta_{d_1}\le -L.\end{cases}\tag{2.17}$$

Hence $d_1^2\widehat{m}-d_1^2\theta_{d_1}\le\Theta_{d_1}^L$, $\Theta_{d_1}^L$ changes sign and furthermore $\lambda_1(\eta,\Theta_{d_1}^L)$ is defined and positive for all d_1 and L sufficiently large. Moreover, by Lemma 1.2, $\lambda_1(\eta,d_1^2\widehat{m}-d_1^2\theta_{d_1})\ge\lambda_1(\eta,\Theta_{d_1}^L)$. Since $\inf_{\Omega}\rho_{m,\eta,P}+C(m,\eta,P)=0$, (2.16) and (2.17) imply that there exist two constants $C_L^2<C_L^1<0$ such that

$$C_L^2<\int_{\Omega}\Theta_{d_1}^L\mathrm{e}^{\eta P}\mathrm{d}x<C_L^1<0\quad\text{for all } d_1>D_{m,\eta,P}.\tag{2.18}$$

Moreover, choosing L and $D_{m,\eta,P}$ even larger if necessary, one can see that $\|\Theta_{d_1}^L\mathrm{e}^{\eta P}\|_\infty=L$ for all $d_1>D_{m,\eta,P}$. Applying a similar approach in [28], one can see that there exists a constant $C_{m,\eta,P}>0$ depending only on m,η,P such that $\lambda_1(\eta,\Theta_{d_1}^L)>C_{m,\eta,P}$ for all $d_1>D_{m,\eta,P}$.

Case 3 $\inf_{\Omega}\rho_{m,\eta,P}+C(m,\eta,P)<0$.

This case yields that $\widehat{m} > \theta_{d_1}$ for sufficiently large d_1 by (2.6). Combined with (1.27), we see that there exists a $D_{m,\eta,P} > 0$ such that $\widehat{m} - \theta_{d_1}$ changes sign in Ω for all $d_1 > D_{m,\eta,P}$. In view of

$$\int_\Omega (-\rho_{m,\eta,P} - C(m,\eta,P))e^{2\eta P} dx = -C(m,\eta,P) \int_\Omega e^{2\eta P} dx < 0, \tag{2.19}$$

by (1.27) and Lemma 1.2, we then observe that both $\lambda_1(\eta, \widehat{m} - \theta_{d_1})$ and $\lambda_1(\eta, (-\rho_{m,\eta,P} - C(m,\eta,P))e^{\eta P})$ are defined and positive for all $d_1 > D_{m,\eta,P}$. Now let $\phi_* > 0$ be the principal eigenfunction corresponding to $\lambda_1(\eta, (-\rho_{m,\eta,P} - C(m,\eta,P))e^{\eta P})$ normalized such that $\max_\Omega \phi_* e^{\eta P} = 1$ and define

$$\Pi_{m,\eta,P} = -\frac{\int_\Omega (\gamma_{m,\eta,P} + K(m,\eta,P))\phi_*^2 e^{2\eta P} dx}{\int_\Omega |\nabla \phi_*|^2 e^{2\eta P} dx}. \tag{2.20}$$

Clearly, $\lambda_1(\eta, \widehat{m} - \theta_{d_1}) = d_1 \lambda_1(\eta, d_1\widehat{m} - d_1\theta_{d_1})$.

We next verify that in fact

$$\lambda_1(\eta, d_1\widehat{m} - d_1\theta_{d_1}) = \frac{1}{\Lambda_{m,\eta,P}} - \frac{\Pi_{m,\eta,P}}{\Lambda^2_{m,\eta,P} d_1} + O\Big(\frac{1}{d_1^2}\Big) \tag{2.21}$$

for all $d_1 > D_{m,\eta,P}$, where $\Lambda_{m,\eta,P} := \dfrac{1}{\lambda_1(\eta, (-\rho_{m,\eta,P} - C(m,\eta,P))e^{\eta P})}$. Since by (2.6), we have

$$d_1\widehat{m} - d_1\theta_{d_1} = -(\rho_{m,\eta,P} + C(m,\eta,P))e^{\eta P} + O\Big(\frac{1}{d_1}\Big) \quad \text{for all } d_1 > D_{m,\eta,P}, \tag{2.22}$$

the continuity of $\lambda_1(\cdot)$ in Lemma 1.2 then leads to

$$\lambda_1(\eta, d_1\widehat{m} - d_1\theta_{d_1}) = \frac{1}{\Lambda_{m,\eta,P}} + O(1) \quad \text{as } d_1 \to +\infty. \tag{2.23}$$

Denote $\varphi > 0$ the principal eigenfunction corresponding to $\lambda_1(\eta, d_1\widehat{m} - d_1\theta_{d_1})$. Then one can easily check that

$$\varphi = \phi_* + o(1) \tag{2.24}$$

as $d_1 \to +\infty$. Rewrite that $\varphi = \phi_* + \dfrac{\psi}{d_1} + \dfrac{\omega}{d_1^2}$, where ψ is the unique solution of

$$\begin{cases} \Delta\psi + \eta\nabla P \cdot \nabla\psi + \dfrac{1}{\Lambda_{m,\eta,P}}(-\rho_{m,\eta,P} - C(m,\eta,P))e^{\eta P}\psi \\ -e^{\eta P}\phi_*\Big[-\dfrac{\Pi_{m,\eta,P}}{\Lambda^2_{m,\eta,P}}(\rho_{m,\eta,P} + C(m,\eta,P)) \\ +\dfrac{1}{\Lambda_{m,\eta,P}}(\gamma_{m,\eta,P} + K(m,\eta,P))\Big] = 0, \quad x \in \Omega, \\ \dfrac{\partial\psi}{\partial n} = 0, \quad x \in \partial\Omega, \\ \displaystyle\int_\Omega \psi\phi_* e^{2\eta P} dx = 0. \end{cases} \tag{2.25}$$

Denote

$$d_1(\widehat{m}-\theta_{d_1})+\mathrm{e}^{\eta P}(\rho_{m,\eta,P}+C(m,\eta,P)) \quad \text{by } F. \tag{2.26}$$

We then obtain that

$$F=O\Big(\frac{1}{d_1}\Big) \quad \text{for all } d_1>D_{m,\eta,P}, \tag{2.27}$$

and it follows from some calculations that ω satisfies the following equations:

$$\begin{cases} \Delta\omega+\eta\nabla P\cdot\nabla\omega+\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})\cdot(d_1\widehat{m}-d_1\theta_{d_1})\omega \\ +d_1^2\mathrm{e}^{\eta P}\phi_*(-\rho_{m,\eta,P}-C(m,\eta,P))\Big(\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})-\dfrac{1}{\Lambda_{m,\eta,P}}+\dfrac{\Pi_{m,\eta,P}}{d_1\Lambda_{m,\eta,P}^2}\Big) \\ +d_1^2\Big(\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})\cdot F+\dfrac{\mathrm{e}^{\eta P}}{d_1\Lambda_{m,\eta,P}}(\gamma_{m,\eta,P}+K(m,\eta,P))\Big)\phi_* \\ +d_1\Big(\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})-\dfrac{1}{\Lambda_{m,\eta,P}}\Big)(-\rho_{m,\eta,P}-C(m,\eta,P))\mathrm{e}^{\eta P}\psi \\ +d_1\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})\cdot F\psi=0, \quad x\in\Omega, \\ \dfrac{\partial\omega}{\partial n}=0, \quad x\in\partial\Omega, \\ \displaystyle\int_\Omega \omega\phi_*\mathrm{e}^{2\eta P}\mathrm{d}x=0. \end{cases} \tag{2.28}$$

Using (2.6), one sees

$$F+\frac{\mathrm{e}^{\eta P}(\gamma_{m,\eta,P}+K(m,\eta,P))}{d_1}=O\Big(\frac{1}{d_1^2}\Big) \quad \text{for all } d_1>D_{m,\eta,P}. \tag{2.29}$$

Multiplying the equation for ω by $\varphi\mathrm{e}^{\eta P}$ and the equation for φ by $\omega\mathrm{e}^{\eta P}$, integrating over Ω and subtracting, we deduce that

$$\begin{aligned} &-d_1\Big(\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})-\frac{1}{\Lambda_{m,\eta,P}}\Big)\int_\Omega \mathrm{e}^{2\eta P}\phi_*\varphi(-\rho_{m,\eta,P}-C(m,\eta,P))\varphi\phi_*\mathrm{d}x \\ =&\frac{\Pi_{m,\eta,P}}{\Lambda_m^2}\int_\Omega \mathrm{e}^{2\eta P}(-\rho_{m,\eta,P}-C(m,\eta,P))\varphi\phi_*\mathrm{d}x \\ &+d_1\int_\Omega \mathrm{e}^{\eta P}\phi_*\varphi\Big(\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})\cdot F+\frac{\mathrm{e}^{\eta P}(\gamma_{m,\eta,P}+K(m,\eta,P))}{d_1\Lambda_{m,\eta,P}}\Big)\mathrm{d}x \\ &+\Big(\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})-\frac{1}{\Lambda_{m,\eta,P}}\Big)\int_\Omega \mathrm{e}^{2\eta P}(-\rho_{m,\eta,P}-C(m,\eta,P))\psi\varphi\mathrm{d}x \\ &+\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})\int_\Omega F\varphi\psi\mathrm{e}^{\eta P}\mathrm{d}x. \end{aligned} \tag{2.30}$$

Combining (2.23), (2.27) and (2.29) together, we see

$$\begin{aligned} &\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})\cdot F+\frac{\mathrm{e}^{\eta P}(\gamma_{m,\eta,P}+K(m,\eta,P))}{d_1\Lambda_{m,\eta,P}} \\ =&\Big(\lambda_1(\eta,d_1\widehat{m}-d_1\theta_{d_1})-\frac{1}{\Lambda_{m,\eta,P}}\Big)F+\frac{1}{\Lambda_{m,\eta,P}}\Big(F+\frac{\mathrm{e}^{\eta P}(\gamma_{m,\eta,P}+K(m,\eta,P))}{d_1}\Big) \end{aligned}$$

$$= o\Big(\frac{1}{d_1}\Big) \tag{2.31}$$

as $d_1 \to +\infty$. By (2.24), we have

$$\int_\Omega \mathrm{e}^{2\eta P}\varphi\phi_*(-\rho_{m,\eta,P} - C(m,\eta,P))\mathrm{d}x = \int_\Omega \mathrm{e}^{2\eta P}\phi_*^2(-\rho_{m,\eta,P} - C(m,\eta,P))\mathrm{d}x + o(1) \tag{2.32}$$

as $d_1 \to +\infty$. Hence dividing both sides of (2.30) by $\int_\Omega(-\rho_{m,\eta,P} - C(m,\eta,P))\varphi\phi_*\mathrm{e}^{\eta P}\mathrm{d}x$ and letting $d_1 \to +\infty$, using (2.23)–(2.24), (2.27), (2.29) and the above estimate again, we derive that

$$-d_1\Big(\lambda_1(\eta, d_1\widehat{m} - d_1\theta_{d_1}) - \frac{1}{\Lambda_{m,\eta,P}}\Big) = \frac{\Pi_{m,\eta,P}}{\Lambda^2_{m,\eta,P}} + o(1) \quad \text{for all } d_1 > D_{m,\eta,P}, \tag{2.33}$$

which implies that $\lambda_1(\eta, d_1\widehat{m} - d_1\theta_{d_1}) - \frac{1}{\Lambda_{m,\eta,P}} = O\big(\frac{1}{d_1}\big)$. This together with (2.27), (2.29) and (2.31) implies that

$$\lambda_1(\eta, d_1\widehat{m} - d_1\theta_{d_1})F + \frac{\mathrm{e}^{\eta P}(\gamma_{m,\eta,P} + K(m,\eta,P))}{d_1\Lambda_{m,\eta,P}} = O\Big(\frac{1}{d_1^2}\Big) \quad \text{for all } d_1 > D_{m,\eta,P}. \tag{2.34}$$

Therefore dividing both sides of (2.30) by $d_1\int_\Omega(-\rho_{m,\eta,P} - C(m,\eta,P))\varphi\phi_*\mathrm{e}^{2\eta P}\mathrm{d}x$ and letting $d_1 \to +\infty$, we obtain (2.21). This in turn indicates that $\lambda_1(\eta, \widehat{m}-\theta_{d_1}) = \frac{d_1}{\Lambda_{m,\eta,P}} - \frac{\Pi_{m,\eta,P}}{\Lambda^2_{m,\eta,P}} + O\big(\frac{1}{d_1}\big)$ and

$$d_2^*(d_2,\eta) = \frac{1}{\lambda_1(\eta, \widehat{m} - \theta_{d_1})} = \frac{\Lambda_{m,\eta,P}}{d_1} + \frac{\Pi_{m,\eta,P}}{d_1^2} + O\Big(\frac{1}{d_1^3}\Big). \tag{2.35}$$

The proof of Theorem 1.3 is thus finished.

Proof of Theorem 1.4 Let (U,V) be the coexistence steady state of (1.6). Then $(\widetilde{U},\widetilde{V}) = (U\mathrm{e}^{-\eta P}, V\mathrm{e}^{-\eta P})$ satisfies

$$\begin{cases} d_1\Delta\widetilde{U} + \alpha_1\nabla P\cdot\nabla\widetilde{U} + \widetilde{U}(m(x) - U - V) & \text{in } \Omega,\\ d_2\Delta\widetilde{V} + \alpha_2\nabla P\cdot\nabla\widetilde{V} + \widetilde{V}(\widehat{m}(x) - U - V) & \text{in } \Omega,\\ \dfrac{\partial\widetilde{U}}{\partial n} = \dfrac{\partial\widetilde{V}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.36}$$

By the maximum principle, we have that

$$\|\widetilde{U}\|_\infty \le \|\widetilde{\theta}_{d_1}\|_\infty < \max_\Omega(m\mathrm{e}^{-\eta P}), \quad \|\widetilde{V}\|_\infty \le \|\widetilde{\theta}_{d_2}\|_\infty < \frac{\int_\Omega m\mathrm{e}^{-\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}. \tag{2.37}$$

Integrating the equation for $\widetilde{V}$ over Ω, we obtain from Hölder inequality that

$$\begin{aligned} 0 &= \int_\Omega V(U+V+\widehat{m})\mathrm{d}x\\ &= \int_\Omega V\Big(\mathrm{e}^{\eta P}\frac{\int_\Omega U\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} + \mathrm{e}^{\eta P}\frac{\int_\Omega V\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} - \widehat{m}\Big)\mathrm{d}x\\ &\quad+ \int_\Omega\Big(V - \mathrm{e}^{\eta P}\frac{\int_\Omega V\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}\Big)\Big(U - \mathrm{e}^{\eta P}\frac{\int_\Omega U\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} + V - \mathrm{e}^{\eta P}\frac{\int_\Omega V\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}\Big)\mathrm{d}x \end{aligned}$$

$$> d_1 \frac{\int_\Omega V\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}\frac{\int_\Omega \mathrm{e}^{\eta P}|\nabla\widetilde{U}|^2\mathrm{d}x}{\|m\mathrm{e}^{-\eta P}\|_\infty} - \frac{1}{2}\int_\Omega \Big(U - \mathrm{e}^{\eta P}\frac{\int_\Omega U\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}\Big)^2\mathrm{d}x$$
$$+ \frac{1}{2}\int_\Omega \Big(V - \mathrm{e}^{\eta P}\frac{\int_\Omega V\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}\Big)^2\mathrm{d}x, \tag{2.38}$$

where we have used the identity $\int_\Omega \mathrm{e}^{\eta P}(m-U-V)\mathrm{d}x = -d_1\int_\Omega \frac{\mathrm{e}^{\eta P}|\nabla\widetilde{U}|^2}{\widetilde{U}^2}\mathrm{d}x$ obtained by multiplying the equation of $\widetilde{U}$ by $\mathrm{e}^{\eta P}$, and dividing by $\widetilde{U}$, integrating over Ω. Since there exists a constant $C>0$ such that

$$C\int_\Omega \mathrm{e}^{\eta P}|\nabla\widetilde{U}|^2\mathrm{d}x \geq \Big\|U - \mathrm{e}^{\eta P}\frac{\int_\Omega U\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}\Big\|_2^2, \tag{2.39}$$

which can be derived by a similar method of the proof of Poincaré's inequality in [11], then (2.38) gives rise to

$$0 > \Big(d_1\frac{\int_\Omega V\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}\frac{\int_\Omega \mathrm{e}^{\eta P}|\nabla\widetilde{U}|^2\mathrm{d}x}{\|m\mathrm{e}^{-\eta P}\|_\infty} - \frac{C}{2}\Big)\int_\Omega \mathrm{e}^{\eta P}|\nabla\widetilde{U}|^2\mathrm{d}x. \tag{2.40}$$

The inequality above implies

$$\frac{\int_\Omega V\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} = O\Big(\frac{1}{d_1}\Big). \tag{2.41}$$

Since $\|V\|_\infty < \frac{\int_\Omega m\mathrm{e}^{-\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}\|\mathrm{e}^{\eta P}\|_\infty$, by (1.32)–(1.33) and (2.41), there exists a $D_{m,\eta,P}>0$ depending only on m,η,P such that both $\|\rho_{m-V,\eta,P}\|_\infty$ and $C(m-V,\eta,P)$ are uniformly bounded in d_2 for all $d_1 > D_{m,\eta,P}$. Therefore, similar to (2.6), together with (2.41), we obtain that

$$\widetilde{U} = \frac{\int_\Omega (m-V)\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} + \frac{\rho_{m-V,\eta,P} + C(m-V,\eta,P)}{d_1} + O\Big(\frac{1}{d_1^2}\Big)$$
$$\text{uniformly in } d_2, \text{ for all } d_1 > D_{m,\eta,P}, \tag{2.42}$$

$$\frac{\int_\Omega (m-V)\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} - \widetilde{U} = O\Big(\frac{1}{d_1}\Big) \quad \text{uniformly in } d_2, \text{ for all } d_1 > D_{m,\eta,P} \tag{2.43}$$

and

$$\widetilde{U} = \frac{\int_\Omega m\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} + O\Big(\frac{1}{d_1}\Big) \quad \text{uniformly in } d_2, \text{ for all } d_1 > D_{m,\eta,P}. \tag{2.44}$$

On the other hand, from the equation for $\widetilde{V}$ and the maximum principle, we have

$$\|V\|_\infty \leq \|\widehat{m} - U\|_\infty \leq \Big\|\Big(\frac{\int_\Omega (m-V)\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} - \widetilde{U}\Big)\mathrm{e}^{\eta P}\Big\|_\infty$$
$$+ \Big\|\frac{\int_\Omega V\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x}\mathrm{e}^{\eta P}\Big\|_\infty = O\Big(\frac{1}{d_1}\Big). \tag{2.45}$$

Thus we have verified (1.36). It only remains to prove (1.37). We claim that

$$\begin{cases} C(m-V,\eta,P) = C(m,\eta,P) + O\Big(\dfrac{1}{d_1}\Big), \\ \rho_{m-V,\eta,P} = \rho_{m,\eta,P} + O\Big(\dfrac{1}{d_1}\Big), \end{cases} \quad \text{uniformly in } d_2, \text{ for all } d_1 > D_{m,\eta,P}. \tag{2.46}$$

Let r be the unique solution to

$$\begin{cases} \Delta r + \eta \nabla P \cdot \nabla r + \Big(\mathrm{e}^{\eta P} \frac{\int_\Omega V \mathrm{e}^{\eta P} \mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x} - V \Big) = 0 & \text{in } \Omega, \\ \frac{\partial r}{\partial n} = 0 & \text{on } \partial\Omega, \\ \frac{\int_\Omega r \mathrm{e}^{2\eta P} \mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x} = 0. \end{cases} \tag{2.47}$$

Multiplying the equation for r by $r\mathrm{e}^{\eta P}$ and integrating over Ω, we obtain that

$$\begin{aligned} \int_\Omega \mathrm{e}^{\eta P} |\nabla r|^2 \mathrm{d}x &= \int_\Omega r \mathrm{e}^{\eta P} \Big(\mathrm{e}^{\eta P} \frac{\int_\Omega V \mathrm{e}^{\eta P} \mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x} - V \Big) \mathrm{d}x \\ &\le \varepsilon \int_\Omega r^2 \mathrm{e}^{2\eta P} \mathrm{d}x + C_\varepsilon \int_\Omega \Big(\mathrm{e}^{\eta P} \frac{\int_\Omega V \mathrm{e}^{\eta P} \mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x} - V \Big)^2 \mathrm{d}x, \end{aligned} \tag{2.48}$$

where we have used Young's inequality. By means of a similar inequality to (2.39), we have that

$$\int_\Omega \mathrm{e}^{\eta P} |\nabla r|^2 \mathrm{d}x \le \varepsilon C \int_\Omega \mathrm{e}^{\eta P} |\nabla r|^2 \mathrm{d}x + C_\varepsilon \int_\Omega \Big(\mathrm{e}^{\eta P} \frac{\int_\Omega V \mathrm{e}^{\eta P} \mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x} - V \Big)^2 \mathrm{d}x. \tag{2.49}$$

Choosing $\varepsilon > 0$ small enough, we then derive from (1.36) and the above estimate that

$$\int_\Omega \mathrm{e}^{\eta P} |\nabla r|^2 \mathrm{d}x = O\Big(\frac{1}{d_1^2} \Big) \quad \text{uniformly in } d_2, \text{ for all } d_1 > D_{m,\eta,P}. \tag{2.50}$$

Since

$$\|r\|_\infty = O\Big(\Big\| \mathrm{e}^{\eta P} \frac{\int_\Omega V \mathrm{e}^{\eta P} \mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x} - V \Big\|_\infty \Big) = O\Big(\frac{1}{d_1} \Big) \tag{2.51}$$

by (2.47) and (1.36), it is easy to see from (1.32) and (2.47) that

$$\frac{\rho_{m-V,\eta,P}}{\int_\Omega (m-V) \mathrm{e}^{\eta P} \mathrm{d}x} = \frac{\rho_m}{\int_\Omega m \mathrm{e}^{\eta P} \mathrm{d}x} + \frac{r}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x}. \tag{2.52}$$

This together with (2.51) and the estimate of V in (1.36) implies the second equality of (2.46). By (1.33) and the above identity, we have

$$\begin{aligned} C(m-V,\eta,P) = C(m,\eta,P) &+ \frac{\int_\Omega \mathrm{e}^{\eta P} \mathrm{d}x \int_\Omega \mathrm{e}^{\eta P} |\nabla r|^2 \mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x} \\ &- \frac{2}{\int_\Omega m \mathrm{e}^{\eta P} \mathrm{d}x} \int_\Omega \rho_{m,\eta,P} \mathrm{e}^{\eta P} \Big(V - \mathrm{e}^{\eta P} \frac{\int_\Omega V \mathrm{e}^{\eta P} \mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x} \Big) \mathrm{d}x. \end{aligned} \tag{2.53}$$

By (1.36), $\frac{2}{\int_\Omega m \mathrm{e}^{\eta P} \mathrm{d}x} \int_\Omega \rho_m \mathrm{e}^{\eta P} \big(V - \mathrm{e}^{\eta P} \frac{\int_\Omega V \mathrm{e}^{\eta P} \mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P} \mathrm{d}x} \big) \mathrm{d}x = O\big(\frac{1}{d_1} \big)$ uniformly in d_2, for all $d_1 > D_{m,\eta,P}$. Hence we obtain the first equality of (2.46).

We now assume that $\inf_\Omega \rho_{m,\eta,P} + C(m,\eta,P) < 0$ and prove (1.37). By (1.36), it suffices to show that

$$\liminf_{\substack{d_1 \to +\infty \\ d_1 d_2 \to p}} d_1 \|V\|_\infty > 0. \tag{2.54}$$

The equation for V reads that $\dfrac{1}{d_2} = \lambda_1(\eta, \widehat{m} - U - V)$, i.e.,

$$\frac{1}{d_1 d_2} = \lambda_1(\eta, d_1(\widehat{m} - U - V)). \tag{2.55}$$

By (2.42) and (2.46),

$$d_1(\widehat{m} - U - V) = -\mathrm{e}^{\eta P}(\rho_{m,\eta,P} + C(m,\eta,P)) + d_1\Big(\mathrm{e}^{\eta P}\frac{\int_\Omega V\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} - V\Big) + O\Big(\frac{1}{d_1}\Big) \tag{2.56}$$

uniformly in d_2, for all $d_1 > D_{m,\eta,P}$. Assuming for contradiction that (2.54) does not hold. By (1.36), passing to a subsequence of d_1 and d_2 if necessary, we get that $d_1\|V\|_\infty \to 0$ as $d_1 \to +\infty$ and $d_1 d_2 \to p \in [0, \Lambda_{m,\eta,P})$, which further implies that

$$\Big\|d_1\Big(\mathrm{e}^{\eta P}\frac{\int_\Omega V\mathrm{e}^{\eta P}\mathrm{d}x}{\int_\Omega \mathrm{e}^{2\eta P}\mathrm{d}x} - V\Big)\Big\|_\infty \to 0. \tag{2.57}$$

However, taking limits on both sides of (2.55) as $d_1 \to +\infty$ and $d_1 d_2 \to p \in [0, \Lambda_{m,\eta,P})$, we obtain from the above estimate, (2.56) and Lemma 1.2 that $\dfrac{1}{p} = \lambda_1(\eta, -\mathrm{e}^{\eta P}(\rho_{m,\eta,P} + C(m,\eta,P))) = \Lambda_{m,\eta,P}$, which is a contradiction. This finishes the proof of (1.37).

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