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On the Refined Esitmates of All Homogeneous Expansions for a Subclass of Biholomorphic Starlike Mappings in Several Complex Variables*

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Abstract The refined estimates of all homogeneous expansions for a subclass of biholomorphic starlike mappings are mainly established on the unit ball in complex Banach spaces or the unit polydisk in $\mathbb{C}^n$ with a unified method. Especially the results are sharp if the above mappings are further k-fold symmetric starlike mappings or k-fold symmetric starlike mappings of order α. The obtained results unify and generalize the corresponding results in some prior literatures.

Keywords Refined estimates of all homogeneous expansions, Starlike mapping, Starlike mapping of order α, k-fold symmetric, Unified method
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1 Introduction

In one complex variable of geometric function theory, MacGregor [1] originally established the refined coefficient estimates of biholomorphic starlike functions. Boyd [2] subsequently derived the refined coefficient estimates of starlike functions of order α. They showed that the above refined estimates of [1] and [2] are sharp if these functions are further k-fold symmetric functions. However, the refined coefficient estimates of other subclasses of biholomorphic starlike functions are scarcely discussed. In several complex variables of geometric function theory, Gong [3] posed the profound Bieberbach conjecture in several complex variables, which is that the sharp estimates of all homogeneous expansions for biholomorphic starlike mappings on the unit polydisk in $\mathbb{C}^n$ hold. The sharp estimate of the second homogeneous expansion for biholomorphic starlike mappings was proved completely (see [3]). After that Hamada and Honda [4] and Liu and Liu [5] investigated the sharp estimate of the third homogeneous expansion for biholomorphic starlike mappings and starlike mappings of order α on the unit polydisk in $\mathbb{C}^n$ by different methods. In addition, Liu [6] obtained the sharp estimates of all homogeneous expansions for quasi-convex mappings (include quasi-convex mappings of type $\mathbb{A}$ and quasi-convex mappings of type $\mathbb{B}$) on the unit polydisk in $\mathbb{C}^n$ with some additional assumptions. Subsequently, Liu and Liu [7] extended the corresponding results of [6] to a general case. Liu, Liu and Xu [8] derived the sharp estimates of all homogeneous expansions for a subclass of biholomorphic starlike mappings in several complex variables as well. With respect to the estimates

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of homogeneous expansions for a subclass of biholomorphic mappings f which have parametric representation ($z = 0$ is a zero of order $k + 1$ of $f(z) - z$) on the unit polydisk in $\mathbb{C}^n$, Hamada and Honda [4] and Xu and Liu [9] established the estimates of the $m(m = k+1, k+2, \cdots, 2k)$-th homogeneous expansions independently. They both stated that the estimate is only sharp for $m = k+1$. Furthermore, Hamada and Honda [4] investigated the third homogeneous expansion for the above mappings. Recently, Liu and Liu [10] obtained the estimates of all homogeneous expansions for a subclass of biholomorphic mappings which have parametric representation. Many interesting results concerning the estimates of homogeneous expansions may be found in references [11–16].

A natural question arouse great interest of many people: Whether the refined estimates of all homogeneous expansions for a subclasses of biholomorphic starlike mappings which have a concrete parameter representation in several complex variables hold or not? We now provide an affirmative answer partly in this article. That is, we shall establish the refined estimates of all homogeneous expansions for a subclass of biholomorphic starlike mappings which have concrete parametric representation on the unit ball of complex Banach spaces, and also obtain the estimates of all homogeneous expansions for the above generalized mappings on the unit polydisk in $\mathbb{C}^n$.

Throughout this article, we denote by X a complex Banach space with the norm $\|\cdot\|$, X^* the dual space of X, B the open unit ball in X, and U the Euclidean open unit disk in $\mathbb{C}$. Also let U^n denote the open unit polydisk in $\mathbb{C}^n$, let $\mathbb{N}^+$ be the set of all positive integers, and let $\mathbb{R}$ denote the set of all real numbers. Let the symbol$'$ represent transpose. For each $x \in X\backslash\{0\}$,

$$T(x) = \{T_x \in X^* : \|T_x\| = 1,\ T_x(x) = \|x\|\}$$

is well defined. We denote by $H(B)$ the set of all holomorphic mappings from B into X. It is known that

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y-x)^n)$$

for all y in some neighborhood of $x \in B$ if $f \in H(B)$, where $D^n f(x)$ is the nth-Fréchet derivative of f at x, and for $n \geq 1$,

$$D^n f(x)((y-x)^n) = D^n f(x)(\underbrace{y-x, \cdots, y-x}_{n}).$$

We say that a holomorphic mapping $f : B \to X$ is biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. If $f : B \to X$ is a holomorphic mapping, then f is said to be normalized if $f(0) = 0$ and $Df(0) = I$, where I means the identity operator from X into X.

A normalized biholomorphic mapping $f : B \to X$ is said to be a starlike mapping if $f(B)$ is a starlike domain with respect to the origin.

Let $S^*(B)$ be the set of all starlike mappings on B.

We now state the following definitions.

Definition 1.1 (see [4]) *Suppose that $g \in H(U)$ is a biholomorphic function such that $g(0) = 1$, $g(\overline{\xi}) = \overline{g(\xi)}$, $\operatorname{Re} g(\xi) > 0$, $\xi \in U$ (so, g has real coefficients in its power series expansion), and assume that g satisfies the conditions*

$$\begin{cases} \min\limits_{|\xi|=r} |g(\xi)| = \min\limits_{|\xi|=r} \operatorname{Re} g(\xi) = g(r), \\ \max\limits_{|\xi|=r} |g(\xi)| = \max\limits_{|\xi|=r} \operatorname{Re} g(\xi) = g(-r). \end{cases}$$

It is not difficult to check that $g(\xi)=\frac{1+A_1\xi}{1+A_2\xi}$ satisfies the condition of Definition 1.1 for $-1\le A_1<A_2\le 1$, and $g(\xi)=\frac{1+A_1\xi}{1+A_2\xi}$ is a biholomorphic function on U if $A_1,A_2\in\mathbb{R},\ |A_1|\le 1,\ |A_2|\le 1,\ A_1\ne A_2$. However, g is not a biholomorphic function on U obviously if $A_1=A_2$.

To consider more general cases, we now assume that $g(\xi)=\frac{1+A_1\xi}{1+A_2\xi}$, $\xi\in U$ ($A_1,A_2\in\mathbb{R},\ |A_1|\le 1,\ |A_2|\le 1$).

We denote by $\mathcal{M}_g$ the set

$$\mathcal{M}_g=\Big\{p\in H(B):p(0)=0,\quad Dp(0)=I,\quad \frac{T_x(p(x))}{\|x\|}\in g(U),\ x\in B\setminus\{0\},\ T_x\in T(x)\Big\}.$$

Definition 1.2 (see [15]) *Suppose that $f:B\to X$ is a normalized locally biholomorphic mapping. If $\alpha\in(0,1)$ and*

$$\Big|\frac{1}{\|x\|}T_x[(Df(x))^{-1}f(x)]-\frac{1}{2\alpha}\Big|<\frac{1}{2\alpha},\quad x\in B\setminus\{0\},$$

then we say that f is a starlike mapping of order α.

Let $S^*_\alpha(B)$ be the set of all starlike mappings of order α on B.

Definition 1.3 (see [9]) *Suppose that $f:B\to X$ is a normalized locally biholomorphic mapping. If $\alpha\in[0,1)$ and*

$$\operatorname{Re}e\{T_x[(Df(x))^{-1}f(x)]\}\ge\alpha\|x\|,\quad x\in B\setminus\{0\},$$

then we say that f is an almost starlike mapping of order α on B.

We denote by $AS^*_\alpha(B)$ the set of all almost starlike mappings of order α on B.

Definition 1.4 (see [9]) *Suppose that $f:B\to X$ is a normalized locally biholomorphic mapping. If $c\in(0,1)$ and*

$$\Big|\frac{1}{\|x\|}T_x[(Df(x))^{-1}f(x)]-\frac{1+c^2}{1-c^2}\Big|<\frac{2c}{1-c^2},\quad x\in B\setminus\{0\},$$

then we say that f is a strongly starlike mapping on B.

Let $SS^*(B)$ denote the set of all strongly starlike mappings on B.

Definition 1.5 (see [17]) *Let $f\in H(B)$. It is said that f is k-fold symmetric if*

$$\mathrm{e}^{-\frac{2\pi\mathrm{i}}{k}}f(\mathrm{e}^{\frac{2\pi\mathrm{i}}{k}}x)=f(x)$$

for all $x\in B$, where $k\in\mathbb{N}^+$ and $\mathrm{i}=\sqrt{-1}$.

Definition 1.6 (see [18]) *Suppose that Ω is a domain (connected open set) in X which contains 0. It is said that $x=0$ is a zero of order k of $f(x)$ if $f(0)=0,\cdots,D^{k-1}f(0)=0$, but $D^kf(0)\ne 0$, where $k\in\mathbb{N}^+$.*

We denote by $S^*_g(B)$ the subset of $S^*(B)$ consisting of normalized locally biholomorphic mappings f which satisfy $(Df(x))^{-1}f(x)\in\mathcal{M}_g$, and $S^*_{g,k+1}(B)$ the subset of $S^*_g(B)$ such that $x=0$ is a zero of order $k+1$ of $f(x)-x$. Let $S^*_{k+1}(B)$ (resp. $S^*_{\alpha,k+1}(B)$, $AS^*_{\alpha,k+1}(B)$, $SS^*_{k+1}(B)$) denote the subset of $S^*(B)$ (resp. $S^*_\alpha(B)$, $AS^*_\alpha(B)$, $SS^*(B)$) which satisfies that $x=0$ is a zero of order $k+1$ of $f(x)-x$.

2 Some Lemmas

In order to establish our main theorems, in this section, it is necessary to provide some lemmas as follows.

Lemma 2.1 *Suppose that* $k \in \mathbb{N}^+,\ A \geq 0$. *Then*

$$A^2 + A\sum_{m=1}^{q-1}(2mk+A)\Big(\frac{1}{m!}\prod_{\mu=0}^{m-1}\Big(\mu+\frac{A}{k}\Big)\Big)^2 = \Big(\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\Big(\mu+\frac{A}{k}\Big)\Big)^2 \quad \text{for } q = 2, 3, \cdots . \tag{2.1}$$

Proof It is readily shown that (2.1) first holds if $q = 2$. We next assume that

$$A^2 + A\sum_{m=1}^{q-1}(2mk+A)\Big(\frac{1}{m!}\prod_{\mu=0}^{m-1}\Big(\mu+\frac{A}{k}\Big)\Big)^2 = \Big(\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\Big(\mu+\frac{A}{k}\Big)\Big)^2, \quad q = 2, 3, \cdots, l . \tag{2.2}$$

It suffice to prove that (2.1) holds for $q = l + 1$. A direct computation shows that

$$\begin{aligned} &A^2 + A\sum_{m=1}^{l}(2mk+A)\Big(\frac{1}{m!}\prod_{\mu=0}^{m-1}\Big(\mu+\frac{A}{k}\Big)\Big)^2 \\ &= \Big(\frac{k}{(l-1)!}\prod_{\mu=0}^{l-1}\Big(\mu+\frac{A}{k}\Big)\Big)^2 + A(2lk+A)\Big(\frac{1}{l!}\prod_{\mu=0}^{l-1}\Big(\mu+\frac{A}{k}\Big)\Big)^2 \\ &= \Big(\frac{k}{l!}\prod_{\mu=0}^{l}\Big(\mu+\frac{A}{k}\Big)\Big)^2 \end{aligned}$$

holds from (2.2). It follows the desired result. This completes the proof.

A direct calculation shows that the following lemma holds (the details are omitted here).

Lemma 2.2 *Suppose that* $k \in \mathbb{N}^+,\ s = 1, 2, \cdots,$ *and* $A \geq 0$. *Then*

$$(m-1)^2 \geq \frac{(sk)^2(m-1+A)}{sk+A}$$

for $m \geq sk + 1$.

Lemma 2.3 *Suppose that* $k \in \mathbb{N}^+,\ f(z) = z + \sum\limits_{m=k+1}^{\infty} a_m z^m \in S^*_{g,k+1}(U)$, *where* $g(z) = \frac{1+A_1 z}{1+A_2 z},\ z \in U,\ A_1, A_2 \in \mathbb{R},\ |A_1| \leq 1,\ |A_2| \leq 1$. *Then*

$$\sum_{m=k+1}^{2k}(m-1)^2|a_m|^2 \leq |A_2 - A_1|^2 . \tag{2.3}$$

Proof Since $\frac{f(z)}{f'(z)z} \prec g(z)$, there exists $\varphi \in H(U,U)$ which statisfies

$$g(\varphi(z)) = \frac{1+A_1\varphi(z)}{1+A_2\varphi(z)} = \frac{f(z)}{zf'(z)}, \quad z \in U .$$

A straightforward computation shows that

$$\varphi(z) = \frac{f'(z)z - f(z)}{A_2 f(z) - A_1 f'(z)z} = b_k z^k + b_{k+1} z^{k+1} + \cdots, \quad z \in U.$$

Hence the above relation yields that

$$(m-1)a_m = (A_2 - A_1)b_{m-1}, \quad m = k+1, k+2, \cdots, 2k. \tag{2.4}$$

Note that

$$f'(z)z - f(z) = \varphi(z)(A_2 f(z) - A_1 f'(z)z), \quad z \in U \tag{2.5}$$

and (2.4). We obtain that

$$\sum_{m=k+1}^{2k} (m-1)^2 |a_m|^2 = |A_2 - A_1|^2 \sum_{m=k}^{2k-1} |b_m|^2 \le |A_2 - A_1|^2 \sum_{m=k}^{\infty} |b_m|^2 \le |A_2 - A_1|^2.$$

This completes the proof.

Lemma 2.4 *Suppose that* $k \in \mathbb{N}^+$, $f(z) = z + \sum_{m=k+1}^{\infty} a_m z^m \in S^*_{g,k+1}(U)$, *where* $g(z) = \frac{1+A_1 z}{1+A_2 z}$, $z \in U$, $A_1, A_2 \in \mathbb{R}$, $|A_1| \le 1$, $|A_2| \le 1$. *Then*

$$|a_m| \le \frac{k}{(m-1)(s-1)!} \prod_{r=0}^{s-1} \Big(r + \frac{|A_2 - A_1|}{k} \Big), \quad sk+1 \le m \le (s+1)k,\ s = 1, 2, \cdots.$$

Especially, if $k = 1$, *then*

$$|a_m| \le \frac{1}{(m-1)!} \prod_{r=0}^{m-2} (r + |A_2 - A_1|).$$

Proof In view of (2.5), it follows that

$$\begin{aligned}\sum_{m=k+1}^{\infty} (m-1)a_m z^m &= \varphi(z)\Big((A_2 - A_1)z + \sum_{m=k+1}^{\infty} (A_2 - mA_1)a_m z^m \Big) \\ &= \varphi(z)\Big((A_2 - A_1)z + \sum_{m=k+1}^{p-k} (A_2 - mA_1)a_m z^m \Big) + \sum_{m=p+1}^{\infty} c_m z^m.\end{aligned}$$

This implies that

$$\sum_{m=k+1}^{p} (m-1)a_m z^m + \sum_{m=p+1}^{\infty} d_m z^m = \varphi(z)\Big((A_2 - A_1)z + \sum_{m=k+1}^{p-k} (A_2 - mA_1)a_m z^m \Big).$$

Similar to the proof of [1, Theorem 1], it yields that

$$\sum_{m=k+1}^{p} (m-1)^2 |a_m|^2 \le |A_2 - A_1|^2 + \sum_{m=k+1}^{p-k} |A_2 - mA_1|^2 |a_m|^2.$$

Hence,

$$\sum_{m=p-k+1}^{p}(m-1)^2|a_m|^2 \le 2|A_2-A_1|\Big(\frac{|A_2-A_1|}{2}+\sum_{m=k+1}^{p-k}\Big(m-1+\frac{|A_2-A_1|}{2}\Big)|a_m|^2\Big). \quad (2.6)$$

Applying an inductive method, we will prove the two following inequalities

$$\sum_{m=sk+1}^{(s+1)k}(m-1)^2|a_m|^2 \le \Big(\frac{k}{(s-1)!}\prod_{r=0}^{s-1}\Big(r+\frac{|A_2-A_1|}{k}\Big)\Big)^2 \quad (2.7)$$

and

$$\sum_{m=sk+1}^{(s+1)k}\Big(m-1+\frac{|A_2-A_1|}{2}\Big)|a_m|^2 \le \Big(sk+\frac{|A_2-A_1|}{2}\Big)\Big(\frac{1}{s!}\prod_{r=0}^{s-1}\Big(r+\frac{|A_2-A_1|}{k}\Big)\Big)^2 \quad (2.8)$$

hold for $s=1,2,3,\cdots$.

When $s=1$, (2.7) holds from (2.3). Also in view of Lemma 2.2 and (2.3), we deduce that

$$\begin{aligned}\sum_{m=k+1}^{2k}\Big(m-1+\frac{|A_2-A_1|}{2}\Big)|a_m|^2 &= \frac{k+\frac{|A_2-A_1|}{2}}{k^2}\sum_{m=k+1}^{2k}\frac{k^2}{k+\frac{|A_2-A_1|}{2}}\Big(m-1+\frac{|A_2-A_1|}{2}\Big)|a_m|^2\\ &\le \frac{k+\frac{|A_2-A_1|}{2}}{k^2}\sum_{m=k+1}^{2k}(m-1)^2|a_m|^2\\ &\le \frac{k+\frac{|A_2-A_1|}{2}}{k^2}|A_2-A_1|^2\\ &= \Big(k+\frac{|A_2-A_1|}{2}\Big)\Big(\frac{|A_2-A_1|}{k}\Big)^2.\end{aligned}$$

Consequently (2.8) is valid for $s=1$ as well. Assume that (2.7) and (2.8) are valid for $s=1,2,\cdots,q-1$. Letting $p=(q+1)k$ in (2.6), it yields that

$$\begin{aligned}&\sum_{m=qk+1}^{(q+1)k}(m-1)^2|a_m|^2\\ &\le 2(|A_2-A_1|)\Big(\frac{|A_2-A_1|}{2}+\sum_{m=k+1}^{qk}\Big(m-1+\frac{|A_2-A_1|}{2}\Big)|a_m|^2\Big)\\ &= 2(|A_2-A_1|)\Big(\frac{|A_2-A_1|}{2}+\sum_{s=1}^{q-1}\sum_{m=sk+1}^{(s+1)k}\Big(m-1+\frac{|A_2-A_1|}{2}\Big)|a_m|^2\Big)\\ &\le 2(|A_2-A_1|)\Big(\frac{|A_2-A_1|}{2}+\sum_{s=1}^{q-1}\Big(sk+\frac{|A_2-A_1|}{2}\Big)\Big(\frac{1}{s!}\prod_{r=0}^{s-1}\Big(r+\frac{|A_2-A_1|}{k}\Big)\Big)^2\Big)\\ &= \Big(\frac{k}{(q-1)!}\prod_{r=0}^{q-1}\Big(r+\frac{|A_2-A_1|}{k}\Big)\Big)^2\end{aligned}$$

from (2.1). It is shown that (2.7) holds for $s=q$. On the other hand, when $s=q$, we prove that

$$\sum_{m=qk+1}^{(q+1)k}\Big(m-1+\frac{|A_2-A_1|}{2}\Big)|a_m|^2$$

$$
\begin{aligned}
&= \frac{qk+\frac{|A_2-A_1|}{2}}{(qk)^2}\sum_{m=qk+1}^{(q+1)k}\frac{(qk)^2}{qk+\frac{|A_2-A_1|}{2}}\Big(m-1+\frac{|A_2-A_1|}{2}\Big)|a_m|^2\\
&\le \frac{qk+\frac{|A_2-A_1|}{2}}{(qk)^2}\sum_{m=qk+1}^{(q+1)k}(m-1)^2|a_m|^2\\
&\le \frac{qk+\frac{|A_2-A_1|}{2}}{(qk)^2}\Big(\frac{k}{(q-1)!}\prod_{r=0}^{q-1}\Big(r+\frac{|A_2-A_1|}{k}\Big)\Big)^2\\
&\le \Big(qk+\frac{|A_2-A_1|}{2}\Big)\Big(\frac{1}{q!}\prod_{r=0}^{q-1}\Big(r+\frac{|A_2-A_1|}{k}\Big)\Big)^2.
\end{aligned}
$$

This implies that (2.8) holds for $s=q$. Hence we derive the desired result from (2.7) readily. This completes the proof.

Remark 2.1 Let $g_1(z)=\frac{1-z}{1+z}$, $g_2(z)=\frac{1-z}{1+(1-2\alpha)z}$ $(\alpha\in(0,1))$, $g_3(z)=\frac{1-(1-2\alpha)z}{1+z}$ $(\alpha\in[0,1))$, $g_4(z)=\frac{1-cz}{1+cz}$ $(c\in(0,1))$ in Lemma 2.4. Then $f\in S^*_{k+1}(U)$ $(S^*_{\alpha,k+1}(U), AS^*_{\alpha,k+1}(U), SS^*_{k+1}(U))$, and we get the corresponding results of Lemma 2.4. It is readily shown that the estimates of Lemma 2.4 are sharp if f is a k-fold symmetric starlike function or a k-fold symmetric starlike function of order α.

Remark 2.2 From the proofs of Lemmas 2.3–2.4, it is shown that Lemmas 2.3–2.4 are still valid if the assumptions of $A_1,A_2\in\mathbb{R}$ are replaced with $A_1,A_2\in\mathbb{C}$. However, the function f must not be a biholomorphic starlike function (even a biholomorphic function).

3 Refined Estimates of All Homogeneous Expansions for a Subclass of Biholomorphic Starlike Mappings in Several Complex Variables

We now present the desired theorems in this section.

Theorem 3.1 *Let* $f:B\to\mathbb{C}\in H(B)$, $F(x)=xf(x)\in S^*_{g,k+1}(B)$, $g(\xi)=\frac{1+A_1\xi}{1+A_2\xi}$, $\xi\in U$, $A_1,A_2\in\mathbb{R}$, $|A_1|\le 1$, $|A_2|\le 1$. *Then*

$$
\frac{\|D^mF(0)(x^m)\|}{m!}\le\frac{k}{(m-1)(s-1)!}\prod_{r=0}^{s-1}\Big(r+\frac{|A_2-A_1|}{k}\Big)\|x\|^m,\quad x\in B,
$$
$$
sk+1\le m\le(s+1)k,\quad s=1,2,\cdots.
$$

In particular, if $k=1$, *then*

$$
\frac{\|D^mF(0)(x^m)\|}{m!}\le\frac{1}{(m-1)!}\prod_{r=0}^{m-2}(r+|A_2-A_1|)\|x\|^m,\quad x\in B,\ m=2,3,\cdots.
$$

Proof Let $x\in B\setminus\{0\}$ be fixed, and we denote by $x_0=\frac{x}{\|x\|}$. Define

$$
h(\xi)=\xi f(\xi x_0),\quad \xi\in U. \tag{3.1}
$$

It yields that

$$
\frac{h(\xi)}{\xi h'(\xi)}=\frac{f(\xi x_0)}{Df(\xi x_0)\xi x_0+f(\xi x_0)}=\frac{T_{\xi x_0}((DF(\xi x_0))^{-1}F(\xi x_0))}{\|\xi x_0\|}\in g(U)
$$

by a direct calculation, and $\xi = 0$ is at least a zero of order $k+1$ of $h(\xi) - \xi$ if $x = 0$ is a zero of order $k+1$ of $F(x) - x$.

On the other hand, we conclude that

$$\xi + \sum_{m=k+1}^{\infty} a_m \xi^m = \xi + \sum_{m=k+1}^{\infty} \frac{D^{m-1} f(0)(x_0^{m-1})}{(m-1)!} \xi^m$$

from (3.1). Compare the coefficients of the two sides in the above equality. It is shown that

$$\frac{D^{m-1} f(0)(x_0^{m-1})}{(m-1)!} = a_m, \quad m = k+1, k+2, \cdots. \tag{3.2}$$

For $k \in \mathbb{N}^+$, we mention that

$$\frac{D^m F(0)(x^m)}{m!} = x \frac{D^{m-1} f(0)(x^{m-1})}{(m-1)!}, \quad x \in B, \ m = k+1, k+2, \cdots$$

if $F(x) = x f(x)$. From Lemma 2.4 and (3.2), it follows the result, as desired. This completes the proof.

Putting $g(\xi) = \frac{1-\xi}{1+\xi}$ in Theorem 3.1, then we get the following corollary readily.

Corollary 3.1 *Let $f : B \to \mathbb{C} \in H(B)$, $F(x) = x f(x) \in S^*_{k+1}(B)$. Then*

$$\frac{\|D^m F(0)(x^m)\|}{m!} \le \frac{k}{(m-1)(s-1)!} \prod_{r=0}^{s-1} \Big(r + \frac{2}{k} \Big) \|x\|^m, \quad x \in B,$$
$$sk + 1 \le m \le (s+1)k, \quad s = 1, 2, \cdots.$$

The above estimates are sharp for $m = sk+1$, $s = 1, 2, \cdots$. In particular, if $k = 1$, then

$$\frac{\|D^m F(0)(x^m)\|}{m!} \le m \|x\|^m, \quad x \in B, \ m = 2, 3, \cdots.$$

The example which shows that the sharpness of estimates of Corollary 3.1 for $m = sk+1$ ($s = 1, 2, \cdots$) is the same as that of [8, Theorem 2.1].

Set $g(\xi) = \frac{1-\xi}{1+(1-2\alpha)\xi}$, $\alpha \in (0,1)$ in Theorem 3.1. Then we obtain the following corollary immediately.

Corollary 3.2 *Let $f : B \to \mathbb{C} \in H(B)$, $\alpha \in (0,1)$, $F(x) = x f(x) \in S^*_{\alpha,k+1}(B)$. Then*

$$\frac{\|D^m F(0)(x^m)\|}{m!} \le \frac{k}{(m-1)(s-1)!} \prod_{r=0}^{s-1} \Big(r + \frac{2(1-\alpha)}{k} \Big) \|x\|^m, \quad x \in B,$$
$$sk + 1 \le m \le (s+1)k, \quad s = 1, 2, \cdots.$$

The above estimates are sharp for $m = sk+1$, $s = 1, 2, \cdots$. In particular, when $k = 1$, then

$$\frac{\|D^m F(0)(x^m)\|}{m!} \le \frac{\prod_{r=2}^{m} (r - 2\alpha)}{(m-1)!} \|x\|^m, \quad x \in B, \ m = 2, 3, \cdots.$$

The example which states that the sharpness of estimates of Corollary 3.2 for $m = sk+1$ ($s = 1, 2, \cdots$) is similar to that of [14, Theorem 2.1].

Setting $g(\xi) = \frac{1-(1-2\alpha)\xi}{1+\xi}$, $\alpha \in [0,1)$ in Theorem 3.1, then the following corollary is derived easily.

Corollary 3.3 *Let $f : B \to \mathbb{C} \in H(B)$, $\alpha \in (0,1)$, $F(x) = xf(x) \in AS^*_{\alpha,k+1}(B)$. Then*

$$\frac{\|D^m F(0)(x^m)\|}{m!} \le \frac{k}{(m-1)(s-1)!} \prod_{r=0}^{s-1} \Big(r + \frac{2(1-\alpha)}{k}\Big)\|x\|^m, \quad x \in B,$$
$$sk+1 \le m \le (s+1)k, \quad s = 1,2,\cdots.$$

In particular, if $k = 1$, then

$$\frac{\|D^m F(0)(x^m)\|}{m!} \le \frac{\prod\limits_{r=2}^{m}(r-2\alpha)}{(m-1)!}\|x\|^m, \quad x \in B,\ m = 2,3,\cdots.$$

Putting $g(\xi) = \frac{1-c\xi}{1+c\xi}$, $c \in (0,1)$ in Theorem 3.1, then the following corollary is given directly.

Corollary 3.4 *Let $f : B \to \mathbb{C} \in H(B)$, $c \in (0,1)$, $F(x) = xf(x) \in SS^*_{k+1}(B)$. Then*

$$\frac{\|D^m F(0)(x^m)\|}{m!} \le \frac{k}{(m-1)(s-1)!} \prod_{r=0}^{s-1} \Big(r + \frac{2c}{k}\Big)\|x\|^m, \quad x \in B,$$
$$sk+1 \le m \le (s+1)k, \quad s = 1,2,\cdots.$$

In particular, when $k = 1$, then

$$\frac{\|D^m F(0)(x^m)\|}{m!} \le \frac{\prod\limits_{r=0}^{m-2}(r+2c)}{(m-1)!}\|x\|^m, \quad x \in B,\ m = 2,3,\cdots.$$

Theorem 3.2 *Let $F(z) = (F_1(z), F_2(z), \cdots, F_n(z))' \in H(U^n)$, and $z = 0$ is a zero of order $k+1$ of $F(z) - z$. If $\frac{DF_j(z)z}{F_j(z)} \in \frac{1}{g}(U)$, $z \in U^n \setminus \{0\}$, where j satisfies the condition $|z_j| = \|z\| = \max\limits_{1\le l\le n} |z_l|$, and $g(\xi) = \frac{1+A_1\xi}{1+A_2\xi}$, $\xi \in U$, $A_1, A_2 \in \mathbb{R}$, $|A_1| \le 1$, $|A_2| \le 1$, then*

$$\frac{\|D^m F(0)(z^m)\|}{m!} \le \frac{k}{(m-1)(s-1)!} \prod_{r=0}^{s-1} \Big(r + \frac{|A_2 - A_1|}{k}\Big)\|z\|^m, \quad z \in U^n,$$
$$sk+1 \le m \le (s+1)k, \quad s = 1,2,\cdots.$$

Especially, if $k = 1$, then

$$\frac{\|D^m F(0)(z^m)\|}{m!} \le \frac{1}{(m-1)!} \prod_{r=0}^{m-2} (r + |A_2 - A_1|)\|z\|^m, \quad z \in U^n,\ m = 2,3,\cdots.$$

Proof Fix $z \in U^n \setminus \{0\}$, and denote $z_0 = \frac{z}{\|z\|}$. Let

$$h_j(\xi) = \frac{\|z\|}{z_j} F_j(\xi z_0), \quad \xi \in U, \tag{3.3}$$

where j satisfies the condition $|z_j| = \|z\| = \max\limits_{1\le l\le n}\{|z_l|\}$. In view of $\frac{DF_j(z)z}{F_j(z)} \in \frac{1}{g}(U)$, $z \in U^n \setminus \{0\}$, we see that

$$\frac{h_j'(\xi)\xi}{h_j(\xi)} = \frac{DF_j(\xi z_0)\xi z_0}{F_j(\xi z_0)} \in \frac{1}{g}(U), \quad \xi \in U \setminus \{0\}$$

by a direct calculation. Hence it is shown that $h_j \in S_g^*(U)$, and $\xi = 0$ is at least a zero of order $k+1$ for $h_j(\xi) - \xi$.

We also show that

$$\xi + \sum_{m=k+1}^{\infty} c_m \xi^m = \xi + \frac{\|z\|}{z_j} \sum_{m=k+1}^{\infty} \frac{D^m F_j(0)(z_0^m)}{m!} \xi^m$$

from (3.3). It yields that

$$\frac{\|z\|}{z_j} \frac{D^m F_j(0)(z_0^m)}{m!} = c_m, \quad m = k+1, k+2, \cdots$$

by comparing the coefficients of the two sides in the above equality. Therefore, it is shown that

$$\frac{|D^m F_j(0)(z_0^m)|}{m!} \leq \frac{k}{(m-1)(s-1)!} \prod_{r=0}^{s-1} \Big(r + \frac{|A_2 - A_1|}{k} \Big), \quad z_0 \in \partial U^n, \ m = k+1, k+2, \cdots$$

from Lemma 2.4. In a way similar to that in the proof of [8, Theorem 3.3], we derive the desired result. This completes the proof.

Let $g(\xi) = \frac{1-\xi}{1+\xi}$ in Theorem 3.2. Then the following corollary is given readily.

Corollary 3.5 *Let $F(z) = (F_1(z), F_2(z), \cdots, F_n(z))' \in H(U^n)$, and $z = 0$ is a zero of order $k+1$ of $F(z) - z$. If $\operatorname{Re} \frac{DF_j(z)z}{F_j(z)} > 0$, $z \in U^n \setminus \{0\}$, where j satisfies the condition $|z_j| = \|z\| = \max_{1 \leq l \leq n} |z_l|$, then*

$$\frac{\|D^m F(0)(z^m)\|}{m!} \leq \frac{k}{(m-1)(s-1)!} \prod_{r=0}^{s-1} \Big(r + \frac{2}{k} \Big) \|z\|^m, \quad z \in U^n,$$
$$sk + 1 \leq m \leq (s+1)k, \quad s = 1, 2, \cdots.$$

The above estimates are sharp for $m = sk+1$, $s = 1, 2, \cdots$. Especially, when $k = 1$, then

$$\frac{\|D^m F(0)(z^m)\|}{m!} \leq m \|z\|^m, \quad z \in U^n, \ m = 2, 3, \cdots.$$

The example which states that the sharpness of estimates of Corollary 3.5 for $m = sk+1$ ($s = 1, 2, \cdots$) is the same as that of [8, Theorem 3.3].

Putting $g(\xi) = \frac{1-\xi}{1+(1-2\alpha)\xi}$, $\alpha \in (0,1)$ in Theorem 3.1, then the following corollary follows immediately.

Corollary 3.6 *Let $\alpha \in (0,1)$, $F(z) = (F_1(z), F_2(z), \cdots, F_n(z))' \in H(U^n)$, and $z = 0$ is a zero of order $k+1$ of $F(z) - z$. If $\left| \frac{F_j(z)}{DF_j(z)z} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}$, $z \in U^n \setminus \{0\}$, where j satisfies the condition $|z_j| = \|z\| = \max_{1 \leq l \leq n} |z_l|$, then*

$$\frac{\|D^m F(0)(z^m)\|}{m!} \leq \frac{k}{(m-1)(s-1)!} \prod_{r=0}^{s-1} \Big(r + \frac{2(1-\alpha)}{k} \Big) \|z\|^m, \quad z \in U^n,$$
$$sk + 1 \leq m \leq (s+1)k, \quad s = 1, 2, \cdots.$$

The above estimates are sharp for $m = sk+1$, $s = 1, 2, \cdots$. In particular, when $k = 1$, then

$$\frac{\|D^m F(0)(z^m)\|}{m!} \leq \frac{\prod_{r=2}^{m} (r - 2\alpha)}{(m-1)!} \|z\|^m, \quad z \in U^n, \ m = 2, 3, \cdots.$$

The example which shows that the sharpness of estimates of Corollary 3.6 for $m=sk+1$ $(s=1,2,\cdots)$ is similar to that of [14, Theorem 3.5].

Setting $g(\xi)=\frac{1-(1-2\alpha)\xi}{1+\xi},\ \alpha\in[0,1)$ in Theorem 3.2, then the following corollary is derived directly.

Corollary 3.7 *Let* $\alpha\in[0,1), F(z)=(F_1(z),F_2(z),\cdots,F_n(z))'\in H(U^n)$, *and* $z=0$ *is a zero of order* $k+1$ *of* $F(z)-z$. *If* $\operatorname{Re}\frac{F_j(z)}{DF_j(z)z}>\alpha, z\in U^n\setminus\{0\}$, *where* j *satisfies the condition* $|z_j|=\|z\|=\max\limits_{1\le l\le n}|z_l|$, *then*

$$\frac{\|D^mF(0)(z^m)\|}{m!}\le\frac{k}{(m-1)(s-1)!}\prod_{r=0}^{s-1}\Big(r+\frac{2(1-\alpha)}{k}\Big)\|z\|^m,\quad z\in U^n,$$
$$sk+1\le m\le(s+1)k,\quad s=1,2,\cdots.$$

Especially, when $k=1$, *then*

$$\frac{\|D^mF(0)(z^m)\|}{m!}\le\frac{\prod\limits_{r=2}^{m}(r-2\alpha)}{(m-1)!}\|z\|^m,\quad z\in U^n,\ m=2,3,\cdots.$$

Putting $g(\xi)=\frac{1-c\xi}{1+c\xi},\ c\in(0,1)$ in Theorem 3.2, then the following corollary is given readily.

Corollary 3.8 *Let* $c\in(0,1),\ F(z)=(F_1(z),F_2(z),\cdots,F_n(z))'\in H(U^n)$, *and* $z=0$ *is a zero of order* $k+1$ *of* $F(z)-z$. *If* $|\frac{F_j(z)}{DF_j(z)z}-\frac{1+c^2}{1-c^2}|<\frac{2c}{1-c^2},\ z\in U^n\setminus\{0\}$, *where* j *satisfies the condition* $|z_j|=\|z\|=\max\limits_{1\le l\le n}|z_l|$, *then*

$$\frac{\|D^mF(0)(z^m)\|}{m!}\le\frac{k}{(m-1)(s-1)!}\prod_{r=0}^{s-1}\Big(r+\frac{2c}{k}\Big)\|z\|^m,\quad z\in U^n,$$
$$sk+1\le m\le(s+1)k,\quad s=1,2,\cdots.$$

In particular, when $k=1$, *then*

$$\frac{\|D^mF(0)(z^m)\|}{m!}\le\frac{\prod\limits_{r=0}^{m-2}(r+2c)}{(m-1)!}\|z\|^m,\quad z\in U^n,\ m=2,3,\cdots.$$

Remark 3.1 Theorem 3.1 is the corollary of Theorem 3.2 if $B=U^n$.

Remark 3.2 Corollaries 3.1 and 3.5 are the same as [8, Theorem 2.1] and [8, Theorem 3.3] respectively if $m=sk+1,\ s=1,2,\cdots$.

Remark 3.3 Corollaries 3.2 and 3.6 reduce to [14, Theorem 2.1] and [14, Theorem 3.5] respectively if $m=sk+1,\ s=1,2,\cdots$.

According to Theorems 3.1–3.2, we naturally propose the open problem as follows.

Open Problem 3.1 Let $F(z)\in S^*_{g,k+1}(U^n),\ g(\xi)=\frac{1+A_1\xi}{1+A_2\xi},\ \xi\in U,\ A_1,A_2\in\mathbb{R},\ |A_1|\le 1,\ |A_2|\le 1$. Then

$$\frac{\|D^mF(0)(z^m)\|}{m!}\le\frac{k}{(m-1)(s-1)!}\prod_{r=0}^{s-1}\Big(r+\frac{|A_2-A_1|}{k}\Big)\|z\|^m,\quad z\in U^n,$$
$$sk+1\le m\le(s+1)k,\quad s=1,2,\cdots.$$

The above estimates are sharp for $A_1 = -1,\ A_2 = 1,\ m = sk+1,\ s = 1, 2, \cdots$ and $A_1 = -1,\ A_2 = 1-2\alpha\ (\alpha \in (0,1)),\ m = sk+1,\ s = 1, 2, \cdots$. In particular, if $k = 1$, then

$$\frac{\|D^m F(0)(z^m)\|}{m!} \le \frac{1}{(m-1)!} \prod_{r=0}^{m-2} (r + |A_2 - A_1|) \|z\|^m, \quad z \in U^n,\ m = 2, 3, \cdots .$$

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