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Global Stability of Multi-wave Configurations for the Compressible Non-isentropic Euler System*

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Abstract This paper is contributed to the structural stability of multi-wave configurations to Cauchy problem for the compressible non-isentropic Euler system with adiabatic exponent $\gamma \in (1, 3]$. Given some small BV perturbations of the initial state, the author employs a modified wave front tracking method, constructs a new Glimm functional, and proves its monotone decreasing based on the possible local wave interaction estimates, then establishes the global stability of the multi-wave configurations, consisting of a strong 1-shock wave, a strong 2-contact discontinuity, and a strong 3-shock wave, without restrictions on their strengths.

Keywords Structural stability, Multi-wave configuration, Shock, Contact discontinuity, Compressible non-isentropic Euler system, Wave front tracking method

2010 MR Subject Classification 35L65, 35B35, 35L45, 35Q31, 76N10

1 Introduction

As is well known, Glimm has proved a global existence of weak solutions for strict hyperbolic conservation laws when the total variation of the initial data is sufficiently small via Glimm scheme in [7]. Bressan has established global existence, uniqueness of solutions to one-dimensional Cauchy problem for the general hyperbolic conservation laws when the total variation of the initial data is sufficiently small by wave front tracking method, and also proved the continuous dependence on the initial data in book [1]. In early works, Chern [4] initially studied Cauchy problem for general hyperbolic conservation laws, and proved the stability of a single large shock wave by Glimm scheme and wave front tracking method. Schochet [16] has shown the BV stability of the multi-wave configurations (a strong 1-shock wave, a strong 2-contact discontinuity, and a strong 3-shock wave) for the non-isentropic gas dynamics for $\gamma \geq \gamma_0 = \frac{6+4\sqrt{2}}{3+6\sqrt{2}} \approx 1.015$, while for $\gamma \in (1, \gamma_0)$, there exist Riemann problems for which BV stability condition fails. Lewicka [11] solved the well-posedness of the solutions for Cauchy problem to the general hyperbolic conservation laws when the initial data is a small perturbation of wave patterns of large non-interacting waves. The result in [11] includes the following facets:

1. If the background wave pattern $\overline{U}(x,t)$ satisfies the finite conditions, then Riemann

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problem with initial data close to $\overline{U}(x,0)$ admits a self-similar solution in the vicinity of the state $\overline{U}(x,t)$.

2. If BV stability condition is satisfied, and the initial data is under a small perturbation of $\overline{U}(x,0)$ with possibly large data, then Cauchy problem of general hyperbolic conservation laws admits a global entropy admissible solution. Section 8 has applied the general framework to non-isentropic Euler equations, and achieved the BV stability of wave patterns consisting of a strong 1-shock, strong 2-contact discontinuity, and strong 3-shock wave for $\gamma \in [1.05576, 8.7577]$. Meanwhile, L^1 stability condition holds for $\gamma \in [1.05576, 8.7577]$, then there exists a Lipschitz continuous semigroup of global entropy admissible solutions.

In short, comparing with the previous references, Schochet [16] was the first to introduce the finite condition, and formulate the stability of M strong shocks, $2 \le M \le n$, by means of matrix analysis and Glimm scheme. Bressan and Colombo [2] considered the general Riemann problem for systems of two equations and derived the corresponding L^1 stability condition of the large solutions. Lewicka [12] proved the BV and L^1 stability conditions for non-interacting two large shocks of general conservation laws. Later, Lewicka [10] has shown that BV stability conditions in [9] is equivalent to Schochet BV finite condition, as well as the equivalent of L^1 stability condition from [9] with the one introduced in [2] for 2×2 system. For a single strong rarefaction wave, the stability of a strong rarefaction wave to Cauchy problem for the general hyperbolic conservation laws has been proved by Lewicka in [14], also see [13]. The structural stability of steady four-wave configurations for two-dimensional steady supersonic Euler flow has been established in Chen-Rigby [3] by Glimm scheme.

Remark 1.1 Here strong or large wave means the strength of the wave is not sufficiently small, or else, it is a weak wave.

In this paper, we are concerned with the BV stability of the multi-wave configurations of Cauchy problem to the compressible non-isentropic Euler equations for $1 < \gamma \le 3$, and to some extent, it fixes up the result of BV stability of such system for $\gamma \in (1, \gamma_0)$. To overcome the difficulties, some new nonlinear weights have been introduced and assigned to each perturbation wave, the total amount of weighted perturbation waves decreases at each interaction with any of the strong waves.

The system composes of the conservation laws of mass, momentum and energy which can be read as (see [5, 15])

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t\Big(\rho\Big(\frac{1}{2}u^2 + e\Big)\Big) + \partial_x\Big(u\Big(\frac{1}{2}\rho u^2 + \rho e + p\Big)\Big) = 0, \end{cases} \tag{1.1}$$

where ρ, p and u stand for the density, the pressure and the speed of the fluid, respectively, and e is the internal energy. The constitutive relations for the polytropic gas are given by

$$p = \kappa \rho^\gamma \exp\Big(\frac{S}{c_v}\Big), \quad e = \frac{p}{(\gamma - 1)\rho},$$

where S represents the entropy, and κ, c_v are positive constants, the adiabatic exponent $\gamma \in (1, 3]$.

For simplicity, system (1.1) can be written in the general conservation law form:

$$\partial_t W(U) + \partial_x H(U) = 0, \tag{1.2}$$

where $U = (\rho, u, p)^\top$, and

$$\begin{aligned} W(U) &= \Big(\rho,\ \rho u,\ \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}\Big)^\top, \\ H(U) &= \Big(\rho u,\ \rho u^2 + p,\ \frac{1}{2}\rho u^3 + \frac{\gamma p u}{\gamma - 1}\Big)^\top. \end{aligned}$$

By solving the polynomial $\det(\lambda \nabla_U W(U) - \nabla_U H(U)) = 0$, the eigenvalues of system (1.2) are respectively

$$\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c,$$

where $c = \sqrt{\frac{\gamma p}{\rho}}$ is the local sonic speed and the corresponding right eigenvectors are

$$r_1 = -\frac{2}{(\gamma+1)c}(\rho, -c, \gamma p)^\top, \quad r_2 = (1, 0, 0)^\top, \quad r_3 = \frac{2}{(\gamma+1)c}(\rho, c, \gamma p)^\top,$$

where r_j is normalized by

$$\nabla \lambda_j \cdot r_j = 1, \quad j = 1, 3.$$

The entropy-entropy flux pair $(\eta, q)(U)$ of system (1.1) is a pair of C^1 functions satisfying

$$\nabla_W \eta(W(U)) \nabla_U H(U) = \nabla_U q(W(U)).$$

In particular, if

$$\nabla^2 \eta(U) \geq 0 \quad \text{for any } U,$$

then $\eta(U)$ is called a convex entropy.

We consider an unperturbed multi-wave configuration, called the background solution denoted by $\overline{U}$, consisting of the four constant states (see Figure 1)

$$\overline{U}(x,t) = \begin{cases} U_1 = (\rho_1, u_1, p_1)^\top, & x < s_{10} t, \\ U_2 = (\rho_2, u_2, p_2)^\top, & s_{10} t < x < u_2 t, \\ U_3 = (\rho_3, u_3, p_3)^\top, & u_2 t < x < s_{30} t, \\ U_4 = (\rho_4, u_4, p_4)^\top, & x > s_{30} t, \end{cases} \tag{1.3}$$

where the constant state U_1 connects to the state U_2 by a strong 1-shock with the speed s_{10}, and the state U_2 is separated from the state U_3 by a strong 2-contact discontinuity with the strength $|\sigma_{20}|$, and the state U_3 joins to the state U_4 by a strong 3-shock wave with the speed s_{30}, satisfying (2.9)–(2.11).

Suppose that the initial data are given by

$$U(x, 0) = U_0(x) = (\rho_0(x), u_0(x), p_0(x))^\top, \tag{1.4}$$

which is a small BV perturbation of the state $\overline{U}(x, 0)$.

Next, we define the entropy solutions to problem (1.2) and (1.4) as follows.

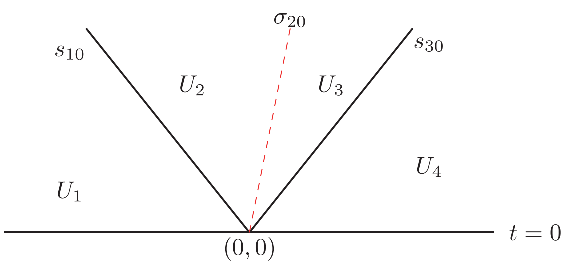

Figure 1 Background solution.

Definition 1.1 *A measurable function* $U(x,t) \in L^\infty(\mathbb{R}_+^2)$ *is an entropy solution to problem* (1.2) *and* (1.4)*, provided that*

(i) $U(x,t)$ *is a weak solution to problem* (1.2) *and* (1.4)*, if for any* $\varphi(x,t) \in C_c^\infty(\mathbb{R}_+^2)$*, it holds that*

$$\int_{-\infty}^{\infty}\int_0^{\infty}(W(U)\varphi_t + H(U)\varphi_x)\mathrm{d}x\mathrm{d}t + \int_0^{+\infty} W(U_0)\varphi(x,0)\mathrm{d}x = 0.$$

(ii) *The Clausius inequality holds in the sense of distributions*:

$$\partial_t(\rho a(S)) + \partial_x(\rho u a(S)) \le 0$$

for any $a(S) \in C^1$ *and* $a'(S) \ge 0$.

The main result of this paper is given by the following theorem.

Theorem 1.1 *There are some positive constants* $\varepsilon > 0$ *and* $C > 0$ *such that if*

$$\|U_0(x) - \overline{U}(x,0)\|_{L^\infty(-\infty,+\infty)} + \mathrm{TV.}\{U_0(\cdot) - \overline{U}(\cdot,0)\} < \varepsilon,$$

then there exists a global existence of entropy solution $U(x,t)$ *to problem* (1.1) *and* (1.4)*, including a strong* 1*-shock wave, a strong* 2*-contact discontinuity, and a strong* 3*-shock wave, which is a small perturbation of solution* (1.3)*. In addition, for all* $t > 0$*, there exists a positive constant* M_0 *such that*

$$\mathrm{TV.}\big\{U(\cdot,t) : (-\infty,+\infty)\big\} \le M_0. \tag{1.5}$$

Moreover, denote the curves of the strong 1*-shock wave, the strong* 2*-contact discontinuity, and the strong* 3*-shock wave by* $x = \chi_i(t)$*,* $i = 1, 2, 3$*, respectively. Then, it satisfies*

$$\begin{aligned} &|U(x,t)|_{\{x<\chi_1(t)\}} - U_1| < C\varepsilon, \\ &|U(x,t)|_{\{\chi_1(t)<x<\chi_2(t)\}} - U_2| < C\varepsilon, \\ &|U(x,t)|_{\{\chi_2(t)<x<\chi_3(t)\}} - U_3| < C\varepsilon, \\ &|U(x,t)|_{\{x>\chi_3(t)\}} - U_4| < C\varepsilon. \end{aligned} \tag{1.6}$$

Our motivation is to study the structural stability of multi-wave configurations to Cauchy problem of the compressible non-isentropic Euler system (1.1) under BV perturbation of initial data. Different from the results involving a single strong wave (shock, rarefaction wave or contact discontinuity), we not only need trace the locations of the strong waves, but also control the change of the strengths for the strong waves after each wave interaction with weak waves from left and right. In account of all the possible local wave interactions, we introduce some weights for the approaching waves, and construct a new Glimm functional, which measures the difference between the total variations of the approximate solutions and background solutions. We observe that the weak 1-waves collide with the strong 1-shock waves from the right, that the weak 3-waves interact with the strong 2-contact discontinuities from left, that the weak 1-waves collide with the strong 2-contact discontinuities from right and that the weak 3-waves interact with the strong 3-shock waves from left, among which the reflection coefficients are less than 1, which is essential for proving the monotone decreasing of Glimm functional.

This paper is organized as follows. In Section 2, we recall some basic properties of elementary waves (shock, rarefaction waves and contact discontinuities), and give the solvability of the Riemann problem for system (1.1), which is discussed in four cases. In Section 3, we construct the approximate solutions to the Cauchy problem by wave front tracking method. In Section 4, we consider all the local wave interaction estimates between weak waves, their reflections on the strong 1-shock waves, strong 2-contact discontinuities and strong 3-shock waves, and so on. In Section 5, we introduce some weighted strengths for the approaching waves, construct a new Glimm functional, and then prove the monotone decreasing of the functional. In Section 6, we derive some further estimates to show that the total strengths of strong wave fronts are bounded. The compactness and the convergence of the approximate solutions follow from the standard procedure.

2 Riemann Solutions

In this section, we study the Riemann problems and analyze the properties of the Riemann solutions to system (1.1), which are essential not only for the interaction estimates between the weak waves, but also for those involving the strong shock waves or strong contact discontinuities, etc..

Consider the Riemann problem of (1.1) with initial data

$$U|_{t=t_0} := (\rho, u, p)^\top|_{t=t_0} = \begin{cases} U_L, & x < x_0, \\ U_R, & x > x_0, \end{cases} \tag{2.1}$$

where $U_L = (\rho_L, u_L, p_L)^\top$ and $U_R = (\rho_R, u_R, p_R)^\top$ represent the left and right states, respectively. The solvability of the Riemann problem can be found in [8, 17] when $|U_L - U_R|$ is sufficiently small.

For any given left state U_l, the set of all possible states U can be connected to U_l on the right by a 1 or 3-shock wave, the wave curves of which can be denoted by S_1 or S_3, respectively. Similarly, we denote by R_1 or R_3 the 1 or 3-rarefaction wave curves. The rarefaction wave curve $R_1(U_l)$ through U_l satisfies

$$p\rho^{-\gamma} = p_l\rho_l^{-\gamma}, \quad u + \frac{2c}{\gamma - 1} = u_l + \frac{2c_l}{\gamma - 1}. \tag{2.2}$$

Similarly, the rarefaction wave curve $R_3(U_l)$ through U_l is given by

$$p\rho^{-\gamma} = p_l\rho_l^{-\gamma}, \quad u - \frac{2c}{\gamma-1} = u_l - \frac{2c_l}{\gamma-1}. \tag{2.3}$$

The second characteristic field is linearly degenerate satisfying $\nabla\lambda_2 \cdot r_2 \equiv 0$. The 2-contact discontinuity through U_l satisfies

$$C_2(U_l) : \rho = \rho_l + \sigma_2, \quad u = u_l, \quad p = p_l.$$

The Rankine-Hugoniot conditions across the shock are given by

$$s[\rho] = [\rho u], \tag{2.4}$$

$$s[\rho u] = [\rho u^2 + p], \tag{2.5}$$

$$s\Big[\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\Big] = \Big[\frac{1}{2}\rho u^3 + \frac{\gamma p u}{\gamma-1}\Big], \tag{2.6}$$

where $[h] = h - h_l$ stands for the jump of function h across the shock, and s is the speed of the shock. Therefore, eliminating s from (2.4)–(2.6), the shock wave curves through U_l can be respectively parameterized by

$$S_1(U_l) : \begin{cases} \frac{p}{p_l} = \exp(-z), \quad \frac{\rho}{\rho_l} = \frac{1+\beta\exp(-z)}{\beta+\exp(-z)}, \\ \frac{u-u_l}{c_l} = \frac{2\sqrt{\zeta}}{\gamma-1}\frac{1-\exp(-z)}{\sqrt{1+\beta\exp(-z)}}, \quad z \le 0 \end{cases}$$

and

$$S_3(U_l) : \begin{cases} \frac{p}{p_l} = \exp(z), \quad \frac{\rho}{\rho_l} = \frac{1+\beta\exp(z)}{\beta+\exp(z)}, \\ \frac{u-u_l}{c_l} = \frac{2\sqrt{\zeta}}{\gamma-1}\frac{\exp(z)-1}{\sqrt{1+\beta\exp(z)}}, \quad z \le 0, \end{cases}$$

where $\beta = \frac{\gamma+1}{\gamma-1}$ and $\zeta = \frac{\gamma-1}{2\gamma}$.

In addition, the Lax entropy conditions across the shock are

$$\lambda_j(U) < s_j < \lambda_j(U_l), \quad j = 1, 3, \tag{2.7}$$

$$s_1 < \lambda_2(U), \quad s_3 > \lambda_2(U_l). \tag{2.8}$$

2.1 Background solution

In this subsection, we will present the unperturbed multi-wave configurations, consisting of a strong 1-shock, a strong 2-contact discontinuity and a strong 3-shock as the Riemann solutions for the compressible full Euler equations (1.1), called background solutions $\overline{U}$ composing of four constant states shown in (1.3). From the parameterizations of the nonlinear elementary waves, it holds that

(i) Along the strong 1-shock wave curve, it satisfies that

$$\begin{cases} s_{10}(\rho_2 - \rho_1) = \rho_2 u_2 - \rho_1 u_1, \\ s_{10}(\rho_2 u_2 - \rho_1 u_1) = \rho_2 u_2^2 + p_2 - \rho_1 u_1^2 - p_1, \\ s_{10}\Big(\frac{1}{2}\rho_2 u_2^2 + \frac{p_2}{\gamma-1} - \frac{1}{2}\rho_1 u_1^2 - \frac{p_1}{\gamma-1}\Big) = \frac{1}{2}\rho_2 u_2^3 + \frac{\gamma p_2 u_2}{\gamma-1} - \frac{1}{2}\rho_1 u_1^3 - \frac{\gamma p_1 u_1}{\gamma-1}. \end{cases} \tag{2.9}$$

(ii) The second characteristic family is linearly degenerate, and the strong 2-contact discontinuity through U_2 is given by

$$C_2(U_2) : \rho_3 = \rho_2 + \sigma_{20}, \quad u_3 = u_2, \quad p_3 = p_2. \tag{2.10}$$

(iii) Along the strong 3-shock wave curve, it satisfies that

$$\begin{cases} s_{30}(\rho_4 - \rho_3) = \rho_4 u_4 - \rho_3 u_3, \\ s_{30}(\rho_4 u_4 - \rho_3 u_3) = \rho_4 u_4^2 + p_4 - \rho_3 u_3^2 - p_3, \\ s_{30}\Big(\frac{1}{2}\rho_4 u_4^2 + \frac{p_4}{\gamma-1} - \frac{1}{2}\rho_3 u_3^2 - \frac{p_3}{\gamma-1}\Big) = \frac{1}{2}\rho_4 u_4^3 + \frac{\gamma p_4 u_4}{\gamma-1} - \frac{1}{2}\rho_3 u_3^3 - \frac{\gamma p_3 u_3}{\gamma-1}. \end{cases} \tag{2.11}$$

Due to some BV perturbations of the initial state, we will take into account of four types of solvers to the Riemann problem (1.1) and (2.1).

2.2 Riemann problem involving only weak waves

In this subsection, we give the solvability of problem (1.1) and (2.1). As shown in [17], when $|U_L - U_R| \ll 1$, we can parameterize physically admissible wave curves in a neighborhood of $U_k (k = 1, 2, 3, 4)$ by C^2 curves: $\alpha_i \mapsto \Phi_i(U_L, \alpha_i)$ satisfying

$$\Phi(U_L; \alpha_1, \alpha_2, \alpha_3) := \Phi_3(\Phi_2(\Phi_1(U_L, \alpha_1), \alpha_2), \alpha_3) = U_R, \tag{2.12}$$

which represents the left state U_L and the right state U_R can be connected by a 1-wave α_1, a 2-wave α_2 and a 3-wave α_3. Moreover, it holds that

$$\Phi(U_L; \alpha_1, \alpha_2, \alpha_3)\Big|_{\alpha_1=\alpha_2=\alpha_3=0} = U_L, \quad \frac{\partial \Phi}{\partial \alpha_i}(U_L; \alpha_1, \alpha_2, \alpha_3)\Big|_{\alpha_1=\alpha_2=\alpha_3=0} = r_i(U_L), \quad i = 1, 2, 3.$$

In addition, $\alpha_i > 0$ along the rarefaction wave curve $R_i(U_L)$, while $\alpha_i < 0$ along the shock wave curve $S_i(U_L)$.

Hereinafter, we denote by $\alpha_i, \beta_i, \gamma_i$ the parameters of the corresponding i-waves, $i = 1, 2, 3$, while by their absolute values the corresponding strengths of the waves. We also use the parameters to represent the i-waves provided no confusion occurs. In the sequel, we parameterize the strong shock by its velocity s, and $\dot{U}(s)$ denotes the derivative of U with respect to s along the strong shock wave curve. For convenience, let $A(U, s) = \nabla_U H(U) - s\nabla_U W(U)$. O_ε stands for a small neighbourhood, which will be used frequently later.

2.3 Riemann problem involving only a strong 1-shock wave

In this subsection, when $|U_L - U_R|$ is not sufficiently small, we consider the Riemann problem (1.1) and (2.1), where $U_L \in O_\varepsilon(U_1)$ and $U_R \in O_\varepsilon(U_2)$. The solvability of this Riemann problem can be given by the following lemma.

Lemma 2.1 *For any $U_L \in O_\varepsilon(U_1)$ and $U_R \in O_\varepsilon(U_2)$, there exists a strong 1-shock wave, joining the left state U_L to the right state U_R with the speed s_1. Moreover, $s_1 \in O_\varepsilon(s_{10})$.*

Proof From Rankine-Hugoniot conditions (2.4)–(2.6), we have

$$s_1[W(U)] = [H(U)]. \tag{2.13}$$

Differentiating (2.13) with respect to s_1, we can obtain that

$$A(U, s_1)\dot{U}(s_1) = [W] = W(U) - W(U_L).$$

By direct calculations, it holds that

$$\det A|_{U=U_2,s_1=s_{10}} = \begin{vmatrix} u - s_1 & \rho & 0 \\ u(u - s_1) & \rho(2u - s_1) & 1 \\ \frac{1}{2}u^2(u - s_1) & \rho u(\frac{3}{2}u - s_1) + \frac{\gamma}{\gamma-1}p & \frac{\gamma u - s_1}{\gamma - 1} \end{vmatrix}_{U=U_2,s=s_{10}}$$

$$= \frac{1}{\gamma - 1}\rho_2(u_2 - s_{10})(s_{10} - \lambda_3(U_2))(s_{10} - \lambda_1(U_2)).$$

From the Lax entropy conditions (2.7)–(2.8), we have $\det A|_{U=U_2,s_1=s_{10}} < 0$. We complete the proof of this lemma by the aid of the implicit function theorem.

The following lemma is important to estimate the strengths of the weak waves reflected on the strong 1-shock waves, and to estimate the changes to the strengths of the strong 1-shock waves (see the proofs of Lemmas 4.2–4.3).

Lemma 2.2 *It holds that*

$$\det A(U, s)|_{U=U_2,s=s_{10}} < 0, \quad \det(Ar_1(U), Ar_2(U), Ar_3(U))|_{U=U_2,s=s_{10}} > 0,$$
$$\det(A\dot{U}(s), Ar_2(U), Ar_3(U))|_{U=U_2,s=s_{10}} > 0.$$

Proof From Lemma 2.1, we can obtain that $\det A < 0$. By direct calculations, one has

$$Ar_1(U) = -\frac{2\rho(u - c - s)}{(\gamma + 1)c}\Big(1, u - c, \frac{1}{2}u^2 - uc + \frac{c^2}{\gamma - 1}\Big)^\top,$$

$$Ar_2(U) = (u - s)\Big(1, u, \frac{1}{2}u^2\Big)^\top,$$

$$Ar_3(U) = \frac{2\rho(u + c - s)}{(\gamma + 1)c}\Big(1, u + c, \frac{1}{2}u^2 + uc + \frac{c^2}{\gamma - 1}\Big)^\top.$$

Then, it yields that

$$\det(Ar_1(U), Ar_2(U), Ar_3(U))|_{U=U_2,s_1=s_{10}}$$
$$= \frac{4\rho^2(s - u)((u - s)^2 - c^2)}{(\gamma + 1)^2c^2}\begin{vmatrix} 1 & 1 & 1 \\ u - c & u & u + c \\ \frac{1}{2}u^2 - uc + \frac{c^2}{\gamma-1} & \frac{1}{2}u^2 & \frac{1}{2}u^2 + uc + \frac{c^2}{\gamma-1} \end{vmatrix}_{\substack{U=U_2 \\ s_1=s_{10}}}$$

$$= \frac{8\rho_2^2c_2(s_{10} - u_2)(\lambda_1(U_2) - s_{10})(\lambda_3(U_2) - s_{10})}{(\gamma + 1)^2(\gamma - 1)} > 0$$

and

$$\det(A\dot{U}(s), Ar_2(U), Ar_3(U))$$

$$
= \frac{2\rho(u-s)(u+c-s)}{(\gamma+1)c}\begin{vmatrix} [\rho] & 1 & 1 \\ [\rho u] & u & u+c \\ [\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}] & \frac{1}{2}u^2 & \frac{1}{2}u^2+uc+\frac{c^2}{\gamma-1} \end{vmatrix}
$$

$$
= \frac{2\rho(s-u)(u+c-s)}{\gamma+1}\Big((s-u)\Big(u+\frac{c}{\gamma-1}\Big)[\rho]-\Big[\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}\Big]+\frac{1}{2}u^2[\rho]\Big).
$$

From the Rankine-Hugoniot conditions (2.4)–(2.5), one has

$$
(s-u)[\rho]=(s-u)(\rho-\rho_L)=\rho_L(u-u_L),\quad p-p_L=\rho_L(u-u_L)(s-u_L).
$$

Therefore, we can derive that

$$
\begin{aligned}
&(s-u)\Big(u+\frac{c}{\gamma-1}\Big)[\rho]-\Big[\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}\Big]+\frac{1}{2}u^2[\rho] \\
&=\rho_L(u-u_L)\Big(u+\frac{c}{\gamma-1}\Big)-\frac{1}{2}\rho_L(u-u_L)(u+u_L)-\frac{\rho_L(u-u_L)(s-u_L)}{\gamma-1} \\
&=\frac{\rho_L(u-u_L)((\gamma-3)(u-u_L)+2(u+c-s))}{2(\gamma-1)}.
\end{aligned}
$$

Since $s<u<u_L$ and $1<\gamma\le 3$, it yields that

$$
\begin{aligned}
&\det(A\dot{U}(s),Ar_2(U),Ar_3(U))|_{U=U_2,s=s_{10}} \\
&=\frac{\rho_1\rho_2(s_{10}-u_2)(\lambda_3(U_2)-s_{10})(u_2-u_1)((\gamma-3)(u_2-u_1)+2(\lambda_3(U_2)-s_{10}))}{\gamma^2-1}>0.
\end{aligned}
$$

Hence, we complete the proof of this lemma.

2.4 Riemann problem involving only a strong 2-contact discontinuity

When $|U_L-U_R|$ is not sufficiently small, where $U_L\in O_\varepsilon(U_2)$ and $U_R\in O_\varepsilon(U_3)$, the solvability of (1.1) and (2.1) can be formulated in the following lemma.

Lemma 2.3 *For any given $U_L\in O_\varepsilon(U_2)$ and $U_R\in O_\varepsilon(U_3)$, there exists a strong 2-contact discontinuity connecting the state U_L to the state U_R with strength $|\sigma_2|$. Moreover, it satisfies that*

$$
U_R=(\rho_L+\sigma_2,u_L,p_L)^\top:=G(U_L;\sigma_2),\quad \nabla_U G(U_L;\sigma_2)=\mathrm{diag}(1,1,1). \tag{2.14}
$$

This lemma can be easily proved by direct calculations. We omit the proof here.

2.5 Riemann problem involving only a strong 3-shock wave

In this subsection, we consider the Riemann problem (1.1) and (2.1), where $U_L\in O_\varepsilon(U_3)$, and $U_R\in O_\varepsilon(U_4)$. The solvability of Riemann problem can be given by the following lemma.

Lemma 2.4 *For any $U_L\in O_\varepsilon(U_3)$ and $U_R\in O_\varepsilon(U_4)$, there exists a strong 3-shock wave, separating the left state U_L from the right state U_R with speed s_3. Moreover, $s_3\in O_\varepsilon(s_{30})$.*

Similar to Lemma 2.1, we can prove this lemma by the implicit function theorem. Thus, we omit the details here.

In the following, we present some properties of the strong 3-shock waves, which are essential to estimate the strengths of the weak waves reflected on the strong 3-shock waves, and to estimate the changes of the strengths of the strong 3-shock waves (see the proofs of Lemmas 4.6–4.7).

Lemma 2.5 *The following statements hold*

$$\det A(U,s)|_{U=U_4,s=s_{30}}<0,\quad \det(Ar_1(U),Ar_2(U),A\dot{U}(s))|_{U=U_3,s=s_{30}}>0.$$

Proof One can calculate directly to obtain that

$$\det A(U,s)|_{U=U_4,s=s_{30}}=\frac{1}{\gamma-1}\rho_4(u_4-s_{30})(s_{30}-\lambda_3(U_4))(s_{30}-\lambda_1(U_4)).$$

From the Lax entropy conditions (2.7) and (2.8), $\det A|_{U=U_4,s=s_{30}}<0$. Meanwhile,

$$\begin{aligned}
&\det(Ar_1(U),Ar_2(U),A\dot{U}(s))|_{U=U_3,s=s_{30}}\\
&=\frac{2\rho(s-u)(u-c-s)}{(\gamma+1)c}\begin{vmatrix}1 & 1 & [\rho]\\ u-c & u & [\rho u]\\ \frac12u^2-uc+\frac{c^2}{\gamma-1} & \frac12u^2 & [\frac12\rho u^2+\frac{p}{\gamma-1}]\end{vmatrix}_{U=U_3,s=s_{30}}\\
&=\frac{2\rho(s-u)(\lambda_1(U)-s)}{\gamma+1}\Big(\Big[\frac12\rho u^2+\frac{p}{\gamma-1}\Big]-(s-u)\Big(u-\frac{c}{\gamma-1}\Big)[\rho]-\frac12u^2[\rho]\Big)\Big|_{U=U_3,s=s_{30}}\\
&=\frac{\rho_3\rho_4(s_{30}-u_3)(\lambda_1(U_3)-s_{30})(u_4-u_3)}{\gamma^2-1}(2(s_{30}-\lambda_1(U_3))+(\gamma-3)(u_4-u_3))>0.
\end{aligned}$$

Thus, the proof of this lemma is completed.

3 Approximate Solutions

In this section, we use the Riemann problem as building blocks to construct approximate solutions of Cauchy problem (1.1) and (1.4) by a modified wave front tracking scheme. First, we consider the solvability of Riemann problem (1.1) and (2.1).

As mentioned in Section 2, the solution to the Riemann problem (1.1) and (2.1) is composed of at most four constant states connected by shocks, rarefaction waves or contact discontinuities. By the wave front tracking method, there are two types of Riemann solvers to solve this Riemann problem.

Case 1. Accurate Riemann solver.

The accurate Riemann solver is given in Section 2, except that every rarefaction wave R_i $(i=1,3)$ is divided into ν equal parts.

Suppose that the left state U_L and the middle state U_M are connected by a 1-rarefaction wave α_1. If $\alpha_1>0$, then let $U_{0,0}=U_L$, $U_{0,\nu}=U_M$. For any $1\le k\le\nu$,

$$U_{0,k}=\Phi_1\Big(U_{0,k-1};\frac{\alpha_1}{\nu}\Big).$$

Thus, the 1-rarefaction wave is replaced by

$$U_A^\nu = \begin{cases} U_L, & x < x_{1,1}, \\ U_{0,k}, & x_{1,k} < x < x_{1,k+1}, \\ U_M, & x_{1,\nu} < x < x_0 + (t-t_0)\lambda_1^*, \end{cases} \tag{3.1}$$

where $x_{1,k} = x_0 + (t-t_0)\lambda_1(U_{0,k})$ and $\lambda_1^* \in (\max_{(x,t)} \lambda_1(U), \min_{(x,t)} \lambda_2(U))$.

Similarly, we can approximate 3-rarefaction wave by ν 3-rarefaction wave fronts in the domain $\{(x,t) : x > x_0 + \lambda_2^*(t-t_0)\}$, where $\lambda_2^* \in (\max_{(x,t)} \lambda_2(U), \min_{(x,t)} \lambda_3(U))$.

Case 2. Simplified Riemann solver.

In order to keep the number of the wave fronts be finite for all $t \geq 0$, the simplified Riemann solver is introduced. Exactly speaking, an auxiliary wave, called a non-physical wave, is constructed with a constant speed $\widehat{\lambda}$, which is strictly larger than all the characteristic speeds of system (1.1). The strength of the non-physical wave measures the error of the simplified Riemann solver. It occurs in the following two cases:

Case a. A j-wave β_j and an i-wave α_i interact at (x_0, t_0), $1 \leq i \leq j \leq 3$. Suppose that U_L, U_M and U_R are three constant states, satisfying

$$U_M = \Phi_j(U_L, \beta_j), \quad U_R = \Phi_i(U_M, \alpha_i). \tag{3.2}$$

The auxiliary state is constructed by

$$\widetilde{U}_R = \begin{cases} \Phi_j(\Phi_i(U_L, \alpha_i), \beta_j), & j > i, \\ \Phi_j(U_L, \alpha_j + \beta_j), & j = i, \end{cases} \tag{3.3}$$

then, the simplified Riemann solver $U_S(U_L, U_R)$ to Riemann problem (1.1) and (2.1) at (x_0, t_0) is given by

$$U_S(U_L, U_R) = \begin{cases} U_A^\nu(U_L, \widetilde{U}_R), & x - x_0 < \widehat{\lambda}(t-t_0), \\ U_R, & x - x_0 > \widehat{\lambda}(t-t_0), \end{cases} \tag{3.4}$$

where $U_A^\nu(U_L, \widetilde{U}_R)$ is constructed by the accurate Riemann solver as shown in case 1. The non-physical wave can be defined by

$$U_{np} = \begin{cases} \widetilde{U}_R, & x - x_0 < \widehat{\lambda}(t-t_0), \\ U_R, & x - x_0 > \widehat{\lambda}(t-t_0), \end{cases} \tag{3.5}$$

whose strength is $|U_R - \widetilde{U}_R|$.

Case b. A non-physical wave ε_{np} collides with a weak i-wave front α_i $(i = 1, 2, 3)$ from left at the point (x_0, t_0). Suppose that the three states U_L, U_M and U_R satisfy

$$|U_M - U_L| = \varepsilon_{np}, \quad U_R = \Phi_i(U_M, \alpha_i).$$

Then, the simplified Riemann solver $U_S(U_L, U_R)$ to problem (1.1) and (2.1) is defined by

$$U_S(U_L, U_R) = \begin{cases} U_L, & x - x_0 < \lambda_i(U_L)(t-t_0), \\ \Phi_i(U_L, \alpha_i), & \lambda_i(U_L)(t-t_0) < x - x_0 < \widehat{\lambda}(t-t_0), \\ U_R, & x - x_0 > \widehat{\lambda}(t-t_0). \end{cases}$$

3.1 Wave front tracking algorithm

The wave front tracking algorithm to construct the approximate solutions is given by:

Case 1. There are no more than two wave fronts interacting at one point by changing the speed of a single wave front with a quantity $O(1)2^{-\nu}$.

Case 2. If two wave fronts α_i and β_j interact, then the generated Riemann problem is solved by the following rules.

Rule 1. If $|\alpha_i\beta_j| > \mu_\nu$ and both are physical, where μ_ν is a fixed small parameter satisfying $\mu_\nu \to 0$ as $\nu \to +\infty$, then the accurate Riemann solver is adopted.

Rule 2. If $|\alpha_i\beta_j| < \mu_\nu$ and either both are physical, or one of them is a non-physical wave, then the simplified Riemann solver is adopted.

Case 3. If a weak wave collides with the strong wave fronts, then the accurate Riemann solver is adopted.

Let τ_k be the time when two wave fronts interact for the k-th time, $k \geq 1$. For any sufficiently large $\nu \in \mathbb{N}$, we can construct a ν-approximate solution $U^\nu(x,t)$, and assign each wave front with a generation order inductively as follows:

Step 1. For $0 \leq t < \tau_1$, suppose that $U^\nu(x,t)$ can be constructed by accurate Riemann solver to solve a series of Riemann problems, which can be carried out as shown in Section 2. All the wave fronts generated from Riemann problems at $t = 0$ have generation order 1.

Step 2. By induction, assume that the approximate solution $U^\nu(x,t)$ has been constructed for $t < \tau_k$, and that $U^\nu|_{t<\tau_k}$ consists of a finite number of wave fronts. As shown in Sections 2–3, when two wave fronts interact at $t = \tau_k$, a new Riemann problem is generated. More exactly speaking, let a weak i-wave front α_i of order n_1 interact with a j-wave front β_j of order n_2. Suppose that each front has been assigned a generation order by the following rules.

Rule 1. When $n_1, n_2 < \nu$, the accurate Riemann solver is adopted to construct the outgoing wave front, and the generation order of the outgoing l-wave is assigned by

$$\begin{cases} \max(n_1,n_2)+1 & \text{if } l \neq i,j, \\ \min(n_1,n_2) & \text{if } l = i = j, \\ n_1 & \text{if } l = i \neq j, \\ n_2 & \text{if } l = j \neq i. \end{cases} \tag{3.6}$$

Rule 2. When $\max(n_1,n_2) = \nu$, the simplified Riemann solver is adopted to construct the outgoing wave fronts. The generation order of the outgoing l-wave front is assigned by (3.6), and that of the non-physical wave front is $\nu + 1$.

Rule 3. When $n_1 = \nu + 1$ and $n_2 \leq \nu$, α_i is a non-physical wave front. We adopt the simplified Riemann solver to construct the outgoing wave front. The generation order of the new non-physical wave front is $\nu + 1$, while the generation order of the outgoing physical wave front is the same as that of the incoming wave β_j.

Therefore, repeating the inductive process, we complete the construction of the approximate solutions in the whole domain.

4 Estimates on the Wave Interactions

In this section, we will make exact estimates of the wave interactions between the weak waves, the reflections on the strong shock waves and contact discontinuities, and so on.

Let $(U_l, U_r) = (\alpha_1, \alpha_2, \alpha_3)$ represent that the Riemann problem with the left state U_l and the right state U_r is solved by a 1-wave α_1, 2-wave α_2 and 3-wave α_3.

4.1 Interaction between weak waves

In this subsection, without loss of generality, suppose that a weak j-physical wave β_j interacts with a weak i-physical wave α_i from left, and that U_B, U_M and $U_A \in O_\varepsilon(U_j)$ $(j = 1, 2, 3, 4)$, satisfying

$$U_M = \Phi(U_L; \beta_j), \quad U_R = \Phi(U_M; \alpha_i).$$

As shown in Section 3, if we adopt the accurate Riemann solver to solve the generated Riemann problem, then the outgoing wave fronts are physical waves, denoted by γ_1, γ_2 and γ_3, respectively. Otherwise, if the simplified solver is adopted, then the outgoing physical waves are denoted by γ_i and γ_j. Meanwhile, a non-physical wave is also introduced, denoted by ε_{np}. By a standard process, see [1, p.133], we can obtain the wave interaction estimates for the weak waves by the following lemma.

Lemma 4.1 *If the accurate Riemann solver is adopted, then it holds*

$$\gamma_i = \alpha_i + O(1)|\alpha_i||\beta_j|, \quad \gamma_j = \beta_j + O(1)|\alpha_i||\beta_j| \quad \text{for } i \neq j \tag{4.1}$$

and

$$\gamma_i = \alpha_i + \beta_j + O(1)|\alpha_i||\beta_j| \quad \text{for } i = j, \tag{4.2}$$

$$\gamma_\ell = O(1)|\alpha_i||\beta_j|, \quad \ell \neq i, j. \tag{4.3}$$

If the simplified Riemann solver is adopted, then it satisfies that

$$\begin{aligned} &\gamma_i = \alpha_i, \quad \gamma_j = \beta_j \quad \text{for } 1 \leq i \neq j \leq 3, \\ \text{or } &\gamma_i = \alpha_i + \beta_i, \quad 1 \leq i = j \leq 3, \\ &\varepsilon_{np} = O(1)|\alpha_i||\beta_j|. \end{aligned} \tag{4.4}$$

4.2 Interaction between the strong 1-shock and a weak 1-wave from right

In this case, we assume that a 1-weak wave α_1 interacts with the strong 1-shock wave s_1 from the right. Let $(U_l, U_m) = (s_1, 0, 0)$ and $(U_m, U_r) = (\alpha_1, 0, 0)$, where $U_l \in O_\varepsilon(U_1)$, $U_m, U_r \in O_\varepsilon(U_2)$. From the construction of the wave front tracking algorithm, the accurate Riemann solver is adopted, and the generated wave fronts are denoted by s_1', γ_2 and γ_3, respectively (see Figure 2). Then we can obtain the following estimates.

Lemma 4.2 *Assume that U_l, U_m and U_r are described as above. Then the generated Riemann problem with the left state U_l and right state U_r is solved by a strong 1-shock s_1', a weak 2-wave γ_2 and a weak 3-wave γ_3. Moreover, it holds*

$$s_1' = s_1 + K_{s_1}\alpha_1, \quad \gamma_2 = K_{s_2}\alpha_1, \quad \gamma_3 = K_{s_3}\alpha_1, \tag{4.5}$$

where K_{s_1} and K_{s_2} are bounded, depending only on the system and background solution, moreover, $K_{s_3} \in (-1, 1)$.

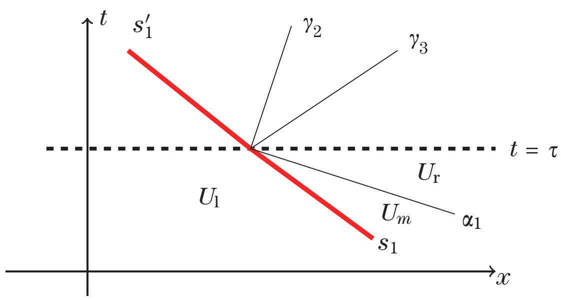

Figure 2 A weak 1-wave interacts with a strong 1-shock wave from right.

Proof From the definition of Φ, it yields that

$$\Phi(\Phi(U_l; s_1, 0, 0); \alpha_1, 0, 0) = \Phi(U_l; s_1', \gamma_2, \gamma_3) = U_r. \tag{4.6}$$

Based on Lemma 2.2, one can obtain that

$$\frac{\partial \Phi(U_l; s_1', \gamma_2, \gamma_3)}{\partial(s_1', \gamma_2, \gamma_3)}\Big|_{s_1'=s_{10}, \gamma_2=\gamma_3=0} = \frac{\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_3(U_2))}{\det A(U_2)} < 0.$$

By the theorem of the implicit function, close to the point $(s_1, \alpha_1) = (s_{10}, 0)$, there exist C^1 functions of (s_1, α_1) such that

$$s_1' = s_1'(s_1, \alpha_1), \quad \gamma_2 = \gamma_2(s_1, \alpha_1), \quad \gamma_3 = \gamma_3(s_1, \alpha_1).$$

Using Taylor's expansion formula, one can derive that

$$\begin{aligned} s_1' &= s_1'(s_1, \alpha_1) - s_1'(s_1, 0) + s_1'(s_1, 0) = K_{s_1}\alpha_1 + s_1, \\ \gamma_i &= \gamma_i(s_1, \alpha_1) - \gamma_i(s_1, 0) + \gamma_i(s_1, 0) = K_{s_i}\alpha_1, \quad i = 2, 3, \end{aligned}$$

where $s_1'(s_1, 0) = s_1$ and $\gamma_2(s_1, 0) = \gamma_3(s_1, 0) = 0$.

Next, we will show that $K_{s_i}|_{i=1,2,3}$ is bounded. Differentiating (4.6) with respect to α_1, it yields that

$$\frac{\partial U_r}{\partial s_1'}\frac{\partial s_1'}{\partial \alpha_1} + \frac{\partial U_r}{\partial \gamma_2}\frac{\partial \gamma_2}{\partial \alpha_1} + \frac{\partial U_r}{\partial \gamma_3}\frac{\partial \gamma_3}{\partial \alpha_1} = \frac{\partial U_r}{\partial \alpha_1}. \tag{4.7}$$

Multiplying (4.7) with $A(U_r)$ from left, letting $s_1 = s_{10}$, $\alpha_1 = 0$ and $U_r = U_2$, one can obtain that

$$A\dot{U}\frac{\partial s_1'}{\partial \alpha_1}\Big|_{s_1=s_{10},\alpha_1=0} + Ar_2(U_2)\frac{\partial \gamma_2}{\partial \alpha_1}\Big|_{s_1=s_{10},\alpha_1=0} + Ar_3(U_2)\frac{\partial \gamma_3}{\partial \alpha_1}\Big|_{s_1=s_{10},\alpha_1=0} = Ar_1(U_2). \tag{4.8}$$

Therefore, from (4.8) and Lemma 2.2, we can formulate

$$K_{s_1}|_{s_1=s_{10},\alpha_1=0} = \frac{\det(Ar_1(U_2), Ar_2(U_2), Ar_3(U_2))}{\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_3(U_2))}$$

$$= \frac{8\rho_2 c_2(\lambda_1(U_2) - s_{10})}{(\gamma+1)\rho_1(u_2-u_1)((\gamma-1)u_2-(\gamma-3)u_1+2c_2-2s_{10})},$$

$$\begin{aligned} K_{s_2}|_{s_1=s_{10},\alpha_1=0} &= \frac{\det(A\dot{U}(s_{10}), Ar_1(U_2), Ar_3(U_2))}{\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_3(U_2))} \\ &= \frac{4\rho_2(\lambda_1(U_2)-s_{10})((\gamma-1)\rho_2(u_2-u_1)^2+2c_2^2(\rho_2-\rho_1)-2p_2+2p_1)}{(\gamma+1)\rho_1 c_2(s_{10}-u_2)(u_2-u_1)((\gamma-1)u_2-(\gamma-3)u_1+2c_2-2s_{10})}, \end{aligned}$$

$$\begin{aligned} K_{s_3}|_{s_1=s_{10},\alpha_1=0} &= \frac{\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_1(U_2))}{\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_3(U_2))} \\ &= \frac{(s_{10}-\lambda_1(U_2))((\gamma-3)u_1-(\gamma-1)u_2+2c_2+2s_{10})}{(\lambda_3(U_2)-s_{10})((\gamma-1)u_2-(\gamma-3)u_1+2c_2-2s_{10})}, \end{aligned}$$

where

$$\begin{aligned} &\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_3(U_2))|_{U=U_2,s=s_{10}} \\ &= \frac{\rho_1\rho_2(s_{10}-u_2)(u_2-u_1)(\lambda_3(U_2)-s_{10})}{(\gamma+1)(\gamma-1)}((\gamma-1)u_2-(\gamma-3)u_1+2c_2-2s_{10}) > 0, \end{aligned}$$

$$\det(Ar_1(U_2), Ar_2(U_2), Ar_3(U_2)) = \frac{8\rho_2^2 c_2(s_{10}-u_2)(\lambda_1(U_2)-s_{10})(\lambda_3(U_2)-s_{10})}{(\gamma+1)^2(\gamma-1)},$$

$$\begin{aligned} &\det(A\dot{U}(s_{10}), Ar_1(U_2), Ar_3(U_2)) \\ &= \frac{4\rho_2^2(\lambda_1(U_2)-s_{10})(\lambda_3(U_2)-s_{10})}{(\gamma+1)^2(\gamma-1)c_2}((\gamma-1)\rho_1(u_2-u_1)^2+2c_2^2(\rho_2-\rho_1)-2p_2+2p_1), \\ &\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_1(U_2)) \\ &= \frac{2\rho(u-s)(s-\lambda_1(U))}{\gamma+1}\Big((s-u)\Big(u-\frac{c}{\gamma-1}\Big)[\rho]-\Big[\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}\Big]+\frac{1}{2}u^2[\rho]\Big)\Big|_{\substack{s_1=s_{10},\\ U=U_2}} \\ &= \frac{\rho_1\rho_2(s_{10}-u_2)(s_{10}-\lambda_1(U_2))(u_2-u_1)((\gamma-3)u_1-(\gamma-1)u_2+2c_2+2s_{10})}{(\gamma+1)(\gamma-1)}. \end{aligned}$$

Since $\lambda_3(U_2) > s_{10} > \lambda_1(U_2)$, $s_{10} < u_2$ and

$$0 < \frac{s_{10}-\lambda_1(U_2)}{\lambda_3(U_2)-s_{10}} < 1, \quad 1 < \gamma \le 3,$$

$$\frac{(\gamma-3)u_1-(\gamma-1)u_2+2c_2+2s_{10}}{(\gamma-1)u_2-(\gamma-3)u_1+2c_2-2s_{10}} + 1 = \frac{4c_2}{(\gamma-3)(u_2-u_1)+2(u_2+c_2-s_{10})} > 0,$$

$$\frac{(\gamma-3)u_1-(\gamma-1)u_2+2c_2+2s_{10}}{(\gamma-1)u_2-(\gamma-3)u_1+2c_2-2s_{10}} - 1 = \frac{2((\gamma-3)(u_1-u_2)+2(s_{10}-u_2))}{(\gamma-3)(u_2-u_1)+2(u_2+c_2-s_{10})} < 0,$$

we have $K_{s_3}|_{\substack{U_r=U_2,\\ s_1=s_{10},\alpha_1=0}} \in (-1,1)$. Since K_{s_i}, $i=1,2,3$ are continuous with respect to s_1, α_1 and U_r, we can demonstrate that $K_{s_3} \in (-1,1)$ and K_{s_1}, K_{s_2} are bounded.

Therefore, the proof of this lemma is completed.

4.3 Interaction between the strong 1-shock wave and a weak *i*-wave from left

Assume that the left state U_l is connected to the middle state U_m by a weak i-wave, and the state U_m is separated from the state U_r by a strong 1-shock with the speed s_1, where

$U_l, U_m \in O_\varepsilon(U_1)$, $U_r \in O_\varepsilon(U_2)$. Then the generated wave fronts are a new strong 1-shock wave s_1', a weak 2-wave γ_2 and a weak 3-wave γ_3, respectively (see Figure 3). Therefore, we can obtain the following estimates.

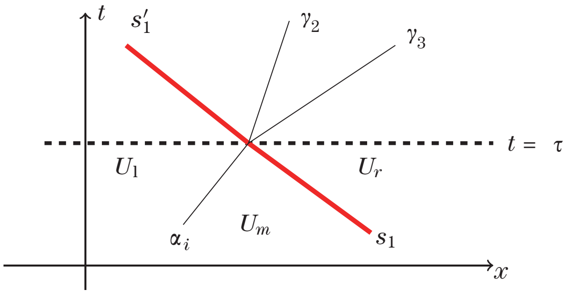

Figure 3 A weak i-wave interacts with a strong 1-shock wave from left.

Lemma 4.3 *It holds that*

$$s_1' = s_1 + \widetilde{K}_{s_1}^i \alpha_i, \quad \gamma_2 = \widetilde{K}_{s_2}^i \alpha_i, \quad \gamma_3 = \widetilde{K}_{s_3}^i \alpha_i, \tag{4.9}$$

where $\widetilde{K}_{s_j}^i$ are bounded, $j = 1, 2, 3$, depending only on the system and background solution.

Proof As we know,

$$\Phi(\Phi_i(U_l; \alpha_i); s_1, 0, 0) = \Phi(U_l; s_1', \gamma_2, \gamma_3) = U_r. \tag{4.10}$$

Since

$$\frac{\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_3(U_2))}{\det A(U_2)} < 0,$$

the condition of the implicit function theorem is satisfied close to the point $(s_1, \alpha_i) = (s_{10}, 0)$, and there exist C^1 functions such that

$$s_1' = s_1'(s_1, \alpha_i), \quad \gamma_2 = \gamma_2(s_1, \alpha_i), \quad \gamma_3 = \gamma_3(s_1, \alpha_i).$$

From the Taylor's expansion formula, one can derive that

$$\begin{aligned} s_1' &= s_1'(s_1, \alpha_i) - s_1'(s_1, 0) + s_1'(s_1, 0) = \widetilde{K}_{s_1}^i \alpha_i + s_1, \\ \gamma_2 &= \gamma_2(s_1, \alpha_i) - \gamma_2(s_1, 0) + \gamma_2(s_1, 0) = \widetilde{K}_{s_2}^i \alpha_i, \\ \gamma_3 &= \gamma_3(s_1, \alpha_i) - \gamma_3(s_1, 0) + \gamma_3(s_1, 0) = \widetilde{K}_{s_3}^i \alpha_i, \end{aligned}$$

where $s_1'(s_1, 0) = s_1$ and $\gamma_2(s_1, 0) = \gamma_3(s_1, 0) = 0$. In the rest, we will show $\widetilde{K}_{s_j}^i|_{j=1,2,3}$ are bounded. Without loss of generality, we suppose that a weak 1-wave α_1 interacts with a strong 1-shock wave from left. Differentiating (4.10) with respect to α_1, one has

$$\frac{\partial U_r}{\partial s_1'} \frac{\partial s_1'}{\partial \alpha_1} + \frac{\partial U_r}{\partial \gamma_2} \frac{\partial \gamma_2}{\partial \alpha_1} + \frac{\partial U_r}{\partial \gamma_3} \frac{\partial \gamma_3}{\partial \alpha_1} = \frac{\partial U_r}{\partial \alpha_1}. \tag{4.11}$$

Multiplying (4.11) with $A(U_r)$ from the left, and letting $s_1 = s_{10}, \alpha_1 = 0, U_r = U_2$, it yields

$$\begin{aligned}
\frac{\partial s_1'}{\partial \alpha_1}\Big|_{\substack{s_1=s_{10}\\ \alpha_1=0}} &= \frac{\det(Ar_1(U_1), A(U_2)r_2(U_2), Ar_3(U_2))}{\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_3(U_2))} \\
&= -\frac{4(\gamma-1)(\lambda_1(U_1)-s_{10})L_1}{(\gamma+1)c_1((\gamma-1)u_2-(\gamma-3)u_1+2c_2-2s_{10})}, \\
\frac{\partial \gamma_2}{\partial \alpha_1}\Big|_{\substack{s_1=s_{10}\\ \alpha_1=0}} &= \frac{\det(A\dot{U}(s_{10}), Ar_1(U_1), Ar_3(U_2))}{\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_3(U_2))} \\
&= \frac{4(\gamma-1)(\lambda_1(U_1)-s_{10})L_2}{(\gamma+1)c_1c_2(s_{10}-u_2)(u_2-u_1)((\gamma-1)u_2-(\gamma-3)u_1+2c_2-2s_{10})}, \\
\frac{\partial \gamma_3}{\partial \alpha_1}\Big|_{\substack{s_1=s_{10}\\ \alpha_1=0}} &= \frac{\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_1(U_1))}{\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_3(U_2))} \\
&= \frac{2(\gamma-1)(\rho_2-\rho_1)(s_{10}-u_2)(\lambda_1(U_1)-s_{10})L_3}{\rho_2c_1(\lambda_3(U_2)-s_{10})(u_2-u_1)((\gamma-2)u_2-(\gamma-3)u_1+2c_2-2s_{10})},
\end{aligned}$$

where

$$\begin{aligned}
&\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_3(U_2)) \\
={}& \frac{\rho_1\rho_2(s_{10}-u_2)(u_2-u_1)(\lambda_3(U_2)-s_{10})}{(\gamma+1)(\gamma-1)}((\gamma-1)u_2-(\gamma-3)u_1+2c_2-2s_{10}) > 0, \\
&\det(Ar_1(U_1), Ar_2(U_2), Ar_3(U_2))|_{s_1=s_{10}} \\
={}& \frac{4\rho_1\rho_2(\lambda_1(U_1)-s_{10})(u_2-s_{10})(\lambda_3(U_2)-s_{10})}{(\gamma+1)^2c_1}L_1,
\end{aligned}$$

$$\det(A\dot{U}(s_{10}), Ar_1(U_1), Ar_3(U_2)) = \frac{4\rho_1\rho_2(\lambda_1(U_1)-s_{10})(\lambda_3(U_2)-s_{10})}{(\gamma+1)^2c_1c_2}L_2,$$

$$\det(A\dot{U}(s_{10}), Ar_2(U_2), Ar_1(U_1)) = \frac{2\rho_1^2(\lambda_1(U_1)-s_{10})(u_2-s_{10})(u_2-u_1)}{(\gamma+1)c_1}L_3$$

with

$$\begin{aligned}
L_1 &= -\frac{1}{2}(u_1-u_2)^2 + (u_1-u_2)\Big(c_1 + \frac{c_2}{\gamma-1}\Big) - \frac{c_1(c_1+c_2)}{\gamma-1}, \\
L_2 &= (s_{10}-\lambda_1(U_1))\Big(\frac{1}{2}u_2^2 + u_2c_2 + \frac{c_2^2-c_1^2}{\gamma-1} - \frac{1}{2}u_1^2 + u_1c_1\Big)(\rho_2-\rho_1) + \\
&\quad -(\lambda_3(U_2)-\lambda_1(U_1))\Big(\frac{1}{2}\rho_2(u_2^2-u_1^2) + \frac{p_2-p_1}{\gamma-1} + (\rho_2-\rho_1)c_1\Big(u_1 - \frac{c_1}{\gamma-1}\Big)\Big), \\
L_3 &= \frac{u_2c_1-u_1c_1}{2} + \frac{c_1^2-(s_{10}-u_1)(u_1-u_2-c_1)}{\gamma-1}.
\end{aligned}$$

Therefore, we can obtain that $\widetilde{K}^i_{s_j}|_{j=1,2,3}$ are bounded, depending only on the system and background solution.

4.4 Interaction between a strong 2-contact discontinuity and a weak 3-wave from left

Assume that the leftmost state U_l is joined to the middle state U_m by a weak 3-wave α_3, and that the states U_m and U_r are connected by a strong 2-contact discontinuity σ_2, where U_l,

$U_m \in O_\varepsilon(U_2)$ and $U_r \in O_\varepsilon(U_3)$. Let $(U_l, U_m) = (0, 0, \alpha_3)$ and $G(U_m; \sigma_2) = U_r$. The outgoing wave fronts are respectively represented by a weak 1-wave γ_1, a strong 2-contact discontinuity σ_2' and a weak 3-wave γ_3, see Figure 4. Meanwhile, we have the following interaction estimates.

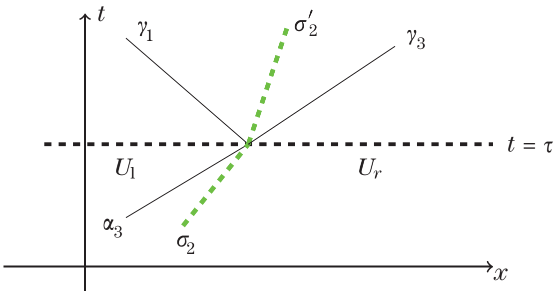

Figure 4 A weak 3-wave interacts with a strong 2-contact discontinuity from left.

Lemma 4.4 *It holds that*

$$\gamma_1 = K_{s_4}\alpha_3, \quad \sigma_2' = \sigma_2 + K_{s_5}\alpha_3, \quad \gamma_3 = K_{s_6}\alpha_3, \tag{4.12}$$

where $K_{s_4} \in (-1, 1)$ and $K_{s_i}|_{i=5,6}$ are bounded, depending only on the system and background solution.

The proofs of Lemmas 4.4–4.5 are omitted in details, see Lemmas 4.2–4.3 in the reference [6].

4.5 Interaction between the strong 2-contact discontinuity and a weak 1-wave from right

Suppose that a strong 2-contact discontinuity σ_2 and a 1-weak wave α_1 interact at time $t = \tau$. Let $U_m = G(U_l; \sigma_2)$ and $(U_m, U_r) = (\alpha_1, 0, 0)$, where $U_l \in O_\varepsilon(U_2)$ and $U_m, U_r \in O_\varepsilon(U_3)$. The outgoing wave fronts are denoted by a weak 1-wave γ_1, a strong 2-contact discontinuity σ_2' and a weak 3-wave γ_3, respectively (see Figure 5). Then we have the following lemma.

Lemma 4.5 *It satisfies that*

$$\gamma_1 = K_{s_7}\alpha_1, \quad \sigma_2' = \sigma_2 + K_{s_8}\alpha_1, \quad \gamma_3 = K_{s_9}\alpha_1, \tag{4.13}$$

where $K_{s_9} \in (-1, 1)$ and $K_{s_i}|_{i=7,8}$ are bounded, depending on background solution.

4.6 Interaction between a strong 3-shock and a weak 3-wave from left

Suppose that the states U_l and U_r are separated by a weak 3-wave α_3 and a strong 3-shock wave. Let $(U_l, U_m) = (0, 0, \alpha_3)$ and $(U_m, U_r) = (0, 0, s_3)$ where $U_l, U_m \in O_\varepsilon(U_3)$, $U_r \in O_\varepsilon(U_4)$. The generated wave fronts are respectively a weak 1-wave γ_1, a weak 2-wave γ_2 and a strong 3-shock wave s_3' (see Figure 6). Then we have the following estimates.

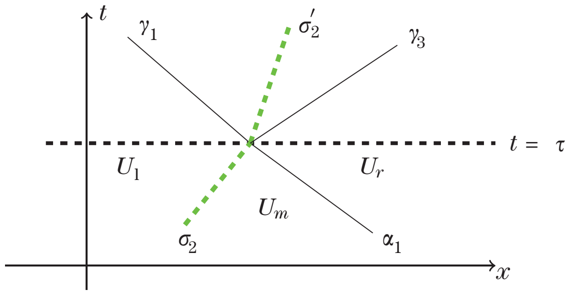

Figure 5 A weak 1-wave interacts with a strong 2-contact discontinuity from right.

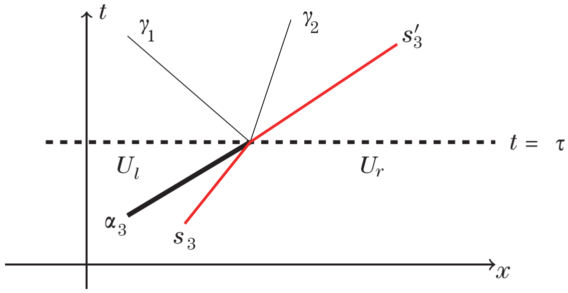

Figure 6 A 3-weak wave collides with 3-strong shock from left.

Lemma 4.6 *It holds that*

$$\gamma_1 = K_{s_{10}}\alpha_3, \quad \gamma_2 = K_{s_{11}}\alpha_3, \quad s_3' = s_3 + K_{s_{12}}\alpha_3,$$

where $K_{s_{10}} \in (-1, 1)$, $K_{s_{11}}$ *and* $K_{s_{12}}$ *are bounded, depending only on the system and background solution.*

Proof As we know, it satisfies that

$$\Phi(\Phi(U_l; 0, 0, \alpha_3), 0, 0, s_3) = \Phi(U_l; \gamma_1, \gamma_2, s_3') = U_r. \tag{4.14}$$

By Lemma 2.5, one can derive that

$$\frac{\partial\Phi(U_l; \gamma_1, \gamma_2, s_3')}{\partial(\gamma_1, \gamma_2, s_3')}\Big|_{\gamma_1=\gamma_2=0, s_3'=s_{30}, U_l=U_3} = \frac{\det(Ar_1(U_3), Ar_2(U_3), A\dot{U}(s_{30}))}{\det A(U_4)} < 0.$$

From the implicit function theorem, close to the point $(s_3, \alpha_3) = (s_{30}, 0)$, there exist C^1 functions of (s_3, α_3) such that

$$\gamma_1 = \gamma_1(s_3, \alpha_3), \quad \gamma_2 = \gamma_2(s_3, \alpha_3), \quad s_3' = s_3'(s_3, \alpha_3).$$

By Taylor's expansion formula, it yields that

$$\begin{aligned}\gamma_1 &= \gamma_1(s_3,\alpha_3) - \gamma_1(s_3,0) + \gamma_1(s_3,0) = K_{s_{10}}\alpha_3,\\ \gamma_2 &= \gamma_2(s_3,\alpha_3) - \gamma_2(s_3,0) + \gamma_2(s_3,0) = K_{s_{11}}\alpha_3,\\ s_3' &= s_3'(s_3,\alpha_3) - s_3'(s_3,0) + s_3'(s_3,0) = K_{s_{12}}\alpha_3 + s_3,\end{aligned}$$

where $\gamma_1(s_3,0) = \gamma_2(s_3,0) = 0$ and $s_3'(s_3,0) = s_3$.

In the following, we will show $K_{s_{10}}$, $K_{s_{11}}$ and $K_{s_{12}}$ are bounded. We differentiate (4.14) with respect to α_3 to derive that

$$\frac{\partial U_r}{\partial \gamma_1}\frac{\partial \gamma_1}{\partial \alpha_3} + \frac{\partial U_r}{\partial \gamma_2}\frac{\partial \gamma_2}{\partial \alpha_3} + \frac{\partial U_r}{\partial s_3'}\frac{\partial s_3'}{\partial \alpha_3} = \frac{\partial U_r}{\partial \alpha_3},$$

Multiply the above equality with $A(U_r)$ from left and let $\alpha_3 = 0, \quad s_3 = s_{30},\ U_l = U_3$, then it yields that

$$A(U_3)r_1(U_3)\frac{\partial \gamma_1}{\partial \alpha_3} + A(U_3)r_2(U_3)\frac{\partial \gamma_2}{\partial \alpha_3} + (W(U_4) - W(U_3))\frac{\partial s_3'}{\partial \alpha_3} = A(U_3)r_3(U_3).$$

Therefore, one can formulate that

$$\begin{aligned}\frac{\partial \gamma_1}{\partial \alpha_3}\Big|_{\substack{\alpha_3=0,s_3=s_{30},\\ U_l=U_3}} &= \frac{\det(A(U_3)r_3(U_3), A(U_3)r_2(U_3), [W])}{\det(A(U_3)r_1(U_3), A(U_3)r_2(U_3), [W])}\\ &= -\frac{(\lambda_3(U_3) - s_{30})((\gamma-3)u_4 - (\gamma-1)u_3 + 2s_{30} - 2c_3)}{(s_{30} - \lambda_1(U_3))((\gamma-3)u_4 - (\gamma-1)u_3 + 2s_{30} + 2c_3)},\\ \frac{\partial \gamma_2}{\partial \alpha_3}\Big|_{\substack{\alpha_3=0,s_3=s_{30},\\ U_l=U_3}} &= \frac{\det(A(U_3)r_1(U_3), A(U_3)r_3(U_3), [W])}{\det(A(U_3)r_1(U_3), A(U_3)r_2(U_3), [W])}\\ &= \frac{4\rho_3(\lambda_3(U_3) - s_{30})L_4}{(\gamma+1)c_3\rho_4(s_{30} - u_3)(u_3 - u_4)((\gamma-3)u_4 - (\gamma-1)u_3 + 2s_{30} + 2c_3)},\\ \frac{\partial s_3'}{\partial \alpha_3}\Big|_{\substack{\alpha_3=0,s_3=s_{30},\\ U_l=U_3}} &= \frac{\det(A(U_3)r_1(U_3), A(U_3)r_2(U_3), A(U_3)r_3(U_3))}{\det(A(U_3)r_1(U_3), A(U_3)r_2(U_3), [W])}\\ &= \frac{8\rho_3 c_3(\lambda_3(U_3) - s_{30})}{(\gamma+1)\rho_4(u_4 - u_3)((\gamma-3)u_4 - (\gamma-1)u_3 + 2s_{30} + 2c_3)},\end{aligned}$$

where

$$\begin{aligned}&\det(Ar_1(U_3), Ar_2(U_3), A\dot{U}(s_{30}))\\ &= \frac{\rho_3\rho_4(s_{30} - u_3)(\lambda_1(U_3) - s_{30})(u_4 - u_3)}{(\gamma+1)(\gamma-1)}(2(s_{30} + c_3 - u_3) + (\gamma-3)(u_4 - u_3)) > 0,\\ &\det(Ar_3(U_3), Ar_2(U_3), A\dot{U}(s_{30}))\\ &= \frac{\rho_3\rho_4(s_{30} - u_3)(\lambda_3(U_3) - s_{30})(u_4 - u_3)((\gamma-3)u_4 - (\gamma-1)u_3 + 2s_{30} - 2c_3)}{(\gamma+1)(\gamma-1)},\\ &\det(Ar_1(U_3), Ar_3(U_3), [W]) = -\frac{4\rho_3^2(\lambda_1(U_3) - s_{30})(\lambda_3(U_3) - s_{30})}{(\gamma+1)^2(\gamma-1)c_3}L_4,\\ &\det(Ar_1(U_3), Ar_2(U_3), Ar_3(U_3)) = \frac{8\rho_3^2 c_3(s_{30} - u_3)(\lambda_1(U_3) - s_{30})(\lambda_3(U_3) - s_{30})}{(\gamma+1)^2(\gamma-1)},\\ &L_4 := \rho_4(u_4 - u_3)((\gamma-3)u_4 - (\gamma-1)u_3 + 2s_{30}) - 2c_3^2(\rho_4 - \rho_3).\end{aligned}$$

Thus, $K_{s_{10}}$, $K_{s_{11}}$ and $K_{s_{12}}$ are bounded. Since $\lambda_3(U_3) > s_{30} > \lambda_1(U_3)$ and $s_{30} > \lambda_2(U_3)$, we have

$$\begin{aligned}
&0 < \frac{u_3 + c_3 - s_{30}}{s_{30} + c_3 - u_3} < 1, \\
&\frac{(\gamma-3)u_4 - (\gamma-1)u_3 + 2s_{30} - 2c_3}{(\gamma-3)u_4 - (\gamma-1)u_3 + 2s_{30} + 2c_3} - 1 = -\frac{4c_3}{2s_{30} + 2c_3 + (\gamma-3)u_4 - (\gamma-1)u_3} < 0, \\
&\frac{(\gamma-3)u_4 - (\gamma-1)u_3 + 2s_{30} - 2c_3}{(\gamma-3)u_4 - (\gamma-1)u_3 + 2s_{30} + 2c_3} + 1 = \frac{2((\gamma-3)(u_4 - u_3) + 2s_{30} - u_3)}{2(s_{30} + c_3 - u_3) + (\gamma-3)(u_4 - u_3)} > 0,
\end{aligned}$$

which implies that $K_{s_{10}} \in (-1, 1)$. This proof of this lemma is finished.

4.7 Interaction between the strong 3-shock and the *i*-weak waves from right

Suppose that a weak i-wave α_i collides with the strong 3-shock wave s_3 from right. Denote the left and right states of the strong 3-shock by U_l and U_m, respectively. The rightmost state U_r is connected to U_m by a weak i-physical wave α_i, where $U_l \in O_\varepsilon(U_3)$, $U_m, U_r \in O_\varepsilon(U_4)$. In other words, $(U_l, U_m) = (0, 0, s_3)$ and $U_r = \Phi_i(U_m, \alpha_i)$. The generated wave fronts are respectively a weak 1-wave γ_1, a weak 2-wave γ_2 and a strong 3-shock wave s'_3 (see Figure 7). Then we have the following estimates.

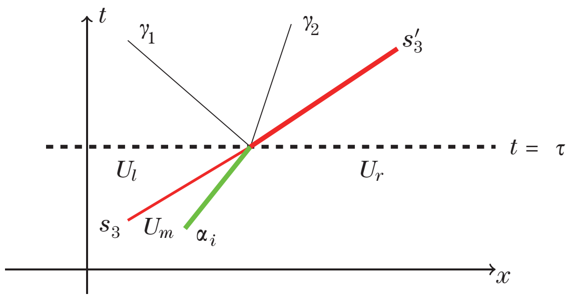

Figure 7 A weak i-wave interacts with strong 3-shock from right.

Lemma 4.7 *It holds that*

$$\gamma_1 = \widetilde{K}^i_{s_4}\alpha_i, \quad \gamma_2 = \widetilde{K}^i_{s_5}\alpha_i, \quad s'_3 = \widetilde{K}^i_{s_6}\alpha_i + s_3,$$

where $\widetilde{K}^i_{s_j}$, $j = 4, 5, 6$, *are bounded, depending only on the system and background solution.*

Proof It is easy to obtain that

$$\Phi_i(\Phi(U_l; 0, 0, s_3); \alpha_i) = \Phi(U_l; \gamma_1, \gamma_2, s'_3) = U_r.$$

Since

$$\frac{\partial \Phi(U_l; \gamma_1, \gamma_2, s'_3)}{\partial(\gamma_1, \gamma_2, s'_3)}\Big|_{\gamma_1=\gamma_2=0, s'_3=s_{30}, U_l=U_3} = \frac{\det(Ar_1(U_3), Ar_2(U_3), W(U_4) - W(U_3))}{\det A(U_4)} > 0,$$

from the theorem of the implicit function, there exist some C^1 functions of (s_3, α_i) close to the point $(s_3, \alpha_i) = (s_{30}, 0)$, such that

$$\gamma_1 = \gamma_1(s_3, \alpha_i), \quad \gamma_2 = \gamma_2(s_3, \alpha_i), \quad s_3' = s_3'(s_3, \alpha_i).$$

We employ Taylor's expansion formula to obtain that

$$\begin{aligned}\gamma_1 &= \gamma_1(s_3, \alpha_i) - \gamma_1(s_3, 0) + \gamma_1(s_3, 0) = \widetilde{K}^i_{s_4}\alpha_i,\\ \gamma_2 &= \gamma_2(s_3, \alpha_i) - \gamma_2(s_3, 0) + \gamma_2(s_3, 0) = \widetilde{K}^i_{s_5}\alpha_i,\\ s_3' &= s_3'(s_3, \alpha_i) - s_3'(s_3, 0) + s_3'(s_3, 0) = \widetilde{K}^i_{s_6}\alpha_i,\end{aligned}$$

where $\gamma_1(s_3, 0) = \gamma_2(s_3, 0) = 0$ and $s_3'(s_3, 0) = s_3$. Similarly to the proof of Lemma 4.3, we can also prove that $\widetilde{K}^i_{s_j}$, $j = 4, 5, 6$, are bounded.

4.8 Interaction between the strong 1-shock and a non-physical wave from left

Suppose that a strong 1-shock wave s_1 interacts with a non-physical wave ε_{np} from right. Assume that the leftmost state is U_l, and the strong 1-shock separates the state U_m from the state U_r. Then $\varepsilon_{np} = |U_m - U_l|$ and $U_r = \Phi_1(U_m, s_1)$. From the construction of the simplified Riemann solver, the outgoing physical wave is a strong 1-shock s_1', and a new auxiliary wave is denoted by ε_{np}', as shown in Figure 8. Meanwhile, we have the following interaction estimates.

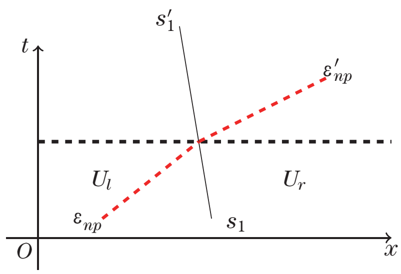

Figure 8 A non-physical wave collides with a strong 1-shock wave from left.

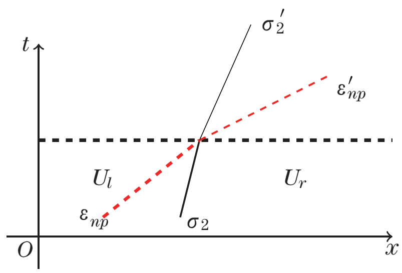

Figure 9 A non-physical wave collides with a strong 2-contact discontinuity from left.

Lemma 4.8 *It holds that*

$$s_1' = s_1, \quad \varepsilon_{np}' = K^1_{np}\varepsilon_{np},$$

where K^1_{np} is bounded, depending only on the system and background solution.

Proof From the definition of simplified Riemann solver, one has

$$s_1' = s_1,$$
$$\varepsilon_{np}' = |\Phi_1(U_l, s_1) - U_r| = |\Phi_1(U_l, s_1) - \Phi_1(U_m, s_1)| = K_p^1 |U_l - U_m| = K_p^1 \varepsilon_{np}.$$

Therefore, the boundness of K_{np}^1 follows easily, and we complete the proof of this lemma.

4.9 Interaction between a non-physical wave and the strong 2-contact discontinuity from left

Suppose that the strong 2-contact discontinuity σ_2 interacts with a non-physical wave ε_{np}, and that the state U_m is joined to the state U_r by a strong 2-contact discontinuity σ_2, i.e., $G(U_m, \sigma_2) = U_r$. Let the leftmost state be U_l, then $\varepsilon_{np} = |U_l - U_m|$. In this case, the outgoing waves are respectively a strong 2-contact discontinuity σ_2' and a non-physical wave ε_{np}', see Figure 9, and the interaction estimates are given by the following lemma.

Lemma 4.9 *It holds that*

$$\sigma_2' = \sigma_2, \quad \varepsilon_{np}' = K_{np}^2 \varepsilon_{np},$$

where K_{np}^2 is bounded, depending only on background solution.

The proof of this lemma can be found in [6, Lemma 4.4].

4.10 Interaction between the strong 3-shock and a non-physical wave from left

Suppose that a non-physical wave ε_{np} interacts with a 3-strong shock wave s_3 from the left. From the construction of simplified Riemann solver, the outgoing physical wave is a strong 3-shock wave s_3', and the auxiliary non-physical wave is denoted by ε_{np}' (see Figure 10). In a similar way to the argument of Lemma 4.8, we can draw the following interaction estimates.

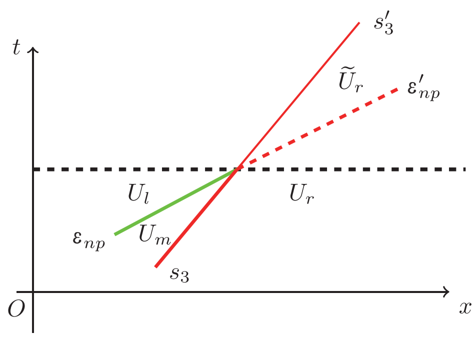

Figure 10 Non-physical wave collides with a strong 3-shock wave from left.

Lemma 4.10 *It holds that*

$$s_3' = s_3, \quad \varepsilon_{np}' = K_{np}^3 \varepsilon_{np},$$

where K_{np}^3 is bounded, depending only on the system.

Due to the construction of the approximate solutions, there is at most one of the following interactions occurring at time τ.

Case 1. Two weak waves, denoted by α_i and β_j, $1 \le i, j \le 3$, interact at time τ.

Case 2. A 1-weak wave, denoted by α_1, interacts with the strong 1-shock wave from right.

Case 3. A weak i-wave α_i collides with a strong 1-shock wave from left, $i = 1, 2, 3$.

Case 4. A weak 3-wave α_3 interacts with a strong 2-contact discontinuity from left.

Case 5. A weak 1-wave α_1 collides with a strong 2-contact discontinuity from right.

Case 6. A weak 3-wave α_3 interacts with a strong 3-shock wave from left.

Case 7. A weak i-wave α_i interacts with a strong 3-shock wave from right.

Case 8. A non-physical wave ε_{np} collides with a strong 1-shock wave from left.

Case 9. A non-physical wave ε_{np} interacts with a strong 2-contact discontinuity from left.

Case 10. A non-physical wave ε_{np} collides with a strong 3-shock wave from left.

Case 11. A non-physical wave ε_{np} interacts with a weak i-wave from left.

We denote

$$E_\nu(\tau) = \begin{cases} |\alpha_i||\beta_j|, & \text{Case 1}, \\ |\alpha_1|, & \text{Case 2 or Case 5}, \\ |\alpha_i|, \quad i = 1, 2, 3, & \text{Case 3 or Case 7}, \\ |\alpha_3|, & \text{Case 4 or Case 6}, \\ |\varepsilon_{np}|, & \text{Case 8–Case 10}, \\ |\alpha_i||\varepsilon_{np}|, & \text{Case 11}, \end{cases} \tag{4.15}$$

which measures the decreasing of the Glimm functional in Section 5.

5 Monotonicity of the Glimm Functional

In this section, we construct a new Glimm functional and prove its monotonicity based on the local wave interaction estimates in Section 4. Then the convergence of the approximate solutions is achieved by a standard procedure. Since the initial state is not a constant, we need to consider the interactions between the strong shock waves (2-contact discontinuity) and weak waves from left and right (see Lemmas 4.2–4.7) in the Glimm functional. We first define the approaching waves as follows.

Definition 5.1 (Approaching Waves)

• $(\alpha_i, \beta_j) \in \mathcal{A}_1$: *Two weak waves* α_i *and* β_j $(i, j \in \{1, 2, 3\})$ *located at points* x_{α_i} *and* x_{β_j} *respectively, with* $x_{\alpha_i} < x_{\beta_j}$, *satisfy at least one of the following conditions*:

(i) $i > j$; (ii) $i = j$ *and one of them is a shock*; (iii) α_i *is a non-physical wave.*

• $\alpha_i \in \mathcal{A}_{s_1}$: *A weak* i*-wave* α_i *is approaching a strong* 1*-shock wave if* $(x_{\alpha_i}, t_{\alpha_i}) \in \Omega_1$, $i = 1, 2, 3$ *or* $i = 1$ *and* $(x_{\alpha_1}, t_{\alpha_1}) \in \Omega_2$;

• $\alpha_i \in \mathcal{A}_{\sigma_2}$: *A weak* i*-wave* α_i *is approaching a strong* 2*-contact discontinuity if* $(x_{\alpha_i}, t_{\alpha_i}) \in \Omega_2$, $i = 3$, *or* $(x_{\alpha_i}, t_{\alpha_i}) \in \Omega_3, i = 1$;

• $\alpha_i \in \mathcal{A}_{s_3}$: *A weak* i*-wave* α_i *is approaching a strong* 3*-shock wave, if* $(x_{\alpha_i}, t_{\alpha_i}) \in \Omega_3$, $i = 3$, *or* $(x_{\alpha_i}, t_{\alpha_i}) \in \Omega_4$, $i = 1, 2, 3$, *with*

$$\Omega_1 = \{(x, t) : x < \chi_1(t), t > 0\},$$

$$\Omega_2 = \{(x,t) : \chi_1(t) < x < \chi_2(t), t > 0\},$$
$$\Omega_3 = \{(x,t) : \chi_2(t) < x < \chi_3(t), t > 0\},$$
$$\Omega_4 = \{(x,t) : x > \chi_3(t), t > 0\},$$

where the curves of the strong 1*-shock wave, strong* 2*-contact discontinuity and the strong* 3*-shock wave are respectively denoted by* $x = \chi_1(t)$*,* $x = \chi_2(t)$ *and* $x = \chi_3(t)$*.*

Remark 5.1 The approaching waves in $\mathcal{A}_1$ are in fact the original approaching waves between weak waves (see Lemma 4.1).

Now we define a weighted Glimm functional $F(t)$ as follows:

$$L_i(t) = \sum \{|\alpha_i| : \alpha_i \text{ is a weak } i\text{-physical wave}\}, \quad i = 1,2,3,$$
$$L_{np}(t) = \sum \{|\varepsilon_{np}| : \varepsilon_{np} \text{ is a non-physical wave}\}, \quad L_w(t) = \sum_{i=1}^{3} L_i(t) + L_{np}(t),$$
$$L_s(t) = \sum \big\{|s_1 - s_{10}| + |\sigma_2 - \sigma_{20}| + |s_3 - s_{30}| : s_i \text{ is the velocity of the strong } i\text{-shock and } \sigma_2 \text{ measures the strength of strong 2-contact discontinuity}\big\},$$
$$Q_0(t) = \sum \{|\alpha_i||\beta_j| : (\alpha_i, \beta_j) \in \mathcal{A}_1\}, \quad Q_{1L}(t) = \sum \{|\alpha_i| : \alpha_i \in \mathcal{A}_{s_1}, (x_{\alpha_i}, t_{\alpha_i}) \in \Omega_1\},$$
$$Q_{1R}(t) = \sum \{|\alpha_1| : \alpha_1 \in \mathcal{A}_{s_1}, (x_{\alpha_1}, t_{\alpha_1}) \in \Omega_2\},$$
$$Q_{2L}(t) = \sum \{|\alpha_3| : \alpha_3 \in \mathcal{A}_{\sigma_2}, (x_{\alpha_3}, t_{\alpha_3}) \in \Omega_2\},$$
$$Q_{2R}(t) = \sum \{|\alpha_1| : \alpha_1 \in \mathcal{A}_{\sigma_2}, (x_{\alpha_1}, t_{\alpha_1}) \in \Omega_3\},$$
$$Q_{3L}(t) = \sum \{|\alpha_3| : \alpha_3 \in \mathcal{A}_{s_3}, (x_{\alpha_3}, t_{\alpha_3}) \in \Omega_3\},$$
$$Q_{3R}(t) = \sum \{|\alpha_i| : \alpha_i \in \mathcal{A}_{s_3}, (x_{\alpha_i}, t_{\alpha_i}) \in \Omega_4\},$$
$$Q_{np}(t) = \sum \{|\varepsilon_{np}| : \varepsilon_{np} \text{ is a non-physical wave}\},$$

and

$$L(t) = L_w(t) + L_s(t),$$
$$Q(t) = K_0 Q_0(t) + K_1 Q_{1L}(t) + K_2 Q_{1R}(t) + K_3 Q_{2L}(t) + K_4 Q_{2R}(t) + K_5 Q_{3L}(t) + K_6 Q_{3R}(t) + Q_{np}(t),$$
$$F(t) = L(t) + KQ(t),$$

where K and K_0 are sufficiently large, and $K_j|_{1\le j\le 6}$ are determined by the following inequalities

$$K_3|K_{s_3}| - K_2 < -M_*, \tag{5.1}$$
$$K_3|\widetilde{K}^i_{s_3}| - K_1 < -M_*, \tag{5.2}$$
$$K_2|K_{s_4}| + K_5|K_{s_6}| - K_3 < -M_*, \tag{5.3}$$
$$K_2|K_{s_7}| + K_5|K_{s_9}| - K_4 < -M_*, \tag{5.4}$$
$$K_4|K_{s_{10}}| - K_5 < -M_*, \tag{5.5}$$
$$K_4|\widetilde{K}^i_{s_4}| - K_6 < -M_*. \tag{5.6}$$

Remark 5.2 M_* is a positive constant, depending on the background solution. Since the reflection coefficients are continuous with respect to U under some small perturbations of the background solution, (5.1)–(5.6) are still valid.

In the following, we prove that the Glimm functional is decreasing based on the local interaction estimates. Before the interaction time τ, we give the inductive hypotheses:

$\mathrm{A}_1(\tau-)$: Before τ, there exist a strong 1-shock wave $s_1^{(k)}$, a strong 2-contact discontinuity $\sigma_2^{(k)}$ and a strong 3-shock wave $s_3^{(k)}$, satisfying $s_1^{(k)} \in O_\varepsilon(s_{10})$, $\sigma_2^{(k)} \in O_\varepsilon(\sigma_{20})$ and $s_3^{(k)} \in O_\varepsilon(s_{30})$, whose curves are respectively denoted by $\chi_1^{(k)}(t)$, $\chi_2^{(k)}(t)$ and $\chi_3^{(k)}(t)$, which divide the whole domain into four regions: $\Omega_1^{(k)}$, $\Omega_2^{(k)}$, $\Omega_3^{(k)}$ and $\Omega_4^{(k)}$, where

$$\begin{aligned}
\Omega_1^{(k)} &= \{(x,t)|x < \chi_1^{(k)}(t), \tau_* < t < \tau\},\\
\Omega_2^{(k)} &= \{(x,t)|\chi_1^{(k)}(t) < x < \chi_2^{(k)}(t), \tau_* < t < \tau\},\\
\Omega_3^{(k)} &= \{(x,t)|\chi_2^{(k)}(t) < x < \chi_3^{(k)}(t), \tau_* < t < \tau\},\\
\Omega_4^{(k)} &= \{(x,t)|x > \chi_3^{(k)}(t), \tau_* < t < \tau\}.
\end{aligned}$$

$\mathrm{A}_2(\tau-)$: Before τ, $U^{(k)}(x,t)|_{\Omega_1^{(k)}} \in O_\varepsilon(U_1)$, $U^{(k)}(x,t)|_{\Omega_2^{(k)}} \in O_\varepsilon(U_2)$, $U^{(k)}(x,t)|_{\Omega_3^{(k)}} \in O_\varepsilon(U_3)$ and $U^{(k)}(x,t)|_{\Omega_4^{(k)}} \in O_\varepsilon(U_4)$.

Remark 5.3 Suppose that two waves interact for the k-th time at time τ, and let τ_* be the last interaction time to τ.

Theorem 5.1 *Suppose that* $\mathrm{A}_1(\tau-)$ *and* $\mathrm{A}_2(\tau-)$ *hold for any interaction time* τ*. Then, there exists a positive constants* δ_1 *such that if*

$$F(\tau-) < \delta_1, \tag{5.7}$$

then,

$$Q(\tau+) - Q(\tau-) < -\frac{1}{2}E_\nu(\tau) \tag{5.8}$$

and

$$F(\tau+) - F(\tau-) \le -\frac{1}{4}E_\nu(\tau). \tag{5.9}$$

Proof Based on all the possible local interaction estimates, the proof can be divided into the following several cases.

Case 1. Interaction between the weak waves.

Suppose that the two weak waves α_i and β_j interact at time $t = \tau$. If the accurate Riemann solver is adopted, then the generated fronts are physical waves, denoted by γ_l, $l = 1,2,3$, respectively. If the simplified Riemann solver is adopted, then the auxiliary non-physical wave is denoted by ε_{np}. Based on Lemma 4.1, we have

$$\begin{aligned}
&\sum_{l=1}^{3} L_l(\tau+) - L_l(\tau-) = O(1)|\alpha_i||\beta_j|, \quad L_{np}(\tau+) - L_{np}(\tau-) = O(1)|\alpha_i||\beta_j|,\\
&L_w(\tau+) - L_w(\tau-) = O(1)|\alpha_i||\beta_j|, \quad L_s(\tau+) - L_s(\tau-) = 0,\\
&F(\tau+) - F(\tau-) = K_0(O(1)|\alpha_i||\beta_j|L_\omega(\tau-) - |\alpha_i||\beta_j|) + O(1)|\alpha_i|\beta_j|.
\end{aligned}$$

When $L_w(\tau-)$ is sufficiently small, one can choose K_0 and K large enough such that

$$F(\tau+) - F(\tau-) \le -\frac{1}{4}|\alpha_i||\beta_j|.$$

Case 2. Interaction between the strong 1-shock and the weak 1-wave from right.

Based on Lemma 4.2, one has

$$\begin{aligned}
&L_1(\tau+) - L_1(\tau-) = -|\alpha_1|, \quad \sum_{i=2}^{3} L_i(\tau+) - \sum_{i=2}^{3} L_i(\tau-) = (|K_{s_2}| + |K_{s_3}|)|\alpha_1|,\\
&L_{np}(\tau+) - L_{np}(\tau-) = 0, \quad L_s(\tau+) - L_s(\tau-) \le |s_1' - s_1| = |K_{s_1}||\alpha_1|,\\
&Q(\tau+) - Q(\tau-) = K_0(|\gamma_2| + |\gamma_3| - |\alpha_1|)L_w(\tau-) - K_2|\alpha_1| + K_3|\gamma_3|\\
&\qquad \le O(1)|\alpha_1|L_w(\tau-) + (K_3|K_{s_3}| - K_2)|\alpha_1|,
\end{aligned}$$

provided that $L_w(\tau-)$ is sufficiently small and (5.1) holds. It satisfies

$$Q(\tau+) - Q(\tau-) \le -\frac{1}{2}|\alpha_1|.$$

When K is chosen suitably large, we have

$$Q(\tau+) - Q(\tau-) \le -\frac{1}{4}|\alpha_1|.$$

Case 3. Interaction between the strong 1-shock wave and the weak i-waves from left.

Based on Lemma 4.3, one has

$$\begin{aligned}
&\sum_{i=1}^{3} L_i(\tau+) - \sum_{i=1}^{3} L_i(\tau-) = (|\widetilde{K}^i_{s_2}| + |\widetilde{K}^i_{s_3}| - 1)|\alpha_i|,\\
&L_{np}(\tau+) - L_{np}(\tau-) = 0, \quad L_s(\tau+) - L_s(\tau-) \le |s_1' - s_1| = |\widetilde{K}^i_{s_1}||\alpha_i|,\\
&Q(\tau+) - Q(\tau-)\\
&= K_0(|\gamma_2| + |\gamma_3| - |\alpha_i|)L_w(\tau-) - K_1|\alpha_i| + K_3|\gamma_3|\\
&\le O(1)|\alpha_i|L_w(\tau-) + (K_3|\widetilde{K}^i_{s_3}| - K_1)|\alpha_i|.
\end{aligned}$$

Therefore, when $L_w(\tau-)$ is sufficiently small and K_1, K_3 are chosen by (5.2), it yields

$$Q(\tau+) - Q(\tau-) < -\frac{1}{2}|\alpha_i|.$$

So, when K is suitably large, one can obtain that

$$F(\tau+) - F(\tau-) < -\frac{1}{4}|\alpha_i|.$$

Case 4. Interaction between the strong 2-contact discontinuity and the weak 3-wave from left.

Based on Lemma 4.4, it satisfies

$$\begin{aligned}
&L_1(\tau+) - L_1(\tau-) = |\gamma_1| = |K_{s_4}||\alpha_3|, \quad L_2(\tau+) - L_2(\tau-) = 0,\\
&L_3(\tau+) - L_3(\tau-) = (|K_{s_6}| - 1)|\alpha_3|, \quad L_{np}(\tau+) - L_{np}(\tau-) = 0,
\end{aligned}$$

$$
\begin{aligned}
L_s(\tau+)-L_s(\tau-) &= |\sigma_2'-\sigma_{20}|-|\sigma_2-\sigma_{20}| \le |\sigma_2'-\sigma_2| = |K_{s_5}||\alpha_3|,\\
Q(\tau+)-Q(\tau-) &= K_0(|\gamma_1|+|\gamma_3|-|\alpha_3|)L_w(\tau-)+K_2|\gamma_1|+K_5|\gamma_3|-K_3|\alpha_3|\\
&\le O(1)|\alpha_3|L_w(\tau-)+(K_2|K_{s_4}|+K_5|K_{s_6}|-K_3)|\alpha_3|.
\end{aligned}
$$

When $L_w(\tau-)$ is sufficiently small and K_2, K_3, K_5 satisfy (5.3), one has

$$
Q(\tau+)-Q(\tau-) \le -\frac{1}{2}|\alpha_3|. \tag{5.10}
$$

Thus, when K is suitably large, it yields

$$
F(\tau+)-F(\tau-) \le -\frac{1}{4}|\alpha_3|.
$$

Case 5. Interaction between the strong 2-contact discontinuity and the weak 1-wave from right.

Based on Lemma 4.5, one can obtain

$$
\begin{aligned}
&L_1(\tau+)-L_1(\tau-) = (|K_{s_7}|-1)|\alpha_1|, \quad L_2(\tau+)-L_2(\tau-) = L_{np}(\tau+)-L_{np}(\tau-) = 0,\\
&L_3(\tau+)-L_3(\tau-) = |K_{s_9}||\alpha_1|, \quad L_s(\tau+)-L_s(\tau-) = |K_{s_8}||\alpha_1|,\\
&\quad Q(\tau+)-Q(\tau-) = K_0(|\gamma_1|+|\gamma_3|-|\alpha_1|)L_w(\tau-)+K_2|\gamma_1|+K_5|\gamma_3|-K_4|\alpha_1|\\
&\qquad\qquad\qquad\qquad \le O(1)|\alpha_1|L_w(\tau-)+(K_2|K_{s_7}|+K_5|K_{s_9}|-K_4)|\alpha_1|.
\end{aligned}
$$

Therefore, when $L_w(\tau-)$ is sufficiently small and (5.4) holds, one can derive

$$
F(\tau+)-F(\tau-) \le -\frac{1}{4}|\alpha_1|,
$$

provided that K is sufficiently large.

Case 6. Interaction between the strong 3-shock and the weak 3-wave from left.

Based on Lemma 4.6, one can obtain

$$
\begin{aligned}
&L_1(\tau+)-L_1(\tau-) = |\gamma_1| = |K_{s_{10}}||\alpha_3|, \quad L_2(\tau+)-L_2(\tau-) = |\gamma_2| = |K_{s_{11}}||\alpha_3|,\\
&L_3(\tau+)-L_3(\tau-) = -|\alpha_3|, \quad L_{np}(\tau+)-L_{np}(\tau-) = 0,\\
&L_s(\tau+)-L_s(\tau-) = |s_3'-s_{30}|-|s_3-s_{30}| \le |s_3'-s_3| = |K_{s_{12}}||\alpha_3|,\\
&\quad Q(\tau+)-Q(\tau-) = K_0(|\gamma_1|+|\gamma_2|-|\alpha_3|)L_w(\tau-)+K_4|\gamma_1|-K_5|\alpha_3|\\
&\qquad\qquad\qquad\qquad \le O(1)|\alpha_3|L_w(\tau-)+(K_4|K_{s_{10}}|-K_5)|\alpha_3|.
\end{aligned}
$$

If $L_w(\tau-)$ is sufficiently small and (5.5) holds, then

$$
Q(\tau+)-Q(\tau-) \le -\frac{1}{2}|\alpha_3|.
$$

When K is suitably large, it satisfies

$$
F(\tau+)-F(\tau-) \le -\frac{1}{4}|\alpha_3|.
$$

Case 7. Interaction between the strong 3-shock and a weak i-wave from right.

Based on Lemma 4.7, one can obtain

$$\begin{aligned}
&\sum_{i=1}^{3} L_i(\tau+) - \sum_{i=1}^{3} L_i(\tau-) = |\gamma_1| + |\gamma_2| - |\alpha_i| = (|\widetilde{K}_{s_4}^i| + |\widetilde{K}_{s_5}^i| - 1)|\alpha_i|, \\
&L_{np}(\tau+) - L_{np}(\tau-) = 0, \quad L_s(\tau+) - L_s(\tau-) \leq |\widetilde{K}_{s_6}^i||\alpha_i|, \\
&Q(\tau+) - Q(\tau-) = K_0(|\gamma_1| + |\gamma_2| - |\alpha_i|)L_w(\tau-) + K_4|\gamma_1| - K_6|\alpha_i| \\
&\qquad\qquad\qquad \leq O(1)|\alpha_i|L_w(\tau-) + (K_4|\widetilde{K}_{s_4}^i| - K_6)|\alpha_i|.
\end{aligned}$$

When $L_w(\tau-)$ is sufficiently small and (5.6) holds, it yields

$$Q(\tau+) - Q(\tau-) \leq -\frac{1}{2}|\alpha_i|.$$

If K is suitably large, then it satisfies

$$F(\tau+) - F(\tau-) \leq -\frac{1}{4}|\alpha_i|.$$

Case 8. Interaction between the strong 1-shock wave and a non-physical wave from left.

Based on Lemma 4.8, one has

$$\begin{aligned}
&L_i(\tau+) - L_i(\tau-) = L_s(\tau+) - L_s(\tau-) = 0, \quad i = 1, 2, 3, \\
&L_{np}(\tau+) - L_{np}(\tau-) = (|K_{np}^1| - 1)|\varepsilon_{np}|, \quad Q(\tau+) - Q(\tau-) = -|\varepsilon_{np}|.
\end{aligned}$$

When K is suitably large, it holds

$$F(\tau+) - F(\tau-) \leq -\frac{1}{4}|\varepsilon_{np}|.$$

Case 9. Interaction between the strong 2-contact discontinuity and a non-physical wave from left.

Based on Lemma 4.9, one has

$$\begin{aligned}
&L_i(\tau+) - L_i(\tau-) = L_s(\tau+) - L_s(\tau-) = 0, \quad i = 1, 2, 3, \\
&L_{np}(\tau+) - L_{np}(\tau-) = (|K_{np}^2| - 1)|\varepsilon_{np}|, \quad Q(\tau+) - Q(\tau-) = -|\varepsilon_{np}|.
\end{aligned}$$

When K is suitably large, it holds

$$F(\tau+) - F(\tau-) \leq -\frac{1}{4}|\varepsilon_{np}|.$$

Case 10. Interaction between the strong 3-shock wave and a non-physical wave from left.

Based on Lemma 4.10, it holds

$$\begin{aligned}
&L_i(\tau+) - L_i(\tau-) = L_s(\tau+) - L_s(\tau-) = 0, \quad i = 1, 2, 3, \\
&L_{np}(\tau+) - L_{np}(\tau-) = (|K_{np}^3| - 1)|\varepsilon_{np}|, \quad Q(\tau+) - Q(\tau-) = -|\varepsilon_{np}|.
\end{aligned}$$

When K is suitably large, it yields

$$F(\tau+) - F(\tau-) \leq -\frac{1}{4}|\varepsilon_{np}|.$$

Case 11. Interaction between a weak i-wave and a non-physical wave from left.

As shown in Bressan [1], when a weak i-wave α_i interacts with a non-physical wave ε_{np}, the generated physical i-wave and non-physical wave are respectively denoted by γ_i and ε'_{np}, it holds

$$\gamma_i = \alpha_i, \quad \varepsilon'_{np} = \varepsilon_{np} + O(1)|\alpha_i||\varepsilon_{np}|.$$

Then we can derive

$$\begin{aligned} L_i(\tau+) - L_i(\tau-) &= L_s(\tau+) - L_s(\tau-) = 0, \quad i = 1, 2, 3, \\ L_{np}(\tau+) - L_{np}(\tau-) &= O(1)|\alpha_i||\varepsilon_{np}|, \\ Q(\tau+) - Q(\tau-) &= K_0 Q_0(\tau+) + |\varepsilon'_{np}| - K_0 Q_0(\tau-) - |\varepsilon_{np}| \\ &= -K_0|\alpha_i||\varepsilon_{np}| + O(1)|\alpha_i||\varepsilon_{np}|. \end{aligned}$$

Thus, when K_0 and K are suitably large, it holds

$$Q(\tau+) - Q(\tau-) \le -\frac{1}{2}|\alpha_i||\varepsilon_{np}|$$

and

$$F(\tau+) - F(\tau-) \le -\frac{1}{4}|\alpha_i||\varepsilon_{np}|.$$

In conclusion, we complete the proof of this theorem.

From Theorem 5.1, we can conclude the following proposition.

Proposition 5.1 *Under the assumptions* $\mathrm{A}_1(\tau-)$ *and* $\mathrm{A}_2(\tau-)$ *given by Theorem* 5.1*, if* $F(\tau-) < \delta_1$*, then there exists a positive constant* $\widehat{\varepsilon}$ *such that*

$$\begin{aligned} &|U^{(k+1)}(x,t)|_{\Omega_1^{(k+1)}} - U_1| < \varepsilon, \quad |U^{(k+1)}(x,t)|_{\Omega_2^{(k+1)}} - U_2| < \varepsilon, \\ &|U^{(k+1)}(x,t)|_{\Omega_3^{(k+1)}} - U_3| < \varepsilon, \quad |U^{(k+1)}(x,t)|_{\Omega_4^{(k+1)}} - U_4| < \varepsilon, \\ &|s_1^{(k+1)} - s_{10}| < \widehat{\varepsilon}(\varepsilon), \quad |\sigma_2^{(k+1)} - \sigma_{20}| < \widehat{\varepsilon}(\varepsilon), \\ &|s_3^{(k+1)} - s_{30}| < \widehat{\varepsilon}(\varepsilon). \end{aligned}$$

6 Estimates on the Approximate Strong Fronts

In this section, we make some further estimates to study the total strength of the strong wave fronts of the approximate solution $U^\nu(x,t)$. First, we give the estimates for $E_\nu(\tau)$.

Lemma 6.1 *There exists a positive constant* M_1*, independent of* ν *and* $U^\nu(x,t)$*, such that*

$$\sum_{\tau>0} E_\nu(\tau) < M_1, \tag{6.1}$$

where the summation is taken over all interaction times and $E_\nu(\tau)$ *is defined by* (4.15).

Proof From Theorem 5.1, we know that for any $\tau \in (\tau_{k-1}, \tau_{k+1}),\ k \ge 1$, it holds that

$$F(\tau_{k+1}-) - F(\tau_{k-1}+) \le -\frac{1}{4}\sum_{\tau_{k-1}}^{\tau_{k+1}} E_\nu(\tau). \tag{6.2}$$

Then, summing (6.2) with respect to k, we have

$$\sum_{k\geq 1}\sum_{\tau_{k-1}}^{\tau_{k+1}} E_\nu(\tau) \leq 4\sum_{k\geq 1}(F(\tau_k-) - F(\tau_k+)) \leq 4F(0+) < \infty.$$

Hence, the proof of this lemma is completed.

Next, we will present some estimates of the strong wave fronts as follows.

Lemma 6.2 *There exists a positive constant M_2, independent of ν and $U^\nu(x,t)$, such that*

$$\mathrm{TV}.\{s_j(\cdot) : t \in [0,\infty)\} = \sum_{k\geq 1} |s_j^{(k+1)} - s_j^{(k)}| \leq M_2, \quad j = 1, 3. \tag{6.3}$$

Proof From the local wave interaction estimates involving the strong shock waves as shown in 5, it holds that

$$\sum_{k\geq 1} |s_j^{(k+1)} - s_j^{(k)}| \leq M \sum_{k\geq 1} E_\nu(\tau) \leq M_2,$$

where M and M_2 are positive constants, depending on the system, independent of ν and $U^\nu(x,t)$.

Finally, by Theorem 5.1 and Lemma 6.1, we have the following lemma.

Lemma 6.3 *There exists a positive constant M_3, independent of ν and $U^\nu(x,t)$, such that*

$$\mathrm{TV}.\{\sigma_2^{(k)}(\cdot) : t \in [0,\infty)\} = \sum_{k\geq 1} |\sigma_2^{(k+1)} - \sigma_2^{(k)}| \leq M_3. \tag{6.4}$$

By Theorem 5.1, the proof of the convergence of the approximate solution $U^\nu(x,t)$ is a standard procedure, also see [1, 3]. The proof of Theorem 1.1 is completed.

Finally, we can make the conclusion of this article as follows.

Theorem 6.1 *Under the assumptions of the main theorem, there exist a subsequence $\{U^\nu(x,t)\}$, a* BV *function $U(x,t)$ and $\chi_j(t) \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R})$, satisfying $\chi_j(0) = 0$, $j = 1, 2, 3$, such that*

(i) *$U^\nu(x,t)$ converges to $U(x,t)$ a.e. in $\mathbb{R}_+^2$, and the limit $U(x,t)$ is a global entropy solution to system* (1.1) *and* (1.4);

(ii) *$\chi_j^{(k)}(t)$ converges to $\chi_j(t)$ uniformly in any bounded t interval, $j = 1, 2, 3$;*

(iii) *$s_j^{(k)}$ converges to $s_j \in \mathrm{BV}(\mathbb{R}_+)$ a.e., and $\chi_j(t) = \int_0^t s_j(s)\mathrm{d}s$, $j = 1, 3$.*

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