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Ricci-Bourguignon Flow on Manifolds with Boundary

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Abstract The authors consider the short time existence for Ricci-Bourguignon flow on manifolds with boundary. If the initial metric has constant mean curvature and satisfies some compatibility conditions, they show the short time existence of the Ricci-Bourguignon flow with constant mean curvature on the boundary.

Keywords Ricci-Bourguignon flow, Boundary value problem
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1 Introduction

The Ricci-Bourguignon flow is

$$\frac{\partial}{\partial t}g(x,t) = -2\mathrm{Ric}(x,t) + 2\rho R(x,t)g(x,t), \quad (x,t) \in M \times [0,T), \tag{1.1}$$

where Ric is the Ricci tensor of the manifold, R is the Scalar curvature and ρ is a constant. This flow, which is a generalization of the Ricci flow, was introduced by Bourguignon [2]. For the study of the Ricci-Bourguignon flow, see [4–6, 11]. Catino et. al. [3] proved the short-time existence of solutions to the Ricci Bourguinon flow on closed manifolds.

There are plenty of works on the geometric flows on compact manifolds with boundary. Hamilton [7] showed the short time existence to the harmonic map heat flow from manifolds with Dirichlet, Neumann and mixed boundary by inverse theorem. Shen [13] proved the short time existence of the Ricci flow on compact manifolds with umbilic boundary. Later, Pulemotov [12] obtained a short time existence for Ricci flow on compact manifolds with boundary of constant mean curvature. Gianniostis [9] derived the short-time existence and uniqueness of the Ricci flow prescribing the mean curvature and conformal class of the boundary.

Inspired by the previous works, we attempt to study the corresponding existence problems for the Ricci-Bourguignon flow. We obtain the following short time existence for the Ricci-Bourguignon flow on compact manifolds with boundary.

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Theorem 1.1 *Let (M, g_0) be a Riemannian manifold with constant mean curvature H_0 on the boundary. Suppose that $\mu(t)$ is a smooth real value function on $[0, \infty)$ with $\mu(0) = 1$ and $g_0 \in C^{4+\tilde{\alpha}}(M)$. In addition on $\partial M \times \{0\}$, the metric g_0 satisfies the compatibility conditions*

$$\mathrm{Ric}(g_0)_{\beta n} = 0, \tag{1.2}$$

where β denotes the tangent direction and n denotes the normal direction with respect to metric g_0. Then for $\rho < \frac{1}{2(n-1)}$, there exists a Ricci-Bourguignon flow $g(t) \in C^{2+\tilde{\alpha}, \frac{2+\tilde{\alpha}}{2}}(M \times [0, T))$ such that the mean curvature $H(x, t)$ satisfies the boundary condition

$$H(x, t) = \mu(t) H_0 \tag{1.3}$$

for all $(x, t) \in \partial M \times [0, T]$ and $g(t)$ converges to g_0 in the geometric $C^{2+\tilde{\alpha}}(\overline{M})$ sense as $t \to 0$.

Remark 1.1 When $\rho \equiv 0$, the Ricci-Bourguignon flow becomes the usual Ricci flow. Hence the above Theorem 1.1 generalizes a result in [12].

In [12], the $\mathcal{W}_q^I$-estimate (see [12, Lemma 2.6]) plays a very important role in the proof of the short time existence of the Ricci flow. Therefore it is natural to ask whether a $\mathcal{W}_q^I$-estimate holds for the Ricci Bourguignon flow. However, the case of the Ricci-Bourguignon flow is harder to deal with than the Ricci flow since we now have an additional term Rg. And unfortunately we could not apply the theorem in [12] to the Ricci-Bourguignon flow on manifolds with boundary. Instead, we show the short time existence of the DeTurck Ricci-Bourguignon flow by inverse function theorem and obtain the short time existence of the Ricci-Bourguignon flow by DeTurck's trick (see Section 3 for details). The precise statement of the short time existence for the DeTurck Ricci Bourguignon flow on compact manifolds with boundary is as follows.

Theorem 1.2 *Let $(M^n, g(0))$ be a Riemannian manifold with boundary. Consider an arbitrary family of background metrics $\widetilde{g} \in C^\infty(M \times [0, \infty))$ that satisfies the zeroth-order compatibility condition $\widetilde{g}(0) = g(x, 0)$. Then for $\rho < \frac{1}{2(n-1)}$ there exists a solution $g(t)$, $t \in [0, T]$ for the DeTurck Ricci-Bourguignon equation*

$$\partial_t g = -2\mathrm{Ric} + 2\rho R g + L_{W(g,\widetilde{g})} g, \tag{1.4}$$

with the boundary conditions

$$\text{on } \partial M \begin{cases} W(g, \widetilde{g})_n = 0, \\ g_{\alpha n} = 0, \\ A_{\alpha\beta} = \dfrac{1}{2}\mu(t)(g_{\alpha\gamma}(x,t) g_0^{\gamma\sigma} A(g_0)_{\sigma\beta}(x) + g_{\beta\gamma}(x,t) g_0^{\gamma\sigma}(x) A(g_0)_{\sigma\alpha}(x)), \end{cases} \tag{1.5}$$

where $W(g, \widetilde{g})_l = g_{lr} g^{pq} (\Gamma(g)_{pq}^r - \Gamma(\widetilde{g})_{pq}^r)$. $A_{\alpha\beta}$ is the second fundamental form on the boundary ∂M and $L_{W(g,\widetilde{g})} g$ is the Lie derivative along the vector field W. The solution is C^∞ on $\overline{M}_T - \partial M \times \{0\}$, and is $C^{2+\tilde{\alpha}, \frac{2+\tilde{\alpha}}{2}}(\overline{M} \times [0, T))$ if the $g(0)$ satisfies the compatibility conditions (1.2) and $\mu(0) = 1$.

The organization of this paper is as follows. In Section 3, we introduce the DeTurck Ricci-Bourguignon flow and show the relationship between the Ricci Bourguinon flow and the DeTurck Ricci Bourguinon flow. In Section 4, the solvability of a linear parabolic initial boundary value problem is obtained. In Section 5, by classic inverse function theorem, we prove the short time existence of the DeTurck Ricci-Bourguignon flow on the compact manifold with boundary.

2 Notation

In the following, we use Greek indices for the directions tangent to the boundary and n for the direction of the inner unit normal vector with respect to the metric $g(0)$. We use $\mathcal{T}$ for the symmetric $(0,2)$ tensors on M and $\mathcal{T}_{\partial M}$ for the restriction of the bundle $\mathcal{T}$ to ∂M. Let $\mathcal{F}$ denote the subbundle of $\mathcal{T}_{\partial M}$ consisting of all $\eta \in \mathcal{T}_{\partial M}$ such that $\eta_{\alpha\beta} = 0$ for $\alpha, \beta = 1 \cdots n-1$ and $\eta_{nn} = 0$. Let $\mathcal{F}^{\perp}$ denote the orthogonal complement of $\mathcal{F}$ with respect to the metric $g(0)$. $Pr_{\mathcal{F}}$ is the orthogonal projection on the subbundle $\mathcal{F}$. We use $a * b$ to denote the linear combination of the tensors a and b. M_T denotes $M \times [0, T)$.

3 The DeTurck Ricci-Bourguignon Flow

In this section, we consider the relationship between DeTurck Ricci Bourguignon flow and the Ricci-Bourguignon flow. The DeTurck Ricci Bourguignon flow is

$$\frac{\partial}{\partial t} g = -2Rc + 2\rho R g + L_{W(g,t)} g, \tag{3.1}$$

where $W(g(t),t)_l = g(t)_{lr} g(t)^{pq} (\Gamma(g(t))^r_{pq} - \Gamma(\widetilde{g}(t))^r_{pq})$. In this paper, $\widetilde{g} \in C^{\infty}(M \times [0,\infty))$ is a family of smooth background metrics that satisfies the zeroth-order compatibility condition $\widetilde{g}(0) = g(x,0)$. Suppose that $g(t)$ is a solution to the DeTurck Ricci-Bourguignon flow with boundary condition

$$\text{on } \partial M \begin{cases} Pr_{\mathcal{F}} g(x,t) = 0, \\ A_{\alpha\beta}(g(t)) = \dfrac{\mu(t)}{2} (g(t)_{\alpha\gamma} g_0^{\gamma\eta} A^0_{\eta\beta} + g(t)_{\beta\gamma} g_0^{\gamma\eta} A^0_{\eta\alpha}), \\ W(g(t))_n = 0. \end{cases} \tag{3.2}$$

Since $Pr_{\mathcal{F}} g(x,t) = 0$, we have $g(x,t)_{\alpha n} = 0$. Hence on the boundary, the inverse matrix of g_{ij} is

$$g^{-1} = \begin{pmatrix} g^{\alpha\beta} & 0 \\ 0 & g^{nn} \end{pmatrix}.$$

So the mean curvature is

$$H_{g(t)}(x,t) = g(t)^{\alpha\beta} A(g(t))_{\alpha\beta} = \mu(t) H_{g_0} = \mu(t) H_0.$$

By the theory of ordinary differential equation, there is a one-parameter transformation $\phi(t) : M \to M$ satisfying

$$\frac{\mathrm{d}\phi(t,x)}{\mathrm{d}t} = -W(t,x)$$

with initial condition $\phi(0,x)=x$. On the boundary, since $W(g(t))_n=0$, we have $\phi(t):\partial M\to\partial M$. Since $g(t)$ is a solution of the DeTurck Ricci-Bourguignon flow, $\phi^*(t)(g(t))$ satisfies the Ricci-Bourguignon equation

$$\begin{aligned}\frac{\partial\phi(t)^*(g(t))}{\partial t}&=\phi(t)^*\Big(\frac{\partial g(t)}{\partial t}\Big)+\phi(t)^*(L_{-W}g(t))\\&=-2\mathrm{Ric}(\phi(t)^*(g(t)))+2\rho R(\phi(t)^*(g(t))).\end{aligned}$$

The mean curvature on the boundary of the metric $\phi^*(g(t))$ is

$$H_{\phi^*(t)(g(t))}(x)=H_{g(t)}(\phi(t)(x))=\mu(t)H_0.$$

So if $g(t)$ is a solution to the DeTurck Ricci-Bourguignon flow with the boundary condition (3.2), then $\phi(t)^*(g(t))$ is a solution to the Ricci-Bourguignon flow with constant mean curvature. As in [12], the boundary condition (3.2) is equivalent to

$$\begin{cases}Pr_{\mathcal{F}}g(t)=0,\\Pr_{\mathcal{F}^\perp}(g^{nn}(x,t)(g(0)_{nn})^{\frac12}\nabla_n^{g(0)}g(x,t))-\zeta(g(x,t))=0,\end{cases}\tag{3.3}$$

where ζ is a symmetric $(0,2)$-tensor

$$\begin{aligned}\zeta_{\alpha\beta}(g(x,t),t)&=-\mu(t)(g(0)_{nn}g^{nn}(x,t))^{\frac12}(g_{\alpha\gamma}(x,t)g(0)^{\gamma\sigma}(x)A^0_{\sigma\beta}(x)+g_{\beta\gamma}(x,t)g(0)^{\gamma\sigma}A^0_{\sigma\alpha})\\&\quad+g(0)_{nn}g^{nn}(x,t)(g_{\alpha\gamma}(x,t)g(0)^{\gamma\sigma}(x)A^0_{\sigma\beta}+g_{\beta\gamma}(x,t)g(0)^{\gamma\sigma}A^0_{\sigma\alpha}),\\\zeta_{nn}(g(x,t),t)&=-2g_{nn}(x,t)(\mu(t)(g(0)_{nn}g^{nn}(x,t))^{\frac12}H(x,0)\\&\quad+g(0)^{\frac12}_{nn}g^{nn}(x,t)(\Gamma^n_{nn}(g_0)-\Gamma^n_{nn}(\widetilde{g}(t)))-g^{\alpha\beta}(t)g(0)^{\frac12}_{nn}\Gamma^n_{\alpha\beta}(\widetilde{g}(t))),\end{aligned}$$

and $\zeta_{\alpha n}(g(x,t))=0,\ x\in\partial M,\ t\in[0,T)$.

In the following, we only consider the DeTurck Ricci-Bourguignon flow (3.1) with the boundary condition (3.2).

4 A Linear Parabolic PDE with Initial Boundary Value Problem

In this section, we consider the existence of the linearized DeTurck Ricci-Bourguignon flow on manifold with boundary. The main theorem is in the following.

Theorem 4.1 *Consider the following linear parabolic initial boundary value problem on symmetric* 2 *tensors on* M*,*

$$\begin{aligned}\mathcal{L}(g(t))u_{ik}=\frac{\partial u_{ik}}{\partial t}&-\Delta u_{ik}+2\rho(\Delta(tru)-\nabla^s\nabla^tu_{st})g_{ik}+M_1(g,\partial_ig)*\nabla u(x,t)\\&+M_2(g,\partial_ig,\partial^2_{ij}g)*u(x,t)=F(x,t),\quad(x,t)\in M\times[0,T]\end{aligned}$$

$$\text{on }\partial M\begin{cases}Pr_{\mathcal{F}}u(x,t)=Pr_{\mathcal{F}}b(x,t),\\Pr_{\mathcal{F}^\perp}(g^{nn}(x,t)(g(0)_{nn})^{\frac12}\nabla_n^{g(0)}u(x,t))+M_3(g,\partial_ig)*u(x,t)=Pr_{\mathcal{F}^\perp}b(x,t).\end{cases}$$

$$u(x,0)=0,\quad x\in M,$$

where $F(x,t)\in C^{l+\tilde\alpha,\frac{l+\tilde\alpha}{2}}(M\times[0,T])$*,* $Pr_{\mathcal F}b(x,t)\in C^{l+2+\tilde\alpha,\frac{l+2+\tilde\alpha}{2}}(\partial M\times[0,T])$*,* $Pr_{\mathcal F^\perp}b(x,t)\in C^{l+1+\tilde\alpha,\frac{l+1+\tilde\alpha}{2}}(\partial M\times[0,T])$ *and* $g(x,t)\in C^{l+2+\tilde\alpha,\frac{l+2+\tilde\alpha}{2}}(M\times[0,T])$*. Suppose* $F(x,t)$ *and* $b(x,t)$ *satisfying the compatibility conditions that are necessary for the existence of a solution from the class* $C^{l+2+\tilde\alpha,\frac{l+2+\tilde\alpha}{2}}(M\times[0,T])$*. Then the initial boundary value problem has a unique solution* $u\in C^{2+l+\tilde\alpha,\frac{2+l+\tilde\alpha}{2}}(M\times[0,T])$ *which satisfies the estimate*

$$\begin{aligned}|u|_{C^{2+l+\tilde\alpha,2+l+\frac{\tilde\alpha}{2}}(M_T)}\le C(&|F|_{C^{l+\tilde\alpha,l+\frac{\tilde\alpha}{2}}(M_T)}+|Pr_{\mathcal F}b(x,t)|_{C^{l+2+\tilde\alpha,\frac{l+2+\tilde\alpha}{2}}(\partial M_T)}\\&+|Pr_{\mathcal F^\perp}b(x,t)|_{C^{l+1+\tilde\alpha,\frac{l+1+\tilde\alpha}{2}}(\partial M_T)}).\end{aligned}\tag{4.1}$$

Proof The proof is based on Theorem 10.1 in Chapter VII of [10] also see [14]. We only need to show that the boundary conditions satisfy the complementing conditions (see in of the book [10, Chapter VII, p. 611]). We fix a point (x_0,t_0) on the boundary $\partial M\times[0,T]$, and choose a coordinate $\{x_i\}$ such that $g_{ij}(x_0,t_0)=\delta_{ij}$, where $\partial_\alpha\in T_p\partial M,\ \alpha\in\{1,\cdots,n-1\}$, and $\partial_n(x_0,t_0)$ is the inward normal vector. Let $\mathcal L_0(g(t))$ denote the principle part of the operator $\mathcal L(g(t))$,

$$\mathcal L_0(g(t))u_{ik}=\frac{\partial u_{ik}}{\partial t}-\Delta u_{ik}+2\rho(\Delta(tru)-\nabla^s\nabla^t u_{st})g_{ik}.$$

The principal symbol of the operator $\mathcal L_0(g(t))$ with coefficients freezing at the point (x_0,t_0) is

$$(\mathcal L_0(x_0,t_0,\mathrm i\xi,p)u)_{ik}=pu_{ik}+\xi^t\xi_t u_{ik}-2\rho\xi^t\xi_t\sum_{j=1}^n u_{jj}\delta_{ik}+2\rho\xi^t\xi^s u_{ts}\delta_{ik}.\tag{4.2}$$

Now we compute the determinant of the principal symbol matrix $\det\mathcal L_0(x_0,t_0,\mathrm i\xi,p)$. Fix a coordinate system

$$\{u_{11},u_{22},\cdots,u_{nn},u_{1n},u_{2n},\cdots,u_{n-1,n},u_{12},\cdots,u_{n-2,n-1}\}.\tag{4.3}$$

Then the matrix is

$$\mathcal L_0(x_0,t_0;\mathrm i\xi,p)=(p+|\xi|^2)E+A+B,\tag{4.4}$$

$$A=-2\rho|\xi|^2\begin{pmatrix}1&\cdots&1&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 1&\cdots&1&0&\cdots&0\\ 0&\cdots&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 0&\cdots&0&0&\cdots&0\end{pmatrix},\tag{4.5}$$

$$B=2\rho\begin{pmatrix}\xi_1^2&\xi_2^2&\cdots&\xi_n^2&\xi_1\xi_n&\cdots&\xi_{n-1}\xi_n&\xi_1\xi_2&\cdots&\xi_{n-2}\xi_{n-1}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \xi_1^2&\xi_2^2&\cdots&\xi_n^2&\xi_1\xi_n&\cdots&\xi_{n-1}\xi_n&\xi_1\xi_2&\cdots&\xi_{n-2}\xi_{n-1}\\ 0&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 0&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&0\end{pmatrix}.\tag{4.6}$$

Hence we have

$$\mathcal{L}_0(x_0, t_0; \mathrm{i}\xi, p) = pE + \begin{pmatrix} F & G \\ 0 & H \end{pmatrix}, \tag{4.7}$$

where F is an $n \times n$ matrix

$$F = \begin{pmatrix} (1-2\rho)|\xi|^2 + 2\rho\xi_1^2 & -2\rho|\xi|^2 + 2\rho\xi_2^2 & \cdots & -2\rho|\xi|^2 + 2\rho\xi_n^2 \\ -2\rho|\xi|^2 + 2\rho\xi_1^2 & (1-2\rho)|\xi|^2 + 2\rho\xi_2^2 & \cdots & -2\rho|\xi|^2 + 2\rho\xi_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ -2\rho|\xi|^2 + 2\rho\xi_1^2 & -2\rho|\xi|^2 + 2\rho\xi_2^2 & \cdots & (1-2\rho)|\xi|^2 + 2\rho\xi_n^2 \end{pmatrix} \tag{4.8}$$

and H is a $\frac{(n-1)n}{2} \times \frac{(n-1)n}{2}$ matrix with $H = |\xi|^2 E$. Now we compute the determinant $\det\big(pE + \left(\begin{smallmatrix} F & G \\ 0 & H \end{smallmatrix}\right)\big) = \det(pE+F)\det(pE+H)$. Obviously,

$$\det(pE+H) = (p + |\xi|^2)^{\frac{(n-1)n}{2}}. \tag{4.9}$$

We can write $pE + F$ as

$$pE + F = (p + |\xi|^2)E + 2\rho\alpha\beta^{\mathrm{T}}, \tag{4.10}$$

where $\alpha = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^{\mathrm{T}}$, and $\beta = \begin{pmatrix} \xi_1^2 - |\xi|^2 & \cdots & \xi_n^2 - |\xi|^2 \end{pmatrix}^{\mathrm{T}}$.

Note that the vector α is an eigenvector of $pE + F$,

$$(pE+F)\begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^{\mathrm{T}} = (p + (1 - 2(n-1)\rho)|\xi|^2)\begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^{\mathrm{T}}. \tag{4.11}$$

Let $V = \{\gamma \in \mathbb{R}^n,\ \gamma^{\mathrm{T}} \cdot \beta = 0\}$. For any $\gamma \in V$, we have

$$(pE+F)\gamma = (p + |\xi|^2)\gamma. \tag{4.12}$$

Note that the dimension of V is $n-1$, and $\alpha \notin V$. Hence the eigenvalues of matrix $pE+F$ are $p+|\xi|^2$ with multiplicity $n-1$ and $p+(1-2(n-1)\rho)|\xi|^2$ with multiplicity 1. The determinant of $pE+F$ is

$$\det(pE+F) = (p + (1-2(n-1)\rho)|\xi|^2)(p+|\xi|^2)^{n-1}. \tag{4.13}$$

Combining (4.9) and (4.13), we have

$$L_0(x_0, t_0; \mathrm{i}\xi, p) = \det \mathcal{L}_0(x_0, t_0; \mathrm{i}\xi, p) = (p + (1-2(n-1)\rho)|\xi|^2)(p+|\xi|^2)^{\frac{n(n+1)}{2}-1}. \tag{4.14}$$

The roots of $L_0(x_0, t_0; \mathrm{i}\xi, p) = 0$ are $p = -|\xi|^2$ and $p = -(1 - 2(n-1)\rho)|\xi|^2$. The matrix differential operator $\mathcal{L}\big(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\big)$ is parabolic if $\rho < \frac{1}{2(n-1)}$ in the sense of Definition 4 in Chapter VII of [10] with $s_k = 0,\ t_k = 2,\ b = 1,\ r = \frac{(n+1)n}{2}$. This result was obtained in [3]. Since we need to verify the boundary condition satisfying the complementary condition, we present the above formulas here.

Now, we compute the adjoint matrix of $\mathcal{L}_0(x_0, t_0; \mathrm{i}\xi, p)$ which is denoted by $\widehat{\mathcal{L}}_0(x_0, t_0; \mathrm{i}\xi, p)$.

Since

$$\mathcal{L}_0(x_0,t_0;\mathrm{i}\xi,p)=\begin{pmatrix}pE+F & G\\ 0 & pE+H\end{pmatrix}, \tag{4.15}$$

the inverse is

$$\mathcal{L}_0^{-1}(x_0,t_0;\mathrm{i}\xi,p)=\begin{pmatrix}(pE+F)^{-1} & -(pE+F)^{-1}G(pE+H)^{-1}\\ 0 & (pE+H)^{-1}\end{pmatrix}. \tag{4.16}$$

Obviously $(pE+H)^{-1}=\frac{1}{p+|\xi|^2}E$. We compute the inverse of the matrix $pE+F$. Since

$$pE+F=(p+|\xi|^2)E+2\rho\alpha\beta^{\mathrm{T}}, \tag{4.17}$$

we suppose

$$(pE+F)^{-1}=\frac{1}{p+|\xi|^2}E+k\alpha\beta^{\mathrm{T}}, \tag{4.18}$$

where k is a constant to be determined.

Since

$$\begin{aligned}E&=(pE+F)(pE+F)^{-1}=((p+|\xi|^2)E+2\rho\alpha\beta^{\mathrm{T}})\Big(\frac{1}{(p+|\xi|^2)}E+k\alpha\beta^{\mathrm{T}}\Big)\\&=E+\Big[k(p+|\xi|^2)+\frac{2\rho}{p+|\xi|^2}+2\rho k(\beta^{\mathrm{T}}\alpha)\Big]\alpha\beta^{\mathrm{T}},\end{aligned}$$

we have $k=-\frac{2\rho}{(p+|\xi|^2)(p+(1-2(n-1)\rho)|\xi|^2)}$.

Note that G can be written as

$$G=\alpha\eta^{\mathrm{T}}, \tag{4.19}$$

where $\eta=\begin{pmatrix}\xi_1\xi_n & \cdots & \xi_{n-1}\xi_n & \xi_1\xi_2 & \cdots & \xi_{n-2}\xi_{n-1}\end{pmatrix}^{\mathrm{T}}$.

Since the vector $\alpha=\begin{pmatrix}1 & \cdots & 1\end{pmatrix}^{\mathrm{T}}$ is a eigenvector of the matrix $pE+F$, we have

$$\begin{aligned}(pE+F)^{-1}G(pE+H)^{-1}&=(pE+F)^{-1}\alpha\eta^{\mathrm{T}}\frac{1}{p+|\xi|^2}E\\&=\frac{1}{(p+|\xi|^2)(p+(1-2(n-1)\rho)|\xi|^2)}\alpha\eta^{\mathrm{T}}.\end{aligned}$$

Combining the above, we have

$$\begin{aligned}&\mathcal{L}_0^{-1}(x_0,t_0;\mathrm{i}\xi,p)\\&=\begin{pmatrix}\frac{1}{p+|\xi|^2}E-\frac{2\rho}{(p+|\xi|^2)(p+(1-2(n-1)\rho)|\xi|^2)}\alpha\beta^{\mathrm{T}} & -\frac{1}{(p+|\xi|^2)(p+(1-2(n-1)\rho)|\xi|^2)}G\\ 0 & \frac{1}{p+|\xi|^2}E\end{pmatrix}.\end{aligned}$$

The adjoint matrix is

$$\widehat{\mathcal{L}_0}(x_0,t_0;\mathrm{i}\xi,p)=L_0(x_0,t_0;\mathrm{i}\xi,p)\cdot\mathcal{L}_0^{-1}(x_0,t_0;\mathrm{i}\xi,p).$$

Next we compute the boundary differential operator matrix $\mathcal{B}\big(x_0,t_0;\frac{\partial}{\partial x},\frac{\partial}{\partial t}\big)$. By the definition of the subbundle F, the boundary condition $Pr_F(u)(x_0,t_0)=0$ is equivalent to

$$u_{\alpha n}(x_0,t_0)=0.$$

In local coordinate, the condition $Pr_{F^\perp}(g^{nn}(x,t)(g(0)_{nn})^{\frac{1}{2}}\nabla_n^{g(0)}u)=\zeta$ can be expressed as

$$g^{nn}(x,t)(g(0)_{nn})^{\frac{1}{2}}\frac{\partial u_{\alpha\beta}}{\partial x_n}+\text{lower order term}=\zeta_{\alpha\beta}$$

and

$$g^{nn}(x,t)(g(0)_{nn})^{\frac{1}{2}}\frac{\partial u_{nn}}{\partial x_n}+\text{lower order term}=\zeta_{nn}.$$

So the indices of the boundary equations in [10, Theorem 10.1] are $\sigma_{\alpha n}=-2,\ \sigma_{nn}=-1,\ \sigma_{\alpha\beta}=-1$. Hence the principal symbols of the boundary differential operator at (x_0,t_0) are

$$\begin{aligned}(\mathcal{B}_0(x_0,t_0;\mathrm{i}\xi,p)u)_{\alpha\beta}&=g^{nn}(x_0,t_0)(g(x_0,0)_{nn})^{\frac{1}{2}}\mathrm{i}\xi_n u_{\alpha\beta},\\(\mathcal{B}_0(x_0,t_0;\mathrm{i}\xi,p)u)_{nn}&=g^{nn}(x_0,t_0)(g(x_0,0)_{nn})^{\frac{1}{2}}\mathrm{i}\xi_n u_{nn},\\(\mathcal{B}_0(x_0,t_0;\mathrm{i}\xi,p)u)_{\alpha n}&=1\cdot u_{\alpha n},\end{aligned}$$

and the matrix of the principal symbol of the boundary operator is

$$\mathcal{B}_0(x_0,t_0;\mathrm{i}\xi,p)=\begin{pmatrix}X&0&0\\0&Y&0\\0&0&Z\end{pmatrix},$$

where $X=C_1\mathrm{i}\xi_nE$ is an $n\times n$ matrix, $Z=C_1\mathrm{i}\xi_nE$ is $\frac{(n-2)(n-1)}{2}$ matrix , $Y=E$ is a $(n-1)\times(n-1)$ type matrix and $C_1=g^{nn}(x_0,t_0)(g(x_0,0)_{nn})^{\frac{1}{2}}$. Denote $\zeta=(\xi_1,\cdots,\xi_{n-1},0)\in T_{x_0}\partial M$, $\tau=\xi_n$ and $(0,\cdots,0,1)=\nu_{x_0}$. Consider the polynomial $L_0(x_0,t_0;\mathrm{i}(\zeta+\tau\nu),p)$ as a function of τ on the whole complex plane. It has positive imaginary roots $\tau=\mathrm{i}\sqrt{p+|\zeta|^2}$ with multiplicity $\frac{(n+1)n}{2}-1$ and $\tau=\mathrm{i}\sqrt{\frac{p}{1-2(n-1)\rho}+|\zeta|^2}$ with multiplicity 1. Denote

$$L^+(x_0,t_0;\zeta,p,\tau)=(\tau-\mathrm{i}\sqrt{p+|\zeta|^2})^{\frac{(n+1)n}{2}-1}\Big(\tau-\mathrm{i}\sqrt{\frac{p}{1-2(n-1)\rho}+|\zeta|^2}\Big).\tag{4.20}$$

Now we prove that the row of the matrix $\mathcal{B}_0(x_0,t_0;\mathrm{i}\xi,p)\cdot\widehat{\mathcal{L}_0}(x_0,t_0;\mathrm{i}\xi,p)$ is independent modulo $L^+(x_0,t_0;\zeta,p,\tau)$ with respect to τ. We observe that the independence of the row of the matrix $\mathcal{B}_0(x_0,t_0;\mathrm{i}\xi,p)\cdot\widehat{\mathcal{L}_0}(x_0,t_0;\mathrm{i}\xi,p)$ modulo $(\tau-\mathrm{i}\sqrt{p+|\zeta|^2})^{\frac{(n+1)n}{2}-1}(\tau-\mathrm{i}\sqrt{\frac{p}{1-2(n-1)\rho}+|\zeta|^2})$ is equivalent to the independence of the row of the matrix

$$\mathcal{B}_0(x_0,t_0;\mathrm{i}\xi,p)\begin{pmatrix}(p+(1-2(n-1)\rho)|\xi|^2)E-2\rho\alpha\beta^{\mathrm{T}}&-G\\0&(p+(1-2(n-1)\rho)|\xi|^2)E\end{pmatrix}$$

modulo $(\tau-\mathrm{i}\sqrt{p+|\zeta|^2})(\tau-\mathrm{i}\sqrt{\frac{p}{1-2(n-1)\rho}+|\zeta|^2})$.

If $\tau=\mathrm{i}\sqrt{p+|\zeta|^2}$, we have

$$\begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix}(p+(1-2(n-1)\rho)|\xi|^2)E=2(n-1)\rho p\begin{pmatrix} E & 0 \\ 0 & -C_1\sqrt{p+|\zeta|^2}E \end{pmatrix}.$$

Since $\mathrm{Re}(p)\geq -\delta|\zeta|^2$ for some $0<\delta<\min\{1,1-2(n-1)\rho\}$ and $|p|+|\zeta|>0$, we have the row of the matrix

$$\begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix}(p+(1-2(n-1)\rho)|\xi|^2)E$$

is independent module $(\tau-\mathrm{i}\sqrt{p+|\zeta|^2})\big(\tau-\mathrm{i}\sqrt{\frac{p}{1-2(n-1)\rho}+|\zeta|^2}\big)$.

Now we prove the rows of the matrix

$$X\cdot((p+(1-2(n-1)\rho)|\xi|^2)E-2\rho\alpha\beta^{\mathrm{T}})$$

are independent module $(\tau-\mathrm{i}\sqrt{p+|\zeta|^2})\big(\tau-\mathrm{i}\sqrt{\frac{p}{1-2(n-1)\rho}+|\zeta|^2}\big)$. If $\tau=\mathrm{i}\sqrt{p+|\zeta|^2}$, the nonzero solution of the linear equation

$$((p+(1-2(n-1)\rho)|\xi|^2)E-2\rho\alpha\beta^{\mathrm{T}})(k_1,\cdots,k_n)^{\mathrm{T}}=0 \tag{4.21}$$

belongs to the set $V_1=\mathrm{span}\{\alpha\}$.

If $\tau=\mathrm{i}\sqrt{\frac{p}{1-2(n-1)\rho}+|\zeta|^2}$, the nonzero solution of the linear equation (4.21) belongs to the set $V_2=\{\gamma\in R^n,\ \beta^{\mathrm{T}}\cdot\gamma=0\}$. Obviously $V_1\cap V_2=0$. So the row of the matrix

$$(p+(1-2(n-1)\rho)|\xi|^2)E-2\rho\alpha\beta^{\mathrm{T}}$$

is independent module $(\tau-\mathrm{i}\sqrt{p+|\zeta|^2})\big(\tau-\mathrm{i}\sqrt{\frac{p}{1-2(n-1)\rho}+|\zeta|^2}\big)$.

Since $X=\mathrm{i}C_1\tau E$, the row of the matrix

$$X\cdot((p+(1-2(n-1)\rho)|\xi|^2)E-2\rho\alpha\beta^{\mathrm{T}})$$

is independent module $(\tau-\mathrm{i}\sqrt{p+|\zeta|^2})\big(\tau-\mathrm{i}\sqrt{\frac{p}{1-2(n-1)\rho}+|\zeta|^2}\big)$, when $\mathrm{Re}(p)\geq-\delta|\zeta|^2$ for some $0<\delta<\min\{1,1-2(n-1)\rho\}$ and $|p|+|\zeta|>0$.

Based on the above analysis, we conclude that

$$\mathcal{B}_0(x_0,t_0;\mathrm{i}(\zeta+\tau\nu),p)\widehat{\mathcal{L}}_0(x_0,t_0;\mathrm{i}(\zeta+\tau\nu),p)$$

are linearly independent modulo the polynomial L^+ as a polynomial in τ if the vector ζ and the number p satisfy

$$\mathrm{Re}(p)\geq-\delta|\zeta|^2,\quad |p|+|\zeta|>0, \tag{4.22}$$

where $0<\delta<\min\{1,1-2(n-1)\rho\}$. Since $g(x,t)\in C^{l+2+\tilde{\alpha},\frac{l+2+\tilde{\alpha}}{2}}(M\times[0,T])$, the coefficients of the operator $\mathcal{L}(g(t))$ belongs to the class $C^{l+\tilde{\alpha},\frac{l+\tilde{\alpha}}{2}}(M\times[0,T])$, the coefficients of the boundary operator $\mathcal{B}(g(t))(u)_{\alpha n}$ are in class $C^{l+2+\tilde{\alpha},\frac{l+2+\tilde{\alpha}}{2}}(\partial M\times[0,T])$, and the coefficients of the boundary operator $\mathcal{B}(g(t))(u)_{\alpha\beta}$ and $\mathcal{B}(g(t))(u)_{nn}$ are in the class $C^{l+1+\tilde{\alpha},\frac{l+1+\tilde{\alpha}}{2}}(\partial M\times[0,T])$.

By [12, Chapter VII, Theorem 10.1], the linear parabolic initial boundary value problem has a unique solution $u_{ij} \in C^{l+2+\tilde{\alpha},\frac{l+2+\tilde{\alpha}}{2}}(M \times [0,T])$ and satisfies the following estimate

$$
\begin{aligned}
|u|_{C^{2+l+\tilde{\alpha},2+l+\frac{\tilde{\alpha}}{2}}(M_T)} \leq C(&|F|_{C^{l+\tilde{\alpha},l+\frac{\tilde{\alpha}}{2}}(M_T)} + |b_{\tilde{\alpha}n}|_{C^{l+2+\tilde{\alpha},\frac{l+2+\tilde{\alpha}}{2}}(\partial M_T)} \\
&+ |b_{\tilde{\alpha}\beta}|_{C^{l+1+\tilde{\alpha},\frac{l+1+\tilde{\alpha}}{2}}(\partial M_T)} + |b_{nn}|_{C^{l+1+\tilde{\alpha},\frac{l+1+\tilde{\alpha}}{2}}(\partial M_T)})
\end{aligned}
$$

if $F(x,t)$ and $b(x,t)$ satisfy the necessary compatible conditions.

5 A Boundary Value Problem for the DeTurck Ricci-Bourguignon Flow

In this section, we use inverse function theorem to prove the short time existence of the initial boundary value problem of the DeTurck Ricci-Bourguignon flow. Firstly recall the inverse function theorem (see [1, Chapter 3])

Theorem 5.1 (see [1, 8]) *Assume that $\mathcal{E} : U \subset B_1 \to B_2$ is a continuous differential map, where B_i, $i = 1, 2$ are Banach spaces and U is an open set in B_1. If there is a continuous linear operator $A : B_2 \to B_1$ such that $\mathcal{E}'(x_0)A = \mathrm{id}_{B_2}$, then there is a C^1 map g from the neighborhood of $y_0 = \mathcal{E}(x_0)$ to the neighborhood of x_0 such that $\mathcal{E}(g(y)) = y$.*

In this section, we denote

$$
E(g(t),t) = 2\mathrm{Ric}(g(t)) - 2\rho R(g(t))g(t) - L_{W(g(t),t)}g(t)
$$

and

$$
\begin{aligned}
B_1 &:= \{h(x,t) \in C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(M_T,\mathcal{T}) \mid h(x,0) = 0,\ \partial_t h(x,0) = 0\},\\
B_3 &:= \{f(x,t) \in C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M_T,\mathcal{T}) \mid f(x,0) = 0,\ x \in M\},\\
B_4 &:= \{b(x,t) \in \Gamma(\mathcal{T}_{\partial M}) \mid b_{\alpha\beta}, b_{nn} \in C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M \times [0,T]),\\
&\qquad b_{\alpha n} \in C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(\partial M \times [0,T]),\ b(x,0) = 0,\ \partial_t b(x,0)_{\alpha n} = 0,\ x \in \partial M\}.
\end{aligned}
$$

Let $B_2 = B_3 \times B_4$. As closed linear subsets of Banach spaces, B_1 and B_2 are Banach spaces.

Now we apply the inverse function theorem to the operator

$$
\mathcal{E} : U \subset B_1 \to B_2,
$$

where $\mathcal{E}(h(x,t)) = \big(\frac{\partial(g_0 - tE(g(0),0) + h(x,t))}{\partial t} + E(g_0 - tE(g(0),0) + h(x,t)\big), B(g_0 - tE(g(0),0) + h(x,t))$ and U is a neighbourhood of 0 in B_1.

On the boundary ∂M, we have

$$
\frac{\partial(g_0 - tE(g(0),0) + h(x,t))_{\alpha n}}{\partial t}\Big|_{t=0} = -2Rc_{\alpha n} + 2\rho R g_{\alpha n} + L_W g_{\alpha n}|_{t=0} = -2Rc_{\alpha n}
$$

since $g_{\alpha n}(x,0) = 0$ and $W|_{t=0} = 0$. So under the condition (1.2), we have

$$
\frac{\partial(g_0 - tE(g(0),0) + h(x,t))_{\alpha n}}{\partial t}(x,0) = 0
$$

for $x \in \partial M$. Hence under the conditions of Theorem 1.2, the range of the map $\mathcal{E}$ is actually in B_2 and $\mathcal{E}$ is well defined.

Assume that T is so small that $g_0 - tE(g_0, 0)$ is a metric on M for $t \in [0, T)$. We also assume that for all $h(x,t) \in U$, $g(t) = g_0 - tE(g(0), 0) + h(x,t)$ is a metric on M. Now we prove that there is a bounded linear operator

$$A : B_2 \to B_1 \tag{5.1}$$

such that $D\mathcal{E}(0) \circ A = \mathrm{id}$, that is for any $(f(x,t), b(x,t)) \in B_2$, there is only one $u \in B_1$, such that

$$D\mathcal{E}(0)(u) = (f(x,t), b(x,t)),$$

and

$$\begin{aligned}\|u\|_{C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}(M\times[0,T])} \le C(&\|f(x,t)\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M\times[0,T])} + \|b_{\alpha n}(x,t)\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(\partial M\times[0,T])}\\ &+ \|b_{\alpha\beta}(x,t)\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])} + \|b_{nn}(x,t)\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}).\end{aligned}$$

We compute the linearization of the Deturck Ricci-Bourguignon flow at $g(t) = g_0 - tE(g(0), 0)$. Denote $g_\lambda(t) = g(t) + \lambda u(t)$, $\lambda \in (-\epsilon, \epsilon)$. The linearized operator $-DE(g(t), t)$ is

$$\begin{aligned}-DE(g(t),t)(u)_{ik} &= -\frac{\partial E(g_\lambda(t))}{\partial \lambda}\Big|_{\lambda=0}\\ &= \Delta u_{ik} - 2\rho(\Delta(tru) - \nabla^s\nabla^t u_{st})g_{ik}\\ &\quad + M_1(g, \partial_i g, t) * \nabla u(x,t) + M_2(g, \partial_i g, \partial^2_{ij} g, t) * u(x,t).\end{aligned}$$

Here Δ and ∇ are Laplace operator and covariant differential operator respectively with respect to the metric $g(t)$. M_1 and M_2 are smooth functions.

Next we compute the linearization of the boundary operator $B(g(t), t)$. Recall the boundary operator is

$$\begin{cases} Pr_{\mathcal{F}} B(g(t),t) = Pr_{\mathcal{F}} g(t),\\ Pr_{\mathcal{F}^\perp} B(g(t),t) = Pr_{\mathcal{F}^\perp}(g^{nn}(x,t)(g(0)_{nn})^{\frac{1}{2}} \nabla_n^{g(0)} g(x,t)) - \zeta(g(x,t),t),\end{cases}$$

where ζ is a (0,2)-tensor

$$\begin{aligned}\zeta_{\alpha\beta}(g(x,t),t) &= -\mu(t)(g(0)_{nn}g^{nn}(x,t))^{\frac{1}{2}}(g_{\alpha\gamma}(x,t)g(0)^{\gamma\sigma}A^0_{\sigma\beta}(x) + g_{\beta\gamma}(x,t)g(0)^{\gamma\sigma}A^0_{\sigma\alpha})\\ &\quad + g(0)_{nn}g^{nn}(x,t)(g_{\alpha\gamma}(x,t)g(0)^{\gamma\sigma}A^0_{\sigma\beta} + g_{\beta\gamma}(x,t)g(0)^{\gamma\sigma}A^0_{\sigma\alpha}),\\ \zeta_{nn}(g(x,t),t) &= -2g_{nn}(x,t)(\mu(t)(g(0)_{nn}g^{nn}(x,t))^{\frac{1}{2}}H(x,0)\\ &\quad + g(0)^{\frac{1}{2}}_{nn}g^{nn}(x,t)(\Gamma^n_{nn}(g_0) - \Gamma^n_{nn}(\widetilde{g}(t))) - g^{\alpha\beta}(t)g(0)^{\frac{1}{2}}_{nn}\Gamma^n_{\alpha\beta}(\widetilde{g}(t))),\end{aligned}$$

and $\zeta_{\alpha n}(g(x,t),t) = 0,\ x \in \partial M,\ t \in [0,T)$.

By computation, the linearization of the boundary operator is

$$\begin{cases} Pr_{\mathcal{F}} DB(g(t),t)(u(x,t)) = Pr_{\mathcal{F}} u(x,t),\\ Pr_{\mathcal{F}^\perp} DB(g(t),t)(u(x,t)) = Pr_{\mathcal{F}^\perp}(g^{nn}(x,t)(g(0)_{nn})^{\frac{1}{2}}\nabla_n^{g(0)} u(x,t) + M_3(g, \partial_i g, \partial\widetilde{g}, t) * u(x,t)),\end{cases}$$

where M_3 is a smooth function.

Now we consider the solvability of linear equation

$$\begin{aligned}\frac{\partial u}{\partial t} - DE(g(t),t)u &= \frac{\partial u_{ik}}{\partial t} - \Delta u_{ik} + 2\rho(\Delta(tru) - \nabla^s\nabla^t u_{st})g_{ik} \\ &\quad + M_1(g,\partial_i g,t) * \nabla u(x,t) + M_2(g,\partial_i g,\partial^2_{ij} g,t) * u(x,t) \\ &= f(x,t)\end{aligned}$$

with boundary condition

$$\begin{cases} Pr_{\mathcal{F}} u(x,t) = Pr_{\mathcal{F}} b(x,t), \\ Pr_{\mathcal{F}^\perp}(g^{nn}(x,t)(g(0)_{nn})^{\frac{1}{2}}\nabla_n^{g(0)} u(x,t) + M_3(g,\partial g,\partial\widetilde{g},t) * u(x,t)) = Pr_{\mathcal{F}^\perp} b(x,t) \end{cases}$$

and initial condition

$$u(x,0) = 0, \quad x \in M,$$

where $b(x,t) \in B_4$, $f(x,t) \in B_3$. From the conditions $u(x,0) = 0$ and $f(x,0) = 0$, we have $\frac{\partial}{\partial t}|_{t=0} u = 0$. Since $b(x,0) = 0$, $\partial_t u_{\alpha n}(x,0) = 0 = \partial_t b_{\alpha n}(x,0)$, $x \in \partial M$, the necessary compatibility conditions for $C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}$ regularity are satisfied on $\partial M \times \{t = 0\}$. Since $g(t) = g(0) - tE(g(0)) \in C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(M_T)$, the regularity assumptions about the coefficients in Theorem 4.1 are satisfied. By Theorem 4.1, there is only one solution $u(x,t) \in B_1$ with

$$\begin{aligned}\|u\|_{C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}} &\le C(\|f(x,t)\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M_T)} + \|b_{\alpha\beta}(x,t)\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M_T)} + \|b_{nn}(x,t)\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M_T)} \\ &\quad + \|b_{\alpha n}(x,t)\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(\partial M_T)}).\end{aligned}$$

In the following, we verify that the map $\mathcal{E}$ is continuously differentiable.

Lemma 5.1 *For* $\|g_i - g_0\|_{C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}} \le R$, $i = 1,2$,

$$\|(DE(g_1,t) - DE(g_2,t))v_{ij}\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M_T)} \le C(R,g_0)\|g_1 - g_2\|_{C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}(M_T)}\|v\|_{C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}(M_T)}.$$

Proof By computation, we have

$$DE(g(t),t)v_{ij} = \Delta v_{ij} + 2\rho\nabla_l\nabla_k v_{kl} g_{ij} - 2\rho\Delta trv g_{ij} + M_1 * \nabla v + M_2 * v, \tag{5.2}$$

where $M_1(g(t),\widetilde{g}(t))$ is a smooth function of $g, \partial g, \widetilde{g}, \partial\widetilde{g}$ and M_2 is smooth function of $g, \partial g, \partial^2 g, \widetilde{g}$, $\partial\widetilde{g}, \partial^2\widetilde{g}$. Since $DE(g(t),t)v_{ij}$ is a linear operator, we can write

$$DE(g(t),t)v_{ij} = a(g,g^{-1})\partial^2 v_{ij} + M_1\partial v_{ij} + M_2 v_{ij},$$

where a, b, c are smooth functions. So we have

$$\begin{aligned}&\|(DE(g_1,t) - DE(g_2),t)v_{ij}\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M_T)} \\ &\le \|a(g_1,g_1^{-1}) - a(g_2,g_2^{-1})\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M_T)}\|v\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(M_T)}\end{aligned}$$

$$+\|M_1(g_1(t),\widetilde{g}(t))-M_1(g_2(t),\widetilde{g}(t))\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M_T)}\|v\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(M_T)}$$
$$+\|M_2(g_1(t),\widetilde{g}(t))-M_2(g_1(t),\widetilde{g}(t))\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M_T)}\|v\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(M_T)}.$$

Since a,b,c are smooth function, we have

$$\|(DE(g_1,t)-DE(g_2),t)v_{ij}\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M_T)}$$
$$\le C(R,g_0,\|\widetilde{g}(t)\|_{C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}})\|g_1-g_2\|_{C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}(M_T)}\|v\|_{C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}(M_T)}.$$

Lemma 5.2 *If* $\|g_i-g(0)\|_{C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}(M_T)}\le R$, $i=1,2$, *for the boundary operator* B,

$$\|(DB(g_1,t)-DB(g_2,t))v\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}\le C(R,g_0,\widetilde{g})\|g_1-g_2\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}}\|v\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}}.$$

Proof By the definition of the boundary operator, we have

$$(DB(g_1,t)-DB(g_2,t))v_{n\alpha}=0.$$

Hence

$$\|(DB(g_1,t)-DB(g_2,t))v_{n\alpha}\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(\partial M\times[0,T])}=0$$

and

$$((DB(g_1,t)-DB(g_2,t))v)_{\alpha\beta}=(g_1^{nn}(x,t)-g_2^{nn}(x,t))(g(0)_{nn})^{\frac{1}{2}}\nabla_n^{g(0)}v(x,t)$$
$$+(M(g_1,\partial g_1,\widetilde{g},\partial\widetilde{g},\mu(t))-M(g_2,\partial g_2,\widetilde{g},\partial\widetilde{g},\mu(t)))*v,$$

where M is a smooth function. So we have

$$\|((DB(g_1,t)-DB(g_2,t))v)_{\alpha\beta}\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}\le C(R,g_0,\widetilde{g})\|g_1-g_2\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}}\|v\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}}.$$

Similarly, we have

$$\|((DB(g_1)-DB(g_2))v)_{nn}\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}\le C(R,g_0,\widetilde{g})\|g_1-g_2\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}}\|v\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}}.$$

We now prove the short time existence of the Deturck Ricci Bourguignon flow on manifold with boundary. Let $g(t)=g_0-tE(g_0,0)$. We have

$$\begin{aligned}R(E)&=\frac{\partial g(t)}{\partial t}+E(g(t),t)\\&=E(g_0-tE(g_0,0),t)-E(g_0,t)+E(g_0,t)-E(g_0,0)\\&=-t\int_0^1 DE(g_0-\theta tE(g_0,0),t)\mathrm{d}\theta\cdot E(g_0,0)+L_{W(g_0,\widetilde{g}(t))}g_0-L_{W(g_0,\widetilde{g}(0))}g_0,\end{aligned}$$

where $L_{W(g_0,\widetilde{g}(0))}g_0=0$. By choosing smooth

$$\widetilde{g}(t)=g_0(x)+t\widehat{g}(x,t),$$

where

$$\widehat{g}(x,t)\in C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(M_T)$$

is a symmetric 2 tensor, we have

$$\|L_{W(g_0,\widetilde{g}(t))}g_0\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M_T)}\to 0 \quad \text{as } T\to 0.$$

On the boundary $\partial M\times[0,T]$, we also have for $i,j=1,\cdots,n-1$ or $i=j=n$,

$$\begin{aligned}R(B,t)_{ij}&=B(g_0-tE(g_0,0),t)\\&=B\Big(\frac{1}{\mu(t)^2}g_0,t\Big)-\int_0^1 DB\Big(s(g_0-tE(g_0,0))+(1-s)\frac{1}{\mu(t)^2}g_0,t\Big)\mathrm{d}s\\&\quad\cdot\Big(g_0-tE(g_0,0)-\frac{1}{\mu(t)^2}g_0\Big).\end{aligned}$$

Obviously, $B\big(\frac{1}{\mu(t)^2}g_0,t\big)_{\alpha\beta}=0$. As for $B\big(\frac{1}{\mu(t)^2}g_0,t\big)_{nn}$, we have

$$\Big\|B\Big(\frac{1}{\mu(t)^2}g_0,t\Big)_{nn}\Big\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}\to 0\quad \text{as } T\to 0.$$

For $\alpha=1,\cdots,n-1$,

$$R(B,t)_{\alpha n}=B(g_0-tE(g_0,0),t)_{\alpha n}=0.$$

Hence

$$\|Pr_{\mathcal{F}}(R(B,t))\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(\partial M\times[0,T])}=0.$$

Now we estimate $\|R(E,t)\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M\times[0,T])}$ and $\|Pr_{\mathcal{F}^\perp}(R(B,t))\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}$. Since $g(0)\in C^{4+\tilde{\alpha}}(M,\mathcal{T})$, we have $E(g(0))\in C^{2+\alpha}(M,\mathcal{T})$. Choosing T^* so small that $g(0)-tE(g(0))$ is a $C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}$ metric for $t\in[0,T^*]$. By choosing R large, we can assume that

$$\|tE(g(0))\|_{C^{2+\tilde{\alpha},2+\frac{\tilde{\alpha}}{2}}(M_T)}\le R.$$

Since the boundary operator $B(g(t))$ is continuously differential, we have the estimate

$$\begin{aligned}&\Big\|DB\Big(s(g_0-tE(g_0,0))+(1-s)\frac{1}{\mu(t)^2}g_0,t\Big)\cdot\Big(g_0-tE(g_0)-\frac{1}{\mu(t)^2}g_0\Big)\Big\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}\\ \le{}&\|t\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}\Big\|DB\Big(s(g_0-tE(g_0,0))+(1-s)\frac{1}{\mu(t)^2}g_0,t\Big)E(g_0)\Big\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}\\ &+\Big\|1-\frac{1}{\mu(t)^2}\Big\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}\Big\|DB\Big(s(g_0-tE(g_0,0))+(1-s)\frac{1}{\mu(t)^2}g_0,t\Big)g_0\Big\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}.\end{aligned}$$

Hence $\|Pr_{\mathcal{F}^\perp}(R(B,t))\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])}\to 0$ as $T\to 0$.

Similarly, we have

$$\begin{aligned}&\|DE(g_0+\theta tE(g_0,0),t)E(g_0,0)\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M\times[0,T])}\\ \le{}&\|(DE(g_0+\theta tE(g_0,0),t)-DE(g_0,t))E(g_0,0)\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M\times[0,T])}\\ &+\|DE(g_0,t)E(g_0,0)\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M\times[0,T])}\\ \le{}&C(g_0,R).\end{aligned}$$

So for any $\epsilon > 0$, we can choose a small $0 < T < T^*$ such that

$$\|R(E,t)\|_{C^{\tilde{\alpha},\frac{\tilde{\alpha}}{2}}(M\times[0,T])} + \|Pr_{\mathcal{F}^\perp}(R(B,t))\|_{C^{1+\tilde{\alpha},\frac{1+\tilde{\alpha}}{2}}(\partial M\times[0,T])} + \|Pr_{\mathcal{F}}(R(B,t))\|_{C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(\partial M\times[0,T])} \le \epsilon.$$

By classic inverse function theorem, there is an $h(x,t) \in B_1$, such that $g(x,t) = g(x,0) - tE(g(x,0),0) + h(x,t)$ satisfies

$$\frac{\partial g(t)}{\partial t} + E(g(t),t) = 0, \quad (x,t) \in M \times [0,T]$$

and

$$B(g(t),t) = 0, \quad (x,t) \in \partial M \times [0,T].$$

Now we have the local existence for the DeTurck Ricci-Bourguignon flow in $C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(M_T)$. By standard interior regularity and boundary regularity estimate for the strictly parabolic type PDE systems, we obtain the following theorem.

Theorem 5.2 *Let $g(t) \in C^{2+\tilde{\alpha},\frac{2+\tilde{\alpha}}{2}}(M_T)$ be a solution to the DeTurck Ricci-Bourguignon flow with boundary value* (3.2). *Let $l = k + \alpha$. Then the following hold*:

(1) (Interior regularity) *Suppose that $\widetilde{g}(t) \in C^{l+2,\frac{l+2}{2}}(M^\circ \times (0,T])$. Then $g(x,t) \in C_{\text{loc}}^{l+2,\frac{l+2}{2}}(M^\circ \times (0,T])$.*

(2) (Boundary regularity) *If $\mu(t) \in C^{\frac{l+1}{2}}([0,T])$, $g_0 \in C^{l+2}(\overline{M})$, $\widetilde{g}(t) \in C^{l+2,\frac{l+2}{2}}(\overline{M} \times [0,T])$ and the data $g_0, \mu(t), \widetilde{g}(t)$ satisfy the necessary compatibility conditions at $\partial M \times \{0\}$, then $g(x,t) \in C^{l+2,\frac{l+2}{2}}(\overline{M}_T)$.*

(3) (Boundary regularity for positive time) *If $\mu(t) \in C^{\frac{l+1}{2}}([0,T])$ and $\widetilde{g}(t) \in C^{l+2,\frac{l+2}{2}}(\overline{M} \times [0,T])$, then $g(x,t) \in C^{l+2,\frac{l+2}{2}}(M \times [\delta,T])$ for any $0 < \delta < T$.*

Since $W_l = g_{lr}g^{pq}(\Gamma(g(t))^r_{pq} - \Gamma(\widetilde{g}(t))^r_{pq})$, the DeTurck vector field W is in $C^{l-1,\frac{l-1}{2}}(\overline{M}_T)$ if $g(t) \in C^{l,\frac{1}{2}}(\overline{M}_T)$ and $\widetilde{g}(t) \in C^{l,\frac{1}{2}}(\overline{M}_T)$.

By Theorem 5.2, the DeTurck vector field $W(g(t),t) \in C^\infty(\overline{M}_T - \partial M \times 0)$ if $\widetilde{g}(x,t)$ and $\mu(t)$ are smooth. By the differentiability property of the flow, we can obtain a unique flow ϕ_t for $t > 0$, which is smooth on $M \times (0,T]$ and C^1 on $M \times [0,T]$, satisfying

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\phi = -W \circ \phi, \\ \phi(0) = \mathrm{id}_M. \end{cases}$$

By results in Section 3, $\widehat{g}(t) = \phi_t^*(g(t))$ solves the Ricci-Bourguignon flow equation. Since $(\phi_t^{-1})^*(\widehat{g}(t)) = g(t)$ and $g(t) \to g(0)$ in the $C^{2+\tilde{\alpha}}(\overline{M})$ sense. So we get $\widehat{g}(t)$ converges to g_0 in the geometric $C^{2+\tilde{\alpha}}$ sense. Since $g(x,t)$ satisfies the boundary condition (1.5), by results in Section 3, we have that $\widehat{g}(t) = \phi_t^*(g(t))$ satisfies the boundary condition (1.3).

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