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Exact Boundary Controllability of Weak Solutions for a Kind of First Order Hyperbolic System — the HUM Method*

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Abstract The exact boundary controllability and the exact boundary observability for the 1-D first order linear hyperbolic system were studied by the constructive method in the framework of weak solutions in the work [Lu, X. and Li, T. T., Exact boundary controllability of weak solutions for a kind of first order hyperbolic system — the constructive method, *Chin. Ann. Math. Ser. B*, **42**(5), 2021, 643–676]. In this paper, in order to study these problems from the viewpoint of duality, the authors establish a complete theory on the HUM method and give its applications to first order hyperbolic systems. Thus, a deeper relationship between the controllability and the observability can be revealed. Moreover, at the end of the paper, a comparison will be made between these two methods.

Keywords First order linear hyperbolic system, Exact boundary controllability, The HUM method
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1 Introduction

First order linear hyperbolic systems are widely used to model various systems in real life, such as traffic flow, fluids in open channels and light propagation in optical fibers, etc. The exact boundary controllability and the exact boundary observability for 1-D first order hyperbolic systems (even in the quasi-linear case) have been established by the constructive method in the framework of classical solutions (see [4–6]). In [12], we established the exact boundary controllability and the exact boundary observability for 1-D first order linear hyperbolic systems in the framework of weak solutions by the constructive method. In this paper, we are going to study these problems from the viewpoint of duality, namely, by the corresponding HUM method.

The HUM method was proposed by J.-L. Lions for a single wave equation with Dirichlet boundary control or Neumann boundary control (see [10–11]). He used an observability inequality to establish the controllability by the HUM method. Later, for a coupled system of

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wave equations, the exact boundary controllability was set up under Dirichlet boundary controls (see [8]) and Neumann boundary controls (see [9]) in suitable Hilbert spaces by the HUM method, respectively. The HUM method applying to a system of transport equation can be found in [2].

In the linear case, by studying the controllability through the HUM method in the framework of weak solutions, a deeper relationship between the controllability for the original system and the observability for the corresponding adjoint system can be revealed.

For the 1-D first order hyperbolic system, although in the framework of C^1 solutions, the controllability and the observability have been completely established, however, the C^1 space is not a Hilbert space, in which the HUM method can not be adopted to get the controllability by duality even in the linear case. Since the first order hyperbolic system is time irreversible in general, differently from the time reversible system like the wave equation, the corresponding HUM method will have its own difficulty, which was called as the RHUM method by J.–L. Lions in [10]. However, for simplicity, we will still call it as the HUM method in this paper. Here we mention that the authors of [1] considered the boundary controllability from the perspective of duality, but the observability inequality based on energy estimates applied in that paper are valid only for certain class of symmetric hyperbolic systems, which can not be extended to general cases, moreover, the HUM method for the first order hyperbolic system was not explained there.

Thus, to our knowledge, there is still no paper that explains thoroughly the HUM method and its applications for first order hyperbolic systems, then we intend to establish a complete theory on the HUM method for first order linear hyperbolic systems to obtain the controllability, and further apply it to study the problem of synchronization in a forthcoming paper.

The system under consideration is given by

$$U_t + \Lambda U_x + AU = 0, \quad t \in (0, +\infty),\ x \in (0, L) \tag{1.1}$$

with the boundary conditions

$$U^+(t, 0) = G_0 U^-(t, 0) + D_0 H^+(t), \quad t \in (0, +\infty), \tag{1.2}$$

$$U^-(t, L) = G_1 U^+(t, L) + D_1 H^-(t), \quad t \in (0, +\infty) \tag{1.3}$$

and the initial data

$$U(0, x) = U_0(x), \quad x \in (0, L), \tag{1.4}$$

where $U = (u_1, \cdots, u_n)^T : (0, +\infty) \times (0, L) \to \mathbb{R}^n$ denotes the state variable, $\Lambda = \mathrm{diag}\{\Lambda^-, \Lambda^+\}$ is a diagonal matrix of order n,

$$\Lambda^- := \mathrm{diag}\{\lambda_1, \cdots, \lambda_m\}, \quad \Lambda^+ := \mathrm{diag}\{\lambda_{m+1}, \cdots, \lambda_n\}$$

with $\lambda_r < 0$ $(r = 1, \cdots, m)$ and $\lambda_s > 0$ $(s = m + 1, \cdots, n)$, the coupling matrix $A = (a_{ij})$ is of order n. Let $\overline{m} = n - m$. The boundary coupling matrices G_0 and G_1 are of order $\overline{m} \times m$ and $m \times \overline{m}$, respectively, the boundary control matrices D_0 and D_1 are of order $\overline{m} \times M_0$ and $m \times M_1$ $(M_0 \leq \overline{m}, M_1 \leq m)$, respectively, both of them are full column-rank matrices. All the matrices mentioned above are with constant elements. Moreover, $U = (U^-, U^+)^T$ with $U^- = (u_1, \cdots, u_m)^T$ and $U^+ = (u_{m+1}, \cdots, u_n)^T$, $H = (H^-, H^+)^T$ with $H^- = (h_1, \cdots, h_{M_1})^T$ and $H^+ = (h_{M_1+1}, \cdots, h_M)^T$ $(M = M_0 + M_1 \le n)$.

The idea of the HUM method is to provide a boundary control by the solution to the following adjoint system

$$\begin{cases} \Phi_t + \Lambda\Phi_x - A^T\Phi = 0, & t \in (0,T), \quad x \in (0,L), \\ \Phi^-(t,0) = -(\Lambda^-)^{-1}G_0^T\Lambda^+\Phi^+(t,0), & t \in (0,T), \\ \Phi^+(t,L) = -(\Lambda^+)^{-1}G_1^T\Lambda^-\Phi^-(t,L), & t \in (0,T) \end{cases} \tag{1.5}$$

with the final data

$$t = T: \ \Phi(T,x) = \Phi_T(x), \quad x \in (0,L). \tag{1.6}$$

Then, based on a strong observability inequality, we can construct an isomorphism between the final data U_T of system (1.1)–(1.3) and the final data Φ_T of the adjoint system (1.5). Thus, for any given $U_T(x) \in (L^2(0,L))^n$, there exists $\Phi_T(x) \in (L^2(0,L))^n$ such that the corresponding adjoint problem (1.5)–(1.6) admits a weak solution $\Phi = \Phi(t,x)$. This procedure can uniquely determine the boundary control function so that the solution to problem (1.1)–(1.4) satisfies exactly the final state (2.6) below at the time $t = T$. Hence, if the strong observability inequality holds for adjoint system (1.5), then system (1.1)–(1.3) is exactly controllable.

By duality, we can see more clearly the close relationship between the exact boundary controllability for system (1.1)–(1.3) and the strong exact boundary observability for adjoint system (1.5). Moreover, the boundary control provided by the HUM method should possess the least L^2 norm among all the boundary controls realizing the exact boundary controllability, which will be proved in Subsection 2.3 in the case of two-sided controls.

By Theorem 3.3 and Remark 3.3 in [12], we have

Lemma 1.1 *For any given $T > 0$, and any given final data $\Phi_T \in (L^2(0,L))^n$, the backward problem* (1.5)–(1.6) *admits a unique weak solution $\Phi = \Phi(t,x) \in (L^2(0,T;L^2(0,L)))^n$, satisfying*

$$\|\Phi(t,\cdot)\|_{(L^2(0,L))^n} \le C\|\Phi_T\|_{(L^2(0,L))^n}, \quad \forall t \in [0,T], \tag{1.7}$$

$$\|\Phi(\cdot,0)\|_{(L^2(0,T))^n} \le C\|\Phi_T\|_{(L^2(0,L))^n} \tag{1.8}$$

and

$$\|\Phi(\cdot,L)\|_{(L^2(0,T))^n} \le C\|\Phi_T\|_{(L^2(0,L))^n}, \tag{1.9}$$

here and hereafter, $C > 0$ denotes a different positive constant, depending only on T.

We will establish the exact boundary controllability for system (1.1)–(1.3) by the HUM method based on the duality in this paper. The case of two-sided controls and the case of one-sided controls will be successively discussed in Section 2 and Section 3, respectively. Then the constructive method and the HUM method will be compared in Section 4.

2 Two-Sided Exact Boundary Controllability of Weak Solutions by the HUM Method

2.1 Two-sided exact boundary controllability

We first prove the two-sided exact boundary controllability for system (1.1)–(1.3) based on the following lemma of strong observability.

Lemma 2.1 (see [12, Lemma 5.1]) *Let $T \geq T_0$, where*

$$T_0 = L \max_{\substack{1 \leq r \leq m \\ m+1 \leq s \leq n}} \left\{ \frac{1}{|\lambda_r|}, \frac{1}{\lambda_s} \right\} > 0. \tag{2.1}$$

Assume that $\Phi = \Phi(t,x)$ is the weak solution to adjoint problem (1.5)–(1.6), in which

$$M = n, \quad \textit{namely}, \; M_0 = \mathrm{rank}(D_0) = \overline{m}, \; \textit{and} \; M_1 = \mathrm{rank}(D_1) = m. \tag{2.2}$$

Then for any given final data $\Phi_T(x) \in (L^2(0,L))^n$, we have the following strong observability inequality

$$\|\Phi_T\|_{(L^2(0,L))^n} \leq C(\|D_0^T \Lambda^+ \Phi^+(\cdot,0)\|_{(L^2(0,T))^{\overline{m}}} + \|D_1^T \Lambda^- \Phi^-(\cdot,L)\|_{(L^2(0,T))^m}). \tag{2.3}$$

Theorem 2.1 *Assume $T \geq T_0$, where T_0 is given by (2.1). Under assumption (2.2), for any given initial data $U_0(x) \in (L^2(0,L))^n$ and any given final data $U_T(x) \in (L^2(0,L))^n$, system (1.1)–(1.3) is exactly controllable at the time $t = T$, and the boundary control $H \in (L^2(0,T))^n$ satisfies*

$$\|H\|_{(L^2(0,T))^n} \leq C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n}). \tag{2.4}$$

Proof By linearity, we claim that the exact boundary controllability for system (1.1)–(1.3) is equivalent to the fact that system (1.1)–(1.3) with the null initial data

$$t = 0: \; U(0,x) \equiv 0, \quad x \in (0,L) \tag{2.5}$$

is exactly controllable, namely, for any given final data $U_T(x) \in (L^2(0,L))^n$, there exists a boundary control $H \in (L^2(0,T))^n$, such that the weak solution $U = U(t,x)$ to the mixed problem (1.1)–(1.3) with (2.5) satisfies exactly the final state

$$t = T: \; U(T,x) = U_T(x), \quad 0 < x < L. \tag{2.6}$$

We will explain this fact at the end of the proof.

Let $\Phi = \Phi(t,x)$ be the solution to adjoint problem (1.5)–(1.6) with smooth final data $\Phi_T \in (C^\infty[0,L])^n$. Setting

$$H^-(t) = -D_1^T \Lambda^- \Phi^-(t,L) \tag{2.7}$$

and

$$H^+(t) = D_0^T \Lambda^+ \Phi^+(t,0), \tag{2.8}$$

we consider the following forward problem

$$\begin{cases} U_t + \Lambda U_x + AU = 0, & t \in (0,T), \quad x \in (0,L), \\ U^+(t,0) = G_0 U^-(t,0) + D_0 D_0^T \Lambda^+ \Phi^+(t,0), & t \in (0,T), \\ U^-(t,L) = G_1 U^+(t,L) - D_1 D_1^T \Lambda^- \Phi^-(t,L), & t \in (0,T) \end{cases} \tag{2.9}$$

with (2.5). For any given $\Phi_T \in (C^\infty[0,L])^n$, let $U = U(t,x)$ be the weak solution to problem (2.9) with (2.5), and let

$$t = T: \ U(T,x) = U_T(x), \quad x \in (0,L). \tag{2.10}$$

Thus, we get the following linear mapping F:

$$F(\Phi_T) = U_T. \tag{2.11}$$

We will prove that F can be extended to a surjective mapping from $(L^2(0,L))^n$ to $(L^2(0,L))^n$, and this fact is equivalent to the exact boundary controllability of problem (1.1)–(1.3) with (2.5).

Assume that $\Psi = \Psi(t,x)$ is the solution to adjoint problem (1.5)–(1.6) with the final data $\Psi_T = \Psi_T(x) \in (C^\infty[0,L])^n$.

Multiplying $\Psi = \Psi(t,x)$ on both sides of (2.9) and integrating by parts, we get

$$\begin{aligned} \langle F(\Phi_T), \Psi_T \rangle =& \langle \Lambda^+ U^+(t,0), \Psi^+(t,0) \rangle + \langle \Lambda^- U^-(t,0), \Psi^-(t,0) \rangle \\ & - \langle \Lambda^+ U^+(t,L), \Psi^+(t,L) \rangle - \langle \Lambda^- U^-(t,L), \Psi^-(t,L) \rangle. \end{aligned} \tag{2.12}$$

Then, by the boundary conditions in (2.9), we have

$$\langle F(\Phi_T), \Psi_T \rangle = \langle D_0^T \Lambda^+ \Phi^+(t,0), D_0^T \Lambda^+ \Psi^+(t,0) \rangle + \langle D_1^T \Lambda^- \Phi^-(t,L), D_1^T \Lambda^- \Psi^-(t,L) \rangle. \tag{2.13}$$

In particular, taking $\Psi(t,x) = \Phi(t,x)$, we get

$$\begin{aligned} \langle F(\Phi_T), \Phi_T \rangle = & \|D_0^T \Lambda^+ \Phi^+(\cdot,0)\|^2_{(L^2(0,T))^{\overline{m}}} \\ & + \|D_1^T \Lambda^- \Phi^-(\cdot,L)\|^2_{(L^2(0,T))^m}, \quad \forall \Phi_T \in (C^\infty[0,L])^n. \end{aligned} \tag{2.14}$$

Thus, we can define a semi-norm in $(C^\infty[0,L])^n$:

$$\begin{aligned} \|\Phi_T\|_{\mathcal{F}} := & (\|D_0^T \Lambda^+ \Phi^+(\cdot,0)\|^2_{(L^2(0,T))^{\overline{m}}} \\ & + \|D_1^T \Lambda^- \Phi^-(\cdot,L)\|^2_{(L^2(0,T))^m})^{\frac{1}{2}}, \quad \forall\, \Phi_T \in (C^\infty[0,L])^n. \end{aligned} \tag{2.15}$$

However, by Lemma 2.1, we have the observability inequality

$$\begin{aligned} \|\Phi_T\|_{(L^2(0,L))^n} \le & C(\|D_0^T \Lambda^+ \Phi^+(\cdot,0)\|_{(L^2(0,T))^{\overline{m}}} \\ & + \|D_1^T \Lambda^- \Phi^-(\cdot,L)\|_{(L^2(0,T))^m}), \quad \forall \Phi_T \in (C^\infty[0,L])^n. \end{aligned} \tag{2.16}$$

On the other hand, noting that D_0 and D_1 are reversible, by (1.8)–(1.9) we have

$$\begin{aligned} & \|D_0^T \Lambda^+ \Phi^+(\cdot,0)\|_{(L^2(0,T))^{\overline{m}}} + \|D_1^T \Lambda^- \Phi^-(\cdot,L)\|_{(L^2(0,T))^m} \\ \le & C\|\Phi_T\|_{(L^2(0,L))^n}, \quad \forall \Phi_T \in (C^\infty[0,L])^n. \end{aligned} \tag{2.17}$$

By (2.16)–(2.17), the norm (2.15) is equivalent to the $L^2(0,L)$ norm. Thus, the Hilbert space $\mathcal{F}$ as the completion of $(C^\infty[0,L])^n$ with respect to the norm $\|\cdot\|_{\mathcal{F}}$ is just $(L^2(0,L))^n$:

$$\mathcal{F} = (L^2(0,L))^n = \mathcal{F}'.$$

Therefore

$$\langle F(\Phi_T), \Psi_T \rangle = \langle \Phi_T, \Psi_T \rangle_{\mathcal{F}}, \quad \forall\, \Phi_T, \Psi_T \in \mathcal{F} \tag{2.18}$$

defines an inner product on $\mathcal{F}$, and it is easy to see that

$$|\langle F(\Phi_T), \Psi_T \rangle| \leq \|\Phi_T\|_{\mathcal{F}} \|\Psi_T\|_{\mathcal{F}}, \quad \forall\, \Phi_T, \Psi_T \in \mathcal{F}. \tag{2.19}$$

Thus F can be extended to a continuous linear operator

$$F : \mathcal{F} \longrightarrow \mathcal{F}'.$$

Noting (1.8)–(1.9), (2.14) and (2.19), by Lax-Milgram Theorem (see [3, Theorem 6, p.57]), F is an isomorphism from $\mathcal{F}$ to $\mathcal{F}'$. Thus, for any given $U_T \in (L^2(0,L))^n$, there exists $\Phi_T \in (L^2(0,L))^n$, such that $F(\Phi_T) = U_T$, namely, problem (1.1)–(1.3) with (2.5) is exactly controllable. Moreover, since the inverse of F is also continuous, by (2.7)–(2.8) and noting (1.8)–(1.9), we have

$$\begin{aligned}\|H\|_{(L^2(0,T))^n} &\leq C(\|\Phi^-(\cdot,L)\|_{(L^2(0,T))^m} + \|\Phi^+(\cdot,0)\|_{(L^2(0,T))^{\overline{m}}}) \leq C\|\Phi_T\|_{(L^2(0,L))^n} \\ &= C\|F^{-1}(U_T)\|_{(L^2(0,L))^n} \leq C\|U_T\|_{(L^2(0,L))^n}.\end{aligned} \tag{2.20}$$

Finally, we prove that system (1.1)–(1.3) is exactly controllable for any given initial data $U_0(x) \in (L^2(0,L))^n$. By linearity, problem (1.1)–(1.4) can be divided into the following two problems:

$$\begin{cases} U_t + \Lambda U_x + AU = 0, & t \in (0,T), \quad x \in (0,L), \\ U^+(t,0) = G_0 U^-(t,0), & t \in (0,T), \\ U^-(t,L) = G_1 U^+(t,L), & t \in (0,T), \\ t = 0 : \; U(0,x) = U_0(x), & x \in (0,L) \end{cases} \tag{2.21}$$

and

$$\begin{cases} U_t + \Lambda U_x + AU = 0, & t \in (0,T), \quad x \in (0,L), \\ U^+(t,0) = G_0 U^-(t,0) + D_0 H^+(t), & t \in (0,T), \\ U^-(t,L) = G_1 U^+(t,L) + D_1 H^-(t), & t \in (0,T), \\ t = 0 : \; U(0,x) = 0, & x \in (0,L). \end{cases} \tag{2.22}$$

By Theorem 3.1 in [12], problem (2.21) admits a unique weak solution $U = U_1(t,x)$ such that

$$\|U_1(T,\cdot)\|_{(L^2(0,L))^n} \leq C\|U_0\|_{(L^2(0,L))^n}. \tag{2.23}$$

On the other hand, for any given final data $U_T(x) \in (L^2(0,L))^n$, according to the proof mentioned above, there exists a boundary control $H \in (L^2(0,T))^n$, such that the unique weak solution $U = U_2(t,x)$ to problem (2.22) satisfies exactly the final data

$$t = T : \; U = U_T(x) - U_1(T,x), \quad x \in (0,L) \tag{2.24}$$

at the time $t = T$, and, by (2.20) and (2.23) we have

$$\begin{aligned}\|H\|_{(L^2(0,T))^n} &\le C\|U_2(T,x)\|_{(L^2(0,L))^n} = C\|U_T(\cdot) - U_1(T,\cdot)\|_{(L^2(0,L))^n} \\ &\le C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n}).\end{aligned} \tag{2.25}$$

Then, $U(t,x) = U_1(t,x) + U_2(t,x)$ is the unique weak solution to the mixed problem (1.1)–(1.4), satisfying exactly the final state (2.6) at the time $t = T$ under boundary control $H \in (L^2(0,T))^n$, which satsifies (2.4). The proof is complete.

2.2 Relationship between the controllability for the system and the strong observability for the adjoint system

Now, we discuss the relationship between the two-sided exact boundary controllability for system (1.1)–(1.3) and the two-sided strong exact boundary observability for adjoint system (1.5).

For any given full column-rank matrices D_0 of order $\overline{m} \times M_0$ ($M_0 \le \overline{m}$) and D_1 of order $m \times M_1$ ($M_1 \le m$), we have the following theorem.

Theorem 2.2 *If system* (1.1)–(1.3) *is two-sided exactly controllable under boundary control $H(t) \in (L^2(0,T))^M$ ($M = M_0 + M_1 \le n$) in the framework of weak solutions, then the adjoint system* (1.5) *satisfies the following strong D_0/D_1-observability*

$$\text{If } D_0^T \Lambda^+ \Phi^+(t,0) = 0 \text{ and } D_1^T \Lambda^- \Phi^-(t,L) = 0, \quad \text{then } \Phi_T \equiv 0. \tag{2.26}$$

Proof By Definition 3.1 in [12] and noting adjoint system (1.5), if system (1.1)–(1.3) is two-sided exactly controllable, then for the null initial data (2.5) and any given final data $U_T(x) \in (L^2(0,L))^n$, there exists a boundary control $H(t) \in (L^2(0,T))^M$, such that

$$\int_0^L \Phi_T^T(x) U_T(x) \mathrm{d}x = \int_0^T (\Phi^+)^T(t,0)\Lambda^+ D_0 H^+(t)\mathrm{d}t - \int_0^T (\Phi^-)^T(t,L)\Lambda^- D_1 H^-(t)\mathrm{d}t. \tag{2.27}$$

If $D_0^T \Lambda^+ \Phi^+(t,0) = 0$ and $D_1^T \Lambda^- \Phi^-(t,L) = 0$, then the left-hand side of (2.27) is equal to zero for any given final data $U_T(x)$ in $(L^2(0,L))^n$, thus $\Phi_T \equiv 0$. The proof is complete.

The strong D_0/D_1-observability (2.26) is weaker than the strong observability inequality (2.3), but it indicates the relationship between the controllability and the strong observability in a more direct way.

Since both the number of boundary controls and that of boundary observations are equal to $M = M_0 + M_1 = \text{rank}(D_0) + \text{rank}(D_1) (\le n)$, by Theorem 2.1 and Theorem 2.2, we have the following corollary.

Corollary 2.1 *If the number of boundary controls can not be reduced for the two-sided exact boundary controllability for system* (1.1)–(1.3)*, then the number of boundary observations can not be reduced for realizing the two-sided strong exact boundary observability for adjoint system* (1.5)*.*

Conversely, if the number of boundary observations can not be reduced for the two-sided strong exact boundary observability for adjoint system (1.5)*, then the number of boundary controls can not be reduced for the two-sided exact boundary controllability for system* (1.1)–(1.3)*.*

2.3 Optimality

In this subsection we will prove that the boundary control provided by the HUM method possesses the least L^2 norm.

Theorem 2.3 *Under the assumptions of Theorem* 2.1, *for any given initial data* $U_0(x) \in (L^2(0,L))^n$ *and final data* $U_T(x) \in (L^2(0,L))^n$, *the boundary control provided by the HUM method possesses the least* L^2 *norm among all the boundary controls which realize the exact boundary controllability.*

Proof At the end of the proof of Theorem 2.1, we have shown that in order to find the boundary control for realizing the exact boundary controllability of system (1.1)–(1.3) for any given initial data $U_0(x) \in (L^2(0,L))^n$, it suffices to construct the boundary control for the exact boundary controllability of system (1.1)–(1.3) with null initial data (2.5) by the HUM method, thus, we only need to prove Theorem 2.3 for system (1.1)–(1.3) with null initial data (2.5).

Let

$$J(H) = \frac{1}{2}\int_0^T |H|^2 \mathrm{d}t, \quad H \in (L^2(0,T))^n. \tag{2.28}$$

For any given $U_T(x) \in (L^2(0,L))^n$, we want to solve the following minimum problem

$$J(H^*) = \inf_{H \in \mathcal{U}_{ad}} J(H), \quad H^* \in \mathcal{U}_{ad}, \tag{2.29}$$

where $\mathcal{U}_{ad} = \{H \in (L^2(0,T))^n \mid U(T,\cdot) = U_T\}$, in which $U = U(t,x)$ is the weak solution to mixed problem (1.1)–(1.3) with (2.5).

Define the penalty functional as follows: For any given $\varepsilon > 0$,

$$J_\varepsilon(H,U) = J(H) + \frac{1}{\varepsilon}K(H,U), \quad (H,U) \in \mathcal{W}, \tag{2.30}$$

where $J(H)$ is given by (2.28),

$$K(H,U) = \frac{1}{2}\int_0^T \int_0^L (U_t + \Lambda U_x + AU)^2 \mathrm{d}x\mathrm{d}t, \tag{2.31}$$

and $\mathcal{W}$ is the set of (H,U), such that $H \in (L^2(0,T))^n$, and $U \in (L^2(0,T;L^2(0,L)))^n$ satisfies

$$\begin{cases} U_t + \Lambda U_x + AU \in (L^2(0,T;L^2(0,L)))^n, & t \in (0,T), \quad x \in (0,L), \\ U^+(t,0) = G_0 U^-(t,0) + D_0 H^+(t), & t \in (0,T), \\ U^-(t,L) = G_1 U^+(t,L) + D_1 H^-(t), & t \in (0,T), \\ t = 0: \ U(0,x) = 0, & x \in (0,L), \\ t = T: \ U(T,x) = U_T, & x \in (0,L) \end{cases} \tag{2.32}$$

in the sense of weak solutions (see [12, Theorem 3.3]). Since system (1.1)–(1.3) is exactly controllable, we have $\emptyset \neq \mathcal{U}_{ad} \subseteq \mathcal{W}$.

For any given $\varepsilon > 0$, consider the optimal control problem

$$\inf J_\varepsilon(H,U), \quad (H,U) \in \mathcal{W}. \tag{2.33}$$

Noticing that, for each $\varepsilon > 0$, $\mathcal{W}$ is a closed convex set, by Theorem 2 in [3, p.54], this optimal control problem admits a unique minimizer $(H_\varepsilon, U_\varepsilon) \in \mathcal{W}$, such that

$$J_\varepsilon(H_\varepsilon, U_\varepsilon) = \inf J_\varepsilon(H, U), \quad (H, U) \in \mathcal{W}. \tag{2.34}$$

Noting that for all $H \in \mathcal{U}_{ad}$, $J_\varepsilon(H, U) = J(H)$, we have

$$J_\varepsilon(H_\varepsilon, U_\varepsilon) \le J(H), \quad \forall H \in \mathcal{U}_{ad}, \ \forall \varepsilon > 0,$$

then

$$J_\varepsilon(H_\varepsilon, U_\varepsilon) \le \inf_{H \in \mathcal{U}_{ad}} J(H), \quad \forall \varepsilon > 0. \tag{2.35}$$

Thus, noting (2.30), H_ε is bounded in $(L^2(0,T))^n$, and

$$\|(U_\varepsilon)_t + \Lambda (U_\varepsilon)_x + AU_\varepsilon\|_{(L^2(0,T;L^2(0,L)))^n} \le C\sqrt{\varepsilon}, \quad \forall \varepsilon > 0. \tag{2.36}$$

Then, by Theorem 3.3 in [12], we have

$$\|U_\varepsilon\|_{(L^2(0,T;L^2(0,L)))^n} \le C, \quad \forall \varepsilon > 0, \tag{2.37}$$

thus, there exist subsequences, still denoted by $\{H_\varepsilon\}$ and $\{U_\varepsilon\}$, such that

$$H_\varepsilon \rightharpoonup H^* \quad \text{in } (L^2(0,T))^n \text{ as } \varepsilon \to 0 \text{ (weak convergence)} \tag{2.38}$$

and

$$U_\varepsilon \rightharpoonup U^* \quad \text{in } (L^2(0,T;L^2(0,L)))^n \text{ as } \varepsilon \to 0 \text{ (weak convergence).} \tag{2.39}$$

By (2.36) and (2.38)–(2.39), taking $\varepsilon \to 0$ in (2.32), we have

$$\begin{cases} U^*_t + \Lambda U^*_x + AU^* = 0, & t \in (0,T), \quad x \in (0,L), \\ U^{*+}(t,0) = G_0 U^{*-}(t,0) + D_0 H^{*+}(t), & t \in (0,T), \\ U^{*-}(t,L) = G_1 U^{*+}(t,L) + D_1 H^{*-}(t), & t \in (0,T), \\ t = 0: \ U^*(0,x) = 0, & x \in (0,L), \\ t = T: \ U^*(T,x) = U_T, & x \in (0,L) \end{cases} \tag{2.40}$$

in the sense of weak solutions, hence, $H^* \in \mathcal{U}_{ad}$.

On the other hand, by

$$J(H_\varepsilon) \le J_\varepsilon(H_\varepsilon, U_\varepsilon) \tag{2.41}$$

and noting that J is weakly lower semi-continuous, we have

$$J(H^*) \le \liminf J(H_\varepsilon) \le \liminf J_\varepsilon(H_\varepsilon, U_\varepsilon). \tag{2.42}$$

Combining (2.35) and (2.42), we get

$$J(H^*) = \inf_{H \in \mathcal{U}_{ad}} J(H) \tag{2.43}$$

and

$$\lim_{\varepsilon\to 0} J(H_\varepsilon) = J(H^*). \tag{2.44}$$

Noting (2.28), (2.38) and (2.44), we have

$$H_\varepsilon \to H^* \quad \text{in } (L^2(0,T))^n \text{ as } \varepsilon \to 0 \text{ (strong convergence)}. \tag{2.45}$$

Finally, we construct the boundary control H^* with the least norm, and we will see that it is just the boundary control provided by the HUM method. Let

$$p_\varepsilon = \frac{1}{\varepsilon}((U_\varepsilon)_t + \Lambda(U_\varepsilon)_x + AU_\varepsilon), \quad \forall \varepsilon > 0.$$

The Euler equation corresponding to the minimum problem (2.33) is

$$\int_0^T H_\varepsilon^T V \mathrm{d}t + \int_0^T \int_0^L p_\varepsilon^T (W_t + \Lambda W_x + AW)\mathrm{d}x\mathrm{d}t = 0 \tag{2.46}$$

for any given $V \in (L^2(0,T))^n$ and for any given W satisfying

$$\begin{cases} W_t + \Lambda W_x + AW \in (L^2(0,T;L^2(0,L)))^n, & t\in(0,T),\quad x\in(0,L), \\ W^+(t,0) = G_0 W^-(t,0) + D_0 V^+(t), & t\in(0,T), \\ W^-(t,L) = G_1 W^+(t,L) + D_1 V^-(t), & t\in(0,T), \\ t=0:\ W(0,x)=0, & x\in(0,L), \\ t=T:\ W(T,x)=0, & x\in(0,L) \end{cases} \tag{2.47}$$

in the sense of weak solutions (see [3, Theorem 2, p.54]). Integrating by parts, we have

$$\begin{aligned} &\int_0^T \int_0^L p_\varepsilon^T (W_t + \Lambda W_x + AW)\mathrm{d}x\mathrm{d}t \\ = &\int_0^T ((p_\varepsilon^+)^T(t,L)\Lambda^+ + (p_\varepsilon^-)^T(t,L)\Lambda^- G_1) W^+(t,L)\mathrm{d}t \\ &- \int_0^T ((p_\varepsilon^+)^T(t,0)\Lambda^+ G_0 + (p_\varepsilon^-)^T(t,0)\Lambda^-) W^-(t,0)\mathrm{d}t \\ &+ \int_0^T (p_\varepsilon^-)^T(t,L)\Lambda^- D_1 V^-(t)\mathrm{d}t - \int_0^T (p_\varepsilon^+)^T(t,0)\Lambda^+ D_0 V^+(t)\mathrm{d}t \\ &- \int_0^T \int_0^L ((p_\varepsilon)_t + \Lambda(p_\varepsilon)_x - A^T p_\varepsilon)^T W \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.48}$$

Comparing with (2.46), we get

$$\begin{cases} (p_\varepsilon)_t + \Lambda(p_\varepsilon)_x - A^T p_\varepsilon = 0, & t\in(0,T),\quad x\in(0,L), \\ p_\varepsilon^-(t,0) = -(\Lambda^-)^{-1} G_0^T \Lambda^+ p_\varepsilon^+(t,0), & t\in(0,T), \\ p_\varepsilon^+(t,L) = -(\Lambda^+)^{-1} G_1^T \Lambda^- p_\varepsilon^-(t,L), & t\in(0,T) \end{cases} \tag{2.49}$$

and

$$H_\varepsilon^-(t) = -D_1^T \Lambda^- p_\varepsilon^-(t,L), \quad H_\varepsilon^+(t) = D_0^T \Lambda^+ p_\varepsilon^+(t,0). \tag{2.50}$$

By Lemma 2.1 and noting (2.50), it follows from (2.49) that

$$\begin{aligned}\|p_\varepsilon(T,x)\|_{(L^2(0,L))^n} &\le C(\|D_1^T\Lambda^- p_\varepsilon^-(\cdot,L)\|_{(L^2(0,T))^m} + \|D_0^T\Lambda^+ p_\varepsilon^+(\cdot,0)\|_{(L^2(0,T))^{\overline{m}}})\\ &= C(\|H_\varepsilon^-\|_{(L^2(0,T))^m} + \|H_\varepsilon^+\|_{(L^2(0,T))^{\overline{m}}}),\end{aligned} \tag{2.51}$$

where $C>0$ is a constant independent of ε. Noting that H_ε is bounded in $(L^2(0,T))^n$, by Lemma 1.1, $p_\varepsilon(\forall\varepsilon>0)$ is bounded in $(L^2(0,T;L^2(0,L)))^n$, thus there exists a subsequence, still denoted by $\{p_\varepsilon\}$, such that

$$p_\varepsilon \rightharpoonup p \quad \text{in } (L^2(0,T;L^2(0,L)))^n \text{ as } \varepsilon\to 0 \text{ (weak convergence)}, \tag{2.52}$$

$$p_\varepsilon(T,x) \rightharpoonup p_T \quad \text{in } (L^2(0,L))^n \text{ as } \varepsilon\to 0 \text{ (weak convergence)}. \tag{2.53}$$

By (2.49)–(2.50), taking $\varepsilon\to 0$, we get

$$\begin{cases} p_t + \Lambda p_x - A^T p = 0, & t\in(0,T),\quad x\in(0,L),\\ p^-(t,0) = -(\Lambda^-)^{-1}G_0^T\Lambda^+ p^+(t,0), & t\in(0,T),\\ p^+(t,L) = -(\Lambda^+)^{-1}G_1^T\Lambda^- p^-(t,L), & t\in(0,T),\\ t=T:\ p(T,x)=p_T, & x\in(0,L)\end{cases} \tag{2.54}$$

in the sense of weak solutions, and

$$H^{*-}(t) = -D_1^T\Lambda^- p^-(t,L),\quad H^{*+}(t) = D_0^T\Lambda^+ p^+(t,0). \tag{2.55}$$

Let $\Phi=p$, $\Phi_T=p_T$ and $\Psi=U^*$. We have

$$\begin{cases} \Phi_t + \Lambda\Phi_x - A^T\Phi = 0, & t\in(0,T),\quad x\in(0,L),\\ \Phi^-(t,0) = -(\Lambda^-)^{-1}G_0^T\Lambda^+\Phi^+(t,0), & t\in(0,T),\\ \Phi^+(t,L) = -(\Lambda^+)^{-1}G_1^T\Lambda^-\Phi^-(t,L), & t\in(0,T),\\ t=T:\ \Phi(T,x)=\Phi_T(x), & x\in(0,L)\end{cases} \tag{2.56}$$

and

$$\begin{cases} \Psi_t + \Lambda\Psi_x + A\Psi = 0, & t\in(0,T),\quad x\in(0,L),\\ \Psi^+(t,0) = G_0\Psi^-(t,0) + D_0D_0^T\Lambda^+\Phi^+(t,0), & t\in(0,T),\\ \Psi^-(t,L) = G_1\Psi^+(t,L) - D_1D_1^T\Lambda^-\Phi^-(t,L), & t\in(0,T),\\ t=0:\ \Psi(0,x)=0, & x\in(0,L),\\ t=T:\ \Psi(T,x)=U_T, & x\in(0,L).\end{cases} \tag{2.57}$$

Thus we get the following mapping F:

$$F(\Phi_T) = U_T, \tag{2.58}$$

which is an isomorphism from $(L^2(0,L))^n$ to $(L^2(0,L))^n$. Therefore, for any given $U_T\in(L^2(0,L))^n$, the boundary control with the least norm, which realizes the exact boundary controllability for system (1.1)–(1.3), is just the boundary control given by (2.7)–(2.8) according to the HUM method.

3 One-Sided Exact Boundary Controllability of Weak Solutions by the HUM Method

Given much control time, the exact boundary controllability can be realized by controls applying only on one side of the boundary, which will reduce largely the number of boundary controls. However, in this case, we have to add some assumptions on the coupling matrix on the boundary so that the one-sided strong exact boundary observability inequality holds, and, based on which, the HUM method can be applied to establish the one-sided exact boundary controllability for system (1.1)–(1.3).

We first give the corresponding one-sided strong observability.

Lemma 3.1 (see [12, Lemma 5.2]) *Let $T \geq \overline{T}_0$, where*

$$\overline{T}_0 = L\Big(\max_{1\leq r\leq m}\frac{1}{|\lambda_r|} + \max_{m+1\leq s\leq n}\frac{1}{\lambda_s}\Big) > 0. \tag{3.1}$$

Assume that $\Phi = \Phi(t,x)$ is the weak solution to adjoint problem (1.5)–(1.6)*. Assume furthermore that*

$$\overline{m} \leq m \ (i.e.,\ n \leq 2m) \tag{3.2}$$

and

$$\mathrm{rank}(G_0) = \overline{m}. \tag{3.3}$$

If

$$M = M_1 = \mathrm{rank}(D_1) = m, \tag{3.4}$$

then for any given final data $\Phi_T(x) \in (L^2(0,L))^n$, we have the following strong observation inequality

$$\|\Phi_T\|_{(L^2(0,L))^n} \leq C\|D_1^T\Lambda^-\Phi^-(\cdot,L)\|_{(L^2(0,T))^m}. \tag{3.5}$$

Theorem 3.1 *Let $T \geq \overline{T}_0$, where $\overline{T}_0$ is given by* (3.1)*. Assume that $H^+(t) \equiv 0$. Under the assumptions of* (3.2)–(3.4)*, for any given initial data $U_0(x) \in (L^2(0,L))^n$ and any given final data $U_T(x) \in (L^2(0,L))^n$, system* (1.1)–(1.3) *is exactly controllable at the time $t = T$, and the boundary control $H^-(t) \in (L^2(0,T))^m$ satisfies*

$$\|H^-\|_{(L^2(0,T))^m} \leq C(\|U_0\|_{(L^2(0,L))^n} + \|U_T\|_{(L^2(0,L))^n}). \tag{3.6}$$

Proof Similarly, we need only to prove that for any given final data $U_T(x) \in (L^2(0,L))^n$, there exists a boundary control $H^- \in (L^2(0,T))^m$, such that the unique weak solution $U = U(t,x)$ to the mixed problem (1.1)–(1.3) with null initial data (2.5) satisfies exactly final state (2.6) at the time $t = T$.

Setting

$$H^-(t) = -D_1^T\Lambda^-\Phi^-(t,L), \tag{3.7}$$

where $\Phi = \Phi(t,x)$ is a solution to adjoint problem (1.5)–(1.6) with any given final data $\Phi_T \in (C^\infty[0,L])^n$, we consider the following mixed problem

$$\begin{cases} U_t + \Lambda U_x + AU = 0, & t \in (0,T), \quad x \in (0,L), \\ U^+(t,0) = G_0 U^-(t,0), & t \in (0,T), \\ U^-(t,L) = G_1 U^+(t,L) - D_1 D_1^T \Lambda^- \Phi^-(t,L), & t \in (0,T) \end{cases} \tag{3.8}$$

with (2.5). For any given $\Phi_T \in (C^\infty[0,L])^n$, let $U = U(t,x)$ be the weak solution to problem (3.8) with (2.5), and let

$$t = T: \ U(T,x) = U_T(x), \quad x \in (0,L). \tag{3.9}$$

We get the linear mapping F:

$$F(\Phi_T) = U_T. \tag{3.10}$$

It suffices to prove that F can be extended to be surjective from $(L^2(0,L))^n$ to $(L^2(0,L))^n$.

In fact, let $\Psi = \Psi(t,x)$ be a solution to adjoint problem (1.5)–(1.6) with final data $\Psi_T = \Psi_T(x) \in (C^\infty[0,L])^n$. Multiplying $\Psi = \Psi(t,x)$ on both sides of (3.8), and integrating by parts, we get

$$\langle F(\Phi_T), \Psi_T \rangle = \langle D_1^T \Lambda^- \Phi^-(t,L), D_1^T \Lambda^- \Psi^-(t,L) \rangle. \tag{3.11}$$

In particular, taking $\Psi(t,x) = \Phi(t,x)$, we get

$$\langle F(\Phi_T), \Phi_T \rangle = \|D_1^T \Lambda^- \Phi^-(\cdot,L)\|^2_{(L^2(0,T))^m}, \quad \forall\, \Phi_T \in (C^\infty[0,L])^n. \tag{3.12}$$

However, by Lemma 3.1, we have

$$\|\Phi_T\|_{(L^2(0,L))^n} \le C \|D_1^T \Lambda^- \Phi^-(\cdot,L)\|_{(L^2(0,T))^m}, \quad \forall\, \Phi_T \in (C^\infty[0,L])^n. \tag{3.13}$$

Then, we can define a semi-norm in $(C^\infty[0,L])^n$:

$$\|\Phi_T\|_{\mathcal{F}} := \|D_1^T \Lambda^- \Phi^-(\cdot,L)\|_{(L^2(0,T))^m}, \quad \forall\, \Phi_T \in (C^\infty[0,L])^n. \tag{3.14}$$

By (1.9), we have

$$\|D_1^T \Lambda^- \Phi^-(\cdot,L)\|_{(L^2(0,T))^m} \le \|\Phi_T\|_{(L^2(0,L))^n}, \quad \forall\, \Phi_T \in (C^\infty[0,L])^n, \tag{3.15}$$

which together with (3.13) shows that the norm $\|\cdot\|_{\mathcal{F}}$ is in fact an equivalent norm to the $L^2(0,L)$ norm. Therefore, the Hilbert space $\mathcal{F}$ as the completion of $(C^\infty[0,L])^n$ with respect to the norm $\|\cdot\|_{\mathcal{F}}$ is just the $(L^2(0,L))^n$ space

$$\mathcal{F} = (L^2(0,L))^n = \mathcal{F}',$$

and

$$|\langle F(\Phi_T), \Psi_T \rangle| \le \|\Phi_T\|_{\mathcal{F}} \|\Psi_T\|_{\mathcal{F}}, \quad \forall\, \Phi_T, \Psi_T \in \mathcal{F}. \tag{3.16}$$

Then F can be extended to a linear continuous operator on $\mathcal{F}$:

$$F : \mathcal{F} \longrightarrow \mathcal{F}'.$$

Noting (1.9), (3.12) and (3.16), by Lax-Milgram Theorem (see [3, Theorem 6, p.57]), F is an isomorphism from $\mathcal{F}$ to $\mathcal{F}'$. Thus, for any given $U_T \in (L^2(0,L))^n$, there exists $\Phi_T \in (L^2(0,L))^n$, such that $F(\Phi_T) = U_T$. The rest of the proof is similar to that of Theorem 2.1.

Remark 3.1 Under assumptions (3.2)–(3.3), to realize the one-sided exact boundary controllability of system (1.1)–(1.3), boundary controls should be given on $x = L$, where there are more coming characteristics, and the number of boundary controls is reduced to m ($\overline{m} \leq m < n$).

Remark 3.2 Corollary 2.1 and Theorem 2.3 are also true for the one-sided exact boundary controllability.

Moreover, as a direct consequence of Theorem 2.2, for any given full column-rank matrix D_1 of order $m \times M_1$ ($M_1 \leq m$), we have the following theorem.

Theorem 3.2 *If system* (1.1)–(1.3) *with* $H^+(t) \equiv 0$ *is one-sided exactly controllable under boundary control* $H^-(t) \in (L^2(0,T))^M$ ($M = M_1 \leq m < n$), *then adjoint system* (1.5) *satisfies the following strong* D_1*-observability*:

$$\textit{If} \quad D_1^T \Lambda^- \Phi^-(t,L) = 0, \quad \textit{then} \quad \Phi_T \equiv 0. \tag{3.17}$$

Remark 3.3 The two-sided exact boundary controllability with fewer boundary controls can be similarly treated by the HUM method.

4 The Comparison Between the Constructive Method and the HUM Method

Both the constructive method and the HUM method can be used to establish the exact boundary controllability for system (1.1)–(1.3) (see Lemmas 4.1–4.2 in [12] and Theorem 2.1, Theorem 3.1 in this paper). They obtain almost the same results from two different points of view: The constructive method is to construct a solution, satisfying exactly the given initial data and final data, by successively solving some forward problem, backward problem, leftward (resp. rightward) problem of system (1.1)–(1.3) with suitable artificial boundary conditions. It is more direct and easier to be understood. While, the HUM method treats the problem by means of the duality, the problem of controllability is then transformed into a problem of observability for the homogeneous adjoint system. Then, to prove the observability inequality is to make estimates of the solution to this homogeneous system. We will make some comments about these two methods in what follows.

1. The constructive method can be easily used to establish not only the controllability of a system, but also the observability of the adjoint system. While, for the HUM method, to prove the controllability of a system is based on the observability inequality of the adjoint system, and in this paper, the observability inequality is also proved by the constructive method.

2. For the problem with inhomogeneous boundary conditions, the constructive method can be also used to establish its controllability and observability. For example, for one-sided exact boundary controllability, by Lemma 4.2 in [12], system (1.1)–(1.3) is exactly controllable for any given boundary control matrix D_0 and boundary function $H^+(t)$ under some hypotheses. But in order to apply the HUM method, we have to assume that there is no boundary control on $x = 0$, namely, $H^+(t) \equiv 0$ in (1.2). Otherwise, (3.11) becomes

$$\langle F(\Phi_T), \Psi_T \rangle = \langle D_1^T \Lambda^- \Phi^-(t, L), D_1^T \Lambda^- \Psi^-(t, L) \rangle + \langle D_0 H^+, \Lambda^+ \Psi^+(t, 0) \rangle,$$

and (3.12) becomes

$$\langle F(\Phi_T), \Phi_T \rangle = \| D_1^T \Lambda^- \Phi^-(\cdot, L) \|^2_{(L^2(0,T))^m} + \langle D_0 H^+, \Lambda^+ \Phi^+(t, 0) \rangle, \quad \forall\, \Phi_T \in (C^\infty[0, L])^n,$$

then $\langle F(\Phi_T), \Phi_T \rangle$ can not define a semi-norm, hence the HUM method fails.

3. The constructive method can be efficiently applied to deal with problems in one-dimensional space, but very difficult for higher dimensional spaces. However, to consider the problem by means of the duality in the framework of weak solutions will be possible and helpful for the study in higher dimensional spaces.

4. By the HUM method, the close relationship between controllability and observability can be better revealed. Take a coupled system of wave equations as an example, two important and difficult issues can be solved by the duality in the study of synchronization and related problems, one is the non-controllability when there is a lack of boundary controls, while, the other is the necessity of the compatibility conditions of the coupling matrix for synchronization. As to the first issue for first order hyperbolic systems, so far there is no further result except for some special decoupled equations (see [7]). As to the second issue, we will take into consideration together with related subjects of synchronization for first order hyperbolic systems in our future work.

5. In the proof of controllability by the constructive method, although the boundary control can be obtained, however, taking the case of one-sided controls for instance, we have to solve a forward problem, a backward problem, then a rightward (resp. leftward) problem. While, by the HUM method, since the boundary control is determined by the solution to the adjoint problem, we only have to solve a backward problem, and the obtained boundary control has an explicit expression. Moreover, the boundary control provided by the HUM method possesses the least L^2 norm.

6. The HUM method is not available in dealing with the exact boundary null controllability for first order hyperbolic systems in general. In fact, when the corresponding weak exact boundary observability inequality holds, since the first order hyperbolic system may not be time reversible, we do not have similar inequalities as (1.8)–(1.9) with respect to the initial data, then we can not obtain an isomorphic mapping as F, so that the HUM method does not work.

7. In the framework of weak solutions, system (1.1)–(1.3) is still exactly (null) controllable when the control time T is equal to T_0. However, in the framework of classical solutions, in order to guarantee the continuity of the constructed solution, it sharply requires $T > T_0$, which

is determined by the regularity requirement of the classical solution, but not required in the framework of weak solutions. This property is still true for related results on the observability.

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