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# Orbifold Stiefel-Whitney Classes of Real Orbifold Vector Bundles over Right-Angled Coxeter Complexes*

Lisu WU

**Abstract** The author gives a definition of orbifold Stiefel-Whitney classes of real orbifold vector bundles over special q-CW complexes (i.e., right-angled Coxeter complexes). Similarly to ordinary Stiefel-Whitney classes, orbifold Stiefel-Whitney classes here also satisfy the associated axiomatic properties.

**Keywords** Right-Angled Coxeter orbifold, Stiefel-Whitney class, Group representation
**2020 MR Subject Classification** 57R18, 57R20

## 1 Introduction

The definition of characteristic classes of an orbifold vector bundle depends on the cohomology ring of its base space (an orbifold). The de Rham cohomology groups of an orbifold are introduced by Satake [15], so one can define orbifold chern classes of a good complex orbifold vector bundle by Chern-Weil construction, which take values in de Rham cohomology groups of base orbifold. Moreover, this definition can be extended to bad orbifold vector bundles (see [17]).

In addition, one can take the equivalent cohomology ring as the cohomology ring of a quotient orbifold. Now the equivalent characteristic classes can be viewed as orbifold characteristic classes. In the book of Adem-Leida-Ruan [1], the orbifold characteristic classes defined lie in the cohomology rings of classifying spaces of the orbifold groupoids. According to [1, Example 2.11], their orbifold characteristic classes actually correspond to the equivalent characteristic classes.

However, the integral and Mod-two integral cohomology rings of general orbifolds are unclear. So it is difficult to define orbifold characteristic classes in the usual way (see [14]).

Recently, Lü-Wu-Yu [12] introduced integral orbifold cellular homology groups of Coxeter complexes by applying the idea of blow-up. In this paper, we define and study orbifold Stiefel-

Manuscript received January 31, 2021. Revised May 25, 2021.
School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: wulisuwulisu@qq.com
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Whitney classes on right-angled Coxeter complexes based on the cohomology groups of Lü-Wu-Yu.

An $n$-dimensional right-angled Coxeter orbifold is a special $n$-orbifold locally modelled on the quotient $\mathbb{R}^n/(\mathbb{Z}_2)^k$ of the standard $(\mathbb{Z}_2)^k$-action on $\mathbb{R}^n$. Here, a $(\mathbb{Z}_2)^k$-action on $\mathbb{R}^n$ is standard if it can be generated by reflections across some coordinate hyperplanes in $\mathbb{R}^n$. A right-angled Coxeter complex is a special q-CW complex defined by Poddar-Sarkar [14] satisfying that its q-cells (i.e., the quotient of a cell by a finite group) are the orbit of $e^n$ by a standard $(\mathbb{Z}_2)^k$-action, and its all attaching maps $\phi$ preserve local groups. Here, preserving local groups means preserving codimension. So we can define the nerve of a right-angled Coxeter complex. See Section 2.1 for more details.

Firstly, we define the Stiefel-Whitney classes of real orbifold vector bundles over $D^n/(\mathbb{Z}_2)^k$, where $D^n/(\mathbb{Z}_2)^k$ is a quotient orbifold of a standard $(\mathbb{Z}_2)^k$-action on $D^n$. Let $\pi : E \to D^n/(\mathbb{Z}_2)^k$ be an $m$-dimensional orbifold vector bundle over $D^n/(\mathbb{Z}_2)^k$. Then by the definition of orbifold vector bundles, $\pi : E \to D^n/(\mathbb{Z}_2)^k$ is determined by a real linear representation of $(\mathbb{Z}_2)^k$,

$$\rho : (\mathbb{Z}_2)^k \to GL_m(\mathbb{R}).$$

Furthermore, by the equivalence of orbifold vector bundles (see Lemma 3.2), $\pi : E \to D^n/(\mathbb{Z}_2)^k$ is determined by an $m \times k$ matrix,

$$C = \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^k \\ x_2^1 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^k \end{pmatrix}_{m\times k}$$

where $x_j^i = \pm 1$. We call $C$ the characteristic matrix of $\pi : E \to D^n/(\mathbb{Z}_2)^k$. Then the total Stiefel-Whitney class of $\pi : E \to D^n/(\mathbb{Z}_2)^k$ is defined as:

$$w(E) = \prod_{i=1}^{m} \Big(1 + \sum_{j=1}^{k} \frac{1 - x_i^j}{2} s_j\Big) \in H^*_{\text{orb}}(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2), \tag{1.1}$$

where $H^*_{\text{orb}}(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2) = \mathbb{Z}_2[s_1, \cdots, s_k]/(s_i^2, i = 1, \cdots, k)$.

Let $X$ be an arbitrary right-angled Coxeter complex, and $\pi : E \to X$ be a real $m$-dimensional orbifold vector bundle over $X$. Next, we define the Stiefel-Whitney classes of $\pi : E \to X$. Let $X_{\text{reg}}$ be the subcomplex of $X$ consisting of all regular cells (i.e., associated local group is trivial). Then $X/X_{\text{reg}}$ is a wedge sum of some right-angled Coxeter complex,

$$X/X_{\text{reg}} = \bigvee H.$$

Each component $H$ contains no regular cell except the unique 0-cell. Let $\mathcal{H} = \{F_1, \cdots, F_\eta\}$ be the local codimension-one faces of $H$. Then the vector bundle $\pi : E \to X$ induces a vector bundle over $H$, denoted by $\pi_H : E_H \to H$. Then there is also an $m \times \eta$ characteristic matrix, denoted by $C_H$. Notice that there is a simplicial map from the nerve $\mathcal{N}(H)$ to $\Delta^\eta = [v_1, \cdots, v_\eta]$,

$$j : \mathcal{N}(H) \to \Delta^\eta.$$

Assume that $\{s_1, \cdots, s_\eta\}$ is the vertices set of $\mathcal{N}(H)$, then $j(s_i) = v_i$ and $j(\delta)$ is a simplex in $\Delta^\eta$ spanned by $\{j \circ \delta(s_1), \cdots, j \circ \delta(s_k)\}$ for simplex $\delta : [s_{i_1}, \cdots, s_{i_k}] \to \mathcal{N}(H)$. Clearly, $j$ induces a homomorphism

$$j^* : \mathbb{Z}_2[v_1, \cdots, v_\eta] \longrightarrow \mathcal{R}_H.$$

Then the Stiefel-Whitney class of $\pi_H : E_H \to H$ is defined as:

$$w(E_H) = j^*(w(E_{C_H})) \in \mathcal{R}_H < H^*(X; \mathbb{Z}_2),$$

where $\mathcal{R}_H$ is a sub-ring of $H^*(X; \mathbb{Z}_2)$, which is generated by the duals of locally codimension-one faces in $H$. Finally, the total Stiefel-Whitney class of $\pi : E \to X$ is defined as:

$$w(E) = w(E_{\text{reg}}) \cdot \prod_H w(E_H), \tag{1.2}$$

where $\pi |_{X_{\text{reg}}}: E_{\text{reg}} \to X_{\text{reg}}$ is a restriction of $\pi : E \to X$ on $X_{\text{reg}}$.

Similarly to the usual cases, the Stiefel-Whitney classes of right-angled Coxeter complexes satisfy the following axioms.

**Proposition 1.1** *There is a unique sequence of functions $w_1, w_2, \cdots$ assigning to each real orbifold vector bundle $E \to B$ a class $w_i(E) \in H^i(B; \mathbb{Z}_2)$, namely, depending only on the isomorphism type of $E$, such that*

(a) $w_i(f^*(E)) = f^*(w_i(E))$ *for a pullback $f^*(E)$, where $f$ is an orbifold map which preserves local groups.*

(b) $w(E_1 \oplus E_2) = w(E_1)w(E_2)$, *where $w = 1 + w_1 + w_2 + \cdots \in H^*(B; \mathbb{Z}_2)$.*

(c) $w_i(E) = 0$ *if $i > \dim E$.*

(d) *For the canonical line bundle $E \to \mathbb{R}P^\infty$, $w_1(E)$ is the generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$. Meanwhile, for the nontrivial line bundle $\widetilde{E} \to D^1/\mathbb{Z}_2$, $w_1(\widetilde{E})$ is the generator of $H^1_{\text{orb}}(D^1/\mathbb{Z}_2; \mathbb{Z}_2)$.*

**Remark 1.1** Orbifold Stiefel-Whitney classes of right-angled Coxeter complexes are generalizations of ordinary Stiefel-Whitney classes.

As an application, we have the following conclusion.

**Theorem 1.1** *Let $P$ be an $n$-dimensional simple polytope. Then $P$ is the product of two simple polytopes $P_1$ and $P_2$ with dimensions $n_1$ and $n_2$, respectively, if and only if, $w_n(TP) = w_{n_1}(TP_1) \cdot w_{n_2}(TP_2)$.*

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we define Stiefel-Whitney classes of real orbifold vector bundles over $D^n/(\mathbb{Z}_2)^k$. In Section 4, we consider the general cases, that is, Stiefel-Whitney classes of real orbifold vector bundles over a right-angled Coxeter complex. In Section 5, we prove Theorem 1.1 and give some examples.

## 2 Preliminaries

### 2.1 Right-Angled Coxeter orbifolds and right-angled Coxeter complexes

An $n$-dimensional right-angled Coxeter orbifold (see [7]) is a special $n$-orbifold locally modelled on the quotient $\mathbb{R}^n/(\mathbb{Z}_2)^n$ of the standard $(\mathbb{Z}_2)^n$-action on $\mathbb{R}^n$ by reflections across the coordinate hyperplanes. So a right-angled Coxeter $n$-orbifold is naturally a manifold with corners (the notion of manifold with corners can be referred to Davis [6, Chapter 10]).

A right-angled Coxeter complex $X$ is a special q-CW complex defined by Poddar-Sarkar [14] satisfying that its q-cells (i.e. the quotient of a cell by a finite group) are the orbit of $e^n$ by a standard reflective action of $(\mathbb{Z}_2)^k$, and all attaching maps $\phi$ are required to preserve local groups. For a right-angled Coxeter cell $e^n/(\mathbb{Z}_2)^k$, we call $(\mathbb{Z}_2)^k$ the local group of $e^n/(\mathbb{Z}_2)^k$, and let $\Phi : D^n/(\mathbb{Z}_2)^k \to X$ be the characteristic map for $e^n/(\mathbb{Z}_2)^k$. If the local group of a right-angled Coxeter cell is trivial, then this cell is called to be regular, otherwise it is called to be singular. All the regular cells form a sub-complex of $X$ which is denoted by $X_{\text{reg}}$, and we denote the set of singular cells by $X_{\text{sing}}$ (see [12] for more details).

For each right-angled Coxeter cell $e^n/(\mathbb{Z}_2)^k$, the standard $(\mathbb{Z}_2)^k$-action on $e^n$ induces a natural facial structure on the singular point set of $e^n/(\mathbb{Z}_2)^k$. Attaching maps preserving local groups are equivalent to preserve local codimension. So each right-angled Coxeter complex has a facial structure such that each face has a well-defined local codimension. A local codimension-one face in $X$ is called a facet of $X$. Under the setting, we can define the nerve of $X$. In detail, the nerve of $X$, denoted by $\mathcal{N}(X)$, is a poset on the facet set of $X$ satisfying that:

• Each codimension-$k$ face $f \subset F_1 \cap \cdots \cap F_k$ of $X$ determines a $(k-1)$-simplex $[F_1, \cdots, F_k] \to \mathcal{N}(X)$, where $F_1, \cdots, F_k$ are some facets of $X$.

**Remark 2.1** The nerve of a manifold with corners is defined similarly, one can refer to [6].

**Example 2.1** Let $P$ be an $n$-dimensional simple polytope. Then there is a natural right-angled Coxeter orbifold structure on $P$ such that each codimension-$k$ face of $P$ has local group $(\mathbb{Z}_2)^k$ which is generated by reflections associated with $k$ facets. The cone of once barycentric subdivision of the nerve of $P$ gives a cubical decomposition of $P$, denoted by $\mathcal{C}(P)$, which is called the standard cubical decomposition of $P$. Clearly, each $k$-cube $c^k$ in $\mathcal{C}(P)$ can be represented as the orbit of $[-1, 1]^k$ by a standard $(\mathbb{Z}_2)^k$-action. Hence, $\mathcal{C}(P)$ is a right-angled Coxeter complex.

### 2.2 Cohomology rings of right-angled Coxeter complexes

For a right-angled Coxeter complex, one can define a cellular chain complex, by the result in [12].

**Theorem 2.1** (see [12]) *Let $X$ be an right-angled Coxeter complex. Then*

$$H^i_{\text{orb}}(X) = \bigoplus_{f\in T} H^{i-l(f)}(f), \tag{2.1}$$

*where $T$ is the set of faces of $X$ (including its underlying space as an $n$-face), and $l(f)$ is the local codimension of face $f$ in $X$.*

Actually, the orbifold cohomology groups and cup product of a right-angled Coxeter complex are defined as the usual cohomology groups and cup product of its blow-up discussed in [12].

**Example 2.2** (Cohomology of Closed Right-Angled Coxeter Cells) Let $D^n/(\mathbb{Z}_2)^k$ be a closed right-angled Coxeter cell, where $k \le n$. Then $D^n/(\mathbb{Z}_2)^k = D^k/(\mathbb{Z}_2)^k \times D^{n-k} \simeq D^k/(\mathbb{Z}_2)^k = \left(D^1/\mathbb{Z}_2\right)^k$. So

$$H^*_{\text{orb}}(D^n/(\mathbb{Z}_2)^k;\mathbb{Z}_2) \cong \mathbb{Z}_2[s_1, s_2, \cdots, s_k]/(s_i^2, i = 1, \cdots, k).$$

**Example 2.3** Let $X$ be a right-angled Coxeter complex and let $H$ be a connected component of the singular set of $X$ with nerve $\mathcal{N}$. Then each facet in $H$ corresponds to a vertex of $\mathcal{N}$, and so, corresponds to a generator of $H^1(X;\mathbb{Z}_2)$. All facets in $H$, denoted by $\{F_1, \cdots, F_\eta\}$, generate a sub-ring of $H^*(X;\mathbb{Z}_2)$, which is isomorphic to

$$\mathcal{R}_H = \mathbb{Z}_2[s_1, \cdots, s_\eta]/(I_H + J_H),$$

where $I_H$ is the Stanley-Reisner ideal of $H$ and $J_H = (s_i^2, i = 1, \cdots, \eta)$.

### 2.3 Orbifold vector bundle

The definition of orbifold vector bundle can be refered to [5, 16].

**Definition 2.1** (Orbifold Vector Bundle, see [16]) *Let $E$ and $B$ be two orbifolds with orbifold structures $\mathcal{U}^* = \{U^*, G^*, \psi^*\}$ and $\mathcal{U} = \{U, G, \psi\}$, respectively. An $n$-dimensional orbifold vector bundle is a $C^\infty$-orbifold map $\pi : E \to B$ satisfying the following conditions*:

(1) (*Local trivialization*) *There exists an one-to-one correspondence between $\{U^*, G^*, \psi^*\}$ and $\{U, G, \psi\}$ such that $U^* = U \times \mathbb{R}^m$. Denoting by $\overline{\pi}_*$ the projection $U^* \to U$, we have*

$$\pi \circ \psi^* = \psi \circ \overline{\pi}_*.$$

(2) *Let $\{U^*, G^*, \psi^*\}$, $\{U, G, \psi\}$; $\{U^{*\prime}, G^{*\prime}, \psi^{*\prime}\}$, $\{U', G', \psi'\}$ be two pairs of corresponding local uniformizing system satisfying $\psi(U) \subset \psi'(U')$. Then $\psi^*(U^*) \subset \psi^{*\prime}(U^{*\prime})$ and there exists a one-to-one correspondence $\lambda \leftrightarrow \lambda^*$ between injections $\lambda : \{U, G, \psi\} \to \{U', G', \psi'\}$ and $\lambda^* : \{U^*, G^*, \psi^*\} \to \{U^{*\prime}, G^{*\prime}, \psi^{*\prime}\}$ such that for $(p, q) \in U^* = U \times \mathbb{R}^m$ we, have*

$$\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$$

*with $g_\lambda(p) \in GL_m(\mathbb{R})$. The mapping $g_\lambda : U \to GL_m(\mathbb{R})$ is a $C^\infty$-map satisfying the relation*

$$g_{\mu\lambda}(p) = g_\mu(\lambda(p)) \cdot g_\lambda(p) \tag{2.2}$$

*for any injections* $\{U,G,\psi\} \xrightarrow{\lambda} \{U',G',\psi'\} \xrightarrow{\mu} \{U'',G'',\psi''\}$.

*An orbifold vector bundle is a composite concept of map* $\pi : E \to B$ *and the equivalent class of orbifold structure pair* $(\mathcal{U},\mathcal{U}^*)$, *where two pairs of orbifold vector bundles* $(\mathcal{U},\mathcal{U}_1^*)$ *and* $(\mathcal{U},\mathcal{U}_2^*)$ *are directly equivalent if there exists a* $C^\infty$*-map* $\delta_U : U \to GL_m(\mathbb{R})$ *satisfying that:*

(1) *For any* $(U,G,\psi) \in \mathcal{U}$, *there exist* $(U \times \mathbb{R}^m, G_1^*, \psi_1^*) \in \mathcal{U}_1^*$ *and* $(U \times \mathbb{R}^m, G_2^*, \psi_2^*) \in \mathcal{U}_2^*$ *such that*

$$\psi_1^*(p,q) = \psi_2^*(p, \delta_U(p)q).$$

(2) *For any injection* $\lambda : \{U,G,\psi\} \to \{U',G',\psi'\}$,

$$g_\lambda^2(p) = \delta_{U'}(\lambda(p)) g_\lambda^1(p) \delta_U^{-1}(p), \tag{2.3}$$

**Remark 2.2** For each $b \in B$, there is a chart $(U,G,\psi)$ such that $b \in U/G$. For each $\overline{b}$ in $\psi^{-1}(b)$, the $\overline{\pi}^{-1}(\overline{b})$ is a real vector space $\mathbb{R}^n$. Then

$$\pi^{-1}(b) \cong \overline{\pi}^{-1}(\overline{b})/G^*,$$

where $G^*$ is a subgroup of $G$. Thus an orbifold vector bundle is not always a vector bundle in the usual sense because its fiber may not be a vector space. The above definition of orbifold vector bundles can be extended to orbispace. The notion of orbispace can be refered to [3, 4].

**Example 2.4** Let $D^1/\mathbb{Z}_2 = [-1,1]/\mathbb{Z}_2$ be a closed right-angled Coxeter 1-cell. Then there exist two orbifold line bundles over $D^1/\mathbb{Z}_2$.

$$\begin{aligned} \overline{E} &= D^1/\mathbb{Z}_2 \times \mathbb{R} = D^1 \times \mathbb{R}/(x,y) \sim (-x,y), \\ \widetilde{E} &= D^1 \times \mathbb{R}/(x,y) \sim (-x,-y). \end{aligned}$$

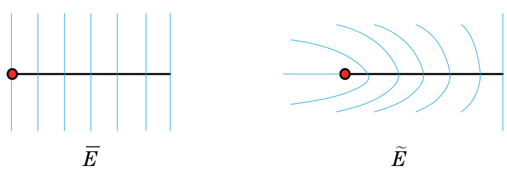


Figure 1 Orbifold line bundles over $D^1/\mathbb{Z}_2$.

Let $\lambda : D^1 \to D^1$ be an injection which maps $x \in D^1$ to $-x$. Let $\overline{g}_\lambda$ be a constant map which corresponds to the line bundle $\overline{E}$. For any $x \in D^1$, $\overline{g}_\lambda(x) = (1) \in GL_1(\mathbb{R})$. Let $\widetilde{g}_\lambda$ be the map which corresponds to the line bundle $\widetilde{E}$. Then for any $x \in D^1$, $\widetilde{g}_\lambda(x) = (-1) \in GL_1(\mathbb{R})$.

The line bundle $\widetilde{E}$ is not a usual vector bundle, which is called the canonical line bundle of $D^1/\mathbb{Z}_2$. Meanwhile, $\widetilde{E}$ is also the orbifold tangent bundle of $D^1/\mathbb{Z}_2$. Moreover, let $q : \overline{E} \to \widetilde{E}$ be a quotient map induced by $(0,y) \sim (0,-y)$, then the bundle $\overline{E}$ can be viewed as a pull back of bundle $\widetilde{E}$.

**Definition 2.2** (Bundle Map, see [16]) *Let $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ be two orbifold vector bundles with orbifold structure $(\mathcal{U}_1, \mathcal{U}_1^*)$ and $(\mathcal{U}_2, \mathcal{U}_2^*)$, respectively. A system of mappings $h^* = \{h_{U_1}\}$ is called a ($C^\infty$-)orbifold bundle map if the following conditions are satisfied:*

(i) *There exists a correspondence $\{U_1, G_1, \psi_1\} \to \{U_2, G_2, \psi_2\}$ from $\mathcal{U}_1$ to $\mathcal{U}_2$, such that for any $\{U_1, G_1, \psi_1\}$, we have a $C^\infty$-map $h^*_{U_1}$ from $U_1 \times \mathbb{R}^m$ to $U_2 \times \mathbb{R}^m$ and a $C^\infty$-map $h_{U_2}$ from $U_1$ to $U_2$ such that*

$$h^*_{U_1}(p, q) = (h_{U_1}(p), r_{U_1}(p)q) \tag{2.4}$$

*with $r_{U_1}(p) \in GL(m)$, where $r_{U_1}$ is a $C^\infty$-map from $U_1$ to $GL(m)$.*

(ii) *Let $\{U_1, G_1, \psi_1\}$, $\{U_1', G_1', \psi_1'\}$ be local uniformizing systems in $\mathcal{U}_1$ such that $\psi_1(U_1) \subset \psi_1'(U_1')$ and $\{U_2, G_2, \psi_2\}$, $\{U_2', G_2', \psi_2'\}$ are the corresponding local uniformizing systems in $\mathcal{U}_2$. Then for any injection $\lambda_1 : \{U_1, G_1, \psi_1\} \to \{U_1', G_1', \psi_1'\}$ there exists an injection $\lambda_2 : \{U_2, G_2, \psi_2\} \to \{U_2', G_2', \psi_2'\}$ such that*

$$\lambda_2^* \circ h^*_{U_1} = h^*_{U_1'} \circ \lambda_1^*$$

*(hence also $\lambda_2 \circ h_{U_1} = h_{U_1'} \circ \lambda_1$). We assume further that*

$$g_{\lambda_2}(h_{U_1}(p)) = r_{U'}(\lambda_1(p)) g_{\lambda_1}(p) r^{-1}_{U_1}(p).$$

Two orbifold vector bundles $E_1, E_2$ over $B$ are equivalent if and only if there exists a bundle-map $h : E_1 \to E_2$ such that $(E_1, \mathcal{U}_1) \cong (E_2, \mathcal{U}_2)$ and $h_U : U \to U'$ is homeomorphism for each $U \in \mathcal{U}_1$.

**Theorem 2.2** (see [16]) *Let $(M, \mathcal{U})$ be an orbifold. If there is a system of $C^\infty$-maps $g_\lambda : U \to GL_m(\mathbb{R})$, $\lambda : \{U, G, \psi\} \to \{U', G', \psi'\}$ being any injections, which satisfies relation (2.2), then there exists a orbifold vector bundle over $(M, \mathcal{U})$. Moreover, if two systems of $\{g^1_\lambda\}$ and $\{g^2_\lambda\}$ satisfy relation (2.3), then the associated orbifold vector bundles are equivalent.*

**Remark 2.3** Since (2.2) holds for the tensor product and direct sum of vector spaces, one can construct the tensor product and direct sum of orbifold vector bundles.

**Example 2.5** (Orbifold Tangent Bundle, see [16]) Let $(M, \mathcal{U})$ be an orbifold. Assuming that each $U$ in a chart $(U, G, \psi)$ is contained in $\mathbb{R}^n$, we fix a coordinate system $\{x^1, \cdots .x^m\}$ in each $U$ once for all. For any injection $\lambda : (U, G, \psi) \to (U^*, G^*, \psi^*)$, let

$$g_\lambda(\overline{p}) = \Big(\frac{\partial x^{*i} \circ \lambda}{\partial x^j}\Big)$$

which is the Jacobian matrix of $\lambda$ at $\overline{p}$, where $\{x^i\}$ and $\{x^{*i}\}$ are the fixed coordinate systems in $U$ and $U^*$, respectively. Then the system $g_\lambda$, satisfying the condition of Theorem 2.2, defines an orbifold vector bundle over $M$. This orbifold vector bundle is called the orbifold tangent vector bundle of $M$.

### 2.4 The linear representation of $(\mathbb{Z}_2)^k$

All group actions in the next are supposed to be locally linear actions. The reflection across a coordinate hyperplane in $\mathbb{R}^n$ is called a standard reflection.

**Lemma 2.1** *Let $\{A_i \mid A_i \in GL_n(\mathbb{R}), i = 1, \cdots, k\}$ be a system of invertible matrices. If for any $i, j$, $A_i^2 = E$, $A_iA_j = A_jA_i$, then $\{A_i\}$ are simultaneously diagonalizable.*

*Moreover, assume that $P$ is an invertible matrix such that $P^{-1}A_iP = \operatorname{diag}(x_1^i, \cdots, x_n^i)$ for each $i$. Then the matrix*

$$C = \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^k \\ x_2^1 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1 & x_n^2 & \cdots & x_n^k \end{pmatrix}_{n\times k}$$

*is unique without considering the order of rows.*

Each linear $G$-action of $\mathbb{R}^n$ corresponds to a linear representation $\rho_G : G \to GL_n(\mathbb{R})$. The two actions of $G$ and $H$ are called conjugate if $\rho_G(G)$ and $\rho_H(H)$ are conjugate in $GL_n(\mathbb{R})$.

**Lemma 2.2** *Each nontrivial linear $(\mathbb{Z}_2)^k$-action on $\mathbb{R}^n$ is conjugate with an action whose generators can be presented as the composites of some standard reflections. Specially, when $k = n$, it is conjugate with a standard reflection action if $(\mathbb{Z}_2)^n$ acts effectively on $\mathbb{R}^n$.*

**Proof** Let $A \in GL_n(\mathbb{R})$. If $A^2 = E$, then $A$ is diagonalizable. Assume that the associated diagonal matrix is $\operatorname{diag}(x_1, \cdots, x_n)$, then $x_i = \pm 1$ for any $i = 1, \cdots, n$.

Let $g_1, \cdots, g_k$ be a set of irreducible generators of $(\mathbb{Z}_2)^k$. A linear $(\mathbb{Z}_2)^k$-action of $\mathbb{R}^n$ determines a linear representation $\rho : \mathbb{Z}_2^k \to GL_n(\mathbb{R})$. By Lemma 2.1, $\{\rho(g_i) \mid i = 1, \cdots, k\}$ are simultaneously diagonalizable. Assume that $D_i \simeq \rho(g_i)$ is the associated diagonal matrix for each $i$. Then the action determined by $\{D_1, \cdots, D_k\}$ is generated by the composites of some standard reflections.

For $k = n$, we assume that $D_1, \cdots, D_n$ are the associated diagonal matrices. Then we can rechoose a set of generators of $(\mathbb{Z}_2)^k$ such that the associated diagonal matrices are standard reflection matrices. Actually, choose $D_i = \operatorname{diag}(x_1^i, \cdots, x_n^i)$ with $x_1^i = -1$, then if $x_1^j = -1$ for $D^j = \operatorname{diag}(x_1^j, \cdots, x_n^j)$, we set $g_jg_i$ to be a new generator. So we can obtain a system of diagonal matrices, denoted by $D_1, \cdots, D_n$ as well, satisfying that $x_1^j = 1$ except $j = i$. By induction, we can get a set of generators which determine $n$ standard reflection matrices.

**Lemma 2.3** *Let $\{s_1, \cdots, s_k\}$ be the generators of group $(\mathbb{Z}_2)^k$. The linear action of $(\mathbb{Z}_2)^k$ on $D^n$ is determined by a linear representation $\rho : (\mathbb{Z}_2)^k \to GL_n(\mathbb{R})$, moreover, is determined by the conjugate class of a series of linear transformations $\rho(s_1), \cdots, \rho(s_k)$. Clearly, $\rho(s_1), \cdots, \rho(s_k)$ are simultaneously diagonalizable.*

## 3 Orbifold Stiefel-Whitney Classes of Real Orbifold Vector Bundles over $D^n/(\mathbb{Z}_2)^k$

In this section, we classify the real orbifold vector bundles over a closed right-angled Coxeter cell $D^n/(\mathbb{Z}_2)^k$ by algebrizing the vector bundle in terms of the representations of local group $\mathbb{Z}_2^k$. First, $D^n/\mathbb{Z}_2^k$ is $\mathbb{Z}_2$-closed, that is, there is a right-angled Coxeter complex structure of $D^n/(\mathbb{Z}_2)^k$ with all the boundary maps being zero when we compute $\mathbb{Z}_2$-homology groups. The cohomology ring of $D^n/(\mathbb{Z}_2)^k$ is

$$H^*_{\text{orb}}(D^n/(\mathbb{Z}_2)^k;\mathbb{Z}_2)\cong\mathbb{Z}_2[s_1,s_2,\cdots,s_k]/(s_i^2,i=1,\cdots,k).$$

**Lemma 3.1** *The $m$-dimensional orbifold vector bundles over a closed right-angled Coxeter cell $D^n/(\mathbb{Z}_2)^k$ one-to-one correspond to the linear actions of $(\mathbb{Z}_2)^k$ on $D^n\times\mathbb{R}^m$ satisfying condition* (1) *in Definition* 2.1.

**Remark 3.1** A real orbifold vector bundle over $D^n/(\mathbb{Z}_2)^k$ may not be a right-angled Coxeter complex. Such as, $\widetilde{E}$ in Example 2.4.

Let $s_1,\cdots,s_k$ be the generators of $(\mathbb{Z}_2)^k$. By Lemma 2.3, a linear $(\mathbb{Z}_2)^k$-action on $D^n\times\mathbb{R}^m$ corresponds to a linear representation

$$\rho:(\mathbb{Z}_2)^k\longrightarrow GL_{n+m}(\mathbb{R}),$$

moreover, corresponds to matrices $\rho(s_1),\cdots,\rho(s_k)$.

By Lemma 2.2 and the equivalence of orbifold structure pair in Definition 2.1, we can assume that $\rho(s_1),\cdots,\rho(s_k)$ are diagonal matrices whose diagonal elements are $\pm1$. According to condition (1) in Definition 2.1, we assume that

$$\rho(s_i)=\text{diag}(1_1,\cdots,-1_i,\cdots,1_n,x_1^i,\cdots,x_m^i).$$

That is the following lemma.

**Lemma 3.2** *Any orbifold vector bundle over $D^n/\mathbb{Z}_2^k$ corresponds to a linear representation $\rho:(\mathbb{Z}_2)^k\to GL_{n+m}(\mathbb{R})$, satisfying that*

$$\rho(s_i)=\text{diag}(1_1,\cdots,-1_i,\cdots,1_n,x_1^i,\cdots,x_m^i),$$

*where $s_1,\cdots,s_k$ is a system of generators of $(\mathbb{Z}_2)^k$.*

Then we can obtain an $m\times k$ matrix

$$C=\begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^k\\ x_2^1 & x_2^2 & \cdots & x_2^k\\ \vdots & \vdots & \ddots & \vdots\\ x_m^1 & x_m^2 & \cdots & x_m^k\end{pmatrix}_{m\times k},$$

which is called the characteristic matrix of $E\to D^n/(\mathbb{Z}_2)^k$. Then the orbifold Stiefel-Whitney class of $E\to D^n/(\mathbb{Z}_2)^k$ is defined by

$$w(E)=\prod_{i=1}^m\Big(1+\sum_{j=1}^k\frac{1-x_i^j}{2}s_j\Big)\in H^*_{\text{orb}}(D^n/(\mathbb{Z}_2)^k;\mathbb{Z}_2). \tag{3.1}$$

If two orbifold vector bundles $E_1,E_2$ over $D^n/(\mathbb{Z}_2)^k$ ($s_1,\cdots,s_k$ are generators of $(\mathbb{Z}_2)^k$) are equivalent, then by Lemma 2.3, the associated characteristic matrices obtained from $\{\rho_1(s_i)\mid i=1,\cdots,k\}$ and $\{\rho_2(s_i)\mid i=1,\cdots,k\}$ are the same if we do not consider the order of rows. So $w(E_1)=w(E_2)$, $w$ is well-defined.

**Remark 3.2** Let $\lambda : D^n \to D^n$ be an injection in Definition 2.1. By [16, Lemma 2], all injections $\lambda : D^n \to D^n$ one-to-one correspond to the elements in $(\mathbb{Z}_2)^k$. Choose an injection $\lambda_0 : D^n \to D^n$ (for example, identity map) which corresponds to the unit element of $(\mathbb{Z}_2)^k$, with the associated $C^\infty$-map $g_{\lambda_0}$. Assume that the generator $s_i$ of $(\mathbb{Z}_2)^k$ corresponds to $\lambda_i : D^n \to D^n$. Then by condition (2) in Definition 2.1, $\rho(s_i)$ here determines a $C^\infty$-map $g_{\lambda_i}$. Hence, each characteristic matrix $C$ determines a system of $g_\lambda : D^n \to GL_m(\mathbb{R})$. By Theorem 2.2, each $C$ determines an orbifold vector bundle over $D^n/(\mathbb{Z}_2)^k$.

**Lemma 3.3** (Whitney Sum of Orbifold Vector Bundles over $D^n/(\mathbb{Z}_2)^k$) *Let $E_1 \oplus E_2$ be the Whitney sum of orbifold vector bundles $\pi_1 : E_1 \to D^n/(\mathbb{Z}_2)^k$ and $\pi_2 : E_2 \to D^n/(\mathbb{Z}_2)^k$. Then*

$$w(E_1 \oplus E_2) = w(E_1)w(E_2).$$

**Proof** Let the characteristic matrices of $\pi_1 : E_1 \to D^n/(\mathbb{Z}_2)^k$ and $\pi_2 : E_2 \to D^n/(\mathbb{Z}_2)^k$ are $M_{m_1\times k}$ and $N_{m_2\times k}$, respectively. Then the characteristic matrix of $E_1 \oplus E_2$ is $\binom{M}{N}_{(m_1+m_2)\times k}$. So

$$w(E_1 \oplus E_2) = \prod_{i=1}^{m_1+m_2} \chi_i\binom{M}{N} = \prod_{i=1}^{m_1} \chi_i(M) \cdot \prod_{i=1}^{m_2} \chi_i(N) = w(E_1)w(E_2).$$

Let $h : D^n/(\mathbb{Z}_2)^k \to D^1/\mathbb{Z}_2$ be an orbifold map. Then by definition of orbifold map, there is a homomorphism between their local groups, denoted by $h_* : (\mathbb{Z}_2)^k \to \mathbb{Z}_2$. Notice that $h_*$ can be characterized by a $k$-column $(y^1, \cdots, y^k) \in (\mathbb{Z}_2)^k$, where $y^i = \pm 1$. And $(y^1, \cdots, y^k)$ determines a line bundle $\pi : E^1 \to D^n/(\mathbb{Z}_2)^k$ over $D^n/(\mathbb{Z}_2)^k$. So we define $[D^n/(\mathbb{Z}_2)^k, D^1/\mathbb{Z}_2] \xrightarrow{f} \mathrm{Vect}_1(D^n/(\mathbb{Z}_2)^k)$ by $f(h) =: E_1$.

**Lemma 3.4** *The composition*

$$[D^n/(\mathbb{Z}_2)^k, D^1/\mathbb{Z}_2] \xrightarrow{f} \mathrm{Vect}_1(D^n/(\mathbb{Z}_2)^k) \xrightarrow{w_1} H^1(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2)$$

*is a bijection.*

**Proof** $D^n/(\mathbb{Z}_2)^k = D^k/(\mathbb{Z}_2)^k \times D^{n-k} \simeq D^k/(\mathbb{Z}_2)^k = (D^1/\mathbb{Z}_2)^k$, where "$\simeq$" stands for "homotopy equivalent to" in the category of orbispaces (see [3, Definition 3.2.2]). So

$$\begin{aligned}
&[D^n/(\mathbb{Z}_2)^k, D^1/\mathbb{Z}_2] = [(D^1/\mathbb{Z}_2)^k, D^1/\mathbb{Z}_2] \cong \mathrm{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k,\\
&\mathrm{Vect}_1(D^n/(\mathbb{Z}_2)^k) = \mathrm{Vect}_1(D^k/(\mathbb{Z}_2)^k) = (\mathbb{Z}_2)^k, \quad \text{(actions of } (\mathbb{Z}_2)^k \text{ on } \mathbb{R}^1),\\
&H^1(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k
\end{aligned}$$

are finite sets. So it is sufficient to prove that $f$ and $w_1$ are injections. Clearly, $f$ is an injections.

Each line bundle $\pi : E^1 \to D^1/(\mathbb{Z}_2)^k$ over $D^1/(\mathbb{Z}_2)^k$ can be characterized by a sequence $(x^1, \cdots, x^k)$, where $x^i = \pm 1$. Then $w_1(E^1) = \sum_{j=1}^{k} \frac{1-x^i}{2} s_i \in H^1(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2)$. Thus $w_1$ is an injection as well.

**Example 3.1** Let $\widetilde{\pi} : \widetilde{E} \to D^1/\mathbb{Z}_2$ be the nontrivial line bundle in Example 2.4. Then

$$w(\widetilde{E}^1) = 1 + s,$$

where $H^*(D^1/\mathbb{Z}_2;\mathbb{Z}_2)=\mathbb{Z}_2[s]/(s^2)$.

There is a bundle map $f:\overline{E}^1\to\widetilde{E}^1$ such that $f^*(w(\widetilde{E}^1))=1$. Furthermore, the orbifold Stiefel-Whitney class of tangent bundle of $D^n/(\mathbb{Z}_2)^k$ is

$$w(T(D^n/(\mathbb{Z}_2)^k))=w(T(D^{n-k})\times\big(T(D^1/\mathbb{Z}_2)\big)^k)=\prod_{i=1}^{k}(1+s_i)\in H^*(D^n/(\mathbb{Z}_2)^k;\mathbb{Z}_2).$$

**Example 3.2** There exist two different real vector bundles with the same orbifold Stiefel-Whitney classes. Let $E\to D^1/\mathbb{Z}_2$ be a 2-dimensional real orbifold vector bundle whose characteristic matrix is $(-1,-1)^T$. Then

$$w=(1+s)^2=1.$$

Clearly, $E$ is nontrivial.

## 4 The Real Orbifold Vector Bundle and Orbifold Stiefel-Whitney Classes over Right-Angled Coxeter Complexes

Let $X$ be an $n$-dimensional right-angled Coxeter complex. Then there is a natural open cover of $X$ described as follows.

Firstly, we take a slightly shrunk open set $U(e^n/W)$ for each Coxeter $n$-cell $e^n/W$ in $X$. Then $X-\cup U(e^n/W)$ is homotopic to the $(n-1)$-skeleton of $X$. Furthermore, taking an open set $U(e^{n-1}/W)$ for the neighborhood of each shrunk Coxeter $(n-1)$-cell $e^{n-1}/W$ in $X$, we have $X-\cup U(e^n/W)-\cup U(e^{n-1}/W)$ is homotopic to the $(n-2)$-skeleton of $X$. Inductively, we can obtain an open cover of $X$ indexed by right-angled Coxeter complex structure of $X$ (every open sets associated with $e^k/W$ can deformation retract on $e^k/W$). And we always assume that there is a well-defined orbispace structure on this open cover (see [3] for the notion of orbispace structure).

Let $\pi:E\to X$ be a real orbifold vector bundle over $X$, and $\pi|_{X_{\mathrm{reg}}}:E|_{X_{\mathrm{reg}}}\to X_{\mathrm{reg}}$ be the restriction of $\pi:E\to X$ on the regular subcomplex $X_{\mathrm{reg}}$ of $X$. Then $\pi|_{X_{\mathrm{reg}}}:E|_{X_{\mathrm{reg}}}\to X_{\mathrm{reg}}$ is a usual vector bundle over CW complex $X_{\mathrm{reg}}$.

**Lemma 4.1** *Any real orbifold vector bundle over $X$ can be viewed as the extension of a usual vector bundle over $X_{\mathrm{reg}}$.*

**Proof** Let $\pi'=\pi|_{X_{\mathrm{reg}}}:E|_{X_{\mathrm{reg}}}\to X_{\mathrm{reg}}$ be a usual vector bundle over $X_{\mathrm{reg}}$. First, let $\pi_{D^1/\mathbb{Z}_2}:E(D^1/\mathbb{Z}_2)\to D^1/\mathbb{Z}_2$ be an orbifold vector bundle over $D^1/\mathbb{Z}_2$. By Lemma 2.3, we can assume that the characteristic matrix of $p=\pi_{D^1/\mathbb{Z}_2}:E(D^1/\mathbb{Z}_2)\to D^1/\mathbb{Z}_2$ is a diagonal matrix $A(p)$. Then, gluing the bundles $E|_{X_{\mathrm{reg}}}$ and $E(D^1/\mathbb{Z}_2)$ together along $E(D^1/\mathbb{Z}_2)|_{\partial D^1/\mathbb{Z}_2}$ and $E'|_{\partial\overline{\Phi(D^1/\mathbb{Z}_2)}}$ to obtain an orbifold vector bundle over $X_{\mathrm{reg}}\cup D^1/\mathbb{Z}_2$. Moreover, we can get an orbifold vector bundle over $X_{\mathrm{reg}}\bigcup X^1_{\mathrm{sing}}$, where $X^1_{\mathrm{sing}}$ is the set of singular 1-cells of $X$.

Let $D^2/W$ be a singular 2-cell of $X$ with two singular 1-cells $D^1_1/\mathbb{Z}_2$ and $D^1_2/\mathbb{Z}_2$ in its boundary ($D^1_1/\mathbb{Z}_2$ and $D^1_2/\mathbb{Z}_2$ may correspond to the same singular 1-cell, see Example 5.1 and Example 5.3). Let $\pi_{D^2/W}:E(D^2/W)\to D^w/W$ be an orbifold vector bundle over $D^2/W$.

Then the orbifold vector bundle over $X_{\text{reg}} \cup X^1_{\text{sing}}$ can be extended to an orbifold vector bundle over $X_{\text{reg}} \cup X^1_{\text{sing}} \cup D^2/W$, if and only if the characteristic matrix of bundle $E(D^2/W)$ is $(A_1; A_2)$, where $A_1, A_2$ is the characteristic matrices of bundles $\pi|_{e^1_1}/\mathbb{Z}_2$ and $\pi|_{D^1_2}/\mathbb{Z}_2$, respectively. Specially, when $D^1_1/\mathbb{Z}_2$ and $D^1_2/\mathbb{Z}_2$ correspond to the same singular 1-cell, if $W = \mathbb{Z}_2$, then the characteristic matrix of bundle $E(D^2/W)$ is $A_1 = A_2$ (see Example 5.3). If $W = (\mathbb{Z}_2)^2$, then the characteristic matrix of bundle $E(D^2/W)$ is $(A_1; A_2)$, where $A_1 = A_2$ (see Example 5.1). The orbifold vector bundle $E(D^2/W)$ is compatible with the bundles $\pi|_{e^1/\mathbb{Z}_2}$ and $\pi|_{e^2/(\mathbb{Z}_2)^2}$, which implies that the linear transformations determined by $A_1, A_2$ are simultaneously diagonalizable. That is, the characteristic matrix of $\pi|_{e^2/W}$ can be presented as $(A_1; A_2)$.

By inductions, we obtain an orbifold vector bundle $\pi : E \to X$ over right-angled Coxeter complex $X = X_{\text{reg}} \cup X^1_{\text{sing}} \cup \cdots \cup X^n_{\text{sing}}$ by extending $\pi' = \pi|_{X_{\text{reg}}} : E|_{X_{\text{reg}}} \to X_{\text{reg}}$. For any right-angled Coxeter cell $e^l/(\mathbb{Z}_2)^k$ in $X$, the characteristic matrix of $\pi|_{e^l/(\mathbb{Z}_2)^k}$ can be presented as $(A_1; \cdots; A_k)$. Now $A_1, \cdots, A_k$ are the characteristic matrices of the restrictions of $\pi$ on some associated singular 1-cells in $X$.

Notice that $X_{\text{reg}} \simeq |X|$ and $X/X_{\text{reg}}$ are the wedge sum of some right-angled Coxeter complexes. Each component of $X/X_{\text{reg}}$ contains no regular cell except the unique 0-cell. Let $H$ be a component of $X/X_{\text{reg}}$, and $\mathcal{N}(H)$ be the nerve of $H$. The orbifold vector bundle $\pi : E \to X$ induces an orbifold vector bundle over $H$, denoted by $\pi_H : E_H \to H$. According to the construction of $\pi : E \to X$, there is a system of diagonal matrices $\{A_1, \cdots, A_\eta\}$ corresponding to the codimension-one faces of $H$, such that for each singular cell $e^l/(\mathbb{Z}_2)^k$ in $H$, the characteristic matrix of $\pi|_{e^l/(\mathbb{Z}_2)^k}$ is combined by some $A_i$. So we have the following conclusion.

**Lemma 4.2** *For each $H$, there is an $m \times \eta$ characteristic matrix*

$$C_H = \begin{pmatrix} x^1_1 & x^2_1 & \cdots & x^\eta_1 \\ x^1_2 & x^2_2 & \cdots & x^\eta_2 \\ \vdots & \vdots & \ddots & \vdots \\ x^1_m & x^2_m & \cdots & x^\eta_m \end{pmatrix}_{m\times\eta},$$

*where $A_i = \operatorname{diag}(x^i_1, \cdots, x^i_m)$ for $F_i \in \mathcal{N}(H)$ and $\eta$ is the number of vertices of $\mathcal{N}(H)$.*

Then $C_H$ can determine a real orbifold vector bundle over $D^\eta/(\mathbb{Z}_2)^\eta$, denoted by $\pi_{C_H} : E_{C_H} \to D^\eta/(\mathbb{Z}_2)^\eta$. If $H$ is a simple polytope, then $H$ can be embedded into $D^\eta/(\mathbb{Z}_2)^\eta$ (see [2, Page 93]). Now $\pi_H : E_H \to H$ is a restriction of $\pi_{C_H} : E_{C_H} \to D^\eta/(\mathbb{Z}_2)^\eta$.

**Lemma 4.3** *Let $P$ be a simple polytope with $\eta$ facets and a right-angled Coxeter orbifold structure. Then any real orbifold vector bundle over $P$ can be viewed as the restriction of a real vector bundle over $D^\eta/(\mathbb{Z}_2)^\eta$, where $\eta$ is the number of facets of $P$.*

**Proof** Let $\pi : E \to P$ be an $m$-dimensional real orbifold vector bundle over $P$ characterized by an $m \times \eta$ matrix $C$. Then $C$ determines a real orbifold vector bundle over $D^\eta/(\mathbb{Z}_2)^\eta$, denoted by $\pi_C : E_C \to D^\eta/(\mathbb{Z}_2)^\eta$.

Let $i : P \hookrightarrow D^\eta/(\mathbb{Z}_2)^\eta$ be an orbifold embedding. Then $\pi : E \to P$ is the restriction of $\pi_C : E_C \to D^\eta/(\mathbb{Z}_2)^\eta$ on $P$.

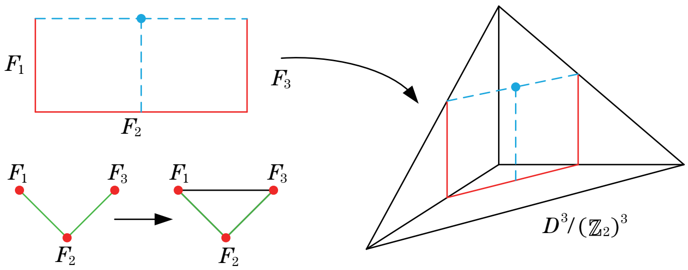


Figure 2 Embedding of right-angled orbifold complexes.

In general, let $j : \mathcal{N}(H) \to \mathcal{N}(D^k/(\mathbb{Z}_2)^k) = \Delta^{k-1}$ be a simplicial map. Then there is an induced map

$$j^* : H^*(D^k/(\mathbb{Z}_2)^k; \mathbb{Z}_2) \to \mathcal{R}_H < H^*(X; \mathbb{Z}_2).$$

The orbifold Stiefel-Whitney class of the part of $H$ is defined as

$$w(E(H)) = j^*(w(E_{C_H})) \in \mathcal{R}_H < H^*(X; \mathbb{Z}_2). \tag{4.1}$$

**Lemma 4.4** *$w(E(H))$ is well-defined.*

**Proof** If there are three facets $F_1, F_2, F_3$ such that $F_1 \cap F_2 \neq \emptyset, F_2 \cap F_3 \neq \emptyset$ and $F_1 \cap F_3 = \emptyset$ (as shown in Figure 2), then $s_1 s_3 = 0$ in $\mathcal{R}_H$. If there is a sub-matrix $\left(\begin{smallmatrix}-1 & 1 & 1\\ 1 & 1 & -1\end{smallmatrix}\right)$ $\Big($or $\left(\begin{smallmatrix}-1 & -1 & 1\\ 1 & -1 & -1\end{smallmatrix}\right)\Big)$ in $(A_1; A_2; A_3)$, it is possible that there is an another matrix of $E(H)$ with the sub-matrix $\left(\begin{smallmatrix}-1 & 1 & -1\\ 1 & 1 & 1\end{smallmatrix}\right)$ $\Big($or $\left(\begin{smallmatrix}-1 & -1 & -1\\ 1 & -1 & 1\end{smallmatrix}\right)\Big)$ in $(A_1; A_2; A_3')$. Thus, the characteristic matrix of $E_H$ is not unique even if we do not consider the order of rows.

However, $s_1 s_3 = 0$ implies that

$$(1 + f(s) + s_1)(1 + f(s) + s_3) = (1 + f(s) + s_1 + s_3)(1 + f(s)),$$

where $f(s)$ is a degree one polynomial which does not contain $s_1, s_3$.

So the choice of matrices does not affect the value of $w(E(H)) = j^*(w(E_C))$. Thus, $w(E(H))$ is well-defined.

**Definition 4.1** *The total orbifold Stiefel-Whitney class of real orbifold vector bundle $\pi : E \to X$ is defined as*

$$w(E) = w(E|_{X_{\text{reg}}}) \cdot \prod_H w(E(H)), \tag{4.2}$$

*where $E|_{X_{\text{reg}}}$ is the restriction of $\pi : E \to X$ on the regular subcomplex $X_{\text{reg}}$ of $X$, and $w(E(H)) = j^*(w(E_{C_H}))$ is defined in Equation (4.1).*

**Remark 4.1** (1) When $X$ is a CW complex, $w(E)$ is the ordinary Stiefel-Whitney class of $\pi : E \to X$.

(2) Let $f$ be a face of $X$. If $f$ is not contractible, then there may exist a vector bundle over $X$ such that some $w_i(E)$ is non-trivial in $H^i(f)$ for $i > 0$ (see Example 5.3).

(3) More generally, the definition of orbifold Stiefel-Whitney classes of real orbifold vector bundles over a Coxeter complex depends on the linear representation of general finite Coxeter group. There are some recommended literatures [8–10, 18].

**Proposition 4.1** *Let $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ be two orbifold vector bundles over a right-angled Coxeter orbifold $X$. Then*

$$w(E_1 \oplus E_2) = w(E_1)w(E_2).$$

**Proof** Let $E = E_1 \oplus E_2$. Then by the definition of orbifold Stiefel-Whitney classes,

$$\begin{aligned} w(E) &= w(E|_{X_{\mathrm{reg}}}) \cdot \prod_H w(E(H)) \\ &= w(E_1|_{X_{\mathrm{reg}}})w(E_2|_{X_{\mathrm{reg}}}) \cdot \prod_H w(E_1(H))w(E_2(H)) \\ &= w(E_1|_{X_{\mathrm{reg}}}) \prod_H w(E_1(H)) \cdot w(E_2|_{X_{\mathrm{reg}}}) \prod_H w(E_2(H)) \\ &= w(E_1)w(E_2), \end{aligned}$$

where $w(E|_{X_{\mathrm{reg}}}) = w(E_1|_{X_{\mathrm{reg}}})w(E_2|_{X_{\mathrm{reg}}})$ is the case of usual CW complex, and the proof of $w(E(H)) = w(E_1(H))w(E_2(H))$ is similar to that of Lemma 3.3.

The orbifold Stiefel-Whitney classes of right-angled Coxeter complexes satisfy the following axioms.

**Proposition 4.2** *There is a unique sequence of functions $w_1, w_2, \cdots$, each of which assigns to each real orbifold vector bundle $E \to B$ a class $w_i(E) \in H^i(B; \mathbb{Z}_2)$, depending only on the isomorphism type of $E$, such that*

(a) *$w_i(f^*(E)) = f^*(w_i(E))$ for a pullback $f^*(E)$, where $f$ is an orbifold map which preserves local groups.*

(b) *$w(E_1 \oplus E_2) = w(E_1)w(E_2)$, where $w = 1 + w_1 + w_2 + \cdots \in H^*(B; \mathbb{Z}_2)$.*

(c) *$w_i(E) = 0$ if $i > \dim E$.*

(d) *For the canonical line bundle $E \to \mathbb{R}P^\infty$, $w_1(E)$ is the generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$. Meanwhile, for the nontrivial line bundle $\widetilde{E} \to D^1/\mathbb{Z}_2$, $w_1(\widetilde{E})$ is the generator of $H^1(D^1/\mathbb{Z}_2; \mathbb{Z}_2)$.*

**Remark 4.2** Orbifold Stiefel-Whitney classes of right-angled Coxeter complexes are generalizations of ordinary Stiefel-Whitney classes.

## 5 Application and Examples

Next, we give an application of orbifold Stiefel-Whitney classes.

According to [11], if a simplicial sphere $K$ can be realized as the nerve of a simple polytope, then $K$ is called a polytopal sphere.

**Lemma 5.1** (see [11, Lemma 3.6]) *If $K = K_1 * K_2$ is a polytopal sphere, then $K_1$ and $K_2$ are polytopal spheres as well.*

**Theorem 5.1** *Let $P$ be an $n$-dimensional simple polytope. Then $P$ is the product of two simple polytopes $P_1$ and $P_2$ with dimensions $n_1$ and $n_2$, respectively, if and only if, $w_n(TP) = w_{n_1}(TP_1) \cdot w_{n_2}(TP_2)$.*

**Proof** By Example 2.1, there is a right-angled Coxeter complex given by the standard cubical decomposition of $P$. Now all boundary maps in the cellular chain complex of $P$ are zero, so

$$H^i(P;\mathbb{Z}_2) \cong (\mathbb{Z}_2)^{f_{n-i}}$$

and

$$H^*(P;\mathbb{Z}_2) \cong \mathbb{Z}_2[s_1,\cdots,s_m]/(I_P + J_P),$$

where $I_P$ is the Stanley-Reisner ideal of $P$, $J_P = (s_i^2, \forall i)$ and $m$ is the number of facets of $P$. Then

$$w(TP) = \prod_{i=1}^{m}(1+s_i)$$

and

$$w_n(TP) = \sum_{F_{i_1}\cap\cdots\cap F_{i_n}\in \mathrm{Vert}(P)} s_{i_1}\cdots s_{i_n}.$$

If $w_n(TP) = f(s_1,\cdots,s_m)\cdot g(s_1,\cdots,s_m)$. $w_n(TP)$ is homogeneous, so are $f$ and $g$. Assume that $\deg(f) = n_1$ and $\deg(g) = n_2$, where $n_1 + n_2 = n$. Then

$$w_n(TP) = \sum_{F_{i_1}\cap\cdots\cap F_{i_n}\in \mathrm{Vert}(P)} s_{i_1}\cdots s_{i_n} = \Big(\sum s_{j_1}\cdots s_{j_{n_1}}\Big)\cdot\Big(\sum s_{k_1}\cdots s_{k_{n_2}}\Big).$$

Notice that $s_j \neq s_k$ and $s_j s_k \neq 0$ for any $s_j$ of $f(s_1,\cdots,s_m)$ and $s_k$ of $g(s_1,\cdots,s_m)$. Let $\mathcal{F}_f, \mathcal{F}_g$ be the facet sets corresponding to the variables of $f$ and $g$, respectively. Then $\mathcal{F}(P) = \mathcal{F}_f \bigsqcup \mathcal{F}_g$. Then the nerve $\mathcal{N}(P)$ of $P$ can be realized as the join of two subcomplexes $K_1$ and $K_2$ of $\mathcal{N}(P)$ which are spanned by $\mathcal{F}_f$ and $\mathcal{F}_g$, respectively. By Lemma 5.1, $K_1$ and $K_2$ can be realized as the nerves of two right-angled Coxeter orbifolds. So $P$ can be realized as the product of two simple polytopes.

Conversely, if $P^n \cong P_1^{n_1} \times P_2^{n_2}$, then $H^*(P;\mathbb{Z}_2) \cong H^*(P_1;\mathbb{Z}_2) \otimes H^*(P_2;\mathbb{Z}_2)$. Now $w(TP) = w(TP_1)\cdot w(TP_2)$, so $w_n(TP) = w_{n_1}(TP_1)\cdot w_{n_2}(TP_2)$.

Finally, some examples are listed.

**Example 5.1** (Orbifold Stiefel-Whitney Class and Bad Orbifold) Let $B = e^0 \cup e^1/\mathbb{Z}_2 \cup e^2/(\mathbb{Z}_2)^2$ be a teardrop, and $\pi : E \to B$ be an $m$-dimensional vector bundle over $B$. Assume that the characteristic matrix of $\pi|_{e^1/\mathbb{Z}_2}$ is $A = (x_1,\cdots,x_m)^T$, where $x_i = \pm 1$. Then the characteristic matrix of $\pi|_{e^2/(\mathbb{Z}_2)^2}$ is $(A;A)$. Now,

$$w(E) = \prod_{i=1}^{m}\Big(1 + \frac{1-x_i}{2}s\Big) = 1 \quad \text{or} \quad 1 + s \in \mathbb{Z}_2[s]/(s^2).$$

**Example 5.2** Let $S^2/\mathbb{Z}_2$ be a quotient orbifold given by a reflection action. Then the blow-up of $S^2/\mathbb{Z}_2$ is a quotient of $S^2$ by identifying a pair of antipodal points. So we have

$$H^i_{\mathrm{orb}}(S^2/\mathbb{Z}_2;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & i=0,1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\pi : E \to S^2/\mathbb{Z}_2$ be an arbitrary vector bundle over $S^2/\mathbb{Z}_2$, then

$$w(E) = w(E|_{X_{\mathrm{reg}}}) \cdot w(E_H) = w(E_H) = 1 \quad \text{or} \quad 1+s,$$

where $s$ is the generator of $H^1_{\mathrm{orb}}(S^2/\mathbb{Z}_2;\mathbb{Z}_2)$. So $w_2 = 0$ for any vector bundle over $S^2/\mathbb{Z}_2$.

**Example 5.3** Let $Q$ be the quotient orbifold of $\mathbb{Z}_2$ on $S^1 \times [-1,1]$ by $(x,y) \to (x,-y)$. We can decompose $S^1 \times [-1,1]/\mathbb{Z}_2$ into $\{v, e_1, e_2, f\}$, one 0-cell, one regular 1-cells, one singular 1-cell and one 2-cell as shown in Figure 3.

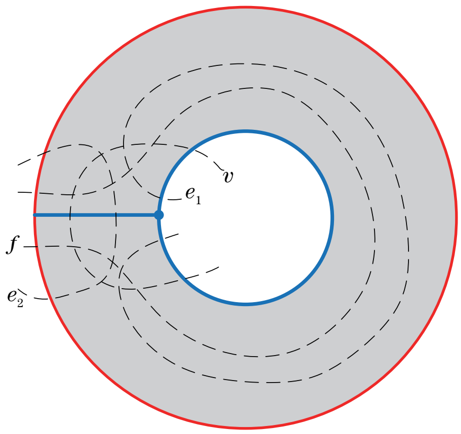


Figure 3 $S^1 \times [-1,1]/\mathbb{Z}_2$.

The blow-up complex of $Q$ is a torus, so

$$H^i_{\mathrm{orb}}(S^1 \times [-1,1]/\mathbb{Z}_2;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & i=0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & i=1 \\ \mathbb{Z}_2, & i=2 \\ 0, & \text{otherwise.} \end{cases}$$

Let $S^1_1$ and $S^1_2$ be blue circle and red circle respectively in Figure 3. By Theorem 2.1, we have

$$H^i_{\mathrm{orb}}(S^1 \times [-1,1]/\mathbb{Z}_2;\mathbb{Z}_2) \cong \begin{cases} H^0(S^1_1;\mathbb{Z}_2), & i=0 \\ H^1(S^1_1;\mathbb{Z}_2) \oplus H^0(S^1_2;\mathbb{Z}_2), & i=1 \\ H^1(S^1_2;\mathbb{Z}_2), & i=2 \\ 0, & \text{otherwise.} \end{cases}$$

Moreover,

$$H^*_{\mathrm{orb}}(S^1 \times [-1,1]/\mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2[s_1,s_2]/(s_1^2,s_2^2),$$

where $s_1$ and $s_2$ are generators of $H^1(S^1_1;\mathbb{Z}_2)$ and $H^0(S^1_2;\mathbb{Z}_2)$, respectively. So $s_1s_2$ in $H^2_{\text{orb}}(S^1\times[-1,1]/\mathbb{Z}_2;\mathbb{Z}_2)$ is non-trivial.

Let $\{U(v),U(e_1),U(e_2),U(f)\}$ be an open cover of $S^1\times[-1,1]/\mathbb{Z}_2$ as shown in Figure 3. Let $T(S^1\times[-1,1]/\mathbb{Z}_2)$ be the orbifold tangent bundle of $S^1\times[-1,1]/\mathbb{Z}_2$. First, the tangent bundle of its underlying space $S^1\times[0,1]\simeq U(v)\cup U(e_1)$ is trivial. The orbifold tangent bundle of $U(e_2)\simeq e^1/\mathbb{Z}_2\times e^1$ is $TU(e_2)\cong\widetilde{E}\times(\mathbb{R}\times(0,1))$, the product of $\widetilde{E}$ in Example 2.4 and the trivial line bundle over $e^1$. Then we glue $TU(e_2)$ and $T(U(v)\cup U(e_1))$ together along the trivial bundle over $U(e_2)\cap U(v)$. Furthermore, $TU(f)\cong\widetilde{E}\times Te^1$. Then we can obtain the orbifold tangent bundle of $S^1\times[-1,1]/\mathbb{Z}_2$ by gluing $TU(f)$ to the glued bundle $T(U(v)\cup U(e_1))\cup TU(e_2)$.

$$w(T(S^1\times[-1,1]/\mathbb{Z}_2))=w(TS^1)(1+s)=1+s.$$

In the other way, $S^1\times[-1,1]/\mathbb{Z}_2=[-1,1]\times([-1,1]/\mathbb{Z}_2)/(-1,x)\sim(1,x)$. Let $[-1,1]\times\widetilde{E}\to[-1,1]\times([-1,1]/\mathbb{Z}_2)$ be a line bundle over $[-1,1]\times([-1,1]/\mathbb{Z}_2)$, where $\widetilde{E}=[-1,1]\times\mathbb{R}/(x,y)\sim(-x,-y)$ is the non-trivial bundle over $[-1,1]/\mathbb{Z}_2$ defined in Example 2.4. Then $E=[-1,1]\times\{[-1,1]\times\mathbb{R}/(x,y)\sim(-x,-y)\}/(-1,x,y)\sim(1,x,-y)$ is a line bundle over $S^1\times[-1,1]/\mathbb{Z}_2$. Now

$$w(E)=w(E|_{X_{\text{reg}}})\cdot w(E_H)=(1+s_1)(1+s_2).$$

So $w_2(E)=s_1s_2$ is non-trivial in $H^2_{\text{orb}}(S^1\times[-1,1]/\mathbb{Z}_2;\mathbb{Z}_2)$.

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