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On Mixed Pressure-Velocity Regularity Criteria to the Navier-stokes Equations in Lorentz Spaces, Part II: The Non-slip Boundary Value Problem*

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Abstract This paper is a continuation of the authors recent work [Beirão da Veiga, H. and Yang, J., On mixed pressure-velocity regularity criteria to the Navier-Stokes equations in Lorentz spaces, *Chin. Ann. Math.*, **42**(1), 2021, 1–16], in which mixed pressure-velocity criteria in Lorentz spaces for Leray-Hopf weak solutions of the three-dimensional Navier-Stokes equations, in the whole space $\mathbb{R}^3$ and in the periodic torus $\mathbb{T}^3$, are established. The purpose of the present work is to extend the result of mentioned above to smooth, bounded domains Ω, under the non-slip boundary condition. Let π denote the fluid pressure and v the fluid velocity. It is shown that if $\frac{\pi}{(1+|v|)^\theta} \in L^p(0,T;L^{q,\infty}(\Omega))$, where $0 \le \theta \le 1$, and $\frac{2}{p}+\frac{3}{q} = 2-\theta$ with $p \ge 2$, then v is regular on $\Omega \times (0,T]$.

Keywords Navier-Stokes equations, Pressure-speed links, Regularity criteria, Lorentz spaces, Boundary value problem
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1 Introduction: The Main Result

We are concerned with the regularity of weak solutions to the Navier-Stokes equations

$$\begin{cases} \partial_t v + v\cdot\nabla v - \Delta v + \nabla\pi = f & \text{in } \Omega\times(0,T),\\ \nabla\cdot v = 0 & \text{in } \Omega\times(0,T),\\ v(x,0) = v_0 & \text{in } \Omega,\\ v = 0 & \text{on } \Gamma\times(0,T), \end{cases} \tag{1.1}$$

where v is the flow velocity field, π is the pressure, and the initial data v_0 is divergence free. Ω is a smooth open, bounded, subset of $\mathbb{R}^3$ and Γ denotes its boundary. f is an external force. Throughout the whole paper we assume, without loss of generality, that for almost all $t \in (0,T)$, the pressure π has vanishing mean-value in Ω.

In our recent paper [6], we established new integral criteria for regularity of weak solutions to the Navier-Stokes equations. These criteria generalize the classical, well-known, Ladyzhenskaya-Prodi-Serrin (L-P-S for short) criteria, see the pioneering references [17], [21], [23]. Let's briefly recall these criteria. They established, in their stronger, more recent form, that if a weak

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solution v of (1.1) satisfies

$$v \in L^p(0,T;L^q(\Omega)), \quad \frac{2}{p}+\frac{n}{q}=1, \quad q>n, \tag{1.2}$$

then v is a strong solution:

$$v \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)). \tag{1.3}$$

It is well-known that strong solutions are smooth, if data and domain are also smooth. The above strong result also holds for $q=n$ (see [11, 22]).

Assumptions like (1.2) are called strong, since the equality sign "=" holds. When the equality sign "=" is replaced by the inequality sign "<", the assumption is called mild. We use this same distinction in all the cases treated here. Concerning the strong criteria for regularity (1.2), the first complete proof is due to Giga [13], followed by [1, 12]. See [6] for some comment.

Below, the criteria of type (1.2) relate pressure and velocity. So they are called mixed pressure-velocity criteria, abbreviate simply to P-V criteria. Let's introduce this significant generalization. The well-known "interior" equation

$$-\Delta\pi = \sum_{i,j=1}^{n} \partial_i\partial_j(v_iv_j) \tag{1.4}$$

suggests the formal equivalence $\pi \cong |v|^2$ or, more appropriately, it merely suggests $\pi \lessapprox |v|^2$ (rather than $|v|^2 \lessapprox \pi$) since (1.4) gives information on π in terms of v, but not the reverse. This means that, roughly, $\frac{|\pi|}{|v|} \lessapprox |v|$, but not the reverse. Hence results under the same integrability assumption on the two different quantities present in the above inequality look stronger (more general) for the assumption on the left-hand side term. Further, since $\frac{|\pi|}{1+|v|} \le \frac{|\pi|}{|v|}$, results obtained under integrability conditions on the left-hand side of the last inequality are stronger than results under the same conditions on the right-hand side. This distinction is significant since in the last case, one rules out regions where π is bounded and $|v|$ is arbitrary small.

The formal relation $\pi \cong |v|^2$ suggests the following generalization

$$\frac{|\pi|}{(1+|v|)^\theta} \cong |v|^{2-\theta}. \tag{1.5}$$

Hence it is natural to consider the following P-V problem (see [5]). Positive replies reinforce the significance of the main relation $\pi \cong |v|^2$.

Problem 1.1 Assume that a weak solution (v,π) of the Navier-Stokes equations (1.1) satisfies

$$\frac{|\pi|}{(1+|v|)^\theta} \in L^p(0,T;L^q(\Omega)) \tag{1.6}$$

for some $\theta \in [0,2]$, where

$$\frac{2}{p}+\frac{n}{q}=2-\theta. \tag{1.7}$$

Question: Does (1.2) hold?

Assumption (1.7) will be called the P-V criteria. For a quite complete overview on the P-V problem see [6]. The first result on this problem, where $\theta=1$, was the Theorem 1.1 of [2], proved by appealing to the truncation method. Later on, in Theorem 1.1 of [3], still by appealing to the truncation method, the result was extended to general values of θ, $0 \leq \theta \leq 1$, with $p = q = \gamma$. These results were mild. It was natural to ask whether the mild regularity assumptions can be replaced by corresponding strong regularity assumptions (conditions with the equality sign "="). A first positive answer was given in [4] for $\theta = 1$. The main ideas followed in the proof essentially appeal to the argument developed in [1]. In [4, Theorem I], it was essentially proved that solutions are smooth for $\theta = 1$. Later on, in [5, Theorem 1.1], the above result was extended to the general θ case. For the particular case $\theta = 0$, $\Omega = \mathbb{R}^n$ (see [8]).

The case $\theta > 1$ was treated by Zhou [29]. In this case, there is no evidence of a positive answer to the relation $\pi \cong |v|^2$. On the contrary, see [6], both Zhou's result, and the constraint (51) imposed in the Lemma 3.6 in [5], go in the direction of a negative answer to the equivalence $\pi \cong |v|^2$ in the case $\theta > 1$.

In [6] we have considered the $\mathbb{R}^3$ whole space case, and the torus $\mathbb{T}^3$ case, and proved the following result (for a definition of Lorentz spaces see section 2 below).

Theorem 1.1 (see [6, Theorem 5.2]) *Set $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. Let (v, π) be a weak solution to* (1.1) *with divergence-free initial data $v_0 \in L^2(\Omega) \cap L^4(\Omega)$. Assume that $0 \leq \theta \leq 1$ and that*

$$\frac{\pi}{(\mathrm{e}^{-|x|^2} + |v|)^\theta} \in L^p(0, T; L^{q,\infty}(\Omega)), \tag{1.8}$$

where p and q are finite, and

$$\frac{2}{p} + \frac{3}{q} = 2 - \theta. \tag{1.9}$$

Then v is regular on $(0, T] \times \Omega$. In a bounded domain (the space-periodic case), we may replace $\mathrm{e}^{-|x|^2}$ simply by 1.

The aim of the present work is to extend the above Theorem 1.1 to the non-slip boundary value problem in a bounded domain. Reference [6] should be assumed as Part I of the present paper. Hence we will not repeat here many of the notes, historical remarks and references, given in [6]. We refer to Sections 1 to 6 in this last reference.

Our main result is the following.

Theorem 1.2 *Assume that Ω is a smooth bounded domain in $\mathbb{R}^3$, and $f = 0$. Let (v, π) be a weak solution to* (1.1) *with divergence-free initial data $v_0 \in H_0^1(\Omega) \cap L^4(\Omega)$. Furthermore, assume that*

$$\frac{\pi}{(1 + |v|)^\theta} \in L^p(0, T; L^{q,\infty}(\Omega)), \tag{1.10}$$

where $p \geq 2$,

$$\frac{2}{p} + \frac{3}{q} = 2 - \theta \tag{1.11}$$

and

$$0 \leq \theta \leq 1. \tag{1.12}$$

Then v is regular on $(0, T] \times \Omega$.

An interesting and challenging problem would be to prove that the above conclusion still holds for $p < 2$, even in a weaker but significant form. See the remark below.

For completeness, let's briefly recall some regularity criteria in Lorentz spaces. Concerning the P-V problem, the pioneering result in Lorentz spaces is Theorem 1.1 of [3]. The first strong result in Lorentz spaces (for the velocity alone) was obtained by Sohr [24]. He established a Lorentz spaces' strong version of L-P-S criteria in the three-dimensional domain. Furthermore, in [9], Berselli and Manfrin obtained similar results. Recently, Suzuki [26–27], and Ji-Wang-Wei [16] studied some regularity criteria in terms of the pressure π in Lorentz spaces. In [16] Ji-Wang-Wei extended Suzuki's result to the range $\frac{3}{2} \leq q < \frac{5}{2}$, by partially appealing to ideas in [4–5].

Remark 1.1 Compared with Theorem 1.1 for the whole space (see [6, Theorem 5.2]), the above result in bounded domains is restrictive. Now, as in [5, Theorem 1.1], we need to assume that $p \geq 2$ $\left(\text{or } q \leq \frac{3}{1-\theta}\right)$. It looks suitable to comment on the reasons that lead to this restriction. Our proof start from Lemma 2.2 below. The main point in the proof is to control the term $\int_\Omega |\pi|^2|v|^2 \mathrm{d}x$ on the right-hand side of (2.10). In [6] we estimate the norm $\|\pi\|_{L^{(2-\beta)r_1,2}}$, see the last row of equation (3.2) below, by appealing to the equation $-\Delta\pi = \sum\limits_{i,j=1}^{3} \partial_i\partial_j(v_iv_j)$ together with the boundedness of the Riesz Transform (2.3). This shows that $\|\pi\|_{L^{(2-\beta)r_1,2}} \leq C\||v|^2\|_{L^{(2-\beta)r_1,2}}$. However, for bounded domains, we can not appeal to the boundedness of the Riesz Transform. To overcome this difficulty, after some calculations and by choosing an appropriate exponent r_1, the problem is reduced to control the quantity $\varepsilon \int_0^t \|\nabla\pi\|_{L^2}^2 \mathrm{d}\tau$. By applying Lemma 2.1 below, we show that $\varepsilon \int_0^t \|\nabla\pi\|_{L^2}^2 \mathrm{d}\tau \leq C\varepsilon \int_0^t \|\nabla|v|^2\|_{L^2}^2 \mathrm{d}\tau$, which can be controlled by the left-hand side of (2.10).

Finally, we show why in our proofs the exponent p must be lager than 2. In [6] the choice of r_1 is relatively free since we can choose r_1 such that $\delta_1, \delta_2 \in [0,1]$ for any $p > 1$ satisfying (1.9). However, for a bounded domain under boundary conditions, due to Lemma 2.1 below, the choice of a suitable r_1 is quite restrained. We need $p \geq 2$ to have $\delta \in [0,1]$ (see (3.8) below). Actually this corresponds to the case $\delta_1 = 1$ in [6].

2 Auxiliary Results

We start by recalling the definition of Lorentz spaces.

Definition 2.1 *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$. The Lorentz space $L^{p,q}(\Omega)$ is the set of all functions f such that $\|f\|_{L^{p,q}(\Omega)} < \infty$, where*

$$\|f\|_{L^{p,q}(\Omega)} := \begin{cases} \left(p\displaystyle\int_0^\infty \tau^q |\{x \in \Omega : |f(x)| > \tau\}|^{\frac{q}{p}} \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{q}}, & q < \infty, \\ \sup\limits_{\tau>0} \tau |\{x \in \Omega : |f(x)| > \tau\}|^{\frac{1}{p}}, & q = \infty. \end{cases} \tag{2.1}$$

Actually the quantity $\|f\|_{L^{p,q}(\Omega)}$ is merely a quasi-norm, not a norm. However, it is well-known that there are equivalent norms. Next we collect some useful properties. For the readers' convenience we recall that these properties are listed, for instance, in [16, 28].

(i) Interpolation characteristic of Lorentz spaces (see [7, Theorem 5.3.1]),

$$(L^{p_0,q_0}(\Omega), L^{p_1,q_1}(\Omega))_{\lambda,q} = L^{p,q}(\Omega), \quad \frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}, \ \frac{1}{q} = \frac{1-\lambda}{q_0} + \frac{\lambda}{q_1}, \ 0 < \lambda < 1. \tag{2.2}$$

(ii) Boundedness of Riesz Transform in Lorentz spaces (see [10, Lemma 2.2]),

$$\|R_j f\|_{L^{p,q}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,q}(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{2.3}$$

(iii) Hölder inequality in the Lorentz spaces (see [20, Theorems 3.4–3.5]),

$$\|fg\|_{L^{r,s}(\Omega)} \leq \|f\|_{L^{r_1,s_1}(\Omega)}\|g\|_{L^{r_2,s_2}(\Omega)}, \tag{2.4}$$

where $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$.

(iv) For $1 \le p < \infty$, $1 \le q_1 < q_2 \le \infty$, we have that (see [15, Proposition 1.4.10])

$$\|f\|_{L^{p,q_2}(\Omega)} \le \Big(\frac{q_1}{p}\Big)^{\frac{1}{q_1}-\frac{1}{q_2}} \|f\|_{L^{p,q_1}(\Omega)}. \tag{2.5}$$

(v) Sobolev inequality in Lorentz spaces (see [19, Theorem 6.5]),

$$\Big\|f - \int_\Omega f\mathrm{d}x\Big\|_{L^{\frac{np}{n-p},q}(\Omega)} \le C\|\nabla f\|_{L^{p,q}(\Omega)} \quad \text{with } 1 \le p < n,\ \ 1 \le q \le \infty. \tag{2.6}$$

Next, we consider the Stokes system

$$\begin{cases} \partial_t v - \Delta v + \nabla \pi = f, & \text{in } \Omega \times (0,T), \\ \nabla \cdot v = 0, & \text{in } \Omega \times (0,T), \\ v(x,0) = v_0, & \text{in } \Omega, \\ v = 0, & \text{on } \Gamma \times (0,T), \end{cases} \tag{2.7}$$

where Ω is a smooth domain in $\mathbb{R}^3$. For the above Stokes system, the following Solonnikov's Lemma holds (see [25]). See also the classical Ladyzenskaya's treatise [18, Chapter 4, Theorem 6].

Lemma 2.1 *Let $1 < r < \infty$, and $r \neq \frac{3}{2}$. Suppose that $f \in L^l(0,T;L^r(\Omega))$ and $v_0 \in W^{2-\frac{2}{r},r}(\Omega)$. If the pair (v,π) is a solution of the Stokes system* (2.7)*, then (v,π) satisfies the following estimate*:

$$\|v_t\|_{L^r(0,T;L^r)} + \|\nabla^2 v\|_{L^r(0,T;L^r)} + \|\nabla \pi\|_{L^r(0,T;L^r)} \le C(\|f\|_{L^r(0,T;L^r)} + \|v_0\|_{W^{2-\frac{2}{r},r}}). \tag{2.8}$$

In particular, if $r = 2$, we have

$$\|v_t\|_{L^2(0,T;L^2)} + \|\nabla^2 v\|_{L^2(0,T;L^2)} + \|\nabla \pi\|_{L^2(0,T;L^2)} \le C(\|f\|_{L^2(0,T;L^2)} + \|v_0\|_{H^1}). \tag{2.9}$$

General mixed-norm estimate also holds. See, for instance, [14, Theorem 2.8].

Lemma 5.3 below follows from [4, (2.3)] by setting in this estimate $\alpha = 4$ and dimension $n = 3$. It can be obtained by multiplying both sides of (1.1) by $|v|^2 v$, integrating by parts, using divergence-free condition and Cauchy-Schwarz inequality. See also [1, Lemmas 1.1–1.2] and [5, Lemma 3.1].

Lemma 2.2 *Let (v,π) be a regular solution to equation* (1.1) *in $\Omega \times [0,T]$. Then we have*

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |v|^4 \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla v|^2|v|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla |v|^2|^2 \mathrm{d}x \le \int_\Omega |\pi|^2|v|^2 \mathrm{d}x. \tag{2.10}$$

3 Proof of Theorem 1.2

We start by controlling the right-hand side of (2.10). In the following we set $\beta = \frac{2}{2-\theta}$, and therefore $\beta \in [1,2]$, and $2 + \theta\beta = 2\beta$. Furthermore, we set

$$V = 1 + |v|, \quad \widetilde{\pi} = \frac{|\pi|}{(1+|v|)^\theta}. \tag{3.1}$$

When $0 \le \theta < 1$, by Hölder's inequality in Lorentz spaces (2.4), one has

$$\int_\Omega |\pi|^2|v|^2 \mathrm{d}x = \int_\Omega \Big(\frac{|\pi|}{(1+|v|)^\theta}\Big)^\beta |\pi|^{2-\beta}(1+|v|)^{\beta\theta}|v|^2 \mathrm{d}x$$

$$
\begin{aligned}
&\le \int_\Omega |\widetilde{\pi}|^\beta |\pi|^{2-\beta} V^{2+\beta\theta} \mathrm{d}x \\
&\le \|\widetilde{\pi}^\beta\|_{L^{\frac{q}{\beta},\infty}} \|\pi^{2-\beta}\|_{L^{r_1,\frac{2}{2-\beta}}} \|V^{2\beta}\|_{L^{r_2,\frac{2}{\beta}}} \\
&= \|\widetilde{\pi}\|^\beta_{L^{q,\infty}} \|\pi\|^{2-\beta}_{L^{(2-\beta)r_1,2}} \|V^2\|^\beta_{L^{\beta r_2,2}}, \qquad (3.2)
\end{aligned}
$$

where

$$\frac{\beta}{q} + \frac{1}{r_1} + \frac{1}{r_2} = 1. \qquad (3.3)$$

When $\theta = 1$, i.e., $\beta = 2$, the corresponding estimate is

$$\int_\Omega |\pi|^2 |v|^2 \mathrm{d}x \le \|\widetilde{\pi}\|^2_{L^{q,\infty}} \|V^2\|^2_{L^{\frac{2q}{q-2},2}}. \qquad (3.4)$$

Now, we take $r_1 = \frac{6}{2-\beta}$, and then $r_2 = \frac{6q}{4q+(q-6)\beta}$. By the Sobolev inequality in Lorentz spaces (2.6), we have

$$\|\pi\|_{L^{(2-\beta)r_1,2}} = \|\pi\|_{L^{6,2}} \le C\|\nabla\pi\|_{L^2}, \qquad (3.5)$$

where we have taken into account that π has zero mean-value. By the interpolation characteristic of Lorentz spaces (2.2), and by Sobolev inequality in Lorentz spaces (2.6), it follows that

$$\|V^2\|_{L^{\beta r_2,2}} \le C\|V^2\|^{1-\delta}_{L^{2,2}} \|V^2\|^\delta_{L^{6,2}} \le C\|V^2\|^{1-\delta}_{L^2} (\|V^2\|_{L^2} + \|\nabla V^2\|_{L^2})^\delta, \qquad (3.6)$$

where

$$\frac{1}{\beta r_2} = \frac{1-\delta}{2} + \frac{\delta}{6}, \qquad (3.7)$$

i.e.,

$$\delta = 1 + \frac{3}{q} - \frac{2}{\beta} = 1 + \frac{3}{q} - (2-\theta) = 1 - \frac{2}{p} \in [0,1]. \qquad (3.8)$$

Hence it easily follows from (3.2) and Young's inequality that

$$
\begin{aligned}
\int_\Omega |\pi|^2 |v|^2 \mathrm{d}x &\le C\|\widetilde{\pi}\|^\beta_{L^{q,\infty}} \|\nabla\pi\|^{2-\beta}_{L^2} \|V^2\|^{(1-\delta)\beta}_{L^2} (\|V^2\|_{L^2} + \|\nabla V^2\|_{L^2})^{\delta\beta} \\
&\le C\|\widetilde{\pi}\|^{\frac{2}{1-\delta}}_{L^{q,\infty}} \|V^2\|^2_{L^2} + \varepsilon\|\nabla\pi\|^2_{L^2} + \varepsilon(\|V^2\|^2_{L^2} + \|\nabla V^2\|^2_{L^2}) \\
&\le C(1 + \|\widetilde{\pi}\|^{\frac{2}{1-\delta}}_{L^{q,\infty}})\|V^2\|^2_{L^2} + \varepsilon\|\nabla\pi\|^2_{L^2} + \varepsilon\|\nabla V^2\|^2_{L^2}, \qquad (3.9)
\end{aligned}
$$

where the constant C depends on ε. Noting

$$\|V^2\|^2_{L^2} = \|1 + 2|v| + |v|^2\|^2_{L^2} \le 4(1 + \|v\|^2_{L^2} + \||v|^2\|^2_{L^2}) \qquad (3.10)$$

and

$$\|\nabla V^2\|^2_{L^2} = \|\nabla(1 + 2|v| + |v|^2)\|^2_{L^2} \le 4(\|\nabla v\|^2_{L^2} + \|\nabla|v|^2\|^2_{L^2}), \qquad (3.11)$$

it follows that

$$\int_\Omega |\pi|^2 v^2 \mathrm{d}x$$

$$\leq 4C(1+\|\widetilde{\pi}\|_{L^{q,\infty}}^{\frac{2}{1-\delta}})(1+\|v\|_{L^2}^2+\|v\|_{L^4}^4)+\varepsilon\|\nabla\pi\|_{L^2}^2+4\varepsilon(\|\nabla v\|_{L^2}^2+\|\nabla|v|^2\|_{L^2}^2). \tag{3.12}$$

By this estimate and Lemma 2.2, we have

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L^4}^4+\frac{1}{2}\int_\Omega|\nabla v|^2|v|^2\mathrm{d}x+\frac{1}{2}\|\nabla|v|^2\|_{L^2}^2$$
$$\leq 4C(1+\|\widetilde{\pi}\|_{L^{q,\infty}}^{\frac{2}{1-\delta}})(1+\|v\|_{L^2}^2+\|v\|_{L^4}^4)+\varepsilon\|\nabla\pi\|_{L^2}^2+4\varepsilon(\|\nabla v\|_{L^2}^2+\|\nabla|v|^2\|_{L^2}^2). \tag{3.13}$$

Next we add side by side the classical energy inequality to estimate (3.13). By choosing ε sufficiently small, this allows us to drop the full 4ε term on the right-hand side of (3.13). In particular, it follows that (here, and in the following, we may drop non essential terms)

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\big(1+2\|v\|_{L^2}^2+\|v\|_{L^4}^4\big)+\frac{1}{2}\int_\Omega|\nabla v|^2|v|^2\mathrm{d}x+\frac{1}{2}\|\nabla|v|^2\|_{L^2}^2$$
$$\leq 4C(1+\|\widetilde{\pi}\|_{L^{q,\infty}}^{\frac{2}{1-\delta}})(1+\|v\|_{L^2}^2+\|v\|_{L^4}^4)+\varepsilon\|\nabla\pi\|_{L^2}^2. \tag{3.14}$$

Integrating (3.13) in time from 0 to t, for any given $t\in(0,T)$, we obtain

$$\frac{1}{4}(1+2\|v\|_{L^2}^2+\|v\|_{L^4}^4)(t)+\frac{1}{2}\int_0^t\int_\Omega|\nabla|v|^2|^2\mathrm{d}x\mathrm{d}\tau$$
$$\leq 4C\int_0^t(1+\|\widetilde{\pi}\|_{L^{q,\infty}}^{\frac{2}{1-\delta}})(1+\|v\|_{L^2}^2+\|v\|_{L^4}^4)\mathrm{d}\tau$$
$$+\frac{1}{4}(1+2\|v_0\|_{L^2}^2+\|v_0\|_{L^4}^4+C\|v_0\|_{H^1}^2), \tag{3.15}$$

where the pressure term has been dropped by choosing ε sufficiently small, since, by (2.9) of Lemma 2.1, one has

$$\int_0^t\|\nabla\pi\|_{L^2}^2\mathrm{d}\tau\leq C(\|\nabla|v|^2\|_{L^2(0,t;L^2)}^2+\|v_0\|_{H^1}^2). \tag{3.16}$$

Now we may also eliminate the last term in the left-hand side of (3.15). Next, by using Gronwall's lemma, we show that

$$v\in L^\infty(0,T;L^4(\Omega)), \tag{3.17}$$

since, due to (3.8) and the definition of $\widetilde{\pi}$, condition $\widetilde{\pi}\in L^{\frac{2}{1-\delta}}(0,T;L^{q,\infty})$ is just our main assumption

$$\widetilde{\pi}\in L^p(0,T;L^{q,\infty}). \tag{3.18}$$

Smoothness of the solution in $\Omega\times[0,T]$ follows from (3.18) together with Ladyzhenskaya-Prodi-Serrin regularity criteria (1.2). Thus we have completed the proof of Theorem 1.2.

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