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On the Kernel of Restriction of Characters*

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Abstract Let G be a finite group, H be a proper subgroup of G, and S be a unitary subring of $\mathbb{C}$. The kernel of the restriction map $S[\mathrm{Irr}(G)] \to S[\mathrm{Irr}(H)]$ as a ring homomorphism is studied. As a corollary, the main result in [Isaacs, I. M. and Navarro, G., Injective restriction of characters, *Arch. Math.*, **108**, 2017, 437–439] is reproved.

Keywords Characters, Character ring, Character restriction
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1 Introduction

In [4], by construction of some characters induced from a cyclic subgroup, Isaacs and Navarro proved that the restriction map of generalized characters from a group to any of its proper subgroup is not injective. The construction is very subtle. If we treat the restriction map as a ring homomorphism from the character ring (with coefficients integers) of the group to the character ring of the proper subgroup, then the main result of Isaacs and Navarro is equivalent to saying that the mentioned ring homomorphism is not injective, i.e., the kernel of it is not zero.

In this paper, we will approach this question in a more general context. We use the same technique similar as in [1]. We explain our method in the following paragraph.

One can view the complex irreducible characters of a finite group as a basis of the vector space of complex class functions on the finite group (see [6, Chapter 2, Theorem 6]). The vector space of complex class functions of the finite groups is thus the character ring with coefficients complex numbers of the finite group. This vector space has another basis which is the characteristic class functions of the finite group. Using this basis, one can easily prove that the restriction homomorphism from the complex character ring of the finite group to the complex character ring of a proper subgroup is not injective. Hence, its kernel is not zero. In this kernel we may get some nonzero generalized characters of the finite group and prove the main result of Isaacs and Navarro as a corollary.

Our notations are from [2] and [6].

Let G be a finite group. The set of all complex characters of G is denoted by $\mathrm{Char}(G)$ and the set of all irreducible characters in $\mathrm{Char}(G)$ is denoted by $\mathrm{Irr}(G)$. For a unitary subring S of $\mathbb{C}$, the ring of S-generalized characters of G is denoted by $S[\mathrm{Irr}(G)]$.

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If H is a proper subgroup of G, in [4] Isaacs and Navarro proved that the character restriction from $\mathrm{Char}(G)$ to $\mathrm{Char}(H)$ is not injective. For a unitary subring S of $\mathbb{C}$, it is stratightforward that the original restriction of characters induces a ring homomorphism

$$r_S: \ S[\mathrm{Irr}(G)] \to S[\mathrm{Irr}(H)].$$

In this paper, we will study the rank of the kernel of the ring homomorphism r_S for different choices of S. The main result in [4] will be a corollary of the main theorem, where $S = \mathbb{Z}$.

Let ω be a $|G|$-th primitive root of 1 in $\mathbb{C}$. Denote the Galois group $\mathrm{Gal}(\mathbb{Q}[\omega]/\mathbb{Q})$ by K and all the algebraic integers in $\mathbb{Q}[\omega]$ by A. Set Cla_H to be the set of all conjugacy classes consisting of elements in G which are not conjugate to any element in H. Since H is proper in G, Cla_H is not empty (see [3, Problem 1A.7]). Also, each element of K maps ω to a power of ω with the power coprime to $|G|$. Thus, each element of K induces a natural permutation on G. Thus, each element of K induces a permutation on Cla_H.

2 Main Results

The following is the main result of this paper.

Theorem 2.1 *Let notations be as above. For any unitary subring S of $\mathbb{C}$, the kernel $\ker(r_S)$ of r_S is an ideal and a free S-submodule of $S[\mathrm{Irr}(G)]$. Moreover, the following are true.*

(i) *If $S = \mathbb{C}$, then the dimension of $\ker(r_S)$ as a complex vector space is equal to $|\mathrm{Cla}_H|$.*

(ii) *If $S = A$, then the rank of the S-module $\ker(r_S)$ is equal to $|\mathrm{Cla}_H|$.*

(iii) *If $S = \mathbb{Z}$, then the rank of the S-module $\ker(r_S)$ is not less than the number of orbits of K on Cla_H.*

Proof For the unitary subring S, we consider the S-linear map

$$r_S: \ S[\mathrm{Irr}(G)] \to S[\mathrm{Irr}(H)].$$

One can easily see that r_S is a ring homomorphism. It follows that $\ker(r_S)$ is an ideal and a free S-submodule of $S[\mathrm{Irr}(G)]$. To prove the remaining statements, recall that for each conjugacy class C of G, the characteristic function f_C of C is a class function on G, which is defined as follows:

$$f_C(x) = \begin{cases} 1, & \text{if } \ x \in C, \\ 0, & \text{if } \ x \in G \backslash C. \end{cases}$$

It is well known that, as an element in $\mathbb{C}[\mathrm{Irr}(G)]$,

$$f_C = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|C|\chi(c)}{|G|}\chi,$$

where $c \in C$ (see [1, section 3.2]).

Now assume that $S = \mathbb{C}$. For any function $h \in \ker(r_S)$, one can see that h vanishes on each conjugacy class of G which is not disjoint with H since h is a class function. This implies that

$$\ker(r_S) \subseteq \bigoplus_{C \in \mathrm{Cla}_H} \mathbb{C} f_C.$$

Conversely, it is obvious that every f_C with $C \in \mathrm{Cla}_H$ is contained in $\ker(r_S)$. It follows that

$$\bigoplus_{C \in \mathrm{Cla}_H} \mathbb{C} f_C \subseteq \ker(r_S).$$

We then get the equality, which proves statement (i).

Next, assume that $S = A$. By statement (i), one can see that the rank of the S-module $\ker(r_S)$ is not greater than $|\mathrm{Cla}_H|$. However, by the expression of f_C, we can prove that $|G| f_C$ belongs to $\ker(r_S)$ for each $C \in \mathrm{Cla}_H$. Thus, $\{|G| f_C : C \in \mathrm{Cla}_H\}$ is an S-basis of $\ker(r_S)$. This completes the proof of statement (ii).

Finally, assume that $S = \mathbb{Z}$. For each orbit $\mathcal{O}$ of K on Cla_H, observe that

$$f_{\mathcal{O}} := \sum_{C \in \mathcal{O}} |G| f_C = \sum_{\chi \in \mathrm{Irr}(G)} \Big(\sum_{x \in \mathcal{O}^{\cup}} \chi(x) \Big) \chi,$$

where $\mathcal{O}^{\cup}$ is the union of all conjugacy classes in $\mathcal{O}$. Since for each $\chi \in \mathrm{Irr}(G)$, the coefficient $\sum\limits_{x \in \mathcal{O}^{\cup}} \chi(x)$ is an algebraic integer and K-invariant. This implies that each of the coefficients is also a rational number and hence the coefficient lies in $\mathbb{Z}$ (see [6, Section 6.4]). It follows that each $f_{\mathcal{O}}$ is a nonzero element in $\mathbb{Z}[\mathrm{Irr}(G)]$. As all these $f_{\mathcal{O}}$ are $\mathbb{Z}$-independent, statement (iii) follows.

Remark 2.1 Assume that $S = \mathbb{Z}$. It may happen that the rank of $\ker(r_S)$ is greater than the number of K-orbits of Cla_H : For instance, if we take G to be a cyclic group of order, a prime p with $p > 3$ and H to be the identity subgroup, then the rank of $\ker(r_S)$ is $p - 1$ and the number of K-orbits of Cla_H equals 1. However, if G is a rational group, then both the rank of $\ker(r_S)$ and the number of K-orbits are equal to $|\mathrm{Cla}_H|$.

So far we did not get an exact formula of the rank of $\ker(r_S)$ for $S = \mathbb{Z}$. Also, we cannot completely describe the generalized characters in the kernel of $r_{\mathbb{Z}}$. There are similar results in [5, Theorem B]. In fact, Ferguson and Isaacs completely described the generalized characters which vanish at elements outside any conjugate of the proper subgroup.

Corollary 2.1 *For a proper subgroup H of G, the restriction map from* $\mathrm{Char}(G)$ *to* $\mathrm{Char}(H)$ *is not injective.*

Proof By statement (iii) of Theorem 2.1, there exists a nonzero element h in $\mathbb{Z}[\mathrm{Irr}(G)]$ such that $h(x) = 0$ for any $x \in H$. One can find disjoint subsets M and N of $\mathrm{Irr}(G)$ such that

$$h = \sum_{\chi \in M} a_\chi \chi - \sum_{\phi \in N} b_\phi \phi,$$

where all a_χ and b_ϕ are positive integers. As $h(1) = 0$, one can see that neither M nor N is empty. Now the distinct characters $\sum\limits_{\chi \in M} a_\chi \chi$ and $\sum\limits_{\phi \in N} b_\phi \phi$ have the identical restrictions to H, which proves the corollary.

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