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Spreading Speeds of Time-Dependent Partially Degenerate Reaction-Diffusion Systems*

Jia LIU[1]

Abstract This paper is concerned with the spreading speeds of time dependent partially degenerate reaction-diffusion systems with monostable nonlinearity. By using the principal Lyapunov exponent theory, the author first proves the existence, uniqueness and stability of spatially homogeneous entire positive solution for time dependent partially degenerate reaction-diffusion system. Then the author shows that such system has a finite spreading speed interval in any direction and there is a spreading speed for the partially degenerate system under certain conditions. The author also applies these results to a time dependent partially degenerate epidemic model.

Keywords Partially degenerate, Reaction-diffusion system, Time dependent, Spreading speed
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1 Introduction

Reaction-diffusion models have been widely used to describe the spatial dynamics of populations in biology and ecology, for example, see the benthic-pelgic population model in [14] and man-environment-man epidemics model in [6–7] and so on. In these two models, at least one diffusion coefficients of population are zero and we usually call such models as partially degenerate reaction-diffusion system, in which some diffusion coefficients are zero. Spreading speeds and traveling waves of partially degenerate reaction-diffusion system are two important issues in the study of biological invasions and disease spread and have attracted a lot of attention, see [2, 7–8, 11–12, 26, 28–32] and references therein.

Due to the presence of various temporal and spatial in many biological models, spreading speeds and traveling wave solution for general time and\or space dependent reaction-diffusion models have been widely investigated in many works. For example, see [1, 4–5, 13, 16–17, 19–24, 27] and references therein. Recently, Bao et al. [4] have introduced the notion of spreading speeds for general time dependent cooperative systems with different dispersal types and see [3] for the existence and stability of generalized transition waves for time-dependent reaction-diffusion systems. However, in general time dependent case, there is little understanding about the spatial spread and front propagation dynamics for partially degenerate reaction-diffusion

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[1]School of Science, Chang'an University, Xi'an 710064, China. E-mail: liujia@chd.edu.cn
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systems. The objective of the current paper is to study the spatial spreading speed of partially degenerate reaction-diffusion systems with general time dependent coefficients.

Consider the following time dependent partially degenerate reaction-diffusion system:

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(t,x) = \Delta u_1(t,x) + f(t,u_1,u_2), \\ \dfrac{\partial u_2}{\partial t}(t,x) = g(t,u_1,u_2), \end{cases} \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

where $u_1(t,x)$ and $u_2(t,x)$ are the densities of two species at time $t \in \mathbb{R}$ and location $x \in \mathbb{R}^N$. The reaction terms of (1.1) satisfy the following standard assumptions:

(H1) $f, g : \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}^2$, $f(t,\mathbf{u})$ and $g(t,\mathbf{u})$ are C^2 in $\mathbf{u}$, Hölder continuous in t, $f(t,\mathbf{0}) = g(t,\mathbf{0})$.

(H2) There exists a positive vector $\mathbf{M} = (M_1, M_2)$ such that $f(t,\mathbf{M}) \le 0$ and $g(t,\mathbf{M}) \le 0$ for any $t \in \mathbb{R}$. Moreover, $f_{u_2}(t,u_1,u_2) > 0$ and $g_{u_1}(t,u_1,u_2) > 0$ for any $t \in \mathbb{R}$ and $(u_1,u_2) \in [0,M_1] \times [0,M_2]$.

(H3) $\mathbf{F}(t,\mathbf{u}) := (f(t,\mathbf{u}), g(t,\mathbf{u}))$ is strictly subhomogeneous on $[0,M_1] \times [0,M_2]$ in the sense that $\mathbf{F}(t,\mu\mathbf{u}) > \mu\mathbf{F}(t,\mathbf{u})$ for any $t \in \mathbb{R}$, $\mu \in (0,1)$, $\mathbf{u} \in [0,M_1] \times [0,M_2]$ with $u_i > 0$ for all $i = 1,2$.

Let $\mathcal{C} = BC(\mathbb{R}^N, \mathbb{R}^2)$ be the set of all bounded and continuous functions from $\mathbb{R}^N$ to $\mathbb{R}^2$. For a constant vector in $\mathbb{R}_+^2$, define $[\mathbf{0}, r]_{\mathcal{C}} := \{\mathbf{u} \in \mathcal{C} : \mathbf{0} \le \mathbf{u}(x) \le r, \forall x \in \mathbb{R}^N\}$. Clearly, $\mathcal{C}_+ = \{\mathbf{u} \in \mathcal{C} : \mathbf{u}(x) \ge \mathbf{0}\}$ is a positive cone of $\mathcal{C}$. Let $\mathcal{C}_{++} := \{\mathbf{u} \in \mathcal{C} : \mathbf{u}(x) > \mathbf{0}, \forall x \in \mathbb{R}^N\}$. By general semigroup theory (see [18]), for any $\mathbf{u}_0(\cdot) \in \mathcal{C}$, (1.1) has a unique (local) solution $\mathbf{u}(t,x;\mathbf{u}_0)$ with $\mathbf{u}(0,x;\mathbf{u}_0) = \mathbf{u}_0$. By comparison principle, if $\mathbf{u}_0(\cdot) \in \mathcal{C}_+$, then $\mathbf{u}(t,x;\mathbf{u}_0)$ exists for all $t \ge 0$ and $\mathbf{u}(t,\cdot;\mathbf{u}_0) \in \mathcal{C}_+$.

Consider the linearization of (1.1) at $\mathbf{0}$, namely,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1(t,x) + f_{u_1}(t,\mathbf{0})u_1 + f_{u_2}(t,\mathbf{0})u_2, \\ \dfrac{\partial u_2}{\partial t} = g_{u_1}(t,\mathbf{0})u_1 + g_{u_2}(t,\mathbf{0})u_2. \end{cases} \tag{1.2}$$

For any $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$, the solution $\mathbf{u}(t,x)$ of (1.2) with the initial value $\mathbf{u}(0,x) = \mathrm{e}^{-\mu x\cdot\xi}\boldsymbol{\alpha}$ and $\boldsymbol{\alpha} \in \mathbb{R}^2$ is of the form $\mathbf{u}(t,x) = \mathrm{e}^{-\mu x\cdot\xi}\boldsymbol{\eta}(t)$, where $\boldsymbol{\eta}(t)$ satisfies the following system of ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}\eta_1}{\mathrm{d}t} = \mu^2 + f_{u_1}(t,\mathbf{0})\eta_1(t) + f_{u_2}(t,\mathbf{0})\eta_2(t), \\ \dfrac{\mathrm{d}\eta_2}{\mathrm{d}t} = g_{u_1}(t,\mathbf{0})\eta_1 + g_{u_2}(t,\mathbf{0})\eta_2. \end{cases} \tag{1.3}$$

Let

$$A^\mu(t) = \begin{pmatrix} \mu^2 + f_{u_1}(t,\mathbf{0}) & f_{u_2}(t,\mathbf{0}) \\ g_{u_1}(t,\mathbf{0}) & g_{u_2}(t,\mathbf{0}) \end{pmatrix}.$$

By (H2), we have $f_{u_2}(t,\mathbf{0}) > 0$ and $g_{u_1}(t,\mathbf{0}) > 0$ for all $t \in \mathbb{R}$. Then the matrix $A^\mu(t)$ is quasi-positive for all $t \in \mathbb{R}$. Moreover, we assume that

(H4) The matrix $A^\mu(t)$ is strongly irreducible and unique ergodic.

By the definition of principle Lyapunov exponent and the principal Floquent bundle, also see Definition 2.2, there is a principle Lyapunov exponent $\lambda(A^\mu)$ of (1.3) for any $\mu \geq 0$ and $\{\text{span}(\boldsymbol{\phi}(\sigma_t A^\mu))\}_{t\in\mathbb{R}}$ is the principal Floquent bundle of (1.3) associated to $\lambda(A^\mu)$.

Let $A(t) := A^0(t)$ for $\mu = 0$. Following from Theorem 2.2, if $\lambda(A) > 0$, (1.1) has a unique, globally stable, spatially homogeneous entire solution $\mathbf{u}^*(t)$. Thus we consider the spreading speeds of (1.1) from $\mathbf{u}^*(t)$ to $\mathbf{0}$. Roughly, for any given $\xi \in S^{N-1}$, a finite interval $[c^*_{\text{inf}}, c^*_{\text{sup}}]$ is called the spreading speed interval of (1.1) from $\mathbf{u}^*$ to $\mathbf{0}$ in the direction of ξ if for any $\mathbf{u}_0 \in \mathcal{C}_+$ satisfying $\mathbf{0} \leq \mathbf{u}_0 \ll \mathbf{u}^*(0,\cdot)$, $\mathbf{u}_0(x) = 0$ for $x\cdot\xi \gg 1$ and $\liminf\limits_{x\cdot\xi\to-\infty} \mathbf{u}_0(x) \gg \mathbf{0}$, there holds

$$\limsup_{t\to\infty, x\cdot\xi\leq ct} |\mathbf{u}(t,x;\mathbf{u}_0) - \mathbf{u}^*(t,x)| = 0, \quad \forall\, c < c^*_{\text{inf}},$$
$$\limsup_{t\to\infty, x\cdot\xi\geq ct} |\mathbf{u}(t,x;\mathbf{u}_0)| = 0, \quad \forall\, c > c^*_{\text{sup}},$$

see Definition 3.1 for details.

Throughout this paper, we assume (H1)–(H4) and $\lambda(A) > 0$. Among others, on the spreading speeds of (1.1), we prove

- (1.1) has a finite spreading speed interval $[c^*_{\text{inf}}(\xi), c^*_{\text{sup}}(\xi)]$.
- $c^*(\xi) := c^*_{\text{inf}}(\xi) = c^*_{\text{sup}}(\xi) = \inf\limits_{\mu>0} \frac{\lambda(A^\mu)}{\mu}$ is the spreading speed of (1.1).
- The spreading speed $c^*(\xi)$ is of some important spreading features (see Theorem 3.3 for details).
- Let $c^*_+ = \inf\limits_{\mu>0} \frac{\lambda(A^\mu)}{\mu}$ be the rightward spreading speed of (1.1). Then there exists $c^*_- = \inf\limits_{\mu>0} \frac{\lambda(A^{-\mu})}{\mu}$ being the leftward spreading speed of (1.1). Moreover, $c^*_+ + c^*_- > 0$.

We point out that the spreading speed and traveling wave solution of (1.1) with time periodic have been studied in [12] for monostable nonlinearity case and in [30] for bistable nonlinearity case. For system (1.1) with monostable nonlinearity in space periodic habitat, Wu et al. [29] has established the existence of spreading speed and Wang and Zhao [28] proved the existence and stability of pulsating wave to such partially degenerate reaction-diffusion system. Here, we establish the existence of spreading speed of time dependent partially degenerate reaction-diffusion system (1.1) with monostable nonlinearity and apply the results to a time dependent epidemic model. The existence and stability of generalized transition wave solution of (1.1) will be studied somewhere else.

The rest of this paper is organized as follows. In Section 2, we will present the Lyapunov exponent theory and prove the existence of entire solution for system (1.1). In Section 3, we will establish the existence of the spreading speed and prove some important spreading features. We apply the results to a time dependent epidemic model in Section 4.

2 The Lyapunov Exponent Theory and Entire Solution

In this section, we present the principle Lyapunov exponent theory for time dependent linear system and study entire positive solution of (1.1).

We first define the sub- and supersolution of (1.1) and prove the comparison principle. For any given $\mathbf{u}_0(x) := (u_{10}, u_{20}) \in \mathcal{C}$, consider the following initial value problem:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1(t,x) + f(t,u_1,u_2), \\ \dfrac{\partial u_1}{\partial t} = g(t,u_1,u_2), \quad t>0,\ x\in\mathbb{R}^N, \\ u_i(0,x) = u_{i0}(x), \quad i=1,2,\ x\in\mathbb{R}^N. \end{cases} \tag{2.1}$$

Definition 2.1 *A continuous vector-valued function* (u_1,u_2) *is called a supersolution (subsolution) of* (2.1) *on* $\mathbb{R}\times\mathbb{R}^N$*, if* $u_i(t,\cdot)\in C^2(\mathbb{R}^N)$ *for any* $t\in(0,+\infty)$*,* $u_i(x,\cdot)\in C^1(0,+\infty)$ *for any* $x\in\mathbb{R}^N$ *and* $(u_1(t,x),u_2(t,x))$ *satisfies*

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \Delta u_1(t,x) - f(t,u_1,u_2) \ge 0\,(\le 0), \\ \dfrac{\partial u_1}{\partial t} - g(t,u_1,u_2) \ge 0\,(\le 0) \end{cases}$$

for any $(t,x)\in\mathbb{R}\times\mathbb{R}^N$.

Theorem 2.1 *Recall that* $\mathbf{u}^\pm(t,x)\in[\mathbf{0},\mathbf{M}]_{\mathcal{C}}$ *are supersolution and subsolution of* (2.1) *on* $[0,\infty)$*, respectively, and satisfy* $\mathbf{u}^+(0,x)\ge\mathbf{u}^-(0,x)$ *for any* $x\in\mathbb{R}^N$*. Then one has* $\mathbf{u}^+(t,x)\ge\mathbf{u}^-(t,x)$ *for all* $t\ge 0$ *and* $x\in\mathbb{R}$*.*

Proof Let $\mathcal{C} = BC(\mathbb{R},\mathbb{R}^2)$ be the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$. It is easy to see that $\mathcal{C}_+ := \{\mathbf{u}\in\mathcal{C} : \mathbf{u}(x)\ge\mathbf{0},\ \forall x\in\mathbb{R}^N\}$ is a positive cone of $\mathcal{C}$. Let $T_1(t)$ be a strongly continuous semigroup on $\mathcal{C}$ generated by the operator $-\frac{\partial u_1}{\partial t} + \Delta u_1$. We define

$$[\mathbf{F}(u)](x) = \mathbf{F}(t,\mathbf{u}) = \begin{pmatrix} f(t,u_1,u_2) \\ g(t,u_1,u_2) \end{pmatrix} \quad \text{and} \quad T(t) = \begin{pmatrix} T_1(t) & 0 \\ 0 & 1 \end{pmatrix}.$$

Then a mild solution of (2.1) on $t\in[0,b)$ with the initial value $\mathbf{u}_0\in[\mathbf{0},\mathbf{M}]_{\mathcal{C}}$ means a continuous function $\mathbf{u}(t,x;\mathbf{u}_0) : [0,b)\to X$ satisfying the following integral system

$$\mathbf{u}(t,\cdot) = T(t)\mathbf{u}_0 + \int_0^t T(t-s)\mathbf{F}(s,\mathbf{u})\mathrm{d}s \quad \text{for } 0<b\le+\infty.$$

Let $\mathbf{u}^\pm(t)(x) := \mathbf{u}^\pm(t,x)$. Due to the positivity of $T(t)$, we have

$$\mathbf{u}^+(t) \ge T(t)\mathbf{u}_0^+ + \int_0^t T(t-s)\mathbf{F}(s,\mathbf{u}^+(s,\cdot))\mathrm{d}s$$

and

$$\mathbf{u}^-(t) \le T(t)\mathbf{u}_0^- + \int_0^t T(t-s)\mathbf{F}(s,\mathbf{u}^-(s,\cdot))\mathrm{d}s$$

for all $0\le t<+\infty$. Using (H2), we have that $\mathbf{F}(t,\mathbf{u})$ is quasi-monotone on $[\mathbf{0},\mathbf{M}]_{\mathcal{C}}$ in the sense that

$$\lim_{h\to 0^+}\frac{1}{h}\mathrm{dist}(\phi-\psi+h[\mathbf{F}(t,\phi)-\mathbf{F}(t,\psi)] : \mathcal{C}_+) = 0$$

for all $\phi,\psi\in[\mathbf{0},\mathbf{M}]_{\mathcal{C}}$ with $\phi\ge\psi$.

Applying [15, Corollary 5] with $S(t,s) = T(t,s) = T(t-s)$ and $B(t,\phi) = \mathbf{F}(t,\phi)$, we obtain that $\mathbf{u}^+(t,x) \geq \mathbf{u}^-(t,x)$ for all $t > 0$ and $x \in \mathbb{R}^N$. In particular, for any $\mathbf{u}_0 \in [\mathbf{0}, \mathbf{M}]_{\mathcal{C}}$ with $\mathbf{u}^-(0,x) \leq \mathbf{u}_0(x) \leq \mathbf{u}^+(0,x)$ for all $x \in \mathbb{R}$, (2.1) has a unique mild solution $\mathbf{u}(t,x;\mathbf{u}_0)$ on $t > 0$ satisfying $\mathbf{0} \leq \mathbf{u}^-(t,x) \leq \mathbf{u}(t,x;\mathbf{u}_0) \leq \mathbf{u}^+(t,x) \leq \mathbf{M}$ for $t > 0$ and $x \in \mathbb{R}^N$.

Next, we present the principle Lyapunov exponent theory for time dependent linear system. Consider the following ordinary differential system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = f_{u_1}(t,\mathbf{0})u_1 + f_{u_2}(t,\mathbf{0})u_2, \\ \dfrac{\partial u_2}{\partial t} = g_{u_1}(t,\mathbf{0})u_1 + g_{u_2}(t,\mathbf{0})u_2 \end{cases} \tag{2.2}$$

and let

$$A(t) = \begin{pmatrix} f_{u_1}(t,\mathbf{0}) & f_{u_2}(t,\mathbf{0}) \\ g_{u_1}(t,\mathbf{0}) & g_{u_2}(t,\mathbf{0}) \end{pmatrix}.$$

For each $\mathbf{u}_0 \in \mathbb{R}^2$ and $s \in \mathbb{R}$, (2.2) has a unique solution $\mathbf{u}(t;s,\mathbf{u}_0)$ with $\mathbf{u}(s;s,\mathbf{u}_0) = \mathbf{u}_0$. Put $\mathbf{U}(t,A)\mathbf{u}_0 = \mathbf{u}(t;0,\mathbf{u}_0)$ for any $\mathbf{u}_0 \in \mathbb{R}^2$. By (H2), $A(t)$ is quasi-positive for any $t \in \mathbb{R}$.

Definition 2.2 *Assume that $A(t)$ is strongly irreducible and unique ergodic. Then $\lambda(A) = \lim\limits_{t\to\infty} \frac{1}{t} \ln \|\mathbf{U}(t,A)\|$ is called the principal Lyapunov exponent of* (2.2) *or $A(\cdot)$ and there is principal Floquet bundle $\{\mathrm{span}(\boldsymbol{\phi}(\sigma_t A))\}_{t\in\mathbb{R}}$ of* (2.2) *or $A(\cdot)$ associated to $\lambda(A)$.*

Let

$$\kappa(A) = \langle A(0)\boldsymbol{\phi}(A), \boldsymbol{\phi}(A)\rangle, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product of $\mathbb{R}^2$. By [25, Proposition 4.10], the principal Lyapunov exponent $\lambda(A)$ can be calculated as follows

$$\lambda(A) = \lim_{t-s\to\infty} \frac{1}{t-s} \int_s^t \kappa(\sigma_\tau A)\mathrm{d}\tau. \tag{2.4}$$

For any $\mu \in \mathbb{R}$, consider the following system of ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}\eta_1}{\mathrm{d}t} = \mu^2 + f_{u_1}(t,\mathbf{0})\eta_1(t) + f_{u_2}(t,\mathbf{0})\eta_2(t), \\ \dfrac{\mathrm{d}\eta_2}{\mathrm{d}t} = g_{u_1}(t,\mathbf{0})\eta_1 + g_{u_2}(t,\mathbf{0})\eta_2 \end{cases} \tag{2.5}$$

and

$$A^\mu(t) = \begin{pmatrix} \mu^2 + f_{u_1}(t,\mathbf{0}) & f_{u_2}(t,\mathbf{0}) \\ g_{u_1}(t,\mathbf{0}) & g_{u_2}(t,\mathbf{0}) \end{pmatrix}.$$

By (H2), we have $f_{u_2}(t,\mathbf{0}) > 0$ and $g_{u_1}(t,\mathbf{0}) > 0$ for all $t \in \mathbb{R}$. Here the matrix $A^\mu(t)$ is quasi-positive and strongly irreducible for all $t \in \mathbb{R}$. Assume that $A^\mu(t)$ is unique ergodic, then by Definition 2.2, there is a principle Lyapunov exponent $\lambda(A^\mu)$ of (2.5) for any $\mu \geq 0$ and $\{\mathrm{span}(\boldsymbol{\phi}(\sigma_t A^\mu))\}_{t\in\mathbb{R}}$ is the principal Floquent bundle of (2.5) associated to $\lambda(A^\mu)$.

Lemma 2.1 (1) *For any given $\xi \in S^{N-1}$ and $\mu > 0$,*

$$\mathbf{u}(t,x) = \exp\Big(-\mu\Big(x\cdot\xi - \frac{\int_s^t \kappa(\sigma_\tau A^\mu)\mathrm{d}\tau}{\mu}\Big)\Big)\boldsymbol{\phi}(\sigma_t A^\mu)$$

is a solution of (1.3).

(2) *Let* $\lambda(\mu, A) = \lambda(A^\mu)$. *For any* $\xi \in S^{N-1}$, *if* $\lambda(A) > 0$, *then there exists a* $\mu^* > 0$ *such that*

$$\inf_{\mu>0} \frac{\lambda(\mu, A)}{\mu} = \frac{\lambda(\mu^*, A)}{\mu^*} \tag{2.6}$$

and

$$\frac{\lambda(\mu, A)}{\mu} \geq \frac{\lambda(\mu^*, A)}{\mu^*} \quad \textit{for } 0 < \mu < \mu^*. \tag{2.7}$$

Proof (1) It follows from the definition of principal Lyapunov exponent $\lambda(\mu, A)$ and the principal Floquent bundle $\{\text{span}(\boldsymbol{\phi}(\sigma_t A^\mu))\}_{t\in\mathbb{R}}$. (2) follows from [4, Lemma 3.4].

For given $\mathbf{u}_1(\cdot), \mathbf{u}_2(\cdot) \in \mathcal{C}_{++}$, if there is $\alpha_0 \geq 1$ such that

$$\frac{1}{\alpha_0}\mathbf{u}_1(x) \leq \mathbf{u}_2(x) \leq \alpha_0 \mathbf{u}_1(x), \quad \forall x \in \mathbb{R},$$

then the so-called part metric $\rho(\mathbf{u}_1, \mathbf{u}_2)$ between $\mathbf{u}_1$ and $\mathbf{u}_2$ is defined as

$$\rho(\mathbf{u}_1, \mathbf{u}_2) = \inf\Big\{ \ln \alpha \mid \alpha \geq 1, \frac{1}{\alpha_1}\mathbf{u}_1(\cdot) \leq \mathbf{u}_2(\cdot) \leq \alpha \mathbf{u}_1(\cdot)\Big\}.$$

Lemma 2.2 *For any* $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{C}_{++}$, $\rho(\mathbf{u}(t, \cdot; \mathbf{u}_1), \mathbf{u}(t, \cdot; \mathbf{u}_2))$ *is non-increasing in* t. *Moreover, if* $\boldsymbol{\varepsilon} \leq \mathbf{u}_1(\cdot) \leq \mathbf{M}$, $\boldsymbol{\varepsilon} \leq \mathbf{u}_2(\cdot) \leq \mathbf{M}$ *for all* $x \in \mathbb{R}^N$ *and* $\rho(\mathbf{u}_1, \mathbf{u}_2) \geq \sigma$ *for any positive constants* ε, σ *and* M *with* $\varepsilon < M$ *and* $\sigma \leq \ln \frac{M}{\varepsilon}$, *then there is* $\delta > 0$ *such that*

$$\rho(\mathbf{u}(\tau, \cdot; \mathbf{u}_1), \mathbf{u}(\tau, \cdot; \mathbf{u}_2)) \leq \rho(\mathbf{u}_1, \mathbf{u}_2) - \delta \quad \textit{for any } \tau > 0.$$

This lemma can be proved by the similar arguments as those in [10, Proposition 3.4] and we omit it here.

Theorem 2.2 *Assume* $\lambda(A) > 0$. *Then there is a unique spatially homogeneous entire solution* $\mathbf{u}^*(t)$ *of* (1.1), *which is globally stable in the sense that for any* $\mathbf{u}_0 \in [\mathbf{0}, \mathbf{M}]_{\mathcal{C}} \setminus \{\mathbf{0}\}$,

$$\lim_{t\to\infty} \mathbf{u}(t+s; s, \mathbf{u}_0) = \mathbf{u}^*(t+s) \quad \textit{uniformly in } s \in \mathbb{R}.$$

Proof First, consider

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(t, \mathbf{u}), \quad t \in \mathbb{R}. \tag{2.8}$$

For any $\mathbf{u}_0 \in \mathbb{R}^2$, let $\mathbf{u}(t; s, \mathbf{u}_0)$ be the solution of (2.8) with $\mathbf{u}(s; s, \mathbf{u}_0) = \mathbf{u}_0$. We prove that (2.8) has an entire solution $\mathbf{u}^*(t)$ with $\inf\limits_{t\in\mathbb{R}, i=1,2} u_i^*(t) > 0$. By Lemma 2.1(1), $\mathrm{e}^{\int_s^t \kappa(\sigma_\tau A)\mathrm{d}\tau}\boldsymbol{\phi}(\sigma_t A)$ is the solution of (2.2). For any given $\tau_2 > \tau_1 > 0$, there exists $\beta \in \mathbb{R}^2$ with $0 < \beta_i \ll 1$ such that $\mathbf{F}(t, \mathbf{u}) \geq (1 - \tau_1)A(t)\mathbf{u} - \tau_2\mathbf{u}$ for any $\mathbf{u} \in [0, \beta]$. Since $\lambda(A) > 0$, for any τ_1 and τ_2 small enough, there exists $T > 0$ such that

$$\int_s^{s+T} (1 - \tau_1)\kappa(\sigma_\tau A)\mathrm{d}\tau > \tau_2 T.$$

Note that $e^{\int_s^t (1-\tau_1)\kappa(\sigma_\tau A)d\tau - \tau_2(t-s)}\boldsymbol{\phi}(\sigma_t A)$ is the solution of

$$\frac{d\mathbf{u}}{dt} = (1-\tau_1)A(t)\mathbf{u} - \tau_2\mathbf{u}.$$

For any $\mathbf{u}_0 \gg \mathbf{0}$, there exists $\rho > 0$ such that $\mathbf{u}_0 \geq \boldsymbol{\phi}(\sigma_s A)$ for any $x \in \mathbb{R}^N$. Then by the comparison principle, we have

$$\mathbf{u}(t; s, \mathbf{u}_0) \geq e^{\int_s^t (1-\tau_1)\kappa(\sigma_\tau A)d\tau - \tau_2(t-s)}\boldsymbol{\phi}(\sigma_t A) > 0 \tag{2.9}$$

for any $t \geq s$. By (H2), $\mathbf{F}(t, \mathbf{u}) < 0$ for any $t \in \mathbb{R}$ and $\mathbf{u}(t; s, \mathbf{M}) \geq \mathbf{M}$. Then $\mathbf{u}(t; s, \mathbf{M}) \leq \mathbf{M}$ for $t > s$. Let $\mathbf{u}^*(t) = \mathbf{u}(t; -nT, \mathbf{M})$, $t \geq -nT$. Then

$$\mathbf{u}(t; -(n+1)T, \mathbf{u}_0) < \mathbf{u}^{n+1}(t) < \mathbf{u}^n(t), \quad t \geq -nT.$$

Let

$$\mathbf{u}^*(t) = \lim_{n\to\infty} \mathbf{u}^n(t).$$

Then $\mathbf{u}^*(t)$ is an entire solution of (2.8) and it is also a spatially homogeneous positive entire solution of (1.1).

Next, we show that $\mathbf{u}^*(t)$ is globally stable. Assume that there is $\mathbf{u}_0 \in \mathcal{C}_{++}$ such that $\|\mathbf{u}(t+s; s, \mathbf{u}_0) - \mathbf{u}^*(t+s)\|_\infty$ does not converge to 0 as $t \to \infty$. Then there are $\varepsilon_0 > 0$, $s_n \in \mathbb{R}$, $t_n \in \mathbb{R}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\|\mathbf{u}(t+s; s, \mathbf{u}_0) - \mathbf{u}^*(t+s)\|_\infty \geq \varepsilon_0.$$

By Lemma 2.2, we have $\rho(\mathbf{u}(t+s_n; s_n, \mathbf{u}_0), \mathbf{u}^*(t+s)) < \rho(\mathbf{u}_0, \mathbf{u}^*(s_n))$ for $t > 0$. Together with (2.9), there are $\mathbf{0} < \varepsilon \leq \mathbf{M}$ such that

$$\varepsilon \leq \mathbf{u}(t+s_n; s_n, \mathbf{u}_0) \leq \mathbf{M}, \quad \varepsilon \leq \mathbf{u}^*(t+s) \leq \mathbf{M}, \quad \forall t \geq s_n.$$

Apply Lemma 2.2 again, there are σ_0, $\delta_0 > 0$ and $\tau > 0$ such that

$$\sigma_0 \leq \rho(\mathbf{u}(t_n+s_n; s_n, \mathbf{u}_0), \mathbf{u}^*(t_n+s)) < \rho(\mathbf{u}_0, \mathbf{u}^*(s_n)) - k\delta_0$$

for $n \geq 1$ and $1 \leq k \leq [\frac{t}{\tau}]$. This is a contradiction. Hence, we have $\|\mathbf{u}(t+s; s, \mathbf{u}_0) - \mathbf{u}^*(t+s)\|_\infty \to 0$ as $t \to \infty$, which implies that $\lim_{t\to\infty} \mathbf{u}(t+s; s, \mathbf{u}_0) = \mathbf{u}^*(t+s)$ uniformly in $s \in \mathbb{R}$. This completes the proof.

3 Spreading Speed

In this section, we first give the definition of the spreading speed interval $[c^*_{\inf}, c^*_{\sup}]$ of system (1.1) and establish its basic properties.

Define

$$\mathbf{u}^*_{\inf} := \Big(\inf_{t\in\mathbb{R}} u_1^*(t), \inf_{t\in\mathbb{R}} u_2^*(t)\Big)(\geq \mathbf{0}).$$

For any given $\mathbf{u}_0(\cdot) \in [\mathbf{0}, \mathbf{u}^*]_{\mathcal{C}}$, we define

$$X^+ = \Big\{\mathbf{u}_0 \in \mathcal{C} \mid \mathbf{0} \leq \mathbf{u}_0 \ll \mathbf{u}^*_{\inf}, \liminf_{x\cdot\xi\to-\infty} \mathbf{u}_0(x) \gg \mathbf{0}, \mathbf{u}_0(x) = \mathbf{0}, \forall x\cdot\xi \gg 1\Big\}.$$

Definition 3.1 (Spreading speed interval) *Let*

$$C^*_{\inf} = \Big\{c \in \mathbb{R} \mid \forall \mathbf{u}_0 \in X^+, \limsup_{x\cdot\xi \le ct, t\to\infty} \|\mathbf{u}(t+s, x; s, \mathbf{u}_0) - \mathbf{u}^*(t+s)\| = 0 \textit{ uniformly in } s \in \mathbb{R}\Big\}$$

and

$$C^*_{\sup} = \Big\{c \in \mathbb{R} \mid \forall \mathbf{u}_0 \in X^+, \limsup_{x\cdot\xi \ge ct, t\to\infty} \|\mathbf{u}(t+s, x; s, \mathbf{u}_0)\| = 0 \textit{ uniformly in } s \in \mathbb{R}\Big\}.$$

Define $c^*_{\inf} = \sup\{c \mid c \in C^*_{\inf}\}$ *and* $c^*_{\sup} = \inf\{c \mid c \in C^*_{\sup}\}$. *We call* $[c^*_{\inf}, c^*_{\sup}]$ *as the spreading speed interval of* (1.1).

Let $\eta(s)$ be the function defined by

$$\eta(s) = \frac{1}{2}\Big(1 + \tanh\frac{s}{2}\Big), \quad s \in \mathbb{R}.$$

Note that $\eta'(s) = \eta(s)(1-\eta(s)) > 0$ and $\eta''(s) = \eta(s)(1-\eta(s))(1-2\eta(s))$ for any $s \in \mathbb{R}$. Without loss of generality, we can assume that there exists a vector $\mathbf{u}^- \ll \mathbf{0}$ in $\mathbb{R}^2$ such that $f(t, \mathbf{u})$ and $g(t, \mathbf{u})$ are defined for all $\mathbf{u} \in [\mathbf{u}^-, \infty)$, $f(t, \mathbf{u}^-) \ge 0$ and $g(t, \mathbf{u}^-) \ge 0$ for all $t \in \mathbb{R}$, and the condition (H3) holds for all $\mathbf{u} \in [\mathbf{u}^-, \mathbf{M}]$.

Lemma 3.1 *Let* $\alpha^\pm$ *be given constant vectors with* $\mathbf{u}^- \le \alpha^- \le \mathbf{0} \ll \alpha^+ \le \mathbf{u}^*_{\inf}$. *There is* $C_0 > 0$ *such that for every* $C \ge C_0$, $s \in \mathbb{R}$ *and* $\xi \in S^{N-1}$, *the following statements are valid*:

(1) *Let* $\mathbf{v}^\pm(t, x; s) = \mathbf{u}(t, x; s, \alpha^\pm)\eta(x\cdot\xi + C(t-s)) + \mathbf{u}(t, x; s, \alpha^\mp)[1 - \eta(x\cdot\xi + C(t-s))]$. *Then* $\mathbf{v}^+$ *and* $\mathbf{v}^-$ *are super- and subsolution of* (1.1) *on* $[0, +\infty)$, *respectively.*

(2) *Let* $\mathbf{w}^\pm(t, x; s) = \mathbf{u}(t, x; s, \alpha^\mp)\eta(x\cdot\xi - C(t-s)) + \mathbf{u}(t, x; s, \alpha^\pm)[1 - \eta(x\cdot\xi - C(t-s))]$. *Then* $\mathbf{w}^+$ *and* $\mathbf{w}^-$ *are super- and subsolution of* (1.1) *on* $[0, +\infty)$, *respectively.*

Proof We only prove $\mathbf{v}^+(t, x; s) = (v_1^+(t, x; s), v_2^+(t, x; s))$ is a supersolution of (1.1) and other statements can be proven similarly. Set $\zeta = x\cdot\xi + C(t-s)$ and $\mathbf{w}(t, x) = \mathbf{u}(t, x; s, \alpha^+) - \mathbf{u}(t, x; s, \alpha^-)$. Since $\mathbf{v}^+(t, x; s) = \mathbf{u}(t, x; s, \alpha^+)\eta(\zeta) + \mathbf{u}(t, x; s, \alpha^-)[1 - \eta(\zeta)]$, we obtain

$$\begin{aligned}
&\frac{\partial v_1^+}{\partial t} - \Delta v_1^+(t, x; s) - f(t, v_1^+, v_2^+)\\
&= \eta'(\zeta)[C w_1(t, x) - 2\nabla w_1(t, x)\cdot\xi - w_1(t, x)(1 - 2\eta(\zeta))]\\
&\quad + \eta(\zeta)[f(t, \mathbf{u}(t, x; s, \alpha^+)) - f(t, \mathbf{u}(t, x; s, \alpha^-))] + f(t, \mathbf{u}(t, x; s, \alpha^-)) - f(t, \mathbf{v}^+).
\end{aligned} \tag{3.1}$$

Note that

$$\begin{aligned}
&\eta(\zeta)[f(t, \mathbf{u}(t, x; s, \alpha^+)) - f(t, \mathbf{u}(t, x; s, \alpha^-))] + f(t, \mathbf{u}(t, x; s, \alpha^-)) - f(t, \mathbf{v}^+)\\
&= \eta(\zeta)\sum_{i=1}^{2}\int_0^1 f_{u_i}(t, \mathbf{u}(t, x; s, \alpha^+) + r\mathbf{w}(t, x))w_i(t, x)\mathrm{d}r\\
&\quad - \sum_{i=1}^{2}\int_0^1 f_{u_i}(t, \mathbf{u}(t, x; s, \alpha^-) + r\eta(\zeta)\mathbf{w}(t, x))\eta(\zeta)w_i(t, x)\mathrm{d}r\\
&= \sum_{i=1}^{2}\eta(\zeta)w_i(t, x)\int_0^1 [f_{u_i}(t, \mathbf{u}(t, x; s, \alpha^+) + r\mathbf{w}(t, x)) - f_{u_i}(t, \mathbf{u}(t, x; s, \alpha^-) + r\eta(\zeta)\mathbf{w}(t, x))]\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&= \eta'(\zeta) w_1(t,x) \int_0^1 r[f_{u_1u_1}(t,\mathbf{u}_r(t,x))w_1 + f_{u_1u_2}(t,\mathbf{u}_r(t,x))w_2]\mathrm{d}r \\
&\quad + \eta'(\zeta) w_2(t,x) \int_0^1 r[f_{u_2u_1}(t,\widetilde{\mathbf{u}}_r(t,x))w_1 + f_{u_2u_2}(t,\widetilde{\mathbf{u}}_r(t,x))w_2]\mathrm{d}r, \qquad (3.2)
\end{aligned}
$$

where $\mathbf{u}_r(t,x), \widetilde{\mathbf{u}}_r(t,x)$ are between $\mathbf{u}(t,x;s,\alpha^+)$ and $\mathbf{u}(t,x;s,\alpha^-)$.

Together with (3.1) and (3.2), we have

$$
\begin{aligned}
&\frac{\partial v_1^+}{\partial t} - \Delta v_1^+(t,x;s) - f(t, v_1^+, v_2^+) \\
&= \eta'(\zeta)\Big[Cw_1(t,x) - 2\nabla w_1(t,x)\cdot\xi - w_1(t,x)(1-2\eta(\zeta)) + w_2^2 \int_0^1 r f_{u_2u_2}(t,\widetilde{\mathbf{u}}_r)\mathrm{d}r \\
&\quad + w_1 \int_0^1 r[f_{u_1u_1}(t,\mathbf{u}_r)w_1 + f_{u_1u_2}(t,\mathbf{u}_r)w_2 + f_{u_2u_1}(t,\widetilde{\mathbf{u}}_r)w_2]\Big].
\end{aligned}
$$

Note that the priori estimates for parabolic equation, there exist $\gamma_1, \gamma_2 > 0$ such that $w_1(t,x) \geq \gamma_1$ and $|\nabla w_1(t,x)\cdot\xi| < \gamma_2$ for all $t \geq s$, $x \in \mathbb{R}^N$. Thus there is $C_1 > 0$ such that for $C \geq C_1$,

$$
\frac{\partial v_1^+}{\partial t} - \Delta v_1^+(t,x;s) - f(t, v_1^+, v_2^+) \geq 0.
$$

On the other hand,

$$
\begin{aligned}
\frac{\partial v_2^+}{\partial t} - g(t, v_1^+, v_2^+) &= \eta'(\zeta)\Big[Cw_2(t,x) + w_1^2 \int_0^1 r f_{u_1u_1}(t,\mathbf{u}_r)\mathrm{d}r \\
&\quad + w_2 \int_0^1 r[f_{u_1u_2}(t,\mathbf{u}_r)w_1 + f_{u_2u_1}(t,\widetilde{\mathbf{u}}_r)w_1 + f_{u_2u_2}(t,\widetilde{\mathbf{u}}_r)w_2]\Big].
\end{aligned}
$$

By the same way, there exists $C_2 > 0$ such that for $C \geq C_2$, $\frac{\partial v_2^+}{\partial t} \geq g(t, v_1^+, v_2^+)$. This completes the proof.

Similar to that in [20, Lemmas 3.3–3.4], for $c^*_{\inf}$ and $c^*_{\sup}$, we have the following spreading properties.

Lemma 3.2 *The following statements are valid*:

(1) *If there is* $\mathbf{u}^*(\cdot) \in X^+$ *such that*

$$
\limsup_{x\cdot\xi \leq ct, t\to\infty} \|\mathbf{u}(t+s,x;s,\mathbf{u}_0^*) - \mathbf{u}^*(t+s)\| = 0
$$

uniformly in $s \in \mathbb{R}$. *Then* $c \leq c^*_{\inf}$.

(2) *If* $c < c^*_{\inf}$, *then for any* $\mathbf{u}_0 \in X^+$,

$$
\limsup_{x\cdot\xi \leq ct, t\to\infty} \|\mathbf{u}(t+s,x;s,\mathbf{u}_0) - \mathbf{u}^*(t+s)\| = 0 \quad \textit{uniformly in } s \in \mathbb{R}.
$$

(3) *If there is* $\mathbf{u}^*(\cdot) \in X^+$ *such that*

$$
\limsup_{x\cdot\xi \geq ct, t\to\infty} \|\mathbf{u}(t+s,x;s,\mathbf{u}_0^*)\| = 0
$$

uniformly in $s \in \mathbb{R}$. *Then* $c \geq c^*_{\inf}$.

(4) *If $c > c^*_{\inf}$, then for any $\mathbf{u}_0 \in X^+$,*

$$\limsup_{x\cdot\xi \geq ct, t\to\infty} \|\mathbf{u}(t+s,x;s,\mathbf{u}_0)\| = 0 \quad \textit{uniformly in } s \in \mathbb{R}.$$

Theorem 3.1 *$[c^*_{\inf}, c^*_{\sup}]$ is a finite interval.*

Proof Let $\alpha^\pm$ be given constant vectors with $\mathbf{u}^- \leq \alpha^- \leq \mathbf{0} \ll \alpha^+ \leq \mathbf{u}^*_{\inf}$. Then there is $\mathbf{u}^*_0(\cdot) \in X^+$ such that

$$\mathbf{w}^+(s,x;s) = \alpha^-\eta(x\cdot\xi) + \alpha^+[1-\eta(x\cdot\xi)] \geq \mathbf{u}^*_0(x), \quad \forall x \in \mathbb{R}^N,\ s \in \mathbb{R}.$$

Then it follows from comparison principle and Lemma 3.1 that

$$\begin{aligned}\mathbf{w}^+(t+s,x;s) &= \mathbf{u}(t+s;s,\alpha^-)\eta(x\cdot\xi - C_0t) + \mathbf{u}(t+s;s,\alpha^+)[1-\eta(x\cdot\xi - C_0t)]\\ &\geq \mathbf{u}(t+s,x;s,\mathbf{u}^*_0)\end{aligned}$$

for $t \geq 0$ and $s \in \mathbb{R}$. For any $C > C_0$, the fact $\eta(\infty) = 1$ implies that

$$\begin{aligned}0 &\leq \limsup_{t\to\infty, x\cdot\xi\geq C_1t} \|\mathbf{u}(t+s,x;s,\mathbf{u}^*_0)\|\\ &\leq \limsup_{t\to\infty, x\cdot\xi\geq C_1t} \|\mathbf{w}^+(t+s,x;s)\| = \limsup_{t\to\infty, x\cdot\xi\geq C_1t} \|\mathbf{u}(t+s;s,\alpha^-)\| = 0.\end{aligned}$$

Then we have

$$\limsup_{t\to\infty, x\cdot\xi\geq C_1t} \|\mathbf{u}(t+s,x;s,\mathbf{u}^*_0)\| = 0 \quad \text{uniformly in } s \in \mathbb{R}.$$

By Lemma 3.2(3), it follows that $c^*_{\sup} \leq C_1$.

On the other hand, let $\alpha^\pm$ be given constant vectors with $\mathbf{u}^- \leq \alpha^- \leq 0 \ll \alpha^+ \leq \mathbf{u}^*_{\inf}$. There is $\widetilde{\mathbf{u}}^*_0(\cdot) \in X^+$ such that

$$\mathbf{v}^-(s,x;s) = \alpha^-\eta(x\cdot\xi) + \alpha^+[1-\eta(x\cdot\xi)] \leq \widetilde{\mathbf{u}}^*_0(x)$$

for $s \in \mathbb{R}$. By the comparison principle and Lemma 3.1 again, we have

$$\begin{aligned}\mathbf{v}^-(t+s,x;s) &= \mathbf{u}(t+s;s,\alpha^-)\eta(x\cdot\xi + C_0t) + \mathbf{u}(t+s,s,\alpha^+)(1-\eta(x\cdot\xi + C_0t))\\ &\leq \mathbf{u}(t+s,x;s,\widetilde{u}^*_0)\end{aligned}$$

for $t \geq 0$, $s \in \mathbb{R}$. Then for each $C_2 < -C_0$, the fact $\eta(-\infty) = 0$ implies that

$$\liminf_{t\to\infty, x\cdot\xi\leq C_2t} \|\mathbf{u}(t+s,x;s,\widetilde{\mathbf{u}}^*_0) - \mathbf{u}^*(t+s)\| = 0$$

uniformly in $s \in \mathbb{R}$. Then $c^*_{\inf} \geq C_2$. Hence, $[c^*_{\inf}, c^*_{\sup}]$ is a finite interval.

Lemma 3.3 *Let $c \in \mathbb{R}$ and $\mathbf{u}_0 \in X^+$ be given. If there are $\mathbf{0} \ll \delta^0 \leq \mathbf{u}^*_{\inf}$ and $T_0 > 0$ such that*

$$\lim_{n\to\infty, x\cdot\xi\leq cnT_0} \mathbf{u}(nT_0+s,x;s,\mathbf{u}_0) \geq \delta_0 \quad \textit{uniformly in } s \in \mathbb{R}, \tag{3.3}$$

then for each $c' < c$,

$$\limsup_{t\to\infty, x\cdot\xi\leq c't} \|\mathbf{u}(t+s,x;s,\mathbf{u}_0) - \mathbf{u}^*(t+s)\| = 0 \quad \textit{uniformly in } s \in \mathbb{R}.$$

Proof By (3.3), there is $n_0 \in \mathbb{N}$ such that

$$\mathbf{u}(nT_0 + s, x; s, \mathbf{u}_0) \geq \frac{\delta_0}{2} \quad \text{for } n \geq n_0,\ x \cdot \xi \leq cn_0T. \tag{3.4}$$

Let $\widetilde{\mathbf{u}}_0(x) \equiv \frac{\delta_0}{2}$. Then for each $\varepsilon > 0$, there exists $n_1 \geq n_0$ such that

$$\mathbf{u}(t + s, x; s, \widetilde{u}_0) \geq \mathbf{u}^*(t + s) - \varepsilon \quad \text{for } t \geq n_1T_0,\ x \in \mathbb{R}^N. \tag{3.5}$$

For a given $B > 1$, let $\widetilde{\mathbf{u}}_B(\cdot) \in \left[0, \frac{\delta_0}{2}\right]_{\mathcal{C}}$ be such that $\widetilde{\mathbf{u}}_B(x) = \frac{\delta_0}{2}$ for $x \cdot \xi \leq B - 1$ and $\widetilde{\mathbf{u}}_B(x) = 0$ for $x \cdot \xi \geq B$. By (3.5), there is $\widetilde{B}_0 > 1$ such that for any $B \geq \widetilde{B}_0$,

$$\mathbf{u}(t + s, 0; s, \widetilde{u}_B) \geq \mathbf{u}^*(t + s) - 2\varepsilon \quad \text{for } n_1T_0 \leq t \leq (n_1 + 1)T_0,\ s \in \mathbb{R}. \tag{3.6}$$

For given $c' < c$, $(c - c')nT_0 \to \infty$ as $n \to \infty$. Then there is $n_2 \geq n_1$ such that

$$(c - c')nT_0 \geq B_0 + c'(n_1 + 1)T_0 \quad \text{for } n \geq n_2. \tag{3.7}$$

This together with (3.1), implies that

$$\mathbf{u}(nT_0 + s, x + x' + c'nT_0\xi + c'\tau\xi; s, \mathbf{u}_0) \geq \widetilde{\mathbf{u}}_B(x'), \quad \forall x' \in \mathbb{R}^N,\ \tau \in [n_1T_0, (n_1 + 1)T_0] \tag{3.8}$$

for all $x \in \mathbb{R}^N$ with $x \cdot \xi \leq 0$ and $n \geq n_2$. For any given $n \geq n_2$ and $(n + n_1)T_0 \leq t < (n + n_1 + 1)T_0$, let $\tau = t - nT_0$, by (3.7) and (3.8), we get

$$\begin{aligned}\mathbf{u}(t + s, x + c't\xi; s, \mathbf{u}_0) &= \mathbf{u}(t + s, x + c't\xi; s, \mathbf{u}(nT_0, \cdot; 0, \mathbf{u}_0)) \\ &\geq \mathbf{u}(t + s, 0; s, \widetilde{u}_B(\cdot)) \\ &\geq \mathbf{u}^*(t + s) - 2\varepsilon\end{aligned}$$

for all $x \in \mathbb{R}^N$ with $x \cdot \xi \leq 0$. Thus we obtain

$$\mathbf{u}(t + s, x; s, \mathbf{u}_0) \geq \mathbf{u}^*(t + s) - 2\varepsilon$$

for $x \cdot \xi \leq c't$, $t \geq (n_1 + n_2)T_0$, $s \in \mathbb{R}$, which implies the result.

Theorem 3.2 *The following statements are valid*:

(1) $c^*_{\inf} = c^*_{\sup} = \inf\limits_{\mu>0} \frac{\lambda(A^\mu)}{\mu}$.

(2) *Let* $c^*_+ = \inf\limits_{\mu>0} \frac{\lambda(A^\mu)}{\mu}$ *be the rightward spreading speed of* (1.1). *Then there exists* $c^*_- = \inf\limits_{\mu>0} \frac{\lambda(A^{-\mu})}{\mu}$ *being the leftward spreading speed of* (1.1). *Moreover,* $c^*_+ + c^*_- > 0$.

Proof (1) First, we show that $c^*_{\inf} \geq \inf\limits_{\mu>0} \frac{\lambda(A^\mu)}{\mu}$. For any $s \in \mathbb{R}$, $\mu > 0$ and $T > 0$, let

$$\mathbf{\Phi}(T; s, A, \mu)\mathbf{u}_0 = \mathbf{u}(s + T; s, \mathbf{u}_0, A^\mu), \tag{3.9}$$

where $\mathbf{u}(t; s, \mathbf{u}_0, A^\mu)$ is the solution of (2.5) with initial value $\mathbf{u}(s; s, \mathbf{u}_0, A^\mu) = \mathbf{u}_0 \in \mathbb{R}^2$. Since $\mathbf{\Phi}(T; s, A, \mu)$ is a compact and strongly monotone operator, by the Krein-Rutman theorem, there is a principal eigenvalue of $\mathbf{\Phi}(T; s, A, \mu)$ and denote by $\gamma^T(s,\mu)$. By [4, Lemma 3.5], we have that

$$\lim_{T\to\infty} \frac{\ln \gamma^T(s,\mu)}{T} = \lambda(A^\mu) \quad \text{uniformly in } s \in \mathbb{R}. \tag{3.10}$$

Since there exists $\beta \gg \mathbf{0}$ with $0 < \beta_i \ll 1$ such that $\mathbf{F}(t,\mathbf{u}) \geq (1-\tau^{-1})A(t)\mathbf{u}$ for $\mathbf{u} \in [\mathbf{0}, \beta]$ and $(1-\tau^{-1})A(t)\mathbf{u} \to A(t)\mathbf{u}$ as $\tau \to \infty$ for any $t \in \mathbb{R}$, then by [4, Lemma 4.1], there are $T \geq 1$ and $\mathbf{v}_0(\cdot) \in X^+$ such that

$$u_i(T+s, x; s, \mathbf{v}_0) \geq v_{i0}(x - c\xi T), \quad i = 1, 2$$

for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$ and $c < \inf_{\mu>0} \frac{\lambda(A^\mu)}{\mu} - \varepsilon$. By induction, we have that

$$u_i(s+nT, x; s, \mathbf{v}_0) \geq v_{i0}(x - cnT\xi), \quad i = 1, 2$$

for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$ and $c < \inf_{\mu>0} \frac{\lambda(A^\mu)}{\mu} - \varepsilon$. Then for any $\varepsilon > 0$ and $c < \inf_{\mu>0} \frac{\lambda(A^\mu)}{\mu} - \varepsilon$,

$$\liminf_{x\cdot\xi \leq cnT_0, n\to\infty} u_i(s+nT, x; s, \mathbf{v}_0) > 0, \quad i = 1, 2$$

uniformly in $s \in \mathbb{R}$. Then by Lemmas 3.2(1) and 3.3, we have

$$c^*_{\text{inf}} \geq \inf_{\mu>0} \frac{\lambda(A^\mu)}{\mu} - \varepsilon.$$

Since ε is arbitrary, we have $c^*_{\text{inf}} \geq \inf_{\mu>0} \frac{\lambda(A^\mu)}{\mu}$.

Next, we show that $c^*_{\text{sup}} \leq \inf_{\mu>0} \frac{\lambda(A^\mu)}{\mu}$. By (H4), we know that $\mathbf{F}(t,\mathbf{u}) \leq A(t)\mathbf{u}$ for all $t \in \mathbb{R}$ and $\mathbf{u} \geq \mathbf{0}$. Let $\boldsymbol{\phi}(t,\mu)$ be the principal Floquent bundle associated to $\lambda(A^\mu)$. Let $\mu^* \in (0, +\infty)$ such that $\inf_{\mu>0} \frac{\lambda(A^\mu)}{\mu} = \frac{\lambda(A^{\mu^*})}{\mu^*}$. For any $\mathbf{u}_0(\cdot) \in X^+$, there is a positive constant ρ such that $\mathbf{u}_0(\cdot) \leq \rho \mathrm{e}^{-\mu^* x\cdot\xi} \boldsymbol{\phi}(s,\mu^*)$ for $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$. Since

$$\lambda(A^\mu) = \lim_{t-s\to\infty} \frac{1}{t-s} \int_s^t \kappa(\sigma_\tau A^\mu) \mathrm{d}\tau,$$

then by the comparison principle and Lemma 2.1(1) that

$$\mathbf{u}(t, x; s, \mathbf{u}_0) \leq \rho \exp\Big(-\mu^* \Big(x\cdot\xi - \frac{\lambda(A^{\mu^*}) + \varepsilon}{\mu^*} t \Big) \Big) \boldsymbol{\phi}(t,\mu^*)$$

for $t \gg s$ and $s \in \mathbb{R}$. This implies that for any $c \geq \frac{\lambda(A^{\mu^*})}{\mu^*}$,

$$\limsup_{x\cdot\xi \geq ct, t\to\infty} \|\mathbf{u}(t+s, x; s, \mathbf{u}_0)\| = 0 \quad \text{uniformly in } s \in \mathbb{R}.$$

By Lemma 3.2(3), $c^*_{\text{sup}} \leq \frac{\lambda(A^{\mu^*})}{\mu^*}$. Hence we have $c^*_{\text{sup}} = c^*_{\text{inf}} = \inf_{\mu>0} \frac{\lambda(A^\mu)}{\mu}$.

(2) By the change variable $w(t,x) = u(t,-x)$ and then repeat the same procedure, we have that c^*_- is the leftward spreading speed of (1.1) and $c^*_- = \inf_{\mu>0} \frac{\lambda(A^{-\mu})}{\mu}$. By (3.9) and the Riesz

representation theorem, for any $s \in \mathbb{R}$, there are bounded nonnegative measures $m_{ij}(s, y, \mathrm{d}y)$ such that

$$(\mathbf{\Phi}(T; s, A, \mu)\mathbf{u}_0)_i = \sum_{j=1}^{2} \int_{\mathbb{R}^N} \mathrm{e}^{\mu y \cdot \xi} u_{j0} m_{ij}(s, y, \mathrm{d}y), \quad i = 1, 2.$$

By the arguments of [29, Theorem 2.5], for $\gamma^T(s, \mu)$, we have

$$[\gamma^T(s, \mu_1)]^\alpha [\gamma^T(s, \mu_2)]^{1-\alpha} \geq \gamma^T(s, \alpha\mu_1 + (1-\alpha)\mu_2).$$

Using (3.10), we know that

$$\lim_{T \to \infty} \frac{\ln[\gamma^T(s, \mu_1)]^\alpha [\gamma^T(s, \mu_2)]^{1-\alpha}}{T} \geq \lim_{T \to \infty} \frac{\ln \gamma^T(s, \alpha\mu_1 + (1-\alpha)\mu_2)}{T}.$$

Thus we have

$$\alpha\lambda(A^{\mu_1}) + (1-\alpha)\lambda(A^{\mu_2}) \geq \lambda(A^{\alpha\mu_1 + (1-\alpha)\mu_2}),$$

that is, $\lambda(A^\mu)$ is convex in μ. Let $\mu_1, \mu_2 > 0$ such that $c_+^* = \frac{\lambda(A^{\mu_1})}{\mu_1}$ and $c_-^* = \frac{\lambda(A^{-\mu_2})}{\mu_2}$. Let $\nu = \frac{\mu_1}{\mu_1 + \mu_2}$. Then $\nu \in (0, 1)$ and $(1-\nu)\mu_1 = \nu\mu_2$. Note that $\lambda(A^\mu)$ is convex in μ, we have

$$\begin{aligned} c_+^* + c_-^* &= \frac{1}{\nu} \frac{1}{\mu_2} [(1-\nu)\lambda(A^{\mu_1}) + \nu\lambda(A^{-\mu})] \\ &\geq \frac{1}{\nu\mu_2} \lambda(A^{(1-\nu)\mu_1 - \nu\mu_2}) = \frac{1}{\nu\mu_2} \lambda(A) > 0. \end{aligned}$$

Hence we have $c_+^* + c_-^* > 0$. This completes the proof.

Moreover, by the similar arguments to that in [9, 20], we have the following spreading features for the spreading speeds c_+^* and c_-^*.

Theorem 3.3 *Let c_+^* and c_-^* be defined as in Theorem 3.2. For any $\xi \in S^{N-1}$, we have*
(1) *for any $c < c_+^*$ and $c' < c_-^*$, if $\mathbf{u}_0 \in X^+$ with $\mathbf{u}_0 \neq 0$, then*

$$\limsup_{t \to \infty, -c't \leq x \cdot \xi \leq ct} \|\mathbf{u}(t+s, x; s, \mathbf{u}_0) - \mathbf{u}^*(t+s)\| = 0 \quad \textit{uniformly in } s \in \mathbb{R}.$$

(2) *If $\mathbf{u}_0 \in X^+$ has compact support and $\mathbf{u}_0(\cdot) \ll \mathbf{u}_{\inf}^*$ for any $x \in \mathbb{R}^N$, then for each $c > c_+^*$ and $c' > c_-^*$,*

$$\limsup_{t \to \infty, x \cdot \xi \geq ct} \|\mathbf{u}(t+s, x; s, \mathbf{u}_0)\| = 0 \quad \textit{and} \quad \limsup_{t \to \infty, x \cdot \xi \leq -c't} \|\mathbf{u}(t+s, x; s, \mathbf{u}_0)\| = 0$$

uniformly in $s \in \mathbb{R}$.

4 A Time-Dependent Epidemic Model

We consider the following time dependent reaction diffusion system modeling man-environment-man epidemics

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(t, x) = d\Delta u_1 - a_{11}(t, x)u_1(t, x) + a_{12}u_2(t, x), \\ \dfrac{\partial u_2}{\partial t}(t, x) = -a_{22}(t, x)u_2(t, x) + g(t, u_1(t, x)), \end{cases} \tag{4.1}$$

where d, a_{11}, a_{12} and a_{22} are positive constants. It is well known that (4.1) can be used to model the spread of epidemics with oral-faecal transmission, see [6–7]. Here $u_1(t,x)$ represents the spatial density of the infection agent and $u_2(t,x)$ denotes the spatial density of the infective human population at time $t \in \mathbb{R}$ and point $x \in \mathbb{R}^N$. In (4.1), $\frac{1}{a_{11}}$ is the mean lifetime of the agent in the environment, $\frac{1}{a_{22}}$ is the mean lifetime period of the human infective, a_{12} is the multiplicative factor of the infections agents and function $g(t,u)$ is the force of infection on the human population due to the concentration of the infection agent.

When $g(t,u)$ is time periodic dependence in t, Liang et al. [12] have studied the spreading speed and traveling wave solutions of epidemic model (4.1) in the monostable case. Recently, Wu and Hsu [30] have studied the existence, uniqueness and stability of periodic traveling fronts of (4.1) with bistable and time periodic nonlinearity. In this section, we will consider (4.1) with time dependent in the monstable case and establish the spreading speed of (4.1) by applying our main results.

For simplicity, we consider the following dimensionless epidemic system as (4.1), i.e.,

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(t,x) = d\Delta u_1(t,x) - u_1(t,x) + \alpha u_2(t,x), \\ \dfrac{\partial u_2}{\partial t}(t,x) = -\beta u_2(t,x) + g(t, u_1(t,x)), \end{cases} \tag{4.2}$$

where $\alpha = \frac{a_{12}}{a_{11}^2}$ and $\beta = \frac{a_{22}}{a_{11}}$.

Assume that

(A1) $g \in C^1(\mathbb{R}_+^2, \mathbb{R}_+)$ and $g(t,\cdot)$ is strictly subhomogeneous on $\mathbb{R}_+$, $g(\cdot,0) = 0$ and $\frac{\partial g}{\partial u}(t,u) > 0$ for $(t,u) \in \mathbb{R}_+^2$.

(A2) There exists $\overline{u} > 0$ such that $\frac{\overline{g}(\overline{u})}{\overline{u}} \le \frac{\beta}{\alpha}$, where $\overline{g}(u) = \max\limits_{t\in\mathbb{R}} g(t,u)$.

By (A1), we know that (H1) and (H3) are true for (4.2). Using (A2), $\mathbf{M} = \left(\overline{u}, \frac{\overline{u}}{\alpha}\right)$ satisfies (H2). Note that the linear matrix of (4.2) is

$$A(t) = \begin{pmatrix} -1 & \alpha \\ g_u(t,0) & \beta \end{pmatrix}.$$

Since $\frac{\partial g}{\partial u}(t,u) > 0$ for $(t,u) \in \mathbb{R}_+^2$, α, β are positive constants, $A(t)$ is quasi-positive and strongly irreducible. Hence we assume that $A(t)$ is unique ergodic and then there is a principal Lyapunov exponent $\lambda(A)$ for $A(t)$. Similar to that for system (1.1), there also is a principal Lyapunov exponent $\lambda(A^\mu)$ for (4.2). Furthermore, we assume that $\lambda(A) > 0$. Hence, the assumptions (H1)–(H4) hold for (4.2). Therefore, the conclusions of Theorems 3.1–3.3 are valid for system (4.2).

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