Chin. Ann. Math. Ser. B
43(1), 2022, 95–114
DOI: 10.1007/s11401-022-0307-8

Chinese Annals of Mathematics, Series B

Pythagorean Theorem & Curvature with Lower or Upper Bound*

Xiaole SU[1] Hongwei SUN[2] Yusheng WANG[3]

Abstract In this paper, the authors give a comparison version of Pythagorean theorem to judge the lower or upper bound of the curvature of Alexandrov spaces (including Riemannian manifolds).

Keywords Pythagorean theorem, Alexandrov space, Toponogov's theorem
2000 MR Subject Classification 53C20

1 Introduction

Let $\mathbb{S}_k^n$ be the complete and simply connected n-dimensional space form with constant curvature k. For any minimal geodesics $[pq], [pr] \subset \mathbb{S}_k^n$ which form an angle $\angle qpr$ at p, the Law of Cosine says that

$$\begin{cases} \cos(\sqrt{k}|qr|) = \cos(\sqrt{k}|pq|)\cos(\sqrt{k}|pr|) + \sin(\sqrt{k}|pq|)\sin(\sqrt{k}|pr|)\cos\angle qpr, & k>0, \\ |qr|^2 = |pq|^2 + |pr|^2 - 2|pq||pr|\cos\angle qpr, & k=0, \\ \cosh(\sqrt{-k}|qr|) = \cosh(\sqrt{-k}|pq|)\cosh(\sqrt{-k}|pr|) - \sinh(\sqrt{-k}|pq|)\sinh(\sqrt{-k}|pr|)\cos\angle qpr, & k<0, \end{cases}$$

where $|\cdot\cdot|$ denotes the distance between two given points. In particular, if $\angle qpr = \frac{\pi}{2}$, then

$$\begin{cases} \cos(\sqrt{k}|qr|) = \cos(\sqrt{k}|pq|)\cos(\sqrt{k}|pr|), & k>0, \\ |qr|^2 = |pq|^2 + |pr|^2, & k=0, \\ \cosh(\sqrt{-k}|qr|) = \cosh(\sqrt{-k}|pq|)\cosh(\sqrt{-k}|pr|), & k<0, \end{cases} \tag{1.1}$$

which is the famous Pythagorean theorem on $\mathbb{S}_k^n$, especially the middle one for $k=0$ (the Gougu theorem in China). A fascinating thing is that the Law of Cosine can be derived from Pythagorean theorem, i.e., the Law of Cosine is equivalent to Pythagorean theorem.

For a general Riemannian manifold M, it is well-known that a necessary and sufficient condition of sectional curvature $\sec_M \geq k$ (or $\leq k$) is a local comparison version of the Law of Cosine. Namely,

Manuscript received March 27, 2020. Revised April 18, 2021.
[1]School of Mathematical Sciences (and Lab. math. Com. Sys.), Beijing Normal University, Beijing, 100875, China. E-mail: suxiaole@bnu.edu.cn
[2]School of Mathematics Science, Capital Normal University, Beijing, 100037, China. E-mail: 5598@cnu.edu.cn
[3]Corresponding author. School of Mathematical Sciences (and Lab. math. Com. Sys.), Beijing Normal University, Beijing, 100875, China. E-mail: wyusheng@bnu.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11971057) and BNSF Z190003.

Theorem 1.1 *Let M be a complete Riemannian manifold. Then $\sec_M \geq k$ ($\leq k$) if and only if for any $x \in M$ there exists a neighborhood U_x of x such that for any minimal geodesics $[pq], [pr] \subset U_x$,*

$$\begin{cases} \cos(\sqrt{k}|qr|) \geq (\leq)\ \cos(\sqrt{k}|pq|)\cos(\sqrt{k}|pr|) + \sin(\sqrt{k}|pq|)\sin(\sqrt{k}|pr|)\cos\angle qpr, & k>0, \\ |qr|^2 \leq (\geq)\ |pq|^2 + |pr|^2 - 2|pq||pr|\cos\angle qpr, & k=0, \\ \cosh(\sqrt{-k}|qr|) \leq (\geq)\ \cosh(\sqrt{-k}|pq|)\cosh(\sqrt{-k}|pr|) - \sinh(\sqrt{-k}|pq|)\sinh(\sqrt{-k}|pr|)\cos\angle qpr, & k<0, \end{cases}$$

and the equality holds for all $x \in M$ and all $[pq], [pr] \subset U_x$ if and only if $\sec_M \equiv k$.

Inspired by the relation between Pythagorean theorem and the Law of Cosine, a natural question is as follows.

Question 1.1 Is there a comparison version of Pythagorean theorem to judge the lower or upper bound of $\sec_M$?

In this paper, for three points p, q, r in a metric space, we denote by $\tilde\angle_k qpr$ the angle between $[\widetilde{pq}]$ and $[\widetilde{pr}]$ in $\mathbb{S}^2_k$ with $|\widetilde{pq}| = |pq|$, $|\widetilde{pr}| = |pr|$ and $|\widetilde{qr}| = |qr|$. Note that a comparison version of (1.1),

$$\begin{cases} \cos(\sqrt{k}|qr|) \geq\ (\leq)\ \cos(\sqrt{k}|pq|)\cos(\sqrt{k}|pr|), & k>0, \\ |qr|^2 \leq\ (\geq)\ |pq|^2 + |pr|^2, & k=0, \\ \cosh(\sqrt{-k}|qr|) \leq\ (\geq)\ \cosh(\sqrt{-k}|pq|)\cosh(\sqrt{-k}|pr|), & k<0, \end{cases}$$

is equivalent to

$$\tilde\angle_k qpr \leq\ (\geq)\ \frac{\pi}{2}.$$

The main goal of the paper is to give a positive answer to Question 1.1 not only for a Riemannian manifold but also for an Alexandrov space. In this paper, an Alexandrov space is a locally compact length space and for any $x \in X$, there is a real number k_x such that a neighborhood of x is of curvature $\geq k_x$ or $\leq k_x$; moreover, for the neighborhood we can define a dimension if it is of curvature $\geq k_x$. For convenience, we call x a CBB-type or CBA-type point according to curvature $\geq k_x$ or $\leq k_x$, respectively (refer to Section 2 for details). Note that a Riemannian manifold is an Alexandrov space.

We now formulate our main result.

Theorem A *Let x be a point in a complete Alexandrov space X. Suppose that x has a finite dimensional neighborhood if it is a* CBB*-type point. Then X is of curvature $\geq$ ($\leq$) k around x if and only if there exists a neighborhood U_x of x such that, for any $q \in U_x$ and $[r_1r_2] \subset U_x$ with $q \notin [r_1r_2]$, if there is $p \in [r_1r_2]^\circ$* [1] *satisfying $|qp| = |q[r_1r_2]| \triangleq \min_{s\in[r_1r_2]}\{|qs|\}$, then*

$$\tilde\angle_k qpr_i \leq\ (\geq)\ \frac{\pi}{2}, \quad i=1,2. \tag{1.2}$$

Moreover, if the equality in (1.2) *holds for all such $q \in U_x$, $[r_1r_2] \subset U_x$ and $p \in [r_1r_2]$ (and thus X is of curvature $\geq k$ around x), and if U_x is of finite dimension, then around any interior point* [2] *of U_x there is a neighborhood which is a Riemannian manifold with sectional curvature equal to k.*

[1] In this paper, A° denotes the interior part of A.

[2] Refer to Section 2 for the definition of an interior point of a finite dimensional CBB-type Alexandrov space.

We would like to point out that in proving that X is of curvature $\geq k$ around x when x is a CBB-type point in Theorem A, we in fact do not care whether U_x is of finite dimension or not. Instead, in proving that X is of curvature $\leq k$ around such an x (i.e., Case 4 in Section 3), our proof indeed relies on the finiteness of dimension (which involves Lemmas 2.4, 3.4 and 3.5 below).

Note that if X is a complete CBB-type Alexandrov space, then "$|qp| = |q[r_1r_2]|$" in Theorem A implies that $\angle qpr_i = \frac{\pi}{2}$ for any $[qp]$ and $[pr_i]$ (see Lemma 2.2 below). Thereby, it is clear that Theorem A has the following corollary, a positive answer to Question 1.1.

Corollary B *Let X be a complete finite dimensional* CBB-*type Alexandrov space. Then X is of curvature $\geq$ $(\leq)$ k if and only if for any $x \in X$ there exists a neighborhood U_x of x such that, for all $[pq], [pr] \subset U_x$ with $\angle qpr = \frac{\pi}{2}$,*

$$\widetilde{\angle}_k qpr \leq \ (\geq) \ \frac{\pi}{2}. \tag{1.3}$$

Moreover, if the equality in (1.3) *holds for all $x \in X$ and all such $[pq], [pr] \subset U_x$, then the interior part of X is a Riemannian manifold with sectional curvature equal to k.*

Remark 1.1 For the rigidity part of Corollary B, one can consider the following simple example. Note that a geodesic triangle on $\mathbb{S}^2_k$ separates $\mathbb{S}^2_k$ into two parts with boundary. The smaller one is a complete Alexandrov space with curvature $\geq k$, but not a Riemannian manifold with boundary, and satisfies Pythagorean theorem locally. (However, the larger one is an Alexandrov space with curvature $\leq k$, and does not satisfy Pythagorean theorem around the vertices of the triangle.)

Remark 1.2 For a CBA-type Alexandrov space X, we cannot judge whether X is of curvature $\geq k$ or $\leq k$ in a similar way as Corollary B. For example, the union of three rays in $\mathbb{R}^2$ starting from a common point (with the induced intrinsic metric), a CBA-type Alexandrov space, has no $\frac{\pi}{2}$-angle nor lower curvature bound.

Remark 1.3 If X is a Riemannian manifold, one can give a proof for Theorem A via the second variation formula and the comparison results on index forms (the main tools in proving the well-known Rauch's theorem), which do not work when X is a general Alexandrov space. Of course, in our proof relying on Toponogov's theorem, many arguments can be removed in the case where X is a Riemannian manifold (i.e., the proof can be much shorter).

As an application of Theorem A, we supply a way to judge whether a point in a finite dimensional CBB-type Alexandrov space is a regular one, i.e., its space of directions is a unit sphere (see Section 2).

Theorem C *Let x be an interior point in a complete finite dimensional* CBB-*type Alexandrov space. If there is a function $\chi(\varepsilon)$ with $\varepsilon > 0$ and $\chi(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that, for all $[pq], [pr] \subset B_x(\varepsilon)$ with $\angle qpr = \frac{\pi}{2}$,*

$$\Big|\frac{|qr|^2}{|pq|^2 + |pr|^2} - 1\Big| < \chi(\varepsilon),$$

then x is a regular point.

We would like to point out that the condition for x to be a regular point in Theorem C should just be sufficient, but not necessary.

As an almost immediate corollary of Theorem C, we have the following known result (see [1]).

Corollary D *Let x be an interior point in a complete finite dimensional* CBB*-type Alexandrov space X. If in addition x is a* CBA*-type point, then x is a regular point* (*as a result, if each point in X° is a* CBA*-type one, then X° is a manifold*).

In the rest, the paper is organized as follows. In Section 2, we will recall some basic concepts on Alexandrov spaces. In Sections 3–4, we will give a proof of Theorem A for curvature "$\geq$ ($\leq$) k" and "$= k$", respectively. In Section 5, we shall give proofs for Theorem C and Corollary D.

2 On Alexandrov Spaces

In this section, we will recall the definition and some basic properties of Alexandrov spaces, which will be used in the proof of Theorem A.

First of all, it is well known that Theorem 1.1 has the following twin versions.

Theorem 2.1 (see [3]) *Let M be a complete Riemannian manifold. Then $\sec_M \geq k$ ($\leq k$) if and only if for any $x \in M$ there exists a neighborhood U_x of x such that*

Condition A : *For any $q \in U_x$, $[pr] \subset U_x$, $\widetilde{q} \in \mathbb{S}_k^2$ and $[\widetilde{p}\widetilde{r}] \subset \mathbb{S}_k^2$ with $|\widetilde{q}\widetilde{p}| = |qp|$, $|\widetilde{q}\widetilde{r}| = |qr|$ and $|\widetilde{p}\widetilde{r}| = |pr|$, we have that, for any $s \in [pr]$ and $\widetilde{s} \in [\widetilde{p}\widetilde{r}]$ with $|ps| = |\widetilde{p}\widetilde{s}|$,*

$$|qs| \geq \ (\leq) \ |\widetilde{q}\widetilde{s}|.$$

We now, based on Theorem 2.1, can give the definition of the Alexandrov space in Theorem A.

Definition 2.1 (see [1–2]) *A locally compact length space X is called an Alexandrov space if for any $x \in X$, there is a real number k_x and a neighborhood U_x of x such that the corresponding* Condition A *in Theorem* 2.1 *with respect to $\mathbb{S}_{k_x}^2$ holds, and X is said to be of curvature* $\mathrm{cur}_X \geq k_x$ *or* $\leq k_x$ *on U_x according to "$|qs| \geq |\widetilde{q}\widetilde{s}|$" or "$|qs| \leq |\widetilde{q}\widetilde{s}|$", respectively.*

Given an Alexandrov space X, we call $x \in X$ a CBB*-type* (*resp.* CBA*-type*) *point when* $\mathrm{cur}_X \geq k_x$ (*resp.* $\leq k_x$) *around x; and we call X a* CBB*-type* (*resp.* CBA*-type*) *Alexandrov space if each $x \in X$ is a* CBB*-type* (*resp.* CBA*-type*) *point.*

It is obvious that Alexandrov spaces include Riemannian manifolds. For a general Alexandrov space X, a significant difference with a Riemannian manifold is that a geodesic (locally shortest path) on X might not be prolonged even when X is complete (in the sense of distance topology).

In an Alexandrov space X, we can define an angle $\angle yxz$ between two minimal geodesics $[xy]$ and $[xz]$. Assume that X is of $\mathrm{cur}_X \geq k_x$ around x. Let $a \in [xy]$ and $b \in [xz]$. By Condition A in Theorem 2.1, $\widetilde{\angle}_{k_x}axb$ is non-decreasing when a,b converge to x (see [2]), i.e., $\lim\limits_{a,b\to x}\widetilde{\angle}_{k_x}axb$ exists. And note that, for any $k\neq k_x$, $\lim\limits_{a,b\to x}\widetilde{\angle}_k axb=\lim\limits_{a,b\to x}\widetilde{\angle}_{k_x}axb$. So, we can define

$$\angle yxz \triangleq \lim_{a,b\to x}\widetilde{\angle}_{k_x}axb. \tag{2.1}$$

Similarly, we can also define $\angle yxz$ if X is of $\mathrm{cur}_X\leq k_x$ around x because, in such a situation, $\widetilde{\angle}_{k_x}axb$ is non-increasing when a,b converge to x (see [1]).

By (2.1) and Definition 2.1, it is not hard to see that Theorem 1.1 also holds for X (see [1–2]).

Theorem 2.2 *Let X be a complete* CBB*-type Alexandrov space. Then X is of $cur_X\geq(\leq)\ k$ if and only if for any $x\in X$ there exists a neighborhood U_x of x such that the inequality in Theorem* 1.1 *holds for any $[pq],[pr]\subset U_x$, or equivalently, for any triangle $\triangle pqr\subset U_x$ (i.e., a union of three minimal geodesics $[pq],[qr],[pr]$), we have that*

$$\angle qpr\geq(\leq)\widetilde{\angle}_k qpr,\quad \angle pqr\geq(\leq)\widetilde{\angle}_k pqr,\quad \angle prq\geq(\leq)\widetilde{\angle}_k prq.$$

Theorem 2.2′ *Let X be a complete* CBA*-type Alexandrov space. Then X is of $cur_X\leq k$ (resp. $\geq k$) if and only if (resp. only if) the condition for $cur_X\leq k$ (resp. $\geq k$) in Theorem* 2.2 *holds.*

In Theorem 2.2′, the "if" (i.e., the sufficiency of the condition) for $\mathrm{cur}_X\geq k$ needs an additional condition that $\angle qsp+\angle qsr=\pi$ for all $s\in[pr]^\circ$ (see [1–2]).

Similar to Riemannian case, we have the first variation formula on Alexandrov spaces.

Lemma 2.1 (see [1–2]) *Let X be a complete Alexandrov space. Then for any $x\in X$, there is a neighborhood U_x of x such that, for any $[pq],[pr]\subset U_x$ and $p_i\in[pr]$ with $p_i\to p$ as $i\to\infty$,*

$$\begin{cases}|qp_i|=|qp|-|pp_i|\cdot\cos\angle qpr+o(|pp_i|), & x \textit{ is a } \text{CBA-}\textit{type point};\\ |qp_i|=|qp|-|pp_i|\cdot\cos|\uparrow_p^r\Uparrow_p^q|+o(|pp_i|), & x \textit{ is a } \text{CBB-}\textit{type point}.\end{cases} \tag{2.2}$$

In this paper, for a given $[xy]$, $\uparrow_x^y$ denotes its direction at x (in Rimannian case, $\uparrow_x^y$ is just the unit tangent vector of $[xy]$ from x to y); and $\Uparrow_x^y$ denotes the union of directions of all minimal geodesics from x to y. Note that by Definition 2.1, in the case where X has an upper curvature bound on U, there is a unique minimal geodesic between any two distinct points in U.

Remark 2.1 For $\mathrm{cur}_X\geq k$, Theorem 2.2 (and 1.1) guarantees a global version of itself (see [2, 6–7]), which is the well-known Toponogov's theorem; namely, $\mathrm{cur}_X\geq k$ implies that Condition A in Theorem 2.1 holds for any $q\in X$ and $[pr]\subset X$. However, there is no global version for $\mathrm{cur}_X\leq k$ in general (see [1]). As a result, (2.2) has a global version (the first variation formula) on a complete Alexandrov space X with $\mathrm{cur}_X\geq k$, but does not on X with only an upper curvature bound.

Moreover, the angles on Alexandrov spaces satisfy the following property.

Lemma 2.2 (see [1–2]) *Let X be a complete Alexandrov space. Then for any $x \in X$, there is a neighborhood U_x of x such that, for any $[qp], [r_1r_2] \subset U_x$ with $p \in [r_1r_2]^\circ$,*

$$\begin{cases} \angle qpr_1 + \angle qpr_2 \geq \pi, & x \text{ is a CBA-type point;} \\ \angle qpr_1 + \angle qpr_2 = \pi, & x \text{ is a CBB-type point.} \end{cases} \tag{2.3}$$

As a result, if x is a CBB*-type point and if in addition $|qp| = |q[r_1r_2]|$, then $\angle qpr_1 = \angle qpr_2 = \frac{\pi}{2}$.*

It is easy to see that the angles on Alexandrov spaces have semi-continuity. Namely, given a complete Alexandrov space X which has a lower (resp. upper) curvature bound on a neighborhood U, if $[p_iq_i] \to [pq]$ and $[p_ir_i] \to [pr]$ as $i \to \infty$ in U, then $\angle qpr \leq \liminf\limits_{i\to\infty} \angle q_ip_ir_i$ (resp. $\angle qpr \geq \limsup\limits_{i\to\infty} \angle q_ip_ir_i$). This, together with (2.3), implies the following continuity.

Lemma 2.3 *Let X be a complete Alexandrov space, and let U be a neighborhood in X. Suppose that $[qp], [q_ip_i], [r_1r_2] \subset U$ with $p, p_i \in [r_1r_2]^\circ$ and $[q_ip_i] \to [qp]$ as $i \to \infty$. If X has a lower curvature bound on U, or if X has an upper curvature bound on U and $\angle qpr_1 + \angle qpr_2 = \pi$, then*

$$\lim_{i\to\infty} \angle q_ip_ir_1 = \angle qpr_1, \quad \lim_{i\to\infty} \angle q_ip_ir_2 = \angle qpr_2.$$

We end this section with some concepts which are only applied to CBB-type Alexandrov spaces (see [2]). We can define a dimension for a CBB-type Alexandrov space X; and if it is of finite dimension, we can consider the space of directions Σ_pX at any $p \in X$ which is an Alexandrov space with curvature ≥ 1 and has a dimension one less than X. If Σ_pX is isometric to a unit sphere, we say that p is a regular point.

As a result, by induction we can define p to be a boundary or interior point, if Σ_pX contains boundary or no boundary point, respectively. Usually, we denote by X° and ∂X the set of interior points and boundary points, respectively. Note that ∂X may be not empty even if X is complete.

As another result, for any $p \in X$, we can define the tangent cone C_pX, a metric cone over Σ_pX. C_pX plays an important role in studying finite dimensional CBB-type Alexandrov spaces because of the following result.

Lemma 2.4 (see [2]) *Let X be a complete finite dimensional* CBB*-type Alexandrov space. Then with base point $p \in X$, $(\lambda X, p)$ converges in the Gromov-Hausdorff sense to C_pX as $\lambda \to +\infty$.*

In this paper, λX denotes X endowed with the metric $\lambda \cdot d$, where d is the original metric on X.

3 Proof of Theorem A for Curvature $\geq$ ($\leq$) k

In this section, we will show the former part of Theorem A, i.e., the sufficiency and necessity of the condition for curvature $\geq k$ or $\leq k$ in Theorem A. By Theorems 2.2 and 2.2′, it is enough to verify the sufficiency, and the verification shall be proceeded according to the following cases.

Case 1 For curvature $\geq k$ around a CBB-type point $x \in X$.

Case 2 For curvature $\geq k$ around a CBA-type point $x \in X$.

Case 3 For curvature $\leq k$ around a CBA-type point $x \in X$.

Case 4 For curvature $\leq k$ around a CBB-type point $x \in X$.

For the convenience of readers, we first give a rough idea of our proof. For instance, in Case 1, if the curvature of X is not $\geq k$ around the CBB-type point x, then by Theorem 2.2 there must be a triangle $\triangle pqr$ containing a "bad" angle, say $\angle qpr$, i.e., $\angle qpr < \widetilde{\angle}_k qpr$. A key observation is that such a triangle can be cut into two (smaller) triangles and at least one of them still contains a "bad" angle (see [7]), which is guaranteed by the lemma right below. By repeating such a cutting operation finite times, we will get a triangle which contradicts the condition for curvature $\geq k$ in Theorem A.

In this paper, for a given $\triangle qpr$, its comparison triangle is defined to be a $\triangle\widetilde{p}\widetilde{q}\widetilde{r} \subset \mathbb{S}_k^2$ with $|\widetilde{p}\widetilde{q}| = |pq|$, $|\widetilde{p}\widetilde{r}| = |pr|$ and $|\widetilde{q}\widetilde{r}| = |qr|$.

Lemma 3.1 (see [7]) *Let x be a* CBB*-point in a complete Alexandrov space, and let k be a real number. Then there is a neighborhood U_x of x such that, for any $\triangle qr_1r_2 \subset U_x$ and its comparison triangle $\triangle\widetilde{q}\widetilde{r}_1\widetilde{r}_2 \subset \mathbb{S}_k^2$, if $|qs| - |\widetilde{q}\widetilde{s}|$ with $s \in [r_1r_2], \widetilde{s} \in [\widetilde{r}_1\widetilde{r}_2]$ and $|r_is| = |\widetilde{r}_i\widetilde{s}|$ attains a negative minimum at $s_0 \in [r_1r_2]^\circ$, then any $[qs_0]$ satisfies*

$$\angle qs_0r_1 < \widetilde{\angle}_k qs_0r_1 \quad \text{or} \quad \angle qs_0r_2 < \widetilde{\angle}_k qs_0r_2; \tag{3.1}$$

in particular,

$$\text{if } |r_is_0| \ll |r_iq| \text{ for } i = 1 \text{ or } 2, \text{ then } \angle qs_0r_i < \widetilde{\angle}_k qs_0r_i. \tag{3.2}$$

Note that if k is positive in the lemma, then we need that the larger k is, the smaller U_x should be, to guarantee that $\triangle qr_1r_2$ has a comparison triangle in $\mathbb{S}_k^2$.

Proof Since x is a CBB-point, by Lemmas 2.1–2.2, there is a neighborhood U_x of x such that we can apply (2.2) and (2.3) on it. For any $[qs_0]$, by (2.2), the negative minimum of $|qs| - |\widetilde{q}\widetilde{s}|$ at s_0 implies

$$\angle qs_0r_i \geq \angle\widetilde{q}\widetilde{s}_0\widetilde{r}_i, \quad i = 1, 2.$$

It then has to hold that

$$\angle qs_0r_i = \angle\widetilde{q}\widetilde{s}_0\widetilde{r}_i, \quad i = 1, 2, \tag{3.3}$$

because, by (2.3), we have that

$$\angle qs_0r_1 + \angle qs_0r_2 = \pi. \tag{3.4}$$

On the other hand, since $|qs_0| < |\widetilde{q}\widetilde{s}_0|$, there is $\widetilde{q}' \in [\widetilde{q}\widetilde{s}_0]^\circ$ such that $|\widetilde{q}'\widetilde{s}_0| = |qs_0|$. It is clear that at least one of "$|\widetilde{q}'\widetilde{r}_i| < |\widetilde{q}\widetilde{r}_i|$" holds. It follows that at least one of "$\angle\widetilde{q}\widetilde{s}_0\widetilde{r}_i < \widetilde{\angle}_k qs_0r_i$" holds; especially, if $|r_is_0| \ll |r_iq|$ for $i = 1$ or 2, then $\angle\widetilde{q}\widetilde{s}_0\widetilde{r}_i < \widetilde{\angle}_k qs_0r_i$. As a result, the lemma follows.

Remark 3.1 (1) In Lemma 3.1, if $|qs|-|\widetilde{q}\widetilde{s}|$ attains a positive maximum at $s_0\in[r_1r_2]^\circ$, then similar to (3.1), either $\angle qs_0r_2>\widetilde{\angle}_kqs_0r_2$ for the $[qs_0]$ with $|\uparrow_{s_0}^q\uparrow_{s_0}^{r_1}|=|\Uparrow_{s_0}^q\uparrow_{s_0}^{r_1}|$, or $\angle qs_0r_1>\widetilde{\angle}_kqs_0r_1$ for the $[qs_0]$ with $|\uparrow_{s_0}^q\uparrow_{s_0}^{r_2}|=|\Uparrow_{s_0}^q\uparrow_{s_0}^{r_2}|$. And similar to (3.2), if $\angle\widetilde{r}_i\widetilde{q}\widetilde{s}_0<\frac{\pi}{2}$ for $i=1$ or 2, then $\angle\widetilde{q}\widetilde{s}_0\widetilde{r}_i>\widetilde{\angle}_kqs_0r_i$ and thus the corresponding $\angle qs_0r_i>\widetilde{\angle}_kqs_0r_i$.

(2) In Lemma 3.1, if x is a CBA-point, the lemma is not true unless (3.4) holds; note that it may occur that $\angle qs_0r_1+\angle qs_0r_2>\pi$ by (2.3).

(3) In Lemma 3.1, if x is a CBA-point, and if $|qs|-|\widetilde{q}\widetilde{s}|$ attains a positive maximum at $s_0\in[r_1r_2]^\circ$, then similar to (3.1), we have that $\angle qs_0r_i>\widetilde{\angle}_kqs_0r_i$ for $i=1$ or 2. (Here, there is a unique minimal geodesic between q and s_0 because of the CBA-property.) Similar to (3.2), if $\angle\widetilde{r}_i\widetilde{q}\widetilde{s}_0<\frac{\pi}{2}$ for $i=1$ or 2, then $\angle\widetilde{q}\widetilde{s}_0\widetilde{r}_i>\widetilde{\angle}_kqs_0r_i$ and thus $\angle qs_0r_i>\widetilde{\angle}_kqs_0r_i$.

Now, according to the four cases listed in the beginning of this section, we begin to prove the former part of Theorem A case by case.

Proof for Case 1 In this case, around a CBB-type point $x\in X$, we will prove:

PropertyA : X is of curvature $\geq k$ if there is a neighborhood U_x of x such that,

$$\begin{gathered}\forall\ q\in U_x,\ [r_1r_2]\subset U_x \text{ with } q\notin[r_1r_2], \text{ if } p\in[r_1r_2]^\circ\\ \text{satisfies } |qp|=|q[r_1r_2]|, \text{ then } \widetilde{\angle}_kqpr_i\leq\frac{\pi}{2}.\end{gathered}\tag{3.5}$$

If X is not of curvature $\geq k$ around x, then we claim that there is a triangle $\triangle qpr\subset U_x$ such that

$$|qs|=|q[pr]| \text{ for some } s\in[pr]^\circ,\quad |qs|<|\widetilde{q}\widetilde{s}| \text{ for } \widetilde{s}\in[\widetilde{p}\widetilde{r}] \text{ with } |\widetilde{s}\widetilde{p}|=|sp|,\tag{3.6}$$

where $[\widetilde{p}\widetilde{r}]$ belongs to the comparison triangle $\triangle\widetilde{q}\widetilde{p}\widetilde{r}\subset\mathbb{S}_k^2$ of $\triangle qpr$. Nevertheless, the $\triangle qpr$ contradicts Property A. In fact, note that either $\angle\widetilde{q}\widetilde{s}\widetilde{p}\geq\frac{\pi}{2}$ or $\angle\widetilde{q}\widetilde{s}\widetilde{r}\geq\frac{\pi}{2}$, we say $\angle\widetilde{q}\widetilde{s}\widetilde{p}\geq\frac{\pi}{2}$. Then by applying (3.5) on $\triangle qpr$, it has to hold that $|qs|\geq|\widetilde{q}\widetilde{s}|$, which is a contradiction.

We now need only to verify the claim, i.e., to show the existence of the desired triangle. By Theorem 2.2, if X is not of curvature $\geq k$ around x, there exists a sufficiently small triangle $\triangle pqr\subset U_x$ which contains a "bad" angle, say $\angle qpr$, i.e., $\angle qpr<\widetilde{\angle}_kqpr$.

First of all, observe that the badness of $\angle qpr$ implies that there is $\overline{s}\in[pr]\setminus\{p\}$ and $\overline{t}\in[pq]\setminus\{p\}$ such that, for all $s\in[p\overline{s}]^\circ$ and $t\in[p\overline{t}]^\circ$,

$$|\overline{t}s|<|\widetilde{\overline{t}}\widetilde{s}|,\quad |\overline{s}t|<|\widetilde{\overline{s}}\widetilde{t}|,\quad |\overline{s}\overline{t}|=|\widetilde{\overline{s}}\widetilde{\overline{t}}|,\quad \angle\overline{t}p\overline{s}-\widetilde{\angle}_k\overline{t}p\overline{s}=\angle qpr-\widetilde{\angle}_kqpr,\tag{3.7}$$

where $\widetilde{\overline{s}},\widetilde{s},\widetilde{\overline{t}},\widetilde{t}$ belong to the comparison triangle $\triangle\widetilde{p}\widetilde{q}\widetilde{r}\subset\mathbb{S}_k^2$ of $\triangle qpr$ and correspond to $\overline{s},s,\overline{t},t$, respectively with $\widetilde{s}\in[\widetilde{p}\widetilde{r}]$ and $|\widetilde{s}\widetilde{p}|=|sp|$, etc. In fact, by the badness of $\angle qpr$ and Lemma 2.1, there is $s_1\in[pr]\setminus\{p\}$ such that

$$|qs_1|=|\widetilde{q}\widetilde{s}_1|,\quad |qs|<|\widetilde{q}\widetilde{s}|\tag{3.8}$$

for all $s\in[ps_1]^\circ$ and $\widetilde{s}_1,\widetilde{s}\in[\widetilde{p}\widetilde{r}]^\circ$ with $|\widetilde{s}_1\widetilde{p}|=|s_1p|$ and $|\widetilde{s}\widetilde{p}|=|sp|$. Note that $\triangle\widetilde{q}\widetilde{p}\widetilde{s}_1$ is a comparison triangle of $\triangle qps_1$ with

$$\angle qps_1-\widetilde{\angle}_kqps_1=\angle qpr-\widetilde{\angle}_kqpr,$$

so $\angle qps_1$ is still "bad" in $\triangle qps_1$. Similarly, there is $t_1 \in [pq] \setminus \{p\}$ such that

$$|s_1t_1| = |\widetilde{s}_1\widetilde{t}_1| \text{ and } |s_1t| < |\widetilde{s}_1\widetilde{t}| \tag{3.9}$$

hold for all $t \in [pt_1]^\circ$ and $\widetilde{t}_1, \widetilde{t} \in [\widetilde{p}\widetilde{q}]$ with $|\widetilde{t}_1\widetilde{p}| = |t_1p|$ and $|\widetilde{t}\widetilde{p}| = |tp|$. Furthermore, we can locate an $s_2 \in [ps_1] \setminus \{p\}$ similar to s_1 and a $t_2 \in [pt_1] \setminus \{p\}$ similar to t_1. Then we can get a sequence of $\{s_j\}_{j=1}^\infty$ and $\{t_j\}_{j=1}^\infty$ (it may occur that $s_j = s_{j_0}$ and $t_j = t_{j_0}$ for all $j \geq j_0$) such that

$$\angle s_jpt_{j-1} - \widetilde{\angle}_k s_jpt_{j-1} = \angle s_jpt_j - \widetilde{\angle}_k s_jpt_j = \angle qpr - \widetilde{\angle}_k qpr, \tag{3.10}$$

which, by (2.1), implies that

$$s_j \to \bar{s} \in [pr] \setminus \{p\}, \quad t_j \to \bar{t} \in [pq] \setminus \{p\} \quad \text{as } j \to \infty.$$

By the corresponding (3.8) and (3.9) for each j and (3.10), we can conclude that $\bar{s}$ and $\bar{t}$ must satisfy the two equalities in (3.7). Then up to repeating this process on $\triangle\bar{t}p\bar{s}$, we can assume that $\bar{s}$ and $\bar{t}$ also satisfy the two inequalities in (3.7). Namely, we have found the desired $\bar{s}$ and $\bar{t}$.

Note that (3.7) implies that

$$\widetilde{\angle}_k\bar{t}\bar{s}p \geq |\uparrow_{\bar{s}}^{p}\Uparrow_{\bar{s}}^{\bar{t}}|, \quad \widetilde{\angle}_k\bar{s}\bar{t}p \geq |\uparrow_{\bar{t}}^{p}\Uparrow_{\bar{t}}^{\bar{s}}|. \tag{3.11}$$

On the other hand, note that $\triangle pqr$ is sufficiently small, so is any $\triangle\bar{t}p\bar{s}$; and thus at least one of $\widetilde{\angle}_k\bar{t}\bar{s}p$ and $\widetilde{\angle}_k\bar{s}\bar{t}p$ is an acute angle. This together with (3.11) implies that there is a triangle $\triangle\bar{t}p\bar{s}$ such that

$$\text{at least one of } \angle\bar{t}\bar{s}p \text{ and } \angle\bar{s}\bar{t}p \text{ is acute, say } \angle\bar{t}\bar{s}p. \tag{3.12}$$

In addition, if $\angle qpr < \frac{\pi}{2}$, then by Lemma 2.1 there is $s \in [p\bar{s}]^\circ$ such that $|\bar{t}s| = |\bar{t}[p\bar{s}]|$, i.e., $\triangle\bar{t}p\bar{s}$ is our desired triangle.

Hence, we need only to show that it can be assumed that $\angle qpr < \frac{\pi}{2}$. Since $\angle qpr$ is a "bad" angle, for $s \in [pr]^\circ$ close to p and $\widetilde{s} \in [\widetilde{p}\widetilde{r}]^\circ$ with $|\widetilde{s}\widetilde{p}| = |sp|$ we have that $|qs| < |\widetilde{q}\widetilde{s}|$ (by Lemma 2.1). Then there is $r' \in [pr]^\circ$ and $\widetilde{r}' \in [\widetilde{p}\widetilde{r}]^\circ$ with $|pr'| = |\widetilde{p}\widetilde{r}'|$ such that

$$|qr'| - |\widetilde{q}\widetilde{r}'| = \min_{s\in[pr],\widetilde{s}\in[\widetilde{p}\widetilde{r}],|ps|=|\widetilde{p}\widetilde{s}|}\{|qs| - |\widetilde{q}\widetilde{s}|\} < 0, \tag{3.13}$$

so via Lemma 3.1, we can conclude that

$$\angle qr'p \text{ is a "bad" angle in } \triangle qr'p, \text{ or } \angle qr'r \text{ is a "bad" angle in } \triangle qr'r. \tag{3.14}$$

In this situation, (3.3) means that

$$\angle qr'p = \angle\widetilde{q}\widetilde{r}'\widetilde{p}, \quad \angle qr'r = \angle\widetilde{q}\widetilde{r}'\widetilde{r}. \tag{3.15}$$

Note that if $\angle qpr \geq \frac{\pi}{2}$, then it follows from the badness of $\angle qpr$ that $\angle\widetilde{p}\widetilde{q}\widetilde{r}$ $(= \widetilde{\angle}_k qpr) > \frac{\pi}{2}$. Moreover, note that $\triangle\widetilde{p}\widetilde{q}\widetilde{r}$ is sufficiently small in $\mathbb{S}_k^2$, it is easy to see that

$$\angle\widetilde{q}\widetilde{r}'\widetilde{p} < \frac{\pi}{2}, \quad \angle\widetilde{q}\widetilde{r}'\widetilde{r} > \frac{\pi}{2}. \tag{3.16}$$

By (3.14)–(3.16), $\triangle qr'p$ contains an acute "bad" angle if the angle $\angle qr'p$ is "bad"; otherwise, we can repeat such a cutting oparation on $\triangle qr'r$. For convenience, we also let $\triangle qpr$ denote the $\triangle qr'r$. Note that up to repeating such a cutting oparation finite times, we can assume that $|pr| \ll |pq|$, and together with (3.2) we can conclude that $\angle qr'p$ is an acute "bad" angle in $\triangle qr'p$. This means that we can assume that $\angle qpr < \frac{\pi}{2}$.

Proof for Case 2 In this case, we shall prove Property A around a CBA-type point $x \in X$. Compared with the proof for Case 1, the only difference and difficulty here is why (3.14) holds. Note that Lemma 3.1 fails to work here (see (2) in Remark 3.1). Namely, the proof for Case 2 will be done if one can show that: For a CBA-type point x, if the U_x in Lemma 3.1 satisfies Property A additionally, then the conclusion of Lemma 3.1 still holds. By (2) in Remark 3.1, it suffices to show that $\angle qs_0r_1 + \angle qs_0r_2 = \pi$, i.e., the possible case "$\angle qs_0r_1 + \angle qs_0r_2 > \pi$" does not occur at all here.

Let $q' \in [qs_0]$ be sufficiently close to s_0, and let $s' \in [r_1r_2]$ such that $|q's'| = |q'[r_1r_2]|$ (for $[qs_0]$ and $[r_1r_2]$ refer to the proof of Lemma 3.1). Note that s' is close to s_0, so we can assume that s' lies in $[r_1r_2]^\circ$. Then for $\widetilde{s}' \in [\widetilde{r}_1\widetilde{r}_2]$ ($\subset \triangle\widetilde{q}\widetilde{r}_1\widetilde{r}_2 \subset \mathbb{S}^2_k$) with $|\widetilde{r}_1\widetilde{s}'| = |r_1s'|$, there is $\widetilde{q}' \in \mathbb{S}^2_k$ such that

$$|\widetilde{r}_1\widetilde{q}'| + |\widetilde{q}'\widetilde{r}_2| = |r_1q'| + |q'r_2| \tag{3.17}$$

and

$$\angle\widetilde{q}'\widetilde{s}'\widetilde{r}_1 = \angle\widetilde{q}'\widetilde{s}'\widetilde{r}_2 = \frac{\pi}{2}. \tag{3.18}$$

By Lemma 2.1, it is easy to see that

$$|r_1q'| + |q'r_2| = |r_1r_2| - |q's_0| \cdot (\cos\angle qs_0r_1 + \cos\angle qs_0r_2) + o(|q's_0|), \tag{3.19}$$

and that

$$\begin{aligned}|\widetilde{r}_1\widetilde{q}'| + |\widetilde{q}'\widetilde{r}_2| &= |\widetilde{r}_1\widetilde{r}_2| - |\widetilde{q}'\widetilde{s}'| \cdot (\cos\angle\widetilde{q}'\widetilde{s}'\widetilde{r}_1 + \cos\angle\widetilde{q}'\widetilde{s}'\widetilde{r}_2) + o(|\widetilde{q}'\widetilde{s}'|)\\ &= |r_1r_2| + o(|\widetilde{q}'\widetilde{s}'|).\end{aligned} \tag{3.20}$$

Hence, if $\angle qs_0r_1 + \angle qs_0r_2 > \pi$, then from (3.17), (3.19) and (3.20) we can see that

$$|q's'| \le |q's_0| < |\widetilde{q}'\widetilde{s}'|. \tag{3.21}$$

However, putting $|q's'| = |q'[r_1r_2]|$, $|\widetilde{r}_i\widetilde{s}'| = |r_is'|$, (3.18) and (3.21) together, we can apply (3.5) to conclude that $|r_1q'| + |q'r_2| < |\widetilde{r}_1\widetilde{q}'| + |\widetilde{q}'\widetilde{r}_2|$, which contradicts (3.17).

Proof for Case 3 In this case, around a CBA-type point $x \in X$, we will prove:

Property B : X is of curvature $\le k$ if there is a neighborhood U_x of x such that,

$$\begin{gathered}\forall\ q \in U_x,\ [r_1r_2] \subset U_x \text{ with } q \notin [r_1r_2], \text{ if } p \in [r_1r_2]^\circ\\ \text{satisfies } |qp| = |q[r_1r_2]|, \text{ then } \widetilde{\angle}_k qpr_i \ge \frac{\pi}{2}.\end{gathered} \tag{3.22}$$

The proof is almost a copy of that for Case 1 with reversing the directions of the corresponding inequalities. (Hint: $\angle qpr$ is a "bad" angle which means that $\angle qpr > \widetilde{\angle}_k qpr$. Then by Lemma 2.1, there is $r' \in [pr]^\circ$ and $\widetilde{r}' \in [\widetilde{p}\widetilde{r}]^\circ$, which correspond to r' and $\widetilde{r}'$ respectively satisfying (3.13), such that

$$|qr'| - |\widetilde{q}\widetilde{r}'| = \max_{s\in[pr],\widetilde{s}\in[\widetilde{p}\widetilde{r}],|ps|=|\widetilde{p}\widetilde{s}|}\{|qs| - |\widetilde{q}\widetilde{s}|\} > 0. \tag{3.23}$$

Furthermore, we can apply (3) in Remark 3.1 to see (3.14).) So, we only point out the main two differences here.

One is how to see (3.12). In Case 1, (3.11) is a key, but here the corresponding (3.11) has inverse directions. However, the CBA-property of x here implies (3.12) directly when U_x is small enough.

The other is how to see the acuteness of $\angle qr'p$ as in the end of the proof for Case 1. In Case 1, a key is (3.16) which is due to "$\frac{\pi}{2} \le \angle qpr < \widetilde{\angle}_k qpr$"; but here the "bad" of $\angle qpr$ means that $\angle qpr > \widetilde{\angle}_k qpr$. However, the CBA-property of x here with "$\frac{\pi}{2} \le \angle qpr$" implies the acuteness of $\angle qr'p$ directly as long as U_x is small enough.

Note that our proofs for Cases 1–3 do not involve the dimension, but our proof for Case 4 will rely on the finiteness of the dimension because Lemmas 2.4, 3.4 and 3.5 will be used.

Proof for Case 4 In this case, we should prove Property B around a CBB-type point $x \in X$. If it is not true, then similarly, under the assumption that X is not of curvature $\le k$ around x, in U_x (in Property B) we just need to locate a triangle contradicting (3.22), i.e., a triangle satisfying the lemma right below.

Lemma 3.2 *Let $x \in X$ be a* CBB*-type point, and suppose that x has a finite dimensional neighborhood. If X is not of curvature $\le k$ around x, and if a sufficiently small neighborhood U_x of x satisfies* (3.22)*, then there is a triangle $\triangle qpr \subset U_x$ such that $|qs| = |q[pr]|$ and $|qs| > |\widetilde{q}\widetilde{s}|$ for some $s \in [pr]^\circ$ and $\widetilde{s} \in [\widetilde{p}\widetilde{r}]^\circ$ with $|\widetilde{s}\widetilde{p}| = |sp|$, where $[\widetilde{p}\widetilde{r}]$ belongs to the comparison triangle $\triangle\widetilde{q}\widetilde{p}\widetilde{r} \subset \mathbb{S}_k^2$ of $\triangle pqr$.*

Actually, the proofs for Cases 1–3 mainly show a corresponding Lemma 3.2. Compared with them, a main difficulty here is that we can not conclude (3.12) because we have no inequalities in (3.11) as in Cases 1–2 nor the CBA-property in Case 3. Another main difficulty here appears in looking for a triangle with an acute "bad" angle. In Cases 1–3, a "bad" angle leads to a situation where we can apply Lemma 3.1 or (3) in Remark 3.1 to locate a smaller triangle with a "bad" angle. And step by step, we can locate the desired triangle. However, in Case 4, such a method fails when we try to apply (1) in Remark 3.1 unless there is a unique minimal geodesic between any two points in U_x.

To overcome the second difficulty right above, we have the following key observation from (3.22).

Lemma 3.3 *Let $x \in X$ be a* CBB*-type point, and suppose that x has a finite dimensional neighborhood. And let U_x be a sufficiently small neighborhood of x satisfying* (3.22)*. Then there is a unique minimal geodesic between any two distinct points in U_x.*

In the proof of Lemma 3.3, we need the following property of CBB-type Alexandrov spaces.

Lemma 3.4 (see [2, 5]) *Let $x \in X$ be a* CBB*-type point, and suppose that x has a finite dimensional neighborhood. Then there is a sufficiently small neighborhood U of x such that if there are two minimal geodesics between two points r_1 and r_2 in U, then they form an angle less than π at r_1 or r_2.*

Proof of Lemma 3.3 We argue by contradiction. Let r_1 and r_2 be two points in U_x, and assume that there are two minimal geodesics between them, denoted by $[r_1r_2]_1$ and $[r_1r_2]_2$. By Lemma 3.4, $[r_1r_2]_1$ and $[r_1r_2]_2$ form an angle less than π at r_1 or r_2, say r_1, i.e.,

$$|(\uparrow_{r_1}^{r_2})_1(\uparrow_{r_1}^{r_2})_2| < \pi.$$

Then, by considering $\Sigma_{r_1}X$ (for it refer to Section 2), it is easy to see that there is a minimal geodesic $[r_1q]$ such that

$$|\uparrow_{r_1}^{q} (\uparrow_{r_1}^{r_2})_1| < |\uparrow_{r_1}^{q} (\uparrow_{r_1}^{r_2})_2| < \frac{\pi}{2} \tag{3.24}$$

$\big($in particular, if $|(\uparrow_{r_1}^{r_2})_1(\uparrow_{r_1}^{r_2})_2| < \frac{\pi}{2}$, we can let $[r_1q] = [r_1r_2]_1\big)$. We select $q_j \in [r_1q] \setminus \{r_1\}$ such that $q_j \to r_1$ as $j \to \infty$. Note that, without loss of generality, we can assume that $|\uparrow_{r_1}^{q} (\uparrow_{r_1}^{r_2})_1| = |\uparrow_{r_1}^{q} \Uparrow_{r_1}^{r_2}|$. By Lemma 2.1, it follows that, as $j \to \infty$,

$$|r_2q_j| = |r_1r_2| - |r_1q_j| \cos |\uparrow_{r_1}^{q} (\uparrow_{r_1}^{r_2})_1| + o(|r_1q_j|). \tag{3.25}$$

On the other hand, by Lemma 2.1, "$|\uparrow_{r_1}^{q} (\uparrow_{r_1}^{r_2})_2| < \frac{\pi}{2}$" implies that there is $\overline{q}_j \in [r_1r_2]_2^\circ$ such that $|q_j\overline{q}_j| = |q_j[r_1r_2]_2|$. By Lemma 2.4, we have that, as $j \to \infty$,

$$|r_2\overline{q}_j| = |r_1r_2| - |r_1\overline{q}_j| = |r_1r_2| - |r_1q_j| \cos |\uparrow_{r_1}^{q} (\uparrow_{r_1}^{r_2})_2| + o(|r_1q_j|). \tag{3.26}$$

It follows from (3.24)–(3.26) that $|r_2q_j| < |r_2\overline{q}_j|$ for sufficiently large j. Since U_x can be sufficiently small, "$|r_2q_j| < |r_2\overline{q}_j|$" implies that $\widetilde{\angle}_k q_j\overline{q}_jr_2 < \frac{\pi}{2}$, which contradicts (3.22).

In order to solve the first difficulty mentioned above, we will use the following technical property of CBB-type Alexandrov spaces, especially (b), in Step 4 of the proof of Lemma 3.2 below.

Lemma 3.5 *Let X be a complete finite dimensional Alexandrov space with curvature $\geq \kappa$, and let $p, q_i, r_i \in X$ with $q_i \to p$ and $r_i \to p$ as $i \to \infty$. Then the following holds*:

(a) (see [2]) *As $i \to \infty$, for any triangle $\triangle pq_ir_i$, we have that*

$$\angle q_ipr_i - \widetilde{\angle}_\kappa q_ipr_i \to 0, \quad \angle pq_ir_i - \widetilde{\angle}_\kappa pq_ir_i \to 0, \quad \angle pr_iq_i - \widetilde{\angle}_\kappa pr_iq_i \to 0.$$

(b) *Additionally, given $[pq_i] \ni p_i$ and $[q_ir_i] \ni s_i$, if there are $c_1 \in (0,1)$ and $c_2 > 0$ such that*

$$|q_is_i| < c_1|q_ir_i|, \quad \min\{\widetilde{\angle}_\kappa q_ip_is_i, \widetilde{\angle}_\kappa p_iq_is_i, \widetilde{\angle}_\kappa p_is_iq_i\} > c_2 \tag{3.27}$$

for all i, then as $i \to \infty$, for any triangle $\triangle p_iq_is_i$, we have that

$$\angle q_ip_is_i - \widetilde{\angle}_\kappa q_ip_is_i \to 0, \quad \angle p_iq_is_i - \widetilde{\angle}_\kappa p_iq_is_i \to 0, \quad \angle p_is_iq_i - \widetilde{\angle}_\kappa p_is_iq_i \to 0.$$

Proof We just need to prove (b).

First of all, by the reason for (2.1), we know that $0 \le \angle p_iq_is_i - \tilde{\angle}_\kappa p_iq_is_i \le \angle pq_ir_i - \tilde{\angle}_\kappa pq_ir_i$, so it follows from (a) that

$$\angle p_iq_is_i - \tilde{\angle}_\kappa p_iq_is_i \to 0 \quad \text{as } i \to \infty. \tag{3.28}$$

Next, we show that $\angle q_is_ip_i - \tilde{\angle}_\kappa q_is_ip_i \to 0$ as $i \to \infty$. Consider the point $u_i \in [q_ir_i]$ with $|u_iq_i| = 2|s_iq_i|$ or $u_i = r_i$ when $|q_is_i| < \frac{1}{2}|q_ir_i|$ or $|q_is_i| \ge \frac{1}{2}|q_ir_i|$, respectively. Let $\triangle\tilde{q}_i\tilde{p}_i\tilde{u}_i \subset \mathbb{S}^2_\kappa$ be a comparison triangle of $\triangle q_ip_iu_i$, and let $\tilde{s}_i \in [\tilde{q}_i\tilde{u}_i]$ such that $|\tilde{s}_i\tilde{q}_i| = |s_iq_i|$. By Definition 2.1, we know that $|p_is_i| \ge |\tilde{p}_i\tilde{s}_i|$. On the other hand, by the same reason for (3.28), we have that $\angle p_iq_iu_i - \tilde{\angle}_\kappa p_iq_iu_i \to 0$ as $i \to \infty$. By (3.27) and Theorem 2.2, we get that $|p_is_i|$ is almost equal to $|\tilde{p}_i\tilde{s}_i|$, precisely,

$$\lim_{i\to\infty} \frac{|p_is_i| - |\tilde{p}_i\tilde{s}_i|}{|p_is_i|} = 0. \tag{3.29}$$

Together with (3.27), this implies that

$$\tilde{\angle}_\kappa q_is_ip_i - \angle\tilde{q}_i\tilde{s}_i\tilde{p}_i \to 0, \quad \tilde{\angle}_\kappa u_is_ip_i - \angle\tilde{u}_i\tilde{s}_i\tilde{p}_i \to 0 \quad \text{as } i \to \infty. \tag{3.30}$$

Since $\angle q_is_ip_i \ge \tilde{\angle}_\kappa q_is_ip_i$, $\angle u_is_ip_i \ge \tilde{\angle}_\kappa u_is_ip_i$ (by Theorem 2.2), $\angle q_is_ip_i + \angle u_is_ip_i = \pi$ and $\angle\tilde{q}_i\tilde{s}_i\tilde{p}_i + \angle\tilde{u}_i\tilde{s}_i\tilde{p}_i = \pi$, it follows from (3.30) that

$$\angle q_is_ip_i - \tilde{\angle}_\kappa q_is_ip_i \to 0, \quad \text{and } \angle u_is_ip_i - \tilde{\angle}_\kappa u_is_ip_i \to 0 \quad \text{as } i \to \infty.$$

At last, we show that $\angle q_ip_is_i - \tilde{\angle}_\kappa q_ip_is_i \to 0$ as $i \to \infty$. Similarly, we consider $v_i \in [q_ip]$ with $|v_iq_i| = 2|p_iq_i|$ or $v_i = p$ when $|q_ip_i| < \frac{1}{2}|q_ip|$ or $|q_ip_i| \ge \frac{1}{2}|q_ip|$, respectively, and a comparison triangle $\triangle\tilde{q}_i\tilde{s}_i\tilde{v}_i \subset \mathbb{S}^2_\kappa$ of $\triangle q_is_iv_i$ and $\tilde{p}_i \in [\tilde{q}_i\tilde{v}_i]$ with $|\tilde{q}_i\tilde{p}_i| = |q_ip_i|$. It is easy to see that (3.29) still holds in the situation here. And if we can show

$$\tilde{\angle}_\kappa s_ip_iq_i - \angle\tilde{s}_i\tilde{p}_i\tilde{q}_i \to 0, \quad \tilde{\angle}_\kappa s_ip_iv_i - \angle\tilde{s}_i\tilde{p}_i\tilde{v}_i \to 0 \quad \text{as } i \to \infty \tag{3.31}$$

(similar to (3.30)), then we can conclude that $\angle q_ip_is_i - \tilde{\angle}_\kappa q_ip_is_i \to 0$ as $i \to \infty$. Indeed, we can similarly show (3.31) except possibly when $|q_ip_i| > \frac{1}{2}|q_ip|$. Note that p_i, unlike s_i satisfying $|q_is_i| < c_1|q_ir_i|$, may be sufficiently close to p and even equal to p. When $\frac{1}{2}|q_ip| < |q_ip_i|$ $(\le |q_ip|)$, by the latter part of (3.27), there is $c_3 > 0$ such that $\tilde{\angle}_\kappa s_iq_ip > c_3$ and $\tilde{\angle}_\kappa q_ips_i > c_3$ for all i. Then by "$\angle q_ips_i - \tilde{\angle}_\kappa q_ips_i \to 0$ as $i \to \infty$ (by (a) in Lemma 3.5)" and (3.29), we can apply the Law of Sine to conclude (3.31).

We will end this section with proving Lemma 3.2 (and so the proof for Case 4 is completed).

Proof of Lemma 3.2 First of all, in this proof, we can assume that there is a unique minimal geodesic between any two distinct points in U_x (see Lemma 3.3).

Since it is assumed that X is not of curvature $\le k$ around x, by Theorem 2.2 there is a triangle $\triangle pqr \subset U_x$ containing a "bad" angle, say $\angle qpr$ (i.e., $\angle qpr > \tilde{\angle}_k qpr$). Then similar to the existence of $\bar{s}$ and $\bar{t}$ in the proof for Case 1, there is $\bar{s} \in [pr] \setminus \{p\}$ and $\bar{t} \in [pq] \setminus \{p\}$ such that, for all $s \in [p\bar{s}]^\circ$ and $t \in [p\bar{t}]$,

$$|\bar{t}s| > |\tilde{\bar{t}}\tilde{s}|, \quad |\bar{s}t| > |\tilde{\bar{s}}\tilde{t}|, \quad |\bar{s}\bar{t}| = |\tilde{\bar{s}}\tilde{\bar{t}}|, \quad \angle\bar{t}p\bar{s} - \tilde{\angle}_k\bar{t}p\bar{s} = \angle qpr - \tilde{\angle}_k qpr, \tag{3.32}$$

where $\widetilde{\overline{s}}, \widetilde{s}, \widetilde{\overline{t}}, \widetilde{t}$ belong to the comparison triangle $\triangle\widetilde{p}\widetilde{q}\widetilde{r} \subset \mathbb{S}^2_k$ of $\triangle qpr$ and correspond to $\overline{s}, s, \overline{t}, t$, respectively with $\widetilde{\overline{s}} \in [\widetilde{p}\widetilde{r}]$ and $|\widetilde{\overline{s}}\widetilde{p}| = |\overline{s}p|$, etc.

Our strategy is also to look for a $\triangle pqr$ with a "bad" angle $\angle qpr$ such that the corresponding $\triangle p\overline{t}\overline{s}$ is the desired triangle. We will fulfill the task through the following four steps based on a general $\triangle pqr$ in which $\angle qpr$ is a "bad" angle.

In Cases 1–3, for any $\triangle pqr$ with "bad" angle $\angle qpr$, we can conclude that at least one of $\angle p\overline{t}\overline{s}$ and $\angle p\overline{s}\overline{t}$ is less than $\frac{\pi}{2}$, so it suffices to find a triangle with an acute "bad" angle. Unfortunately, as mentioned above, in the situation here we cannot conclude such a property for a general triangle with a "bad" angle.

Step 1 To show that $\triangle qpr$ can be chosen to satisfy that there is at most one point $t \in [pr]^\circ$ such that $[qt]$ is perpendicular to $[pr]$.

Note that it suffices to consider the case where there are two distinct points $t_1, t_2 \in [pr]^\circ$ such that $\angle qt_ip = \angle qt_ir = \frac{\pi}{2}$. Claim: Up to a new choice, t_i satisfies that either $\angle qtt_1 = \angle qtt_2 = \frac{\pi}{2}$ for all $t \in [t_1t_2]^\circ$, or one of $\angle qtt_i$ $(i = 1, 2)$ is less than $\frac{\pi}{2}$ for all $t \in [t_1t_2]^\circ$. In fact, if there is $t \in [t_1t_2]^\circ$ such that $\angle qtt_1 < \frac{\pi}{2}$, then $\angle qt't_1 < \frac{\pi}{2}$ for t' sufficiently close to t (by Lemma 2.3), which implies the claim.

If $\angle qtt_1 = \angle qtt_2 = \frac{\pi}{2}$ for all $t \in [t_1t_2]^\circ$, then it has to hold that $|qt| = |qt_1| = |qt_2|$ by Lemma 2.1. Since U_x can be sufficiently small, it follows that $\widetilde{\angle}_k qtt_i < \frac{\pi}{2}$, which contradicts (3.22).

If $\angle qtt_1 < \frac{\pi}{2}$ or $\angle qtt_2 < \frac{\pi}{2}$ for all $t \in [t_1t_2]^\circ$, $\triangle qpr$ can be chosen to be $\triangle qt_1t_2$. Note that $\angle qt_1t_2 = \angle qt_2t_1 = \frac{\pi}{2}$, and at least one of $\widetilde{\angle}_k qt_1t_2$ and $\widetilde{\angle}_k qt_2t_1$ is less than $\frac{\pi}{2}$ as long as U_x is small enough, i.e., $\triangle qt_1t_2$ has a "bad" angle.

Step 2 To show that there is $[\hat{p}\hat{r}] \subset [pr]^\circ$ such that $\angle q\hat{p}\hat{r}$ is acute and "bad" in $\triangle q\hat{p}\hat{r}$.

Since $\angle qpr > \widetilde{\angle}_k qpr$, by Lemma 2.1 there is corresponding $r' \in [pr]^\circ$ and $\widetilde{r}' \in [\widetilde{p}\widetilde{r}]$ which satisfy (3.23) (see (3.13)). Then similarly we can apply (1) in Remark 3.1 to conclude that $\triangle qr'p$ or $\triangle qr'r$ contains a "bad" angle $\angle qr'p$ or $\angle qr'r$, respectively (see (3.14)). Moreover, as in Case 1 (see the end of the proof for Case 1), up to repeating such a cutting operation finite times we can assume that $|pr| \ll |pq|$, which implies that $\widetilde{\angle}_k pqr \ll \frac{\pi}{2}$, and thus by (1) in Remark 3.1 again we can conclude that

$$\angle qr'p \text{ and } \angle qr'r \text{ are "bad" angles in } \triangle qr'p \text{ and } \triangle qr'r, \text{ respectively.} \tag{3.33}$$

Since there is at most one point $t \in [pr]^\circ$ such that $\angle qtp = \frac{\pi}{2}$ (by Step 1), we can assume that in $[pr']^\circ$ or $[r'r]^\circ$, say $[r'r]^\circ$, there is no point t such that $\angle qtr' = \frac{\pi}{2}$. Meantime, note that $\angle qtr' + \angle qtr = \pi$ for all $t \in [r'r]^\circ$ (by Lemma 2.2). Then by repeating the cutting operation on $\triangle qr'r$ two more times, we can locate $[\hat{p}\hat{r}] \subset [r'r] \setminus \{r\} \subset [pr]^\circ$ such that $\angle q\hat{p}\hat{r}$ is acute and "bad" in $\triangle q\hat{p}\hat{r}$.

Step 3 To show that there are $\check{r} \in [\hat{p}\hat{r}]$ and $p_i \in [\hat{p}\check{r}]^\circ$ with $p_i \to \check{r}$ as $i \to \infty$ such that $\angle qp_i\check{r}$ is acute and "bad" in $\triangle qp_i\check{r}$.

Let $\triangle\widetilde{q}\widetilde{\widehat{p}}\widetilde{\widehat{r}}\subset\mathbb{S}_k^2$ be a comparison triangle of $\triangle q\widehat{p}\widehat{r}$. Since $\angle q\widehat{p}\widehat{r}$ is "bad" (i.e., $\angle q\widehat{p}\widehat{r}>\angle\widetilde{q}\widetilde{\widehat{p}}\widetilde{\widehat{r}}$), by Lemma 2.1 there is $t_1\in[\widehat{p}\widehat{r}]\setminus\{\widehat{p}\}$ such that $|qt_1|=|\widetilde{q}\widetilde{t}_1|$ and $|qt|>|\widetilde{q}\widetilde{t}|$ for all $t\in[\widehat{p}t_1]^\circ$, where $\widetilde{t}_1,\widetilde{t}\in[\widetilde{\widehat{p}}\widetilde{\widehat{r}}]$ with $|\widetilde{t}_1\widetilde{\widehat{p}}|=|t_1\widehat{p}|$ and $|\widetilde{t}\widetilde{\widehat{p}}|=|t\widehat{p}|$. As a result, by (3.22), it is not hard to see that

$$|qt|\neq|q[\widehat{p}t_1]|\quad\text{for any } t\in[\widehat{p}t_1]^\circ. \tag{3.34}$$

And due to "$\angle q\widehat{p}\widehat{r}<\frac{\pi}{2}$" (see Step 2), (3.34) has a more precise version

$$|qt|>|qt_1|\quad\text{for any } t\in[\widehat{p}t_1]^\circ, \tag{3.35}$$

which, by Lemma 2.1, implies that

$$\angle qt_1\widehat{p}\geq\frac{\pi}{2}. \tag{3.36}$$

Note that $\triangle\widetilde{q}\widetilde{\widehat{p}}\widetilde{t}_1$ is a comparison triangle of $\triangle q\widehat{p}t_1$, so $\angle q\widehat{p}t_1$ is also "bad" in $\triangle q\widehat{p}t_1$. Together with "$\widehat{p}\in[pr]^\circ$" (see Step 2) and by Lemma 2.3, this implies that $\angle qtt_1$ is "bad" in $\triangle qtt_1$ for $t\in[\widehat{p}t_1]^\circ$ sufficiently close to $\widehat{p}$. Hence, one of the following two cases must happen:

- for all $t\in[\widehat{p}t_1]^\circ$, $\angle qtt_1$ is "bad" in $\triangle qtt_1$;
- there is $t_*\in[\widehat{p}t_1]^\circ$ such that $\angle qtt_1$ is "bad" in $\triangle qtt_1$ for all $t\in[\widehat{p}t_*]$ except t_*.

In the former case, we can put $\check{r}=t_1$; otherwise, $\angle qtt_1\geq\frac{\pi}{2}$ for all $t\in[\widehat{p}t_1]^\circ$ sufficiently close to t_1, and thus $|qt|\leq|qt_1|$ by Lemma 2.1, which contradicts (3.35).

In the latter case, we shall put $\check{r}=t_*$ by showing that there is $p_i\in[\widehat{p}t_*]^\circ$ with $p_i\to t_*$ as $i\to\infty$ such that $\angle qp_it_*$ is acute and "bad" in $\triangle qp_it_*$.

We first observe that, for any $t\in[\widehat{p}t_*]\setminus\{t_*\}$, at least one of $\angle qtt_*$ and $\angle qt_*t$ is a "bad" angle of $\triangle qtt_*$. Otherwise, $\angle qtt_*\leq\widetilde{\angle}_kqtt_*$ and $\angle qt_*t\leq\widetilde{\angle}_kqt_*t$. Consider $\triangle\widetilde{q}\widetilde{t}\widetilde{t}_*\subset\mathbb{S}_k^2$, a comparison triangle of $\triangle qtt_*$, and let $[\widetilde{t}\widetilde{t}_*]\subset[\widetilde{t}\widetilde{z}]\subset\mathbb{S}_k^2$ with $|\widetilde{t}\widetilde{z}|=|tt_1|$. Since $\angle qtt_1>\widetilde{\angle}_kqtt_1$ (by the badness of $\angle qtt_1$), "$\angle qtt_*\leq\widetilde{\angle}_kqtt_*$" implies that $|qt_1|<|\widetilde{q}\widetilde{z}|$. Together with "$\angle qt_*t_1\leq\widetilde{\angle}_kqt_*t_1$" (note that $\angle qt_*t_1$ is not "bad" in $\triangle qt_*t_1$), we conclude that $\angle qt_*t_1<\angle\widetilde{q}\widetilde{t}_*\widetilde{z}$, and thus by Lemma 2.2 we have that $\angle qt_*t>\angle\widetilde{q}\widetilde{t}_*\widetilde{t}=\widetilde{\angle}_kqt_*t$, a contradiction.

Based on the observation right above, for t close to t_*, we can conduct a cutting operation on $\triangle qtt_*$ as in Step 2 to locate a $\widehat{t}\in[tt_*]^\circ$ such that $\angle q\widehat{t}t_*$ is "bad" in $\triangle q\widehat{t}t_*$ (see (3.33)). Namely, we can locate $p_i\in[\widehat{p}t_*]^\circ$ with $p_i\to t_*$ as $i\to\infty$ such that $\angle qp_it_*$ is "bad" in $\triangle qp_it_*$. On the other hand, we claim that $\angle qt_*t_1<\frac{\pi}{2}$, which implies that $\angle qp_it_*<\frac{\pi}{2}$ (by Lemma 2.3). In fact, the claim follows from that $\angle qt_*t_1\leq\widetilde{\angle}_kqt_*t_1$ (note that $\angle qt_*t_1$ is not "bad" in $\triangle qt_*t_1$) and $\widetilde{\angle}_kqt_*t_1<\frac{\pi}{2}$ (note that $|qt_*|>|qt_1|$ by (3.35) and U_x is sufficiently small).

As shown in the beginning of the proof, for each $\triangle qp_i\check{r}$, there is $\overline{s}_i\in[p_i\check{r}]\setminus\{p_i\}$ and $\overline{t}_i\in[p_iq]\setminus\{p_i\}$ such that the corresponding (3.32) holds for all $s\in[p_i\overline{s}_i]^\circ$ and $t\in[p_i\overline{t}_i]$.

Step 4 To show that $\triangle p_i\overline{t}_i\overline{s}_i$ is our desired triangle for large i (and thus the proof is done).

Note that if $|\overline{s}_i\overline{t}_i|\geq|\overline{s}_ip_i|$ or $|\overline{s}_i\overline{t}_i|\geq|\overline{t}_ip_i|$, say $|\overline{s}_i\overline{t}_i|\geq|\overline{t}_ip_i|$, then by Lemma 2.1 the acuteness of $\angle\overline{t}_ip_i\overline{s}_i$ $(=\angle qp_i\check{r})$ implies that $|\overline{t}_is|=|\overline{t}_i[p_i\overline{s}_i]|$ for some $s\in[p_i\overline{s}_i]^\circ$. Then, by the inequalities in the corresponding (3.32) for $\overline{s}_i$ and $\overline{t}_i$, we can conclude that $\triangle p_i\overline{t}_i\overline{s}_i$ is our desired triangle.

Hence, in the rest of the proof, we only need to consider the case where

$$|\overline{s}_i\overline{t}_i| < |\overline{s}_ip_i| \text{ and } |\overline{s}_i\overline{t}_i| < |\overline{t}_ip_i|. \tag{3.37}$$

In this case, it suffices to show that one of $\angle\overline{s}_i\overline{t}_ip_i$ and $\angle\overline{t}_i\overline{s}_ip_i$ is less than $\frac{\pi}{2}$ for large i, which together with the acuteness of $\angle\overline{t}_ip_i\overline{s}_i$ also implies that $\triangle p_i\overline{t}_i\overline{s}_i$ is our desired triangle by Lemma 2.1.

The main tool here is (b) in Lemma 3.5. In order to apply it, we need to verify its conditions in the situation here. We first note that (3.37) with $|p_i\overline{s}_i| \leq |p_i\check{r}| \to 0$ as $i\to\infty$ implies that $|p_i\overline{t}_i| \to 0$ as $i \to \infty$, and thus there is $r_i \in [p_iq]$ such that

$$|p_i\overline{t}_i| = \frac{1}{2}|p_ir_i|. \tag{3.38}$$

On the other hand, note that $[p_i\overline{s}_i] \subseteq [p_i\check{r}] \subset [\widehat{p}\widehat{r}] \subset [pr]^\circ$, so by Lemma 2.3,

$$\angle\overline{t}_ip_i\overline{s}_i \ (= \angle qp_i\check{r}) \to \angle q\check{r}r \quad \text{as } i\to\infty. \tag{3.39}$$

Moreover, we can assume that X is of curvature $\geq k_x$ around x, so we have that $\lim\limits_{i\to\infty}(\angle\overline{t}_ip_i\overline{s}_i - \widetilde{\angle}_{k_x}\overline{t}_ip_i\overline{s}_i) = 0$ by (a) in Lemma 3.5, which together with (3.37) and (3.39) implies that there is a $c > 0$ such that

$$\min\{\widetilde{\angle}_{k_x}\overline{s}_ip_i\overline{t}_i, \widetilde{\angle}_{k_x}p_i\overline{s}_i\overline{t}_i, \widetilde{\angle}_{k_x}p_i\overline{t}_i\overline{s}_i\} > c. \tag{3.40}$$

Note that (3.38) and (3.40) enable us to apply (b) in Lemma 3.5 on $\triangle\overline{t}_ip_i\overline{s}_i$ to conclude that

$$|\angle\overline{s}_i\overline{t}_ip_i - \widetilde{\angle}_{k_x}\overline{s}_i\overline{t}_ip_i| + |\angle\overline{t}_ip_i\overline{s}_i - \widetilde{\angle}_{k_x}\overline{t}_ip_i\overline{s}_i| + |\angle\overline{t}_i\overline{s}_ip_i - \widetilde{\angle}_{k_x}\overline{t}_i\overline{s}_ip_i| \to 0 \quad \text{as } i\to\infty.$$

Note that

$$\widetilde{\angle}_{k_x}\overline{s}_i\overline{t}_ip_i + \widetilde{\angle}_{k_x}\overline{t}_ip_i\overline{s}_i + \widetilde{\angle}_{k_x}\overline{t}_i\overline{s}_ip_i \to \pi \quad \text{as } i\to\infty.$$

Then together with (3.39) we can conclude that at least one of $\angle\overline{s}_i\overline{t}_ip_i$ and $\angle\overline{t}_i\overline{s}_ip_i$ is less than $\frac{\pi}{2}$ for large i.

4 Proof of Theorem A for Curvature $\equiv k$ on X°

In this section, we will show the rigidity part of Theorem A, i.e., around any interior point of U_x there is a neighborhood which is a Riemannian manifold with sectional curvature equal to k if

$$\begin{aligned}&\forall\, q\in U_x,\ [r_1r_2]\subset U_x \text{ with } q\notin[r_1r_2], \text{ if } p\in[r_1r_2]^\circ\\ &\text{satisfies } |qp| = |q[r_1r_2]|, \text{ then } \widetilde{\angle}_kqpr_i = \frac{\pi}{2},\end{aligned} \tag{4.1}$$

and if U_x is of finite dimension.

Note that, by the conclusion in Theorem A for curvature $\geq k$, (4.1) implies clearly that U_x is of curvature $\geq k$. Since U_x is of finite dimension, at each $z\in U_x$, we can consider the space of directions and the tangent cone, $\Sigma_z X$ and $C_z X$, which are still Alexandrov spaces of curvature ≥ 1 and ≥ 0, respectively (see Section 2). An easy observation from (4.1) is that $\Sigma_z X$ and $C_z X$ also satisfy a corresponding property of (4.1), i.e., Lemma 4.1 below. This makes it possible to apply the inductive assumption on $\Sigma_z X$, which is of dimension one less than U_x and has an empty boundary if z is an interior point of U_x.

Lemma 4.1 *Let U_x be the neighborhood which satisfies* (4.1) *and is of finite dimension, and let $z \in U_x$. Then for all $\overline{q} \in C_z X$ (resp. $\in \Sigma_z X$) and $[\overline{r}_1\overline{r}_2] \subset C_z X$ (resp. $\subset \Sigma_z X$) with $\overline{q} \notin [\overline{r}_1\overline{r}_2]$,*

$$\text{if there is } \overline{p} \in [\overline{r}_1\overline{r}_2]^\circ \text{ such that } |\overline{q}\,\overline{p}| = |\overline{q}[\overline{r}_1\overline{r}_2]|, \text{ then } \widetilde{\angle}_{0\ (resp.\ 1)}\overline{q}\,\overline{p}\,\overline{r}_i = \frac{\pi}{2}. \tag{4.2}$$

Proof By definition, $C_z X$ is the cone over $\Sigma_z X$ (see [2]) [3]. So, it is not hard to see that the property of (4.2) for $C_z X$ implies that for $\Sigma_z X$. In order to see (4.2) for $C_z X$, one just needs to notice that $C_z X$ is the limit space of $(\lambda X, z)$ as $\lambda \to +\infty$ by Lemma 2.4.

Note that (4.1) is contained in (3.22), so via Lemma 3.3 we have another easy observation.

Lemma 4.2 *Let Y be a cone over a circle with perimeter less than 2π (an Alexandrov space of curvature ≥ 0). Then around its vertex there is no neighborhood such that* (4.1) *holds with respect to $k = 0$.*

Proof The lemma is an easy corollary of Lemma 3.3 because we can find two points sufficiently close to the vertex of Y between which there are two minimal geodesics.

To $q, p, [r_1r_2]$ in (4.1), we associate $\widetilde{q} \in \mathbb{S}^2_k$ and $\widetilde{p} \in [\widetilde{r}_1\widetilde{r}_2] \subset \mathbb{S}^2_k$ with $|\widetilde{r}_1\widetilde{r}_2| = |r_1r_2|$, $|\widetilde{q}\widetilde{r}_i| = |qr_i|$ and $|\widetilde{r}_i\widetilde{p}| = |r_ip|$. Since X is of curvature $\geq k$ on U_x, we know that $|qp| \geq |\widetilde{q}\widetilde{p}|$ (by Definition 2.1) and any $[qp]$ is perpendicular to $[r_1r_2]$ at p (by Lemma 2.2). Then we have the third easy observation from (4.1):

$$|qp| = |\widetilde{q}\widetilde{p}| \text{ (and } [\widetilde{q}\widetilde{p}] \text{ has to be perpendicular to } [\widetilde{r}_1\widetilde{r}_2]). \tag{4.3}$$

Thereby, the following rigidity version of Theorem 2.2 can be applied.

Theorem 4.1 (see [4]) *Let X be a complete Alexandrov space with curvature $\geq k$, and let $q \in X$ and $[pr] \subset X$. If there is $s \in [pr]^\circ$ such that $|qs| = |\widetilde{q}\widetilde{s}|$, where $\widetilde{q} \in \mathbb{S}^2_k$ and $\widetilde{s} \in [\widetilde{p}\widetilde{r}] \subset \mathbb{S}^2_k$ with $|\widetilde{q}\widetilde{p}| = |qp|$, $|\widetilde{p}\widetilde{r}| = |pr|$, $|\widetilde{q}\widetilde{r}| = |qr|$ and $|ps| = |\widetilde{p}\widetilde{s}|$, then there are $[pq]$ and $[qr]$ such that $[pr]$ together with them bounds a convex surface which can be embedded isometrically into $\mathbb{S}^2_k$.*

Based on the three observations above, we can prove the rigidity part of Theorem A.

Proof of the rigidity part of Theorem A As mentioned in the beginning of this section, (4.1) implies that U_x is of curvature $\geq k$. And we can give a proof by induction on $\dim(U_x)$ because U_x is of finite dimension.

[3] The metric on $C_z X$ is defined from the Law of Cosine on $\mathbb{R}^2$ by viewing distances on $\Sigma_z X$ as angles (see [2]).

We first consider the case where $\dim(U_x) = 2$. It suffices to show that, for any $p \in U_x^\circ$ (i.e., its space of directions $\Sigma_p X$ has no boundary), there is a convex surface $\mathcal{S}$ ($\subseteq U_x$) which can be embedded isometrically into $\mathbb{S}_k^2$ such that $p \in \mathcal{S}^\circ$.

Since $p \in U_x^\circ$, $\Sigma_p X$ is a circle with perimeter $\leq 2\pi$ (see [2]). Since C_pX is the cone over $\Sigma_p X$, by Lemma 4.2, the perimeter of $\Sigma_p X$ must be equal to 2π, and thus C_pX is a plane. Let $\xi_i \in \Sigma_p X$, $i = 1, 2, 3$, be $\frac{2\pi}{3}$-separated $\big($i.e., $|\xi_i\xi_j| = \frac{2\pi}{3}$ for $i \neq j\big)$. We know that there are $\{q_{il}\}_{l=1}^\infty \subset U_x$ with $|q_{1l}p| = |q_{2l}p| = |q_{3l}p| \to 0$ and the directions $\uparrow_p^{q_{il}} \to \xi_i$ as $l \to \infty$ (see [2]). On the other hand, we know that C_pX is the limit space of $\big(\frac{1}{|pq_{il}|}U_x, p\big)$ as $l \to \infty$ by Lemma 2.4. It follows that $q_{il} \to q_i$ as

$$\Big(\frac{1}{|pq_{il}|}U_x, p\Big) \to C_pX$$

with $|pq_i| = 1$ and $\uparrow_p^{q_i} = \xi_i$ in C_pX. Note that as

$$\Big(\frac{1}{|pq_{il}|}U_x, p\Big) \to C_pX,$$

any triangle

$$\triangle q_{1l}q_{2l}q_{3l} \to \triangle q_1q_2q_3,$$

an equilateral triangle. It then is easy to see that, for sufficiently large l, $|q_{3l}[q_{1l}q_{2l}]| = |q_{3l}q|$ for some $q \in [q_{1l}q_{2l}]^\circ$. Then by (4.3) and Theorem 4.1, $\triangle q_{1l}q_{2l}q_{3l}$ bounds a convex surface $\mathcal{S}_l$ which can be embedded isometrically into $\mathbb{S}_k^2$ (note that there is a unique minimal geodesic between any two points around p by Lemma 3.3).

We claim that $\mathcal{S}_l$ for large l is just our desired surface. In fact, $\mathcal{S}_l$ converges to the domain bounded by $\triangle q_1q_2q_3$ in the plane C_pX as

$$\Big(\frac{1}{|pq_{il}|}U_x, p\Big) \to C_pX.$$

It follows that, for large l, there is $\overline{p} \in \mathcal{S}_l^\circ$ such that

$$|p\overline{p}| = |p\mathcal{S}_l|.$$

Since $\mathcal{S}_l$ is a convex surface in U_x and $\dim(U_x) = 2$, it must hold that $p = \overline{p}$, i.e.,

$$p \in \mathcal{S}_l^\circ.$$

We now assume that $\dim(U_x) = n \geq 3$. Similarly, for any $p \in U_x^\circ$, it suffices to show that p lies in the interior part of a convex domain (in U_x) which can be embedded isometrically into $\mathbb{S}_k^n$.

We first show that C_pX, the cone over $\Sigma_p X$, is isometric to the Euclidean space $\mathbb{R}^n$, i.e., $\Sigma_p X$ is isometric to the unit sphere $\mathbb{S}_1^{n-1}$. Note that $\Sigma_p X$ is a complete Alexandrov space with curvature ≥ 1, and $(\Sigma_p X)^\circ = \Sigma_p X$ with dimension equal to $n-1$ because $p \in U_x^\circ$. By Lemma 4.1, we can apply the inductive assumption on $\Sigma_p X$ to conclude that it is a complete Rimannian manifold with sectional curvature equal to 1. Then we need only to show that $\Sigma_p X$ is simply connected. If it is not true, then by classical results in Riemannian geometry, there is a closed geodesic, a circle $S^1 \subset \Sigma_p X$, with perimeter less than 2π. Note that the cone over the S^1 is convex in C_pX, which is impossible by Lemma 4.2.

Similarly, since $\Sigma_p X$ is isometric to $\mathbb{S}_1^{n-1}$, we can select an $\left(\arccos \frac{-1}{n}\right)$-separated subset $\{\xi_i \in \Sigma_p X \mid i = 1, 2, \cdots, n+1\}$, and $\{q_{il}\}_{l=1}^{\infty} \subset U_x$ with $|q_{1l}p| = |q_{2l}p| = \cdots = |q_{(n+1)l}p| \to 0$ and the directions $\uparrow_p^{q_{il}} \to \xi_i$ as $l \to \infty$. It follows that $q_{il} \to q_i$ as $\left(\frac{1}{|pq_{il}|}U_x, p\right) \to C_pX$ with $|pq_i| = 1$ and $\uparrow_p^{q_i} = \xi_i$.

Claim: For sufficiently large l, there is a convex simplex Δ_l with vertices $q_{1l}, \cdots, q_{(n+1)l}$ which can be embedded isometrically into $\mathbb{S}_k^n$. By the claim, we need only to show that $p \in \Delta_l^\circ$ for large l. Note that as $\left(\frac{1}{|pq_{il}|}U_x, p\right) \to C_pX$, Δ_l converges to the simplex with vertices $q_1, \cdots, q_{n+1}$ in $C_pX \overset{\text{iso}}{\cong} \mathbb{R}^n$. It follows that, for large l, there is $\overline{p} \in \Delta_l^\circ$ such that

$$|p\overline{p}| = |p\Delta_l|.$$

Since Δ_l is a convex in X and $\dim(\Delta_l) = \dim(X)$, it must hold that $p = \overline{p}$, i.e.,

$$p \in \Delta_l^\circ.$$

To complete the proof, it suffices to verify the claim right above. Note that there is a unique minimal geodesic between any two points around p by (4.1) and Lemma 3.3. And as in the proof for $\dim(U_x) = 2$, for large l, $|q_{3l}[q_{1l}q_{2l}]| = |q_{3l}q|$ for some $q \in [q_{1l}q_{2l}]^\circ$. So, the triangle $\triangle q_{1l}q_{2l}q_{3l}$ bounds a convex surface $\mathcal{S}_l$ which can be embedded isometrically into $\mathbb{S}_k^2$ (by (4.3) and Theorem 4.1). Furthermore, there is $\overline{q}_{4l} \in \mathcal{S}_l^\circ$ (around the center of $\mathcal{S}_l$) such that

$$|q_{4l}\overline{q}_{4l}| = |q_{4l}\mathcal{S}_l|.$$

Let $\widetilde{q}_{il} \in \mathbb{S}_k^3$, $i = 1, 2, 3, 4$, be the vertices of a simplex of dimension 3 with

$$|\widetilde{q}_{il}\widetilde{q}_{jl}| = |q_{il}q_{jl}|.$$

By applying (4.3) and Theorem 4.1 iteratively, it is not hard to see that $\bigcup_{s \in \mathcal{S}_l} [q_{4l}s]$ is convex and isometric to the simplex in $\mathbb{S}_k^3$ with vertices $\widetilde{q}_{1l}, \widetilde{q}_{2l}, \widetilde{q}_{3l}, \widetilde{q}_{4l}$. And step by step, we can eventually get the desired Δ_l in the claim.

5 Proofs of Theorem C and Corollary D

Proof of Theorem C Let x be an interior point in a complete finite dimensional CBB-type Alexandrov space X. As mentioned in Section 2, we can consider the space of directions and the tangent cone at x, $\Sigma_x X$ and $C_x X$. And by Lemma 2.4, we know that C_xX is the limit space of $(\lambda X, x)$ as $\lambda \to +\infty$. It is not hard to see that if we substitute the condition "$\left|\frac{|qr|^2}{|pq|^2+|pr|^2} - 1\right| < \chi(\varepsilon)$ for all $[pq], [pr] \subset B_x(\varepsilon)$ with $\angle qpr = \frac{\pi}{2}$ and $\chi(\varepsilon) \to 0$ as $\varepsilon \to 0$" of Theorem C for the condition of Lemma 4.1, the conclusion of Lemma 4.1 for $\Sigma_x X$ and $C_x X$ still holds. Then from the proof for the rigidity part of Theorem A (in Section 4), we can conclude that $\Sigma_x X$ is isometric to the unit sphere, i.e., x is regular point.

Proof of Corollary D Let x be an interior point in a complete finite dimensional CBB-type Alexandrov space X. If x is also a CBA-type point, then by Theorems 2.2 and 2.2′ it is easy to see that there is a function $\chi(\varepsilon)$ with $\chi(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that for all $[pq], [pr] \subset B_x(\varepsilon)$ with $\angle qpr = \frac{\pi}{2}$,

$$\left|\frac{|qr|^2}{|pq|^2 + |pr|^2} - 1\right| < \chi(\varepsilon).$$

So, by Theorem C, x is regular point.

References

[1] Alexander, S., Kapovitch, V. and Petrunin, A., Alexandrov Geometry, 2010.

[2] Burago, Y., Gromov, M. L. and Perel'man, G., A.D. Aleksandrov spaces with curvature bounded below, *Uspeckhi Mat. Nank*, **47**(2), 1992, 3–51.

[3] Cheeger, J. and Ebin, D., Comparison theorems in Riemannian geometry, American Elsevier Pub. New York, 1975.

[4] Grove, K. and Markvorsen, S., New extremal problems for the Riemannian recognition program via Alexandrov geometry, *J. AMS*, **8**(1), 1995, 1–28.

[5] Li, N. and Rong, X., Bounding geometry of loops in Alexandrov spaces, *J. Diff. Geom.*, **92**(1), 2012, 31–54.

[6] Plaut, C., Spaces of Wald-Berestovskii curvature bounded below, *J. Geom. Anal.*, **6**, 1996, 113–134.

[7] Wang, Y. S., A Schur-Toponogov theorem in Riemannian geometry & a new proof of Toponogov's theorem in Alexandrov geometry, arXiv:1809.09818, 2018.