Chin. Ann. Math. Ser. B
43(1), 2022, 115–124
DOI: 10.1007/s11401-022-0308-7

Chinese Annals of Mathematics, Series B

The Isoperimetric Inequality in Steady Ricci Solitons*

Yuqiao LI[1]

Abstract The author proves that the isoperimetric inequality on the graphic curves over circle or hyperplanes over $\mathbb{S}^{n-1}$ is satisfied in the cigar steady soliton and in the Bryant steady soliton. Since both of them are Riemannian manifolds with warped product metric, the author utilize the result of Guan-Li-Wang to get his conclusion. For the sake of the soliton structure, the author believes that the geometric restrictions for manifolds in which the isoperimetric inequality holds are naturally satisfied for steady Ricci solitons.

Keywords Isoperimetric inequality, Ricci soliton
2010 Mathematics Subject Classification 53C44

1 Introduction and Statement of the Main Results

Let Ω be a bounded domain in the two-dimensional Euclidean space. We know that the isoperimetric inequality

$$L^2 \geq 4\pi A$$

holds, where L and A are the boundary length and the area of Ω, respectively. Equality is attained only when Ω is a ball.

In $\mathbb{R}^n$, if $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ which is an $(n-1)$-dimensional hypersurface, then the corresponding isoperimetric inequality reads

$$\text{Area}(\partial\Omega) \geq c(n)(\text{Vol}(\Omega))^{\frac{n-1}{n}},$$

where $\text{Area}(\partial\Omega)$ denotes the $(n-1)$-Haussdorff measure of the hypersurface $\partial\Omega$ and $\text{Vol}(\Omega)$ is the n-dimensional volume of Ω and $c(n) = n\omega_n^{\frac{1}{n}}$ is a constant depends only on the dimension n. Equality holds only when Ω is a ball.

Ricci soliton is a self-similar solution of the Ricci flow. It is obtained by a family of diffeomorphisms of the initial metric and satisfies the soliton equation

$$-2\text{Ric}(g) = \mathcal{L}_X g + \varepsilon g,$$

where X is a vector field and ε is a constant. A Ricci soliton is called a steady soliton if $\varepsilon = 0$ and a gradient soliton if $X = \nabla f$ for some function f. We always write a triple (M, g, f) to denote a gradient Ricci soliton and we say a Riemannian manifold (M, g) has a gracient soliton structure if there is a function f satisfying the soliton equation.

Manuscript received February 21, 2020. Revised June 1, 2021.
[1]Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.
E-mail: lyq112@mail.ustc.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11721101, 11526212).

We consider the isoperimetric problem on manifolds that have a steady gradient Ricci soliton structure. Because of the Ricci soliton equation, we find that the isoperimetric inequality still holds in the cigar steady soiton and in the Bryant steady soliton. Our main theorems are the followings.

Theorem 1.1 *Let $(\mathbb{R}^2, g, f)$ be the cigar steady Ricci soliton. If $\gamma \subset \mathbb{R}^2$ is a graph over $\mathbb{S}^1$, the length of γ and the area of the compact domain whose boundary is γ are denoted by $L(\gamma)$ and $A(\gamma)$, respectively. Then*

$$L(\gamma) \geq F(A(\gamma))$$

with equality holds if and only if γ is a circle $\{r\} \times \mathbb{S}^1$, where F is a single variable function which represents the relation between the length and the area of circles in the cigar (*see Theorem* 2.4 *in Section* 2).

Theorem 1.2 *Let (M, g, f) be the complete steady Bryant soliton with $M = (0, \infty) \times \mathbb{S}^{n-1}$, $g = \mathrm{d}r^2 + \phi(r)^2 g_{\mathbb{S}^{n-1}}$. Let $\Sigma \subset M$ be a hypersurface which is defined as $r = \rho(p)$, $p \in \mathbb{S}^{n-1}$ for a smooth function ρ on $\mathbb{S}^{n-1}$. Let $S(r_0)$ be the level set of $r = r_0$ and Ω be the domain bounded by Σ and $S(r_0)$. Then*

$$\mathrm{Area}(\Sigma) \geq \xi(\mathrm{Vol}(\Omega)),$$

where ξ is a well-defined single-valued function that relates the area and volume of spheres in (M, g) (*see Theorem* 2.2 *in Section* 2). *Moreover, the equality is attained if and only if Σ is a level set of r.*

2 The Isoperimetric Inequality in Riemannian Manifolds with Warped Product Metric

In the paper of Guan-Li-Wang [4], they proved an isoperimetric inequality by investigating a deformed mean curvature type flow which preserves the volume but decreases the area in manifolds with warped product metric. We will briefly state their results in this section.

Let $(\mathbb{B}^{n-1}, \widetilde{g})$ be a closed Riemannian manifold and $\phi = \phi(r)$ be a smooth positive function defined on the interval $[r_0, r_1]$ for some $r_0 < r_1$. Consider a Riemannian manifold of warped product metric $(\mathbb{N}^n, \overline{g})$,

$$\overline{g} = \mathrm{d}r^2 + \phi^2 \widetilde{g}, \quad r \in [r_0, r_1], \tag{2.1}$$

$X = \phi(r)\partial_r$ is a conformal Killing field of $\mathbb{N}^n$, i.e., $\mathcal{L}_X \overline{g} = 2\phi'(r)\overline{g}$. Let M be a smooth closed embedded hypersurface in $\mathbb{N}^n$ with an embedding F_0. Consider the following evolution equation for a family of embeddings of hypersurfaces with F_0 as an initial data,

$$\frac{\partial F}{\partial t} = ((n-1)\phi' - uH)\nu, \tag{2.2}$$

where ν is the outward unit normal vector field, H is the mean curvature, $u = \langle X, \nu \rangle$. A hypersurface is said to be graphical if it is defined by $r = \rho(p)$, $p \in \mathbb{B}^{n-1}$ for a smooth function ρ on $\mathbb{B}^{n-1}$. In [4], they proved the following theorem.

Theorem 2.1 *Let M_0 be a smooth graphical hypersurface in $(\mathbb{N}^n, \overline{g})$ with $n \geq 3$ and $\overline{g}$ in* (2.1). *If $\phi(r)$ and $\overline{g}$ satisfy the following conditions*:

$$\widetilde{\mathrm{Ric}} \geq (n-2)K\widetilde{g},$$

$$0 \le (\phi')^2 - \phi''\phi \le K \quad on\ [r_0, r_1], \tag{2.3}$$

where $K > 0$ is a constant and $\widetilde{\mathrm{Ric}}$ is the Ricci curvature of $\widetilde{g}$. Then the evolution equation (2.2) *with M_0 as the initial data has a smooth solution for $t \in [0, \infty)$. Moreover, the solution hypersurface converges exponentially to a level set of r as $t \to \infty$.*

From the long-time existence of the flow (2.2), they got an isoperimetric inequality for warped product space. Let $S(r)$ be a level set of r and $B(r)$ be the bounded domain enclosed by $S(r)$ and $S(r_0)$. The volume of $B(r)$ and surface area of $S(r)$ are denoted by $V(r)$ and $A(r)$, respectively. There is a well-defined single variable function $\xi(x)$ that satisfies

$$A(r) = \xi(V(r)) \tag{2.4}$$

for any $r \in [r_0, r_1]$.

Theorem 2.2 *Let $\Omega \subset \mathbb{N}^n$ be a domain bounded by a smooth graphical hypersurface M and $S(r_0)$. Assume* (2.3)*, then*

$$\mathrm{Area}(M) \ge \xi(\mathrm{Vol}(\Omega)), \tag{2.5}$$

where $\mathrm{Area}(M)$ *is the area of M and* $\mathrm{Vol}(\Omega)$ *is the volume of Ω. If either $(\phi')^2 - \phi''\phi < K$ or $\widetilde{\mathrm{Ric}} > (n-2)K\widetilde{g}$ on $[r_0, r_1]$, then " $=$ " is attained in* (2.5) *if and only if M is a level set of r.*

For $n = 2$, similar result is proved by Dylan Cant in [1].

Let $N = (0, R) \times \mathbb{S}^1$ be a surface with metric $g = \mathrm{d}r^2 + \phi(r)^2 \mathrm{d}\theta^2$, $\phi(r) > 0$.

Theorem 2.3 *Let N^2 be a warped product space with warp potential $\phi(r)$ satisfying $(\phi')^2 - \phi\phi'' \ge 0$. If $\gamma_0 \subset N$ is a smooth hypersurface, then there is a unique flow $\gamma(t)$ with* (2.2) *and $\gamma(0) = \gamma_0$.*

Let C_r denote the circle $\{r\} \times \mathbb{S}^1 \subset N$, and $L(r)$ and $A(r)$ denote its length and area, respectively. There is some function F with $L(r) = F(A(r))$.

Theorem 2.4 *If $\gamma_0 \subset N$ is a piecewise C^1 Lipschitz radial graph and $(\phi')^2 - \phi\phi'' \in [0, 1]$, then $L(\gamma_0) \ge F(A(\gamma_0))$, where equality holds if and only if γ_0 is a circle $\{r\} \times \mathbb{S}^1$ if $(\phi')^2 - \phi\phi'' \not\equiv 1$.*

3 Cigar Steady Soliton

We wonder if the isoperimetric inequality is still available on Ricci solitons. First, we consider the gradient steady solitons of dimension $n = 2$. We have the following theorem (see [3, Corollary 4.9]) which implies that the complete steady gradient Ricci soliton with positive curvature is the cigar soliton.

Theorem 3.1 (see [3, Corollary 4.9]) *If $(M^2, g(t))$ is a complete steady gradient Ricci soliton with positive curvature, then $(M^2, g(t))$ is the cigar soliton.*

Let $(\mathbb{R}^2, g)$ be a complete Riemannian manifold with complete metric $g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{1 + x^2 + y^2}$. In polar coordinates, we can write

$$g = \frac{\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2}{1 + r^2}.$$

Let $s = \operatorname{arcsinh} r = \log(r + \sqrt{1+r^2})$, then

$$g = \mathrm{d}s^2 + \tanh^2 s\mathrm{d}\theta^2.$$

By the gradient steady Ricci soliton equation

$$\mathrm{Ric}(g) + \nabla^2 f = 0,$$

we have that $(\mathbb{R}^2, g)$ is a gradient steady Ricci soliton with potential function

$$f(s) = -2\log(\cosh s)$$

and its curvature is

$$\mathrm{Ric}(g) = \frac{2}{\cosh^2 s} g.$$

This soliton is said to be the cigar steady soliton (see [3, Section 3 of Chapter 4]).

Corresponding to Theorems 2.3–2.4, in the cigar steady soliton case,

$$\phi(s) = \tanh s,$$

we can calculate directly to get

$$\phi'(s) = \frac{1}{\cosh^2 s}, \quad \phi''(s) = -2\frac{\sinh s}{\cosh^3 s}.$$

Thus

$$(\phi')^2 - \phi\phi'' = \cosh^{-4} s(1 + 2\sinh^2 s) > 0$$

and

$$(\phi')^2 - \phi\phi'' = \frac{2\cosh^2 s - 1}{\cosh^4 s} = 1 - \frac{(\cosh^2 s - 1)^2}{\cosh^4 s} \leq 1.$$

Therefore, Theorem 2.4 implies that the isoperimetric inequality is still true for cigar steady soliton. Theorem 1.1 is concluded.

4 Bryant Steady Soliton

In this section, we focus on the gradient steady Ricci solitons which is radial symmetric with $n \geq 3$. Let $g_{\mathbb{S}^{n-1}}$ be the standard metric on the unit $(n-1)$-sphere. We will search for gradient steady Ricci solitons on $(0, \infty) \times \mathbb{S}^{n-1}$ which extend to Ricci solitons on $\mathbb{R}^n$ by a one point compactification of one end. Consider the metric

$$g = \mathrm{d}r^2 + \phi^2(r) g_{\mathbb{S}^{n-1}},$$

its Ricci curvature is

$$\mathrm{Ric}(g) = -(n-1)\frac{\phi''}{\phi}\mathrm{d}r^2 + ((n-2)(1-(\phi')^2) - \phi\phi'')g_{\mathbb{S}^{n-1}}.$$

The Hessian of a function f with respect to g is

$$\nabla^2 f(r) = f''(r)\mathrm{d}r^2 + \phi\phi' f' g_{\mathbb{S}^{n-1}}.$$

From the steady Ricci soliton equation

$$\mathrm{Ric}(g) + \nabla^2 f = 0,$$

we get the following ODE system

$$\begin{aligned} f'' &= (n-1)\frac{\phi''}{\phi}, \\ \phi\phi' f' &= -(n-2)(1-(\phi')^2) + \phi\phi''. \end{aligned}$$

Making the transformations

$$x = \phi', \quad y = (n-1)\phi' - \phi f', \quad \mathrm{d}t = \frac{\mathrm{d}r}{\phi},$$

the ODE system becomes

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x(x-y) + (n-2), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x(y-(n-1)x). \end{aligned} \tag{4.1}$$

We will check the condition (2.3). Notice that the first assumption of (2.3) is naturally satisfied because $\mathrm{Ric}(g_{\mathbb{S}^{n-1}}) = (n-2)g_{\mathbb{S}^{n-1}}$ and $K = 1$. We calculate

$$\begin{aligned} (\phi')^2 - \phi\phi'' &= (\phi')^2 - \phi\phi' f' - (n-2)(1-(\phi')^2) \\ &= -\phi\phi' f' + (n-1)(\phi')^2 - (n-2) \\ &= xy - (n-1)x^2 + (n-1)x^2 - (n-2) \\ &= xy - (n-2). \end{aligned} \tag{4.2}$$

Thus, the second condition $0 \le (\phi')^2 - \phi\phi'' \le 1$ of (2.3) is equivalent to

$$n - 2 \le xy \le n - 1. \tag{4.3}$$

We can see the ODE system (4.1) has constant solutions $(1, n-1)$ and $(-1, -n+1)$. They both satisfy (4.3). For the extendability of g to the origin, we need the following lemma of necessary conditions.

Lemma 4.1 (see [2, Lemma A.2]) *Let $0 < L \le \infty$ and let g be a warped product metric on the topological cylinder $(0, L) \times \mathbb{S}^{n-1}$ of the form*

$$g = \mathrm{d}r^2 + w(r)^2 g_{\mathbb{S}^{n-1}},$$

where $w : (0, L) \to \mathbb{R}_+$ is a positive function. Then g extends to a smooth metric as $r \to 0_+$ if and only if

$$\begin{aligned} \lim_{r\to 0_+} w(r) &= 0, \\ \lim_{r\to 0_+} w'(r) &= 1, \\ \lim_{r\to 0_+} \frac{\mathrm{d}^{2k} w}{\mathrm{d}r^{2k}}(r) &= 0 \end{aligned}$$

for all $k \in \mathbb{N}$.

Remark 4.1 For example, when $w(r) = r$, then g represents $\mathbb{R}^n$, and we could compute that

$$\lim_{r\to 0_+} w(r) = \lim_{r\to 0_+} r = 0,$$
$$\lim_{r\to 0_+} w'(r) = \lim_{r\to 0_+} 1 = 1,$$
$$\lim_{r\to 0_+} w^k = 0$$

for any $k \in \mathbb{N}, k > 1$.

When $w(r) = \sin r$, then g represents $\mathbb{S}^n$, and

$$\lim_{r\to 0_+} w(r) = \lim_{r\to 0_+} \sin r = 0,$$
$$\lim_{r\to 0_+} w'(r) = \lim_{r\to 0_+} \cos r = 1,$$
$$\lim_{r\to 0_+} \frac{\mathrm{d}^{2k} w}{\mathrm{d}r^{2k}}(r) = \lim_{r\to 0_+} (-1)^k \sin r = 0$$

for $k \in \mathbb{N}$.

When $w(r) = \sinh r$, then g represents $\mathbb{H}^n$, and

$$\lim_{r\to 0_+} w(r) = \lim_{r\to 0_+} \sinh r = 0,$$
$$\lim_{r\to 0_+} w'(r) = \lim_{r\to 0_+} \cosh r = 1,$$
$$\lim_{r\to 0_+} \frac{\mathrm{d}^{2k} w}{\mathrm{d}r^{2k}}(r) = \lim_{r\to 0_+} \sinh r = 0$$

for $k \in \mathbb{N}$.

Thus, by Lemma 4.1, in the space forms $\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n$, the metric extends smoothly across the origin.

Remark 4.2 In Section 3, the cigar steady soliton has $g = \mathrm{d}s^2 + \tanh^2 s\mathrm{d}\theta^2$, then for $\phi(s) = \tanh s$, we can calculate

$$\lim_{s\to 0_+} \phi(s) = \lim_{s\to 0_+} \tanh s = 0,$$
$$\lim_{s\to 0_+} \phi'(s) = \lim_{s\to 0_+} \frac{1}{\cosh^2 s} = 1.$$

Since $\phi''(s) = -2\phi\phi'$, then $\lim\limits_{s\to 0_+} \phi''(s) = 0$; inductively, we get

$$\lim_{s\to 0_+} \frac{\mathrm{d}^{2k}\phi(s)}{\mathrm{d}s^{2k}} = 0.$$

So the cigar steady soliton extends smoothly across the origin by Lemma 4.1.

Now we return to the ODE system (4.1). Since we want the metric to close up smoothly as $r \to 0$, by Lemma 4.1, we need

$$x \to 1, \quad y \to n-1.$$

From the facts that $\mathrm{d}t = \frac{\mathrm{d}r}{\phi}$ and $\phi > 0$, we see that $t = t(r)$ is an increasing function of r. So, by the implicit function theorem, $r = r(t)$ is also an increasing function of t.

We have

$$x = \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{1}{\phi},$$

$$\frac{\mathrm{d}\phi}{\phi} = x\mathrm{d}t,$$

so

$$\phi(t) = \phi(r(t)) = \phi(r(0))\mathrm{e}^{\int_0^t x(\xi)\mathrm{d}\xi}.$$

Take limit as $r \to 0$, by Lemma 4.1, we obtain

$$\lim_{r\to 0} \phi = 0 = \phi(r(0))\mathrm{e}^{\lim_{r\to 0}\int_0^t x(\xi)\mathrm{d}\xi}.$$

Therefore,

$$\lim_{r\to 0}\int_0^{t(r)} x(\xi)\mathrm{d}\xi = -\infty. \tag{4.4}$$

The linearization of (4.1) at $(1, n-1)$ is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -(n-3)u - v,$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -(n-1)u + v$$

and its eigenvalues are 2 and $2-n$. From phase plane analysis, the right-hand trajectory is incomplete and the left-hand trajectory is complete (see [2, Proposition 1.32]). The Bryant steady soliton is the left-hand trajectory. We see that x is decreasing, y is increasing, and $(y-(n-1)x) > 0$ along this trajectory. Moreover,

$$x \to 0_+, \quad y \to +\infty$$

as t increases. We have $\lim\limits_{r\to 0} t = -\infty$ from the fact that $0 < x < 1$ and (4.4). Let

$$X = \sqrt{n-1}\frac{x}{y}, \quad Y = \frac{\sqrt{(n-1)(n-2)}}{y}, \quad \mathrm{d}s = y\mathrm{d}t.$$

Then as $x \to 0_+$, $y \to +\infty$, we see $X \to 0_+$ decreasingly and $Y \to 0_+$ decreasingly too. The ODE system turns out to be

$$\frac{\mathrm{d}X}{\mathrm{d}s} = X^3 - X + \alpha Y^2,$$

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = Y(X^2 - \alpha X),$$

where $\alpha = \frac{1}{\sqrt{n-1}}$.

As $t \to -\infty$, we have $r \to 0$, $x \to 1$, $y \to n-1$. Because $\mathrm{d}s = y\mathrm{d}t$ and $y > 0$, by the implicit function theorem, we have $s = s(t)$ is an increasing function of t and $t = t(s)$ is also an increasing function of s. There exists a constant T_0, such that $y \le n$ when $t < T_0$; accordingly, we have $y \le n$ when $s < S_0 = s(T_0)$.

From

$$\mathrm{d}t = \frac{\mathrm{d}s}{y(t(s))},$$

we have

$$t(s) = \int_{S_0}^{s} \frac{\mathrm{d}\xi}{y(\xi)} + t(S_0).$$

Take limit as $s \to -\infty$, we get

$$\lim_{s\to-\infty} t(s) = -\int_{-\infty}^{S_0} \frac{\mathrm{d}\xi}{y(\xi)} + t(S_0),$$
$$\int_{-\infty}^{S_0} \frac{\mathrm{d}\xi}{y(\xi)} \geq \int_{-\infty}^{S_0} \frac{1}{n}\mathrm{d}\xi = +\infty.$$

Hence,

$$\lim_{s\to-\infty} t = -\infty.$$

So as $s \to -\infty$, we have $xy \to n-1$.

We have the following lemma from [2, Lemma 1.33].

Lemma 4.2 (see [2, Lemma 1.33]) *We have*

$$\lim_{s\to+\infty} \frac{X}{Y^2} = \alpha,$$

where $\alpha = \frac{1}{\sqrt{n-1}}$.

Then,

$$\lim_{s\to+\infty} xy = (n-2)\sqrt{n-1} \lim_{s\to+\infty} \frac{X}{Y^2} = n-2.$$

We need to check condition (4.3).

Lemma 4.3 *We have*

$$\frac{X}{Y^2} < \frac{1}{(n-2)\alpha}$$

for $s \in \mathbb{R}$, *where* $\alpha = \frac{1}{\sqrt{n-1}}$. *Therefore,*

$$xy = \frac{X}{Y^2}(n-2)\sqrt{n-1} < n-1.$$

Proof From the ODE system, we can calculate directly

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{X}{Y^2}\Big) &= \frac{X^3 - X + \alpha Y^2}{Y^2} - \frac{2X(X^2 - \alpha X)}{Y^2} \\ &= \alpha - \frac{X}{Y^2}(X^2 - 2\alpha X + 1).\end{aligned}$$

Let

$$S = \Big\{ s \in \mathbb{R} \Big| \frac{X}{Y^2} \geq \frac{1}{(n-2)\alpha} \Big\} \subset \mathbb{R}.$$

If the set $S = \emptyset$, then we already get the conclusion. Now we consider the case that $S \neq \emptyset$. Since as $s \to -\infty$, $xy \to n-1$, we can see $\frac{X}{Y^2} \to \frac{1}{(n-2)\alpha}$. Since X, Y are smooth functions of s, we know that S consists of isolated points or finite closed intervals or $(-\infty, s_1]$ for some $s_1 > -\infty$. Let $s_0 \in \overline{S}$ be any minimal point of the finite closed intevals or one of the isolated points, then at s_0, $\frac{X}{Y^2} = \frac{1}{(n-2)\alpha}$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{X}{Y^2}\Big) &= \alpha - \frac{X}{Y^2}((X-\alpha)^2 + 1 - \alpha^2) \\ &= \alpha - \frac{1}{(n-2)\alpha}((X-\alpha)^2 + 1 - \alpha^2) \\ &= -\frac{1}{(n-2)\alpha}(X-\alpha)^2 \\ &\le 0, \end{aligned} \tag{4.5}$$

where the last step follows from the facts that X is decreasing and $X \to \alpha$ as $s \to -\infty$, so $X \le \alpha$. If $s_0 > -\infty$, then $X < \alpha$; this tells us that $\frac{X}{Y^2}$ is decreasing at s_0. Then there is an $\varepsilon > 0$ small, such that

$$\frac{X}{Y^2}(s_0 - \varepsilon) > \frac{X}{Y^2}(s_0) = \frac{1}{(n-2)\alpha}.$$

This means that $s_0 - \varepsilon \in S$ and $s_0 - \varepsilon < s_0$, which contradicts to the definition of s_0. Therefore, S does not contain closed intervals or isolated points. By Lemma 4.2, there exists $s_1 < +\infty$, such that $S = (-\infty, s_1]$. By the middle value theorem, there is $s_2 \in (-\infty, s_1)$, such that at s_2,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{X}{Y^2}\Big) = 0.$$

But from (4.5) and $X < \alpha$ at $s_2 > -\infty$, we have $\frac{\mathrm{d}}{\mathrm{d}t}\big(\frac{X}{Y^2}\big) < 0$ at s_2, which leads to a contradiction.

Then we get the conclusion.

From [2, Lemma 1.37], we know that the Bryant steady soliton has strictly positive curvature away from the origin, and the sectional curvature of the plane tangent to the radial direction is $-\frac{\phi''}{\phi}$, then we obtain that $\phi'' < 0$ and thus $(\phi')^2 - \phi\phi'' > 0$. Therefore, the Bryant steady soliton satisfies the condition (2.3) of Theorem 2.1, which implies the isoperimetric inequality (2.5) by Theorem 2.2. Theorem 1.2 is proved.

Remark 4.3 Ievy also constructed steady solitons on doubly warped product metric (see [5]). Let $(M^n, \mathrm{d}\sigma^2)$ be a compact Einstein manifold with Einstein constant $\varepsilon > 0$, and $\mathrm{d}\theta^2$ be the standard metric on $\mathbb{S}^k$. Consider the metric

$$g = \mathrm{d}r^2 + f(r)^2\mathrm{d}\theta^2 + h(r)^2\mathrm{d}\sigma^2$$

on $\mathbb{R} \times \mathbb{S}^k \times M^n$. We do not know whether the isoperimetric inequality holds in this steady soliton and this is an open question.

Acknowledgement I would like to express my gratitude to my advisor Professor Jiayu Li. He introduced me the problem and gave me so much useful suggestions. I also want to thank the anonymous referee for comments that improved this paper.

References

[1] Cant, D., A curvature flow and applications to an isoperimetric inequality, arXiv, 1610.05844, 2016.

[2] Chow, B., Chu, S.-C., Glickenstein, D., et al., The Ricci Flow: Techniques and Applications, Part I, **135**, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2007.

[3] Chow, B., Lu, P. and Ni, L., Hamilton's Ricci Flow, Graduate Studies in Mathematics, **77**, American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006.

[4] Guan, P., Li, J. and Wang, M.-T., A volume preserving flow and the iosoperimetric problem in warped product spaces, *Trans. Amer. Math. Soc.*, **372**(4), 2019, 2777–2798.

[5] Ivey, T., New example of complete Ricci solitions, *Proc. Amer. Math. Soc.*, **122**(1), 1994, 241–245.