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Diophantine Inequality by Unlike Powers of Primes

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Abstract Suppose that $\lambda_1, \cdots, \lambda_5$ are nonzero real numbers, not all of the same sign, satisfying that $\frac{\lambda_1}{\lambda_2}$ is irrational. Then for any given real number η and $\varepsilon > 0$, the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < \Big(\max_{1 \le j \le 5} p_j^j\Big)^{-\frac{19}{756}+\varepsilon}$$

has infinitely many solutions in prime variables $p_1, \cdots, p_5$. This result constitutes an improvement of the recent results.

Keywords Prime, Davenport-Heilbronn method, Diophantine inequalities
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1 Introduction

In 1953, Prachar [11] showed that for sufficiently large odd integer N, the equation

$$N = p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5 \tag{1.1}$$

is solvable in primes $p_1, \cdots, p_5$. Motivated by the work of Prachar [11], Ge and Li [2] considered the analogous form for Diophantine inequality. Let $\lambda_1, \cdots, \lambda_5$ be nonzero real numbers, not all of the same sign, satisfying that $\frac{\lambda_1}{\lambda_2}$ is irrational and let $0 \le \sigma \le \frac{1}{720}$. Ge and Li [2] proved that for any given real number η and $\varepsilon > 0$, the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < \Big(\max_{1 \le j \le 5} p_j^j\Big)^{-\sigma+\varepsilon} \tag{1.2}$$

is solvable in primes $p_1, \cdots, p_5$. In 2017, the result (1.2) was improved by Mu [9]. By employing the method in Languasco and Zaccagnini [7], Mu [9] enlarged the range of the exponent to $\sigma \le \frac{1}{180}$. Afterwards, Liu [8] further improved the result to $\sigma \le \frac{5}{288}$. Motivated by Wang and Yao [12], by combining the sieve method in Harman [4] and Harman and Kumchev [5], Mu and Qu [10] refined Liu's result and showed that (1.2) holds for $\sigma \le \frac{5}{252}$.

In this paper, by applying a new method to estimating the related integral over the minor arc (see Lemma 3.5), we are able to provide a stronger minor arc estimate and obtain the following sharper result.

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Theorem 1.1 *Suppose that $\lambda_1, \cdots, \lambda_5$ are nonzero real numbers, not all of the same sign with $\frac{\lambda_1}{\lambda_2}$ irrational. Then for any given real number η and $\varepsilon > 0$, the inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < \Big(\max_{1\le j\le 5} p_j^j\Big)^{-\frac{19}{756}+\varepsilon} \tag{1.3}$$

has infinitely many solutions in prime variables $p_1, \cdots, p_5$.

2 Outline of the Method

Throughout the paper, the letter p, with or without subscript, is reserved for a prime number. Let ε be a sufficiently small positive number and $\delta > 0$ is a small constant depending on the coefficients $\lambda_1, \cdots, \lambda_5$. We use $e(\alpha)$ to denote $\mathrm{e}^{2\pi \mathrm{i}\alpha}$. Since $\frac{\lambda_1}{\lambda_2}$ is irrational, we let $\frac{a}{q}$ be a convergent to $\frac{\lambda_1}{\lambda_2}$, with the denominator q sufficiently large. Write

$$X = q^{\frac{21}{11}}, \quad \tau = X^{-\frac{19}{756}+5\varepsilon}, \quad L = \log X, \quad I_j = [(\delta X)^{\frac{1}{j}}, X^{\frac{1}{j}}].$$

Denote

$$K_\tau(\alpha) = \begin{cases} \Big(\dfrac{\sin(\pi\tau\alpha)}{\pi\alpha}\Big)^2, & \text{if } \alpha \ne 0, \\ \tau^2, & \text{if } \alpha = 0. \end{cases}$$

Then

$$K_\tau(\alpha) \ll \min(\tau^2, |\alpha|^{-2}), \tag{2.1}$$

$$\int_{-\infty}^{+\infty} e(yx) K_\tau(x)\mathrm{d}x = \max(0, \tau - |y|). \tag{2.2}$$

We borrow the function $\rho(m)$ defined in [5, (5.2)]. Set

$$\psi(m, z) = \begin{cases} 1, & \text{if } p|m \Rightarrow p \ge z, \\ 0, & \text{otherwise}, \end{cases}$$

$$z(p) = \begin{cases} X^{\frac{5}{28}} p^{-\frac{1}{2}}, & \text{if } p < X^{\frac{1}{7}}, \\ p, & \text{if } X^{\frac{1}{7}} \le p \le X^{\frac{3}{14}}, \\ X^{\frac{5}{14}} p^{-1}, & \text{if } X^{\frac{3}{14}} < p. \end{cases}$$

The function $\rho(m)$ takes the form

$$\rho(m) = \psi(m, X^{\frac{5}{42}}) - \sum_{X^{\frac{5}{42}} \le p < X^{\frac{1}{4}}} \psi\Big(\frac{m}{p}, z(p)\Big). \tag{2.3}$$

For $m \le X^{\frac{1}{2}}$, it follows from the construction of $\rho(m)$ that (see [5, (2.3)])

$$\rho(m) \le \begin{cases} 1, & \text{if } m \text{ is prime}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

Let

$$S_2^*(\alpha)=\sum_{m\in I_2}\rho(m)e(m^2\alpha),\quad S_j(\alpha)=\sum_{p\in I_j}e(p^j\alpha)\log p. \tag{2.5}$$

For any measurable subset $\mathfrak{X}$ of $\mathbb{R}$, write

$$I(\tau,\eta,\mathfrak{X})=\int_{\mathfrak{X}}S_1(\lambda_1\alpha)S_2^*(\lambda_2\alpha)\prod_{3\le j\le 5}S_j(\lambda_j\alpha)K_\tau(\alpha)e(\alpha\eta)\mathrm{d}\alpha. \tag{2.6}$$

From (2.2) and (2.4), we have

$$\begin{aligned}I(\tau,\eta,\mathbb{R})&=\sum_{\substack{p_1\in I_1,m_2\in I_2\\ p_j\in I_j,\,3\le j\le 5}}\rho(m_2)\prod_{\substack{1\le j\le 5\\ j\neq 2}}\log p_j\\&\quad\times\int_{\mathbb{R}}e\Big(\Big(\lambda_1p_1+\lambda_2m_2^2+\sum_{j=3}^{5}\lambda_jp_j^j+\eta\Big)\alpha\Big)K_\tau(\alpha)\mathrm{d}\alpha\\&=\sum_{\substack{p_1\in I_1,m_2\in I_2\\ p_j\in I_j,\,3\le j\le 5}}\rho(m_2)\prod_{\substack{1\le j\le 5\\ j\neq 2}}\log p_j\\&\quad\times\max\Big(0,\tau-\Big|\lambda_1p_1+\lambda_2m_2^2+\sum_{j=3}^{5}\lambda_jp_j^j+\eta\Big|\Big)\\&\le L^4\sum_{\substack{p_j\in I_j\\ 1\le j\le 5}}\max(0,\tau-|\lambda_1p_1+\lambda_2p_2^2+\lambda_3p_3^3+\lambda_4p_4^4+\lambda_5p_5^5+\eta|)\\&\le \tau N_\tau(X)L^4,\end{aligned} \tag{2.7}$$

where $N_\tau(X)$ counts the number of solutions of the inequality

$$|\lambda_1p_1+\lambda_2p_2^2+\lambda_3p_3^3+\lambda_4p_4^4+\lambda_5p_5^5+\eta|<\tau \tag{2.8}$$

with $p_j\in I_j$. Let $\phi=X^{-\frac{1}{8}}$ and $\xi=\tau^{-2}X^{\frac{1}{80}+2\varepsilon}$. We divide the real line into three parts

$$\mathfrak{M}=\{\alpha:|\alpha|\le\phi\},\quad \mathfrak{m}=\{\alpha:\phi<|\alpha|\le\xi\},\quad \mathfrak{t}=\{\alpha:|\alpha|>\xi\}. \tag{2.9}$$

These sets are called the major arc $\mathfrak{M}$, the minor arc $\mathfrak{m}$ and the trivial arc $\mathfrak{t}$, respectively. Thus

$$I(\tau,\eta,\mathbb{R})=I(\tau,\eta,\mathfrak{M})+I(\tau,\eta,\mathfrak{m})+I(\tau,\eta,\mathfrak{t}). \tag{2.10}$$

Following the argument of [10, (3.26) and (5.3)], we can get

$$I(\tau,\eta,\mathfrak{M})\gg\tau^2X^{\frac{77}{60}}L^{-1},\quad |I(\tau,\eta,\mathfrak{t})|=o(\tau^2X^{\frac{77}{60}}L^{-1}). \tag{2.11}$$

In the following, we will prove

$$|I(\tau,\eta,\mathfrak{m})|\ll\tau^2X^{\frac{77}{60}-\varepsilon}. \tag{2.12}$$

3 Some Auxiliary Lemmas

In this section, we collect some auxiliary results required in the proof of Theorem 1.1.

Lemma 3.1 *Let*

$$T(\alpha) \in \{S_1(\lambda_1\alpha)^2,\ S_3(\lambda_3\alpha)^8,\ S_2^*(\lambda_2\alpha)^2 S_5(\lambda_5\alpha)^6,\ S_2^*(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^4,$$
$$S_2^*(\lambda_2\alpha)^2 S_3(\lambda_3\alpha)^4,\ S_2^*(\lambda_2\alpha)^2 S_3(\lambda_3\alpha)^2 S_5(\lambda_5\alpha)^2\}.$$

Then we have

$$\int_{-\infty}^{+\infty} |T(\alpha)| K_\tau(\alpha) \mathrm{d}\alpha \ll \tau X^{-1} T(0)^{1+\varepsilon}. \tag{3.1}$$

Proof It follows easily from [10, Lemma 3.7].

Lemma 3.2 *Suppose that* $X \ge Z_1 \ge X^{\frac{5}{6}+2\varepsilon},\ X^{\frac{1}{2}} \ge Z_2 \ge X^{\frac{3}{7}+2\varepsilon},\ X^{\frac{1}{3}} \ge Z_3 \ge X^{\frac{11}{36}+2\varepsilon}$ *and* $|S_1(\lambda_1\alpha)| > Z_1,\ |S_2^*(\lambda_2\alpha)| > Z_2,\ |S_3(\lambda_3\alpha)| > Z_3$. *Then there are integers* $a_1, q_1,\ a_2, q_2$ *and* a_3, q_3 *satisfying*

$$(a_i, q_i) = 1, \quad q_i \ll \Big(\frac{X^{\frac{1}{i}+\varepsilon}}{Z_i}\Big)^2, \quad |q_i\lambda_i\alpha - a_i| \ll X^{-1}\Big(\frac{X^{\frac{1}{i}+\varepsilon}}{Z_i}\Big)^2, \quad i = 1, 2, 3. \tag{3.2}$$

Proof For $i = 1$, see [8, Lemma 2.1]. For $i = 3$, see [3, Corollary 2.2]. We prove the case for $i = 2$. By Dirichlet's theorem, there exist co-prime integers $a_2,\ q_2$, such that

$$1 \le q_2 \le X\Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^{-4}, \quad |q_2\lambda_2\alpha - a_2| \ll X^{-1}\Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^4. \tag{3.3}$$

It follows from [10, (4.7)] that

$$|S_2^*(\lambda_2\alpha)| \ll X^{\frac{3}{7}+\frac{1}{2}\varepsilon} + X^{\frac{1}{2}+\frac{1}{2}\varepsilon}\Big(\frac{1}{q_2} + \frac{q_2}{X}\Big)^{\frac{1}{4}}. \tag{3.4}$$

Since $|S_2^*(\lambda_2\alpha)| > Z_2 \ge X^{\frac{3}{7}+2\varepsilon}$ and $X^{\frac{3}{7}+\frac{1}{2}\varepsilon} + X^{\frac{1}{4}+\frac{1}{2}\varepsilon} q_2^{\frac{1}{4}} \ll Z_2 X^{-\frac{1}{2}\varepsilon}$, we have

$$Z_2 < |S_2^*(\lambda_2\alpha)| \ll X^{\frac{1}{2}+\frac{1}{2}\varepsilon} q_2^{-\frac{1}{4}}.$$

Hence

$$q_2 \ll \Big(\frac{X^{\frac{1}{2}+\frac{1}{2}\varepsilon}}{Z_2}\Big)^4, \quad |q_2\lambda_2\alpha - a_2| \ll X^{-1}\Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^4. \tag{3.5}$$

Then we can deduce from the proof of [5, Lemma 1] with $Q = X^{\frac{1}{7}-\varepsilon}$ that (or see [6, Lemma 5.6] with $z = X^{\frac{3}{28}}$)

$$\begin{aligned} Z_2 < |S_2^*(\lambda_2\alpha)| &\ll \frac{X^{\frac{1}{2}+\varepsilon}}{(q_2 + X|q_2\lambda_2\alpha - a_2|)^{\frac{1}{2}}} + X^{\frac{1}{7}+\frac{11}{40}+\varepsilon} + X^{\frac{11}{28}+\varepsilon} \\ &\ll \frac{X^{\frac{1}{2}+\varepsilon}}{(q_2 + X|q_2\lambda_2\alpha - a_2|)^{\frac{1}{2}}}. \end{aligned} \tag{3.6}$$

Therefore, by (3.6), we can obtain

$$q_2 \ll \Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^2, \quad |q_2\lambda_2\alpha - a_2| \ll X^{-1}\Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^2.$$

Lemma 3.3 *Suppose that* $|S_3(\lambda_3\alpha)| > X^{\frac{11}{36}+2\varepsilon}$ *and there exist integers* a *and* q *satisfying*

$$(a, q) = 1, \quad q \ll X^{\frac{1}{18}}, \quad |q\lambda_3\alpha - a| \ll X^{-\frac{17}{18}}. \tag{3.7}$$

Then we have

$$|S_3(\lambda_3\alpha)| \ll \frac{X^{\frac{1}{3}+\varepsilon}}{(q + X|q\lambda_3\alpha - a|)^{\frac{1}{2}}}. \tag{3.8}$$

Proof It follows easily from [3, Lemma 2.1].

Lemma 3.4 *Write*

$$\mathfrak{N}_1 = \{\alpha : \alpha \in \mathfrak{m},\ X^{\frac{11}{36}+2\varepsilon} < |S_3(\lambda_3\alpha)|\}.$$

Then we have

$$\int_{\mathfrak{N}_1} |S_3(\lambda_3\alpha)|^2 |S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\mathrm{d}\alpha \ll \tau X^{\frac{1}{6}+4\varepsilon}.$$

Proof We first note that $K_\tau(\frac{\alpha}{\lambda_3}) = \lambda_3^2 K_{\frac{\tau}{\lambda_3}}(\alpha)$. Hence

$$\int_{\mathfrak{N}_1} |S_3(\lambda_3\alpha)|^2 |S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\mathrm{d}\alpha = \lambda_3 \int_{\mathfrak{N}_2} \Big| S_3(\alpha)^2 S_4\Big(\frac{\lambda_4}{\lambda_3}\alpha\Big)^2 \Big| K_{\frac{\tau}{\lambda_3}}(\alpha)\mathrm{d}\alpha, \tag{3.9}$$

where

$$\mathfrak{N}_2 = \Big\{\alpha : \frac{\alpha}{\lambda_3} \in \mathfrak{m},\ X^{\frac{11}{36}+2\varepsilon} < |S_3(\alpha)|\Big\}. \tag{3.10}$$

Denote

$$\mathfrak{N}^*(n) = \bigcup_{1 \le q \le X^{\frac{1}{18}}} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \Big(n + \frac{a}{q} - \frac{1}{qX^{\frac{17}{18}}}, n + \frac{a}{q} + \frac{1}{qX^{\frac{17}{18}}}\Big],$$

$$\mathfrak{N}^* = \bigcup_{n=-\infty}^{+\infty} \mathfrak{N}^*(n). \tag{3.11}$$

Let $V(\alpha)$ be the function of period 1 and defined for $\alpha \in [0, 1)$ by

$$V(\alpha) = \begin{cases} (q + X|q\alpha - a|)^{-1}, & \alpha \in \mathfrak{N}^* \bigcap [0, 1), \\ 0, & \alpha \in [0, 1) \setminus \mathfrak{N}^*. \end{cases}$$

Applying Lemma 3.2 with $Z_3 = X^{\frac{11}{36}+2\varepsilon}$, we get

$$\mathfrak{N}_2 \subseteq \mathfrak{N}^*. \tag{3.12}$$

Then we can deduce from Lemma 3.3 and (3.12) that

$$|S_3(\alpha)| \ll X^{\frac{1}{3}+\varepsilon} V^{\frac{1}{2}}(\alpha) \quad \text{for } \alpha \in \mathfrak{N}_2. \tag{3.13}$$

Write

$$\psi(v)=\sum_{\substack{p_1^4-p_2^4=v\\ (\delta X)^{\frac14}\le p_1,p_2\le X^{\frac14}}}\log p_1\log p_2,$$

$$\Psi(\alpha)=\Big|S_4\Big(\frac{\lambda_4}{\lambda_3}\alpha\Big)^2\Big|=\sum_v\psi(v)e\Big(\frac{\lambda_4}{\lambda_3}v\alpha\Big). \tag{3.14}$$

From (3.12)–(3.14), we have

$$\begin{aligned}&\int_{\mathfrak{N}_2}\Big|S_3(\alpha)^2S_4\Big(\frac{\lambda_4}{\lambda_3}\alpha\Big)^2\Big|K_{\frac{\tau}{\lambda_3}}(\alpha)\mathrm{d}\alpha\\ &\ll\int_{\mathfrak{N}_2}X^{\frac23+2\varepsilon}V(\alpha)\Psi(\alpha)K_{\frac{\tau}{\lambda_3}}(\alpha)\mathrm{d}\alpha\\ &\ll X^{\frac23+2\varepsilon}\int_{\mathfrak{N}^*}V(\alpha)\Psi(\alpha)K_{\frac{\tau}{\lambda_3}}(\alpha)\mathrm{d}\alpha.\end{aligned} \tag{3.15}$$

Applying [1, Lemma 3] with $Q=X^{\frac{1}{18}}$, we get

$$\begin{aligned}&\int_{\mathfrak{N}^*}V(\alpha)\Psi(\alpha)K_{\frac{\tau}{\lambda_3}}(\alpha)\mathrm{d}\alpha\\ &\ll\tau X^{\varepsilon}(1+\tau)^{1+\varepsilon}X^{-1}\Big(\sum_v|\psi(v)|+X^{\frac{1}{18}}\sum_{|\frac{\lambda_4}{\lambda_3}v|\le\frac{\tau}{|\lambda_3|}}|\psi(v)|\Big)\\ &\ll\tau X^{-1+\varepsilon}\Big(\sum_v|\psi(v)|+X^{\frac{1}{18}}\sum_{|v|\le\frac{\tau}{|\lambda_4|}}|\psi(v)|\Big).\end{aligned} \tag{3.16}$$

Since $\tau=X^{-\frac{19}{756}+5\varepsilon}$, we have

$$\sum_{|v|\le\frac{\tau}{|\lambda_4|}}|\psi(v)|\le\sum_{\substack{|p_1^4-p_2^4|\le\frac{\tau}{|\lambda_4|}\\ p_1,p_2\le X^{\frac14}}}L^2=\sum_{\substack{p_1=p_2\\ p_1,p_2\le X^{\frac14}}}L^2\le X^{\frac14+\varepsilon}. \tag{3.17}$$

Moreover, it is easy to find that

$$\sum_v|\psi(v)|=\Psi(0)\le X^{\frac12}L^2. \tag{3.18}$$

Now combining (3.9) and (3.15)–(3.18), we obtain

$$\int_{\mathfrak{N}_1}|S_3(\lambda_3\alpha)^2S_4(\lambda_4\alpha)^2|K_\tau(\alpha)\mathrm{d}\alpha\ll\tau X^{\frac16+4\varepsilon}. \tag{3.19}$$

Lemma 3.5 *Let*

$$\mathfrak{m}_3=\{\alpha\in\mathfrak{m}:\ |S_2^*(\lambda_2\alpha)|\le X^{\frac37+2\varepsilon},\ |S_3(\lambda_3\alpha)|\le X^{\frac{11}{36}+2\varepsilon}\}$$

and

$$J_k=\int_{\mathfrak{m}_3}|S_2^*(\lambda_2\alpha)^2S_3(\lambda_3\alpha)^k|K_\tau(\alpha)\mathrm{d}\alpha.$$

Then we have

$$J_{12}\ll\tau X^{\frac{233}{63}+23\varepsilon}. \tag{3.20}$$

Proof Write $G_k(\alpha) = |S_2^*(\lambda_2\alpha)^2 S_3(\lambda_3\alpha)^{k-2}|$. We have

$$\begin{aligned}
J_k &= \int_{\mathfrak{m}_3} S_3(\lambda_3\alpha)S_3(-\lambda_3\alpha)G_k(\alpha)K_\tau(\alpha)\mathrm{d}\alpha \\
&= \sum_{p\in I_3}(\log p)\int_{\mathfrak{m}_3} e(\alpha\lambda_3 p^3)S_3(-\lambda_3\alpha)G_k(\alpha)K_\tau(\alpha)\mathrm{d}\alpha \\
&\le \sum_{p\in I_3}(\log p)\Big|\int_{\mathfrak{m}_3} e(\alpha\lambda_3 p^3)S_3(-\lambda_3\alpha)G_k(\alpha)K_\tau(\alpha)\mathrm{d}\alpha\Big| \\
&\le \sum_{n\in I_3}\Big|\int_{\mathfrak{m}_3} e(\alpha\lambda_3 n^3)S_3(-\lambda_3\alpha)G_k(\alpha)K_\tau(\alpha)\mathrm{d}\alpha\Big|L. \qquad (3.21)
\end{aligned}$$

By Cauchy's inequality and the obvious facts $G_k(\alpha) = G_k(-\alpha)$, $K_\tau(\alpha) = K_\tau(-\alpha)$, we can get

$$\begin{aligned}
J_k &\ll X^{\frac{1}{6}}L\Big(\sum_{n\in I_3}\Big|\int_{\mathfrak{m}_3} e(\alpha\lambda_3 n^3)S_3(-\lambda_3\alpha)G_k(\alpha)K_\tau(\alpha)\mathrm{d}\alpha\Big|^2\Big)^{\frac{1}{2}} \\
&\ll X^{\frac{1}{6}}L\Big(\int_{\mathfrak{m}_3} S_3(\lambda_3\beta)G_k(\beta)K_\tau(\beta) \\
&\quad\times\Big(\int_{\mathfrak{m}_3} S_3(-\lambda_3\alpha)G_k(\alpha)K_\tau(\alpha)\sum_{n\in I_3} e(\lambda_3 n^3(\alpha-\beta))\mathrm{d}\alpha\Big)\mathrm{d}\beta\Big)^{\frac{1}{2}} \\
&\ll X^{\frac{1}{6}}L\Big(\int_{\mathfrak{m}_3} |S_3(\lambda_3\beta)G_k(\beta)|K_\tau(\beta)F(\beta)\mathrm{d}\beta\Big)^{\frac{1}{2}}, \qquad (3.22)
\end{aligned}$$

where

$$F(\beta) = \int_{\mathfrak{m}_3}\Big|S_3(-\lambda_3\alpha)G_k(\alpha)K_\tau(\alpha)\sum_{n\in I_3} e(\lambda_3 n^3(\alpha-\beta))\Big|\mathrm{d}\alpha. \qquad (3.23)$$

From [3, (7.6)–(7.11)], we obtain

$$\begin{aligned}
F(\beta) &\ll \tau^{\frac{1}{2}}X^{\frac{1}{6}+\varepsilon}\Big(\int_{\mathfrak{m}_3}|G_k(\alpha)|^2K_\tau(\alpha)\mathrm{d}\alpha\Big)^{\frac{1}{2}} \\
&\quad + X^{\frac{1}{4}+\varepsilon}\int_{\mathfrak{m}_3}|G_k(\alpha)S_3(\lambda_3\alpha)|K_\tau(\alpha)\mathrm{d}\alpha. \qquad (3.24)
\end{aligned}$$

Hence, by (3.22) and (3.24), we have

$$\begin{aligned}
J_k &\ll \tau^{\frac{1}{4}}X^{\frac{1}{4}+\varepsilon}\Big(\int_{\mathfrak{m}_3}|G_k(\alpha)|^2K_\tau(\alpha)\mathrm{d}\alpha\Big)^{\frac{1}{4}}\Big(\int_{\mathfrak{m}_3}|G_k(\alpha)S_3(\lambda_3\alpha)|K_\tau(\alpha)\mathrm{d}\alpha\Big)^{\frac{1}{2}} \\
&\quad + X^{\frac{7}{24}+\varepsilon}\int_{\mathfrak{m}_3}|G_k(\alpha)S_3(\lambda_3\alpha)|K_\tau(\alpha)\mathrm{d}\alpha. \qquad (3.25)
\end{aligned}$$

Note that

$$J_{k-2} = \int_{\mathfrak{m}_3}|G_k(\alpha)|K_\tau(\alpha)\mathrm{d}\alpha. \qquad (3.26)$$

Then we can deduce from Cauchy's inequality that

$$\int_{\mathfrak{m}_3}|G_k(\alpha)S_3(\lambda_3\alpha)|K_\tau(\alpha)\mathrm{d}\alpha$$

$$\ll \Big(\int_{\mathfrak{m}_3} |G_k(\alpha)S_3(\lambda_3\alpha)^2|K_\tau(\alpha)\mathrm{d}\alpha\Big)^{\frac{1}{2}}\Big(\int_{\mathfrak{m}_3} |G_k(\alpha)|K_\tau(\alpha)\mathrm{d}\alpha\Big)^{\frac{1}{2}}$$
$$= J_k^{\frac{1}{2}} J_{k-2}^{\frac{1}{2}}. \tag{3.27}$$

Since

$$\max_{\alpha\in\mathfrak{m}_3} |S_2^*(\lambda_2\alpha)| \le X^{\frac{3}{7}+2\varepsilon}$$

and

$$\max_{\alpha\in\mathfrak{m}_3} |S_3(\lambda_3\alpha)| \le X^{\frac{11}{36}+2\varepsilon},$$

we have

$$\begin{aligned}\int_{\mathfrak{m}_3} |G_k(\alpha)|^2 K_\tau(\alpha)\mathrm{d}\alpha &= \int_{\mathfrak{m}_3} |S_2^*(\lambda_2\alpha)|^4 |S_3(\lambda_3\alpha)|^{2k-4} K_\tau(\alpha)\mathrm{d}\alpha \\ &\ll \max_{\alpha\in\mathfrak{m}_3} |S_2^*(\lambda_2\alpha)^2 S_3(\lambda_3\alpha)^{k-4}| J_k \\ &\ll X^{\frac{6}{7}+\frac{11(k-4)}{36}+2k\varepsilon} J_k. \end{aligned} \tag{3.28}$$

Combining (3.25) and (3.27)–(3.28), we conclude that

$$\begin{aligned} J_k \ll\, & \tau^{\frac{1}{4}} X^{\frac{1}{4}+\varepsilon} (X^{\frac{6}{7}+\frac{11(k-4)}{36}+2k\varepsilon} J_k)^{\frac{1}{4}} (J_k^{\frac{1}{2}} J_{k-2}^{\frac{1}{2}})^{\frac{1}{2}} \\ & + X^{\frac{7}{24}+\varepsilon} J_k^{\frac{1}{2}} J_{k-2}^{\frac{1}{2}}. \end{aligned} \tag{3.29}$$

This yields that

$$J_k \ll \tau^{\frac{1}{2}} J_{k-2}^{\frac{1}{2}} X^{\frac{13}{14}+\frac{11(k-4)}{72}+(k+2)\varepsilon} + X^{\frac{7}{12}+2\varepsilon} J_{k-2}. \tag{3.30}$$

Moreover, for $k = 4$, we can deduce from Lemma 3.1 that

$$J_4 \ll \int_{-\infty}^{+\infty} |S_2^*(\lambda_2\alpha)^2 S_3(\lambda_3\alpha)^4| K_\tau(\alpha)\mathrm{d}\alpha \ll \tau X^{\frac{4}{3}+\varepsilon}. \tag{3.31}$$

Applying (3.30) and (3.31), we can obtain

$$\begin{aligned} J_6 &\ll \tau^{\frac{1}{2}} (\tau X^{\frac{4}{3}+\varepsilon})^{\frac{1}{2}} X^{\frac{13}{14}+\frac{11}{36}+8\varepsilon} + X^{\frac{7}{12}+2\varepsilon}(\tau X^{\frac{4}{3}+\varepsilon}) \ll \tau X^{\frac{23}{12}+3\varepsilon}, \\ J_8 &\ll \tau^{\frac{1}{2}} (\tau X^{\frac{23}{12}+3\varepsilon})^{\frac{1}{2}} X^{\frac{13}{14}+\frac{11}{18}+10\varepsilon} + X^{\frac{7}{12}+2\varepsilon}(\tau X^{\frac{23}{12}+2\varepsilon}) \ll \tau X^{\frac{5}{2}+12\varepsilon}, \\ J_{10} &\ll \tau^{\frac{1}{2}} (\tau X^{\frac{5}{2}+12\varepsilon})^{\frac{1}{2}} X^{\frac{13}{14}+\frac{11}{12}+12\varepsilon} + X^{\frac{7}{12}+2\varepsilon}(\tau X^{\frac{5}{2}+12\varepsilon}) \ll \tau X^{\frac{65}{21}+18\varepsilon}, \\ J_{12} &\ll \tau^{\frac{1}{2}} (\tau X^{\frac{65}{21}+18\varepsilon})^{\frac{1}{2}} X^{\frac{13}{14}+\frac{11}{9}+14\varepsilon} + X^{\frac{7}{12}+2\varepsilon}(\tau X^{\frac{65}{21}+18\varepsilon}) \ll \tau X^{\frac{233}{63}+23\varepsilon}. \end{aligned}$$

4 The Minor Arc $\mathfrak{m}$

Now we come to estimate $I(\tau,\eta,\mathfrak{m})$. We first introduce a detailed division of the minor arc $\mathfrak{m}$. Let

$$\begin{aligned} \mathfrak{m}_1 &= \{\alpha \in \mathfrak{m} :\ |S_1(\lambda_1\alpha)| \le X^{\frac{5}{6}+2\varepsilon}\}, \\ \mathfrak{m}_2 &= \{\alpha \in \mathfrak{m} :\ |S_2^*(\lambda_2\alpha)| \le X^{\frac{3}{7}+2\varepsilon},\ |S_3(\lambda_3\alpha)| > X^{\frac{11}{36}+2\varepsilon}\}, \end{aligned}$$

$$\mathfrak{m}_3 = \{\alpha \in \mathfrak{m} : \ |S_2^*(\lambda_2\alpha)| \le X^{\frac{3}{7}+2\varepsilon}, \ |S_3(\lambda_3\alpha)| \le X^{\frac{11}{36}+2\varepsilon}\},$$
$$\mathfrak{m}_4 = \mathfrak{m} \setminus (\mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \mathfrak{m}_3).$$

Thus

$$|I(\tau,\eta,\mathfrak{m})| \le \sum_{1\le i\le 4} |I(\tau,\eta,\mathfrak{m}_i)|. \tag{4.1}$$

Applying Hölder's inequality and Lemma 3.1, we have

$$\begin{aligned}
&|I(\tau,\eta,\mathfrak{m}_1)| \\
&\ll \max_{\alpha\in\mathfrak{m}_1} |S_1(\lambda_1\alpha)|^{\frac{3}{16}} \Big(\int_{-\infty}^{+\infty} |S_2^*(\lambda_2\alpha)^2 S_3(\lambda_3\alpha)^2 S_5(\lambda_5\alpha)^2| K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{1}{8}} \\
&\quad \times \Big(\int_{-\infty}^{+\infty} |S_3(\lambda_3\alpha)|^8 K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{3}{32}} \Big(\int_{-\infty}^{+\infty} |S_2^*(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^4| K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{1}{4}} \\
&\quad \times \Big(\int_{-\infty}^{+\infty} |S_2^*(\lambda_2\alpha)^2 S_5(\lambda_5\alpha)^6| K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{1}{8}} \Big(\int_{-\infty}^{+\infty} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{13}{32}} \\
&\ll \tau X^{\frac{5}{32}+\frac{2}{15}+\frac{5}{32}+\frac{1}{4}+\frac{3}{20}+\frac{13}{32}+2\varepsilon} = \tau X^{\frac{601}{480}+2\varepsilon}.
\end{aligned} \tag{4.2}$$

Let $\mathfrak{N}_1$ be defined as in Lemma 3.4. It is easy to see that $\mathfrak{m}_2 \subseteq \mathfrak{N}_1$. Then we can deduce from Cauchy's inequality, Lemmas 3.1 and 3.4 that

$$\begin{aligned}
&|I(\tau,\eta,\mathfrak{m}_2)| \\
&\ll X^{\frac{1}{5}} \max_{\alpha\in\mathfrak{m}_2} |S_2^*(\lambda_2\alpha)| \Big(\int_{-\infty}^{+\infty} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{1}{2}} \\
&\quad \times \Big(\int_{\mathfrak{N}_1} |S_3(\lambda_3\alpha)|^2 |S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{1}{2}} \\
&\ll \tau X^{\frac{1}{5}+\frac{3}{7}+\frac{1}{2}+\frac{1}{12}+5\varepsilon} = \tau X^{\frac{509}{420}+5\varepsilon}.
\end{aligned} \tag{4.3}$$

Moreover, by Hölder's inequality, Lemmas 3.1 and 3.5, we can get

$$\begin{aligned}
&|I(\tau,\eta,\mathfrak{m}_3)| \\
&\ll \Big(\int_{\mathfrak{m}_3} |S_2^*(\lambda_2\alpha)^2 S_3(\lambda_3\alpha)^{12}| K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{1}{12}} \Big(\int_{-\infty}^{+\infty} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{1}{2}} \\
&\quad \times \Big(\int_{-\infty}^{+\infty} |S_2^*(\lambda_2\alpha)^2 S_5(\lambda_5\alpha)^6| K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{1}{6}} \Big(\int_{-\infty}^{+\infty} |S_2^*(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^4| K_\tau(\alpha)\mathrm{d}\alpha \Big)^{\frac{1}{4}} \\
&\ll \tau X^{\frac{233}{756}+\frac{1}{2}+\frac{1}{5}+\frac{1}{4}+3\varepsilon} = \tau X^{\frac{1189}{945}+3\varepsilon}.
\end{aligned} \tag{4.4}$$

Remark 4.1 We remark that the constraint on the choice $\tau = X^{-\frac{19}{756}+5\varepsilon}$ arises from (4.4). In view of (2.12), the estimate in (4.4) should not exceed $O(\tau^2 X^{\frac{77}{60}-\varepsilon})$. Hence this leads to the constraint

$$\tau \gg X^{-\frac{19}{756}+5\varepsilon}.$$

In the following, we consider the range $\mathfrak{m}_4 = \mathfrak{m} \setminus \big(\mathfrak{m}_1 \bigcup \mathfrak{m}_2 \bigcup \mathfrak{m}_3\big)$. Note that for $\alpha \in \mathfrak{m}_4$, we have

$$|S_1(\lambda_1\alpha)| > X^{\frac{5}{6}+2\varepsilon}, \quad |S_2^*(\lambda_2\alpha)| > X^{\frac{3}{7}+2\varepsilon}.$$

So we can divide $\mathfrak{m}_4$ into disjoint sets $S(Z_1, Z_2, y)$ such that for $\alpha \in S(Z_1, Z_2, y)$, we have

$$Z_1 < |S_1(\lambda_1\alpha)| \le 2Z_1, \quad Z_2 < |S_2^*(\lambda_2\alpha)| \le 2Z_2, \quad y < |\alpha| \le 2y, \tag{4.5}$$

where $Z_1 = 2^{t_1} X^{\frac{5}{6}+2\varepsilon}$, $Z_2 = 2^{t_2} X^{\frac{3}{7}+2\varepsilon}$ and $y = 2^r X^{-\frac{1}{8}}$ for some positive integers t_1, t_2 and r. Thus by Lemma 3.2, there are co-prime integers $(a_1, q_1), (a_2, q_2)$ satisfying

$$\begin{aligned} q_1 &\ll \Big(\frac{X^{1+\varepsilon}}{Z_1}\Big)^2, \quad |q_1\lambda_1\alpha - a_1| \ll X^{-1}\Big(\frac{X^{1+\varepsilon}}{Z_1}\Big)^2, \\ q_2 &\ll \Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^2, \quad |q_2\lambda_2\alpha - a_2| \ll X^{-1}\Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^2. \end{aligned} \tag{4.6}$$

We remark that $a_1a_2 \ne 0$, since otherwise we have $\alpha \in \mathfrak{M}$. Furthermore, we subdivide $S(Z_1, Z_2, y)$ into sets $S(Z_1, Z_2, y, Q_1, Q_2)$, where $Q_j < q_j \le 2Q_j$ on each set. Then

$$\begin{aligned} \Big|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\Big| &= \Big|\frac{a_2(q_1\lambda_1\alpha - a_1) + a_1(a_2 - q_2\lambda_2\alpha)}{\lambda_2\alpha}\Big| \\ &\ll Q_2X^{-1}\Big(\frac{X^{1+\varepsilon}}{Z_1}\Big)^2 + Q_1X^{-1}\Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^2 \\ &\ll \frac{X^{2+4\varepsilon}}{Z_1^2Z_2^2} \ll X^{-\frac{11}{21}-\varepsilon}. \end{aligned} \tag{4.7}$$

Note that $q = X^{\frac{11}{21}}$. Thus

$$\Big|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\Big| = o(q^{-1}). \tag{4.8}$$

We also have

$$|a_2q_1| \ll yQ_1Q_2. \tag{4.9}$$

Hence, if $|a_2q_1|$ take R distinct values, we could deduce the existence of n satisfying

$$\Big\|n\frac{\lambda_1}{\lambda_2}\Big\| \ll X^{-\frac{11}{21}-\varepsilon}, \quad n \ll \frac{yQ_1Q_2}{R}. \tag{4.10}$$

This would contradict $\frac{a}{q}$ being a convergent to $\frac{\lambda_1}{\lambda_2}$ if q is sufficiently large, unless

$$R \ll \frac{yQ_1Q_2}{q}. \tag{4.11}$$

By (4.7) and the well-known bound on the divisor function, we find that each value of a_2q_1 corresponds to $O(X^\varepsilon)$ values of a_2, q_1 and a_1, q_2. Then we obtain that each set of $S(Z_1, Z_2, y, Q_1, Q_2)$ is made up of $O(RX^\varepsilon)$ intervals of length

$$\min\Big(\frac{1}{Q_1X}\Big(\frac{X^{1+\varepsilon}}{Z_1}\Big)^2, \ \frac{1}{Q_2X}\Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^2\Big). \tag{4.12}$$

Let $\mathfrak{L}$ denote such a set $S(Z_1, Z_2, y, Q_1, Q_2)$. We have

$$\int_{\mathfrak{L}} 1\mathrm{d}\alpha \ll yQ_1Q_2q^{-1} \min\Big(\frac{1}{Q_1X}\Big(\frac{X^{1+\varepsilon}}{Z_1}\Big)^2,\ \frac{1}{Q_2X}\Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^2\Big)$$
$$\ll \frac{yX^{2+4\varepsilon}}{qZ_1^2Z_2^2}. \tag{4.13}$$

Recall that $\tau = X^{-\frac{19}{756}+5\varepsilon}$, $y \ll \xi = \tau^{-2}X^{\frac{1}{80}+2\varepsilon}$, $q = X^{\frac{11}{21}}$, $Z_2 \gg X^{\frac{3}{7}+2\varepsilon}$, $Z_1 \gg X^{\frac{5}{6}+2\varepsilon}$ and $K_\tau(\alpha) \ll \tau^2$. Then we can deduce from (4.5) and (4.13) that

$$|I(\tau, \eta, \mathfrak{L})| \ll \tau^2 Z_1Z_2X^{\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}\Big(\int_{\mathfrak{L}} 1\mathrm{d}\alpha\Big)$$
$$\ll \tau^2\frac{yX^{2+\frac{47}{60}+4\varepsilon}}{qZ_1Z_2} \ll \tau^2X^{\frac{16033}{15120}+\varepsilon}. \tag{4.14}$$

Summing over all possible values of y, Q_1, Q_2, Z_1, Z_2, we get

$$|I(\tau, \eta, \mathfrak{m}_4)| \ll |I(\tau, \eta, \mathfrak{L})|L^5 \ll \tau^2X^{\frac{16033}{15120}+2\varepsilon}. \tag{4.15}$$

Now combining (4.1)–(4.4) and (4.15), we have

$$|I(\tau, \eta, \mathfrak{m})| \ll \tau X^{\frac{601}{480}+2\varepsilon} + \tau X^{\frac{509}{420}+5\varepsilon} + \tau X^{\frac{1189}{945}+3\varepsilon} + \tau^2X^{\frac{16033}{15120}+2\varepsilon} \ll \tau^2X^{\frac{77}{60}-\varepsilon}. \tag{4.16}$$

5 Proof of Theorem 1.1

Combining (2.7), (2.10)–(2.11) and (4.16), we can conclude that

$$I(\tau, \eta, \mathbb{R}) \gg \tau^2X^{\frac{77}{60}}L^{-1}, \quad N_\tau(X) \gg \tau X^{\frac{77}{60}}L^{-5}. \tag{5.1}$$

Since $\frac{\lambda_1}{\lambda_2}$ is irrational, there are infinitely many pairs of co-prime integers q and a such that $\frac{a}{q}$ is convergent to $\frac{\lambda_1}{\lambda_2}$. Then we have $X = q^{\frac{21}{11}} \to +\infty$ as $q \to +\infty$. This implies that (5.1) holds for infinite sequence of values X. Thus the proof of the theorem is completed.

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