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Local Unstable Entropy and Local Unstable Pressure for Partially Hyperbolic Endomorphisms*

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Abstract In this paper, local unstable metric entropy, local unstable topological entropy and local unstable pressure for partially hyperbolic endomorphisms are introduced and investigated. Specially, two variational principles concerning relationships among the above mentioned numbers are formulated.

Keywords Partially hyperbolic endomorphism, Local unstable metric entropy, Local unstable topological entropy, Local unstable pressure, Variational principle
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1 Introduction

Entropy including metric entropy and topological entropy plays an important role in dynamical systems, which describes the complexity of a given dynamical system from different points of view. It is well known that there is a variational principle connecting metric entropy and topological entropy, which says that for a given system (X,T), where X is a compact topological space and $T\colon X\to X$ is a surjective and continuous map, its topological entropy is equal to the supremum of all metric entropies over all invariant probability measures with respect to T. As a generalization of entropy, pressure with respect to a potential function is introduced, and a similar variational principle can also be established. The reader can refer to [7] for more details concerning entropy theory.

In order to obtain more information from a dynamical system, various versions of entropy are introduced, among which local entropy including local metric entropy and local topological entropy is an important one. Correspondingly, related variational principle is formulated. In [14], given a T-invariant measure μ, for a given open cover $\mathcal{U}$ of X, Romagnoli introduced two types of metric entropies with respect to $\mathcal{U}$: $h_\mu(T,\mathcal{U})$ and $h_\mu^+(T,\mathcal{U})$ with $h_\mu(T,\mathcal{U})\le h_\mu^+(T,\mathcal{U})$, and gave the following local variational principle:

$$h_{\rm top}(T,\mathcal{U})=\sup_{\mu\text{ is }T\text{-invariant}}\{h_\mu(T,\mathcal{U})\},$$

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where $h_{\text{top}}(T,\mathcal{U})$ is the local topological entropy with respect to $\mathcal{U}$. In addition, if T is invertible, Glasner and Weiss [2] proved that the above variational principle also holds for $h_\mu^+(T,\mathcal{U})$. For more results concerning local entropies, the reader can refer to [1, 5–6]. Specially, in [6], for a given factor map $\pi : X \to Y$ between two topological dynamical systems, a Borel cover $\mathcal{U}$ and an invariant Borel probability measure μ, two notions of measure-theoretical conditional entropy $h_\mu^+(T,\mathcal{U} \mid Y)$ and $h_\mu^-(T,\mathcal{U} \mid Y)$ were introduced. And the authors showed that $h_\mu^+(T,\mathcal{U} \mid Y) = h_\mu^-(T,\mathcal{U} \mid Y)$. Moreover, $\max\limits_{\mu} h_\mu^+(T,\mathcal{U} \mid Y) = h_{\text{top}}(T,\mathcal{U} \mid Y)$ when $\mathcal{U}$ is an open cover. Then as a consequence of the above results, the relative variational principle was given.

Equipping X with additional structure, e.g. Riemannian structure, we can establish more results concerning entropies. Due to the differential structure, we can require that T is C^r ($r \geq 1$). And we often require that T is equipped with some hyperbolicity, e.g. T is a uniformly hyperbolic diffeomorphism or a partially hyperbolic diffeomorphism. Under the above assumptions, the unstable manifold can be introduced. In Ledrappier and Young's papers (see [8–9]), a kind of metric entropy defined via increasing measurable partitions subordinate to unstable manifolds was introduced, which is suitable for developing the relationship between Lyapunov exponents and metric entropy. A new type of entropy including metric and topological versions, focusing on the expansive part of a dynamical system, was introduced by Hu, Hua and Wu in [3] for C^1 partially hyperbolic diffeomorphisms, which is called unstable entropy. The unstable metric entropy was given by a finite partition α and a measurable partition η subordinate to unstable manifolds in the form $\lim\limits_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha|\eta)$, where for two measurable partitions β and γ of X, $H(\beta|\gamma)$ is the conditional entropy of β with respect to γ. It is important to point out that, for establishing a variational principle between unstable metric entropy and unstable topological entropy, the latter form in [3] is easier to take advantage of. In [3], the authors also gave the relationship between unstable metric entropy and Ledrappier-Young's entropy, which implies that the complexity of a partially hyperbolic system is caused by the expansive part. In [4], unstable pressure was introduced, and a variational principle for unstable pressure was also formulated.

In [19], the concepts of unstable entropies and unstable pressure were generalized to local case for partially hyperbolic diffeomorphisms, which bring us a new point of view to investigate the complexity of a partially hyperbolic dynamical system. Variational principles for local unstable topological entropy and local unstable pressure were obtained respectively. Note that in order to give the above variational principles, unstable topological conditional entropy and unstable tail entropy were introduced, which play crucial roles in the proofs.

Noticing that plenty of physical processes are irreversible, in addition, the evolution law dependents on time, some counterparts of the above objects are considered for noninvertible map via preimage structure (see [20]) and some of the above results are generalized to random case (see [16, 18]). It is interesting to investigate corresponding results as in [3–4, 19]. In [17], the authors introduced unstable entropies and unstable pressure for partially hyperbolic endomorphisms, and obtained a corresponding variational principle. The main purpose of this paper is to introduce local unstable entropies and local unstable pressure for partially hyperbolic endomorphisms. However, for endomorphisms, there are some difficulties to establish similar results. Due to non-invertibility, the notion of unstable manifolds is not well defined, in order to overcome this difficulty, in [22], Zhu introduced the inverse limit space (see Section 2, for details), which makes it possible to define the unstable manifolds and borrow some ideas from the smooth ergodic theory of random dynamical systems. Moreover, some techniques and results in [17] can be applied.

This paper is organized as follows. In Section 2, we give some basic knowledge necessary for our goal and state our main results. In Section 3, we give the definitions of two kinds of local unstable metric entropies, and some properties of these two local entropies and relations between them are also obtained. In Section 4, we give the definition of local unstable topological entropy with some important properties of them. In Section 5, we give the definitions of unstable topological conditional entropy and unstable tail entropy and their relations with local unstable entropies and unstable entropies, which are crucial to the proofs of our variational principles. In Section 6, we give the proofs of the variational principles for both local entropies and local pressure.

2 Preliminaries and Main Results

Throughout this paper, let M be a C^∞ Riemannian manifold without boundary endowed with metric $d(\cdot,\cdot)$ and $f: M\to M$ be a C^1 endomorphism. Denote TM the tangent bundle of M with norm $\|\cdot\|$. Both $d(\cdot,\cdot)$ and $\|\cdot\|$ are induced by the Riemannian metric.

For a metric space X, denote $\mathcal{B}(X)$ the Borel σ-algebra of X. Let $M^{\mathbb{Z}}$ be the infinite product space of M endowed with the product topology and the metric $\widetilde{d}(\widetilde{x},\widetilde{y}) = \sum\limits_{n=-\infty}^{\infty} 2^{-|n|} d(x_n,y_n)$ for $\widetilde{x}=\{x_n\}_{n=-\infty}^{\infty}$ and $\widetilde{y}=\{y_n\}_{n=-\infty}^{\infty}$. In order to define unstable manifolds, we need the concept of inverse limit space denoted by M^f, which means it is a subspace of the product space $M^{\mathbb{Z}}$, and $fx_n = x_{n+1}$, $n\in\mathbb{Z}$, for $\widetilde{x}=\{x_n\}_{n=-\infty}^{+\infty}\in M^f$. It is clear that M^f is a closed subspace of $M^{\mathbb{Z}}$. Let $\Pi\colon M^f\to M$ be the projection such that for $\widetilde{x}=\{x_n\}_{n=-\infty}^{+\infty}$, $\Pi(\widetilde{x})=x_0$. Let $\tau\colon M^f\to M^f$ be the left shift operator.

Consider the pull back bundle $E=\Pi^*TM$. The tangent map Df on TM induces a fiber preserving map on E with respect to the left shift operator τ, defined by $\Pi^*\circ Df\circ\Pi_*$, which is still denoted by Df for simplicity.

Now, we give the definition of partial hyperbolicity.

Definition 2.1 *f is said to be (uniformly) partially hyperbolic if there exists a continuous splitting of the pull back bundle E into three subbundles, i.e., for any $\widetilde{x}\in M^f$, $E(\widetilde{x}) = E^s(\widetilde{x})\oplus E^c(\widetilde{x})\oplus E^u(\widetilde{x})$ and constants λ_1, λ_1', λ_2, ${\lambda_2}'$ and C with $0<\lambda_1<1<\lambda_2$, $\lambda_1<{\lambda_1}'\le{\lambda_2}'<\lambda_2$ and $C>0$ such that for each $\widetilde{x}\in M^f$,*

(i) *$D_{\widetilde{x}}f(E^k(\widetilde{x})) = E^k(\tau(\widetilde{x}))$, for $k=s,c,u$;*
(ii) *for $v^s\in E^s(\widetilde{x})$ and $n\in\mathbb{Z}^+$, $\|D_{\widetilde{x}}f^n v^s\|\le C{\lambda_1}^n\|v^s\|$;*
(iii) *for $v^c\in E^c(\widetilde{x})$ and $n\in\mathbb{Z}^+$, $C^{-1}({\lambda_1}')^n\|v^c\|\le\|D_{\widetilde{x}}f^n v^c\|\le C({\lambda_2}')^n\|v^c\|$;*
(iv) *for $v^u\in E^u(\widetilde{x})$ and $n\in\mathbb{Z}^+$, $\|D_{\widetilde{x}}f^n v^u\|\ge C^{-1}{\lambda_2}^n\|v^u\|$.*

Denote $\mathcal{M}(f)$ the set of all f-invariant Borel measures on M, and denote $\mathcal{M}(\tau)$ the set of all τ-invariant measures on M^f. On one hand, for any $\mu\in\mathcal{M}(f)$, there is a unique τ-invariant measure $\widetilde{\mu}$ on M^f corresponding to μ with $\Pi(\widetilde{\mu})=\mu$ (see [12, Proposition I.3.1]); on the other hand, for any $\widetilde{\mu} \in \mathcal{M}(\tau)$, $\mu := \Pi(\widetilde{\mu})$ is an f-invariant measure on M. In addition, μ is ergodic with respect to f if and only if $\widetilde{\mu}$ is ergodic with respect to τ. For more details on the relationship between $\mathcal{M}(f)$ and $\mathcal{M}(\tau)$, the reader can refer to [12, I.3]. In the remaining of this paper, we always denote μ and $\widetilde{\mu}$ the measures on M and M^f, respectively, with $\Pi(\widetilde{\mu}) = \mu$.

From now on, let (f, M, μ) be a dynamical system, where f is a partially hyperbolic endomorphism, and μ is an f-invariant Borel measure. Let $\widetilde{\mu}$ be the corresponding measure on M^f.

For $\widetilde{x} = \{x_n\}_{n=-\infty}^{\infty} \in M^f$ and $\epsilon > 0$ small enough, define

$$\begin{aligned} W_\epsilon^u(\widetilde{x}, f) := \Big\{ & z_0 \in M \colon \text{There exists } \widetilde{z} \in M^f \text{ with } \Pi(\widetilde{z}) = z_0, \\ & d(z_{-n}, x_{-n}) < \epsilon \text{ for } n \in \mathbb{N} \text{ and } \limsup_{n\to\infty} \frac{1}{n} \log d(z_{-n}, x_{-n}) \le -\log \lambda_2 \Big\}, \end{aligned}$$

where λ_2 is the constant in Definition 2.1. $W_\epsilon^u(\widetilde{x}, f)$ is called a local unstable manifold of f at $\widetilde{x}$. Now we have the following theorem, which is stated for hyperbolic endomorphisms, while it is still valid for our partially hyperbolic case. The reader can also refer to [11, 15, 21] for more details.

Theorem 2.1 (see [12, Theorem IV.2.1]) *Let f be a partially hyperbolic endomorphism. Then there exist a continuous family of C^1 embedded disks $\{D^u_{\widetilde{x}}\}_{\widetilde{x}\in M^f}$ in M and constants $0 < \lambda < 1$ and $\epsilon > 0$ such that*

(i) *$T_{x_0} D^u_{\widetilde{x}} = E^u(x_0)$, for any $\widetilde{x} \in M^f$;*

(ii) *for any $z_0 \in D^u_{\widetilde{x}}$, there exists unique $\widetilde{z} \in M^f$ such that $\Pi(\widetilde{z}) = z_0$ and*

$$d(z_{-n}, x_{-n}) \le \lambda^n d(z_0, x_0) \tag{2.1}$$

for $n \in \mathbb{Z}^+$;

(iii) *$D^u_{\widetilde{x}} \cap B(x_0, \epsilon) = W_\epsilon^u(\widetilde{x}, f)$, where $B(x_0, \epsilon) = \{y \in M \colon d(y, x) < \epsilon\}$.*

Then we can define

$$\widetilde{W}_\epsilon^u(\widetilde{x}, f) := \{\widetilde{z} \in M^f \colon \Pi(\widetilde{z}) \in W_\epsilon^u(\widetilde{x}, f) \text{ and } \widetilde{z} \text{ satisfies (2.1)}\}.$$

Sometimes, we will use the notation $W^u_{\rm loc}(\widetilde{x}, f)$ and $\widetilde{W}^u_{\rm loc}(\widetilde{x}, f)$ for $W_\epsilon^u(\widetilde{x}, f)$ and $\widetilde{W}_\epsilon^u(\widetilde{x}, f)$ respectively.

Remark 2.1 According to Theorem 2.1, it is clear that

$$\Pi|_{\widetilde{W}^u_{\rm loc}(\widetilde{x}, f)} \colon \widetilde{W}^u_{\rm loc}(\widetilde{x}, f) \to W^u_{\rm loc}(\widetilde{x}, f)$$

is a bijection, which is crucial for our subsequent proofs.

Now we define

$$\begin{aligned} W^u(\widetilde{x}, f) = \Big\{ & z_0 \in M \colon \text{There exists } \widetilde{z} \text{ with } \Pi(\widetilde{z}) = z_0 \\ & \text{and } \limsup_{n\to+\infty} \frac{1}{n} d(z_{-n}, x_{-n}) \le -\log \lambda_2 \Big\} \end{aligned}$$

and

$$\widetilde{W}^u(\widetilde{x}, f) = \Big\{\widetilde{z} \in M^f : \Pi(\widetilde{z}) \in W^u(\widetilde{x}, f) \text{ with } \limsup_{n\to+\infty} \frac{1}{n} d(z_{-n}, x_{-n}) \le -\log\lambda_2\Big\},$$

where λ_2 is the constant in Definition 2.1. We call $W^u(\widetilde{x}, f)$ the global unstable set at $\widetilde{x}$. It can also be proved as in [22] that there exists a sequence of C^1 embedded disks $\{W_{-n}(\widetilde{x})\}_{n=0}^{+\infty}$ in M such that $fW_{-n}(\widetilde{x}) \supset W_{-(n-1)}(\widetilde{x})$ for $n \in \mathbb{Z}^+$ and

$$W^u(\widetilde{x}, f) = \bigcup_{n=0}^{+\infty} f^n W_{-n}(\widetilde{x}),$$

which shows that $W^u(\widetilde{x}, f)$ is in fact an immersed submanifold of M tangent at $\Pi(\widetilde{x})$ to $E^u(\Pi(\widetilde{x}))$. Then we denote the set $\{W^u(\widetilde{x}, f) \colon \widetilde{x} \in M^f\}$ by W^u, which is called W^u-foliation.

A family of sets $\mathcal{U} = \{U_i\}_{i\in I}$ is called a cover of M^f, if it satisfies

$$\bigcup_{i\in I} U_i \supset M^f,$$

where I is an index set. Denote by $\mathcal{U}(\widetilde{x})$ the element of $\mathcal{U}$ containing $\widetilde{x}$. It is clear that $\Pi(\mathcal{U}) := \{\Pi(U_i)\}_{i\in I}$ is a cover of M. $\mathcal{U}$ is called a Borel cover, if $\Pi(\mathcal{U})$ is a Borel cover of M. Specially, if $\Pi(\mathcal{U})$ is an open cover of $M, \mathcal{U}$ is called an open cover. A cover is said to be finite, if I is a finite set. Denote by $\mathcal{C}_{M^f}$ and $\mathcal{C}^o_{M^f}$ the set of finite Borel covers and the set of finite open covers respectively.

For a Borel cover $\mathcal{U}$ of M^f, define $\mathrm{diam}(\mathcal{U})$ as follows

$$\mathrm{diam}(\mathcal{U}) := \mathrm{diam}(\Pi(\mathcal{U})) = \max_{U\in\mathcal{U}} \mathrm{diam}(\Pi(U)),$$

where $\mathrm{diam}(\Pi(U)) = \sup\limits_{x,y\in\Pi(U)} d(x, y)$.

It is clear that a measurable partition α of M^f can be regarded as a Borel cover of M^f, and $\Pi(\alpha)$ is a Borel cover of M.

Now we give some definitions related to measurable partitions.

Definition 2.2 *A measurable partition η of M^f is said to be subordinate to W^u-foliation if for $\widetilde{\mu}$-a.e. $\widetilde{x}$, $\eta(\widetilde{x})$ has the following properties*:

(i) $\Pi|_{\eta(\widetilde{x})} \colon \eta(\widetilde{x}) \to \Pi(\eta(\widetilde{x}))$ *is bijective*;

(ii) *there exists a $k(\widetilde{x})$-dimensional* (*where $k(\widetilde{x}) = \dim E^u(x_0)$*) *$C^1$ embedded submanifold $W_{\widetilde{x}}$ of M with $W_{\widetilde{x}} \subset W^u(\widetilde{x})$, such that $\Pi(\eta(\widetilde{x})) \subset W_{\widetilde{x}}$, and $\Pi(\eta(\widetilde{x}))$ contains an open neighborhood of x_0 in $W_{\widetilde{x}}$.*

Given $\widetilde{\mu} \in \mathcal{M}(\tau)$. For a measurable partition η of M^f, there exists a canonical system $\{\widetilde{\mu}^{\eta}_{\widetilde{x}}\}_{\widetilde{x}\in M^f}$ of conditional measures of $\widetilde{\mu}$ associated with η, satisfying

(i) for every measurable set $\widetilde{B} \in M^f$, $\widetilde{x} \mapsto \widetilde{\mu}^{\eta}_{\widetilde{x}}(\widetilde{B})$ is measurable;

(ii) $\widetilde{\mu}(\widetilde{B}) = \int_{M^f} \widetilde{\mu}^{\eta}_{\widetilde{x}}(\widetilde{B})\mathrm{d}\widetilde{\mu}(\widetilde{x})$.

See [13] for more details.

For two covers α and β of M^f, $\alpha \ge \beta$ means for any element $A \in \alpha$, there is an element $B \in \beta$ such that $A \subset B$. In the following, we consider a special type of measurable partitions.

Definition 2.3 *A measurable partition ξ of M^f is said to be increasing if $\tau^{-1}\xi \geq \xi$.*

Consider a measurable partition $\xi = \{A_i\}_{i\in I}$ of M^f. A measurable set B is called a ξ-set if $B = \bigcup_{A\in I'} A$, where $I' \subset I$. Denote $\mathcal{B}(\xi)$ the σ-algebra of M^f consisting of all measurable ξ-sets. Given $\widetilde{\mu} \in \mathcal{M}(\tau)$, define

$$\mathcal{B}^u := \{B \in \mathcal{B}_{\widetilde{\mu}}(M^f) \colon \widetilde{x} \in B \text{ implies } \widetilde{W}^u(\widetilde{x}) \subset B\},$$

where $\mathcal{B}_{\widetilde{\mu}}(M^f)$ is the completion of $\mathcal{B}(M^f)$ with respect to $\widetilde{\mu}$.

The following proposition ensures the existence of increasing measurable partitions, the reader can see [12, Section IX.2.2] for details.

Proposition 2.1 *There exists a measurable partition ξ of M^f which has the following properties*:

(i) $\tau^{-1}\xi \geq \xi$;

(ii) $\bigvee_{n=0}^{\infty} \tau^{-n}\xi$ *is equal to the partition into single points*;

(iii) $\mathcal{B}(\bigwedge_{n=0}^{\infty} \tau^{n}(\xi)) = \mathcal{B}^u$, $\widetilde{\mu}$-mod 0;

(iv) ξ *is subordinate to W^u-foliation of f.*

We denote by $\mathcal{Q}^u(M^f)$ the set of all increasing measurable partitions subordinate to W^u-foliation as in Proposition 2.1.

Define $W^u(\widetilde{x}, \delta) = \{y \in W^u(\widetilde{x}) \colon d^u_{\widetilde{x}}(y, x) \leq \delta\}$, where $d^u_{\widetilde{x}}(\cdot,\cdot)$ is the distance along $W^u(\widetilde{x}, f)$. We can choose $\epsilon_1 > 0$ small enough and $C_0 > 1$ such that $d(\cdot,\cdot) \leq d^u_{\widetilde{x}}(\cdot,\cdot) \leq C_0 d(\cdot,\cdot)$ on any local unstable manifold $W^u(\widetilde{x}, \epsilon_1)$.

Choose and fix a variable λ_0 such that $\lambda_0 > \|D_{\Pi(\widetilde{x})} f|_{E^u(\widetilde{x})}\| > 1$ for any $\widetilde{x} \in M^f$. Choose $L > 0$ and $0 < \epsilon_0 \ll \min\{\epsilon_1, L\}$ such that

(i) for any $\widetilde{x} \in M^f$, $W^u(\widetilde{x}, L) \cap B(\Pi(\widetilde{x}), \epsilon_0)$ has only one connected component.

(ii) $\lambda_0 C_0 \epsilon_0 \ll L$.

Let $\mathcal{P}(M^f)$ denote the set of all finite Borel partitions α of M^f with $\operatorname{diam}(\alpha) < \epsilon_0$. For a partition $\alpha \in \mathcal{P}(M^f)$, adapting method in [3], we can construct $\alpha^u \geq \alpha$ satisfying $\alpha^u(\widetilde{x}) = \alpha(\widetilde{x}) \cap \widetilde{W}^u(\widetilde{x}, \overline{\epsilon}_0)$ for any $\widetilde{x} \in M^f$, where $\widetilde{W}^u(\widetilde{x}, \rho) = \{\widetilde{y} \in \widetilde{W}^u_{\mathrm{loc}}(\widetilde{x}) \colon \Pi(\widetilde{y}) \in W^u(\widetilde{x}, \rho)\}$ and $C_0\epsilon_0 < \overline{\epsilon}_0 < L$.

Denote by $\mathcal{P}^u(M^f)$ the set of all partitions constructed by above method.

Remark 2.2 By the definition of $\widetilde{W}^u_\epsilon(\widetilde{x}, f)$ and Theorem 2.1, if $\mu(\partial(\Pi(\alpha))) = 0$, η is a measurable partition subordinate to W^u-foliation, where $\partial(\Pi(\alpha)) = \bigcup_{A\in\alpha} \partial(\Pi(A))$ and for $B \subset M$, ∂B means the boundary of B.

Given $\mathcal{U} \in \mathcal{C}^o_{M^f}$. As generalizations of local unstable metric entropy and local unstable topological entropy in [19], we introduce corresponding notions for endomorphisms, denote them by $h^u_\mu(f, \mathcal{U}|\zeta)$, $h^{u,+}_\mu(f, \mathcal{U}|\zeta)$ (see Section 3) and $h^u_{\mathrm{top}}(f, \mathcal{U})$ (see Section 4), respectively, where $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$.

Now we can give our main results as follows.

Theorem A *Let f be a partially hyperbolic endomorphism, and $\mathcal{U} \in \mathcal{C}^o_{M^f}$ with small enough* *diameter. Then for any* $\mu \in \mathcal{M}(f)$ *and* $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$, *we have*

$$h_\mu^u(f,\mathcal{U}|\zeta) = h_\mu^{u,+}(f,\mathcal{U}|\zeta) = h_\mu^u(f|\zeta) = h_\mu^u(f)$$

and

$$h_{\text{top}}^u(f,\mathcal{U}) = h_{\text{top}}^u(f),$$

where for the definition of $h_\mu^u(f|\zeta)$ *and* $h_\mu^u(f)$, *see Definition 3.2,* $h_{\text{top}}^u(f)$ *is the unstable topological entropy, for more details, we refer the reader to the paper* [17].

As a corollary of [17, Theorems A and D], we have the following theorem.

Theorem B *Let* f *be a partially hyperbolic endomorphism, and* $\mathcal{U} \in \mathcal{C}_{M^f}^o$ *with small enough diameter. Then for any* $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$, *we have*

$$h_{\text{top}}^u(f,\mathcal{U}) = \sup_{\mu \in \mathcal{M}(f)} h_\mu^u(f,\mathcal{U}|\zeta) = \sup_{\mu \in \mathcal{M}(f)} h_\mu^{u,+}(f,\mathcal{U}|\zeta).$$

We can generalize above results to the unstable pressure for endomorphisms. Firstly, we have the following theorem. Denote the set

$$\{\phi\colon M \to \mathbb{R}\colon \phi \text{ is continuous}\}$$

by $C(M)$.

Theorem C *Let* f *be a partially hyperbolic endomorphism, and* $\mathcal{U} \in \mathcal{C}_{M^f}^o$ *with small enough diameter. Then for any* $\phi \in C(M)$, *we have*

$$P^u(f,\phi,\mathcal{U}) = P^u(f,\phi).$$

Applying [17, Theorem C], we can obtain a variational principle as follows.

Theorem D *Let* f *be a partially hyperbolic endomorphism, and* $\mathcal{U} \in \mathcal{C}_{M^f}^o$ *with small enough diameter. Then for any* $\phi \in C(M)$, *we have*

$$\begin{aligned} &P^u(f,\phi,\mathcal{U}) \\ =& \sup_{\mu \in \mathcal{M}(f)} \Big\{h_\mu^u(f,\mathcal{U}|\zeta) + \int_M \phi \mathrm{d}\mu\Big\} \\ =& \sup_{\mu \in \mathcal{M}(f)} \Big\{h_\mu^{u,+}(f,\mathcal{U}|\zeta) + \int_M \phi \mathrm{d}\mu\Big\}. \end{aligned}$$

3 Local Unstable Metric Entropy

In this section, we give the precise definition of local unstable metric entropy for a partially hyperbolic endomorphism.

Firstly we give some knowledge on the information function, which is slightly modified in our context.

Definition 3.1 *Let* α *and* η *be two measurable partitions of* M^f. *The information function of* α *with respect to* $\widetilde{\mu}$ *is defined as*

$$I_{\widetilde{\mu}}(\alpha)(\widetilde{x}) := -\log \widetilde{\mu}(\alpha(\widetilde{x})),$$

and the entropy of α *with respect to* $\widetilde{\mu}$ *is defined as*

$$H_{\widetilde{\mu}}(\alpha) := \int_{M^f} I_{\widetilde{\mu}}(\alpha)(\widetilde{x})\mathrm{d}\widetilde{\mu}(\widetilde{x}) = -\int_{M^f} \log \widetilde{\mu}(\alpha(\widetilde{x}))\mathrm{d}\widetilde{\mu}(\widetilde{x}).$$

The conditional information function of α *with respect to* η *is defined as*

$$I_{\widetilde{\mu}}(\alpha|\eta)(\widetilde{x}) := -\log \widetilde{\mu}^{\eta}_{\widetilde{x}}(\alpha(\widetilde{x})),$$

where $\{\widetilde{\mu}^{\eta}_{\widetilde{x}}\}_{\widetilde{x}\in M^f}$ *is a canonical system of conditional measures of* $\widetilde{\mu}$ *with respect to* η*. Then the conditional entropy of* α *with respect to* η *is defined as*

$$H_{\widetilde{\mu}}(\alpha|\eta) := \int_{M^f} I_{\widetilde{\mu}}(\alpha|\eta)(\widetilde{x})\mathrm{d}\widetilde{\mu}(\widetilde{x}) = -\int_{M^f} \log \widetilde{\mu}^{\eta}_{\widetilde{x}}(\alpha(\widetilde{x}))\mathrm{d}\widetilde{\mu}(\widetilde{x}).$$

Now we can give the definition of unstable metric entropy by finite partitions.

Definition 3.2 *The conditional entropy of* f *for a finite measurable partition* α *of* M^f *with respect to* $\eta \in \mathcal{P}^u(M^f)$ *is defined as*

$$h_\mu(f, \alpha|\eta) = \limsup_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta).$$

The conditional entropy of f *with respect to* η *is defined as*

$$h_\mu(f|\eta) = \sup_{\alpha\in\mathcal{P}(M^f)} h_\mu(f, \alpha|\eta),$$

and the conditional entropy of f *along* W^u*-foliation is defined as*

$$h^u_\mu(f) = \sup_{\eta\in\mathcal{P}^u(M^f)} h_\mu(f|\eta).$$

In the following, we give some useful conclusions from [17].

Lemma 3.1 (see [17, Proposition 2.14]) *For any* $\alpha \in \mathcal{P}(M^f)$ *and* $\eta \in \mathcal{P}^u(M^f)$*, the map* $\widetilde{\mu} \mapsto H_{\widetilde{\mu}}(\alpha|\eta)$ *from* $\mathcal{M}(\tau)$ *to* $\mathbb{R}^+ \cup \{0\}$ *is concave. Moreover, the map* $\widetilde{\mu} \mapsto h^u_\mu(f)$ *from* $\mathcal{M}(\tau)$ *to* $\mathbb{R}^+ \cup \{0\}$ *is affine.*

Lemma 3.2 (see [17, Proposition 2.15]) *Let* $\widetilde{\mu} \in \mathcal{M}(\tau)$ *and* $\eta \in \mathcal{P}^u(M^f)$*. Assume that there exists a sequence of partitions* $\{\beta_n\}_{n=1}^{\infty} \subset \mathcal{P}(M^f)$ *such that* $\beta_1 < \beta_2 < \cdots < \beta_n < \cdots$ *and* $\mathscr{B}(\beta_n) \nearrow \mathscr{B}(\eta)$*, and moreover,* $\mu(\partial(\Pi(\beta_n))) = 0$*, for* $n = 1, 2, \cdots$*. Let* $\alpha \in \mathcal{P}(M^f)$ *satisfy* $\mu(\partial(\Pi(\alpha))) = 0$*. Then the function* $\widetilde{\mu}' \mapsto H_{\widetilde{\mu}'}(\alpha|\eta)$ *is upper semi-continuous at* $\widetilde{\mu}$*, i.e.,*

$$\limsup_{\widetilde{\mu}'\to\widetilde{\mu}} H_{\widetilde{\mu}'}(\alpha|\eta) \le H_{\widetilde{\mu}}(\alpha|\eta).$$

Moreover, the function $\widetilde{\mu}' \mapsto h^u_{\mu'}(f)$ *is upper semi-continuous at* $\widetilde{\mu}$*, i.e.,*

$$\limsup_{\widetilde{\mu}'\to\widetilde{\mu}} h^u_{\mu'}(f) \le h^u_\mu(f).$$

Lemma 3.3 (see [17, Corollary 3.5]) *Suppose that* $\widetilde{\mu} \in \mathcal{M}(\tau)$ *is ergodic, then for any* $\alpha \in \mathcal{P}(M^f)$ *and* $\eta \in \mathcal{P}^u(M^f)$*, we have*

$$h^u_\mu(f) = h_\mu(f, \alpha|\eta) = \lim_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta).$$

We denote $\bigvee_{j=m}^{n} \tau^{-j}\alpha$ by α_m^n, for any $m \le n$, $m, n \in \mathbb{Z} \cup \{\pm\infty\}$. Given $\mu \in \mathcal{M}(f)$, now, we give the definition of $h_\mu^{u,+}(f,\mathcal{U}|\zeta)$. Note that we can still define $h_\mu(f,\alpha|\xi)$ for $\xi \in \mathcal{Q}^u(M^f)$ as in Definition 3.2 with η replaced by ξ.

Definition 3.3 *For any $\mathcal{U} \in \mathcal{C}_{M^f}$, and $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$, define*

$$h_\mu^{u,+}(f,\mathcal{U}|\zeta) = \inf_{\alpha \in \mathcal{P}(M^f), \alpha \ge \mathcal{U}} h_\mu(f,\alpha|\zeta).$$

In order to study properties of $h_\mu^{u,+}(f,\mathcal{U}|\zeta)$, which are crucial to the proofs of our main results, we give some lemmas as follows.

Lemma 3.4 *Let $\alpha \in \mathcal{P}(M^f)$ with diameter smaller than $\frac{\epsilon_0}{\lambda_0}$, and $\eta \in \mathcal{P}^u(M^f)$. Then for any $n \in \mathbb{N}$, we have*

$$\alpha_{-n}^{-1} \vee \tau^n\eta \ge (\tau\alpha)^u.$$

Proof Given $\widetilde{x} \in M^f$, let $\widetilde{y} \in (\alpha_{-n}^{-1} \vee \tau^n\eta)(\widetilde{x})$. Thus, we have $\tau^{-n}(\widetilde{y}) \in \eta(\tau^{-n}(\widetilde{x}))$, which implies that

$$d^u_{\tau^{-n}(\widetilde{x})}(\Pi(\tau^{-n}(\widetilde{y})), \Pi(\tau^{-n}(\widetilde{x}))) < C_0\epsilon_0.$$

Thus, we get

$$d^u_{\tau^{-(n-1)}(\widetilde{x})}(\Pi(\tau^{-(n-1)}(\widetilde{y})), \Pi(\tau^{-(n-1)}(\widetilde{x}))) < C_0\epsilon_0\lambda_0 \ll L.$$

For the choice of $\widetilde{y}$, we have $\Pi(\tau^{-(n-1)}(\widetilde{y})) \in \Pi(\alpha(\tau^{-(n-1)}(\widetilde{x})))$, which implies that

$$d(\Pi(\tau^{-(n-1)}(\widetilde{y})), \Pi(\tau^{-(n-1)}(\widetilde{x}))) < \frac{\epsilon_0}{\lambda_0}.$$

As ϵ_0 is small enough, we have

$$d^u_{\tau^{-(n-1)}(\widetilde{x})}(\Pi(\tau^{-(n-1)}(\widetilde{y})), \Pi(\tau^{-(n-1)}(\widetilde{x}))) < \frac{C_0\epsilon_0}{\lambda_0}.$$

Then, we can obtain

$$d^u_{\tau^{-1}(\widetilde{x})}(\Pi(\tau^{-1}(\widetilde{y})), \Pi(\tau^{-1}(\widetilde{x}))) < \frac{C_0\epsilon_0}{\lambda_0}$$

by induction. Recall that f is uniformly expanding on W^u, we have $d^u_{\widetilde{x}}(\Pi(\widetilde{x}), \Pi(\widetilde{y})) < C_0\epsilon_0 < \overline{\epsilon}_0$, which implies that $\widetilde{y} \in \widetilde{W}^u(\widetilde{x}, \overline{\epsilon}_0)$. Noticing that $\widetilde{y} \in (\tau\alpha)(\widetilde{x})$, we have $\widetilde{y} \in (\tau\alpha)^u(\widetilde{x})$, hence we have

$$(\alpha_{-n}^{-1} \vee (\tau^n\eta)) \ge (\tau\alpha)^u.$$

Because of the arbitrariness of $\widetilde{y}$, we obtain the result we need.

Lemma 3.5 *For any $\eta \in \mathcal{P}^u(M^f)$ and $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$, $H_{\widetilde{\mu}}(\eta|\zeta)$ is finite.*

Proof Let $\beta \in \mathcal{P}(M^f)$ such that $\eta = \beta^u$ satisfying $\alpha^u(\widetilde{x}) = \alpha(\widetilde{x}) \cap \widetilde{W}^u(\widetilde{x}, \overline{\epsilon}_0)$ for any $\widetilde{x} \in M^f$, then

$$\begin{aligned} H_{\widetilde{\mu}}(\eta|\zeta) &= -\int_{M^f} \log \mu^\zeta_{\widetilde{x}}(\eta(\widetilde{x}))\mathrm{d}\widetilde{\mu}(\widetilde{x}) \\ &= -\int_{M^f} \log \mu^\zeta_{\widetilde{x}}(\beta(\widetilde{x}) \cap \widetilde{W}^u(\widetilde{x}, \overline{\epsilon}_0))\mathrm{d}\widetilde{\mu}(\widetilde{x}) \end{aligned}$$

$$
\begin{aligned}
&= -\int_{M^f} \log \mu^{\zeta}_{\widetilde{x}}(\beta(\widetilde{x}))\mathrm{d}\widetilde{\mu}(\widetilde{x}) \\
&= H_{\widetilde{\mu}}(\beta|\zeta) \le H_{\widetilde{\mu}}(\beta) < \infty,
\end{aligned}
$$

the last inequality is due to the definition of fiberwise finite partition.

Lemma 3.6 *$h_\mu(f,\alpha|\eta) = \lim_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta)$ exists for any $\eta \in \mathcal{P}^u(M^f)$ and any $\alpha \in \mathcal{P}(M^f)$ with* $\mathrm{diam}(\alpha) < \frac{\epsilon_0}{\lambda_0}$.

Proof Let $\beta \in \mathcal{P}$ such that $\eta = \beta^u$. Then we have

$$
\begin{aligned}
H_{\widetilde{\mu}}(\alpha_0^{m+n-1}|\eta) &= H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\tau^{-n}\alpha_0^{m-1}|\alpha_0^{n-1} \vee \eta) \\
&= H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\alpha_0^{m-1}|\alpha_{-n}^{-1} \vee \tau^n \eta) \\
&\le H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\alpha_0^{m-1}|(\tau\alpha)^u) \\
&\le H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\alpha_0^{m-1}|\eta) + H_{\widetilde{\mu}}(\eta|(\tau\alpha)^u),
\end{aligned}
$$

in the third inequality, Lemma 3.4 is used. The diameter of $(\tau\alpha)^u(\widetilde{x})$ with respect to $d^u_{\widetilde{x}}$ is no more than $C_0\epsilon_0$, which implies that $(\tau\alpha)^u(\widetilde{x}) \subset \widetilde{W}^u(\widetilde{x}, \overline{\epsilon}_0)$, hence by Lemma 3.5, we have

$$
H_{\widetilde{\mu}}(\eta|(\tau\alpha)^u) \le H_{\widetilde{\mu}}(\beta).
$$

Then we have

$$
H_{\widetilde{\mu}}(\alpha_0^{m+n-1}|\eta) \le H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\alpha_0^{m-1}|\eta) + H_{\widetilde{\mu}}(\beta),
$$

which means that the sequence $\{H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\beta)\}$ is a subadditive sequence. So we have

$$
\lim_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) = \lim_{n\to\infty} \frac{1}{n}(H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\beta)) = \inf_{n\in\mathbb{N}} \frac{1}{n}(H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\beta)).
$$

Lemma 3.7 *Let $\mu \in \mathcal{M}(f)$. Then $h^u_\mu(f) = h^u_\mu(f|\eta) = h_\mu(f,\alpha|\eta)$ for any $\eta \in \mathcal{P}^u(M^f)$ and $\alpha \in \mathcal{P}(M^f)$ with* $\mathrm{diam}(\alpha) < \frac{\epsilon_0}{\lambda_0}$.

Proof First we show that $\alpha_0^\infty \vee \eta = \varepsilon$, where ε is the partition of M^f into points.

Fix $\widetilde{x} \in M^f$. If $\widetilde{x} \ne \widetilde{y}$ and $\widetilde{y} \in (\alpha_0^\infty \vee \eta)(\widetilde{x})$, then we have $\widetilde{y} \in \eta(\widetilde{x})$ and $\tau^j(\widetilde{y}) \in \alpha(\tau^j(\widetilde{x}))$ for any $j \in \mathbb{N}$. Let k be the first number such that

$$
d^u_{\tau^k(\widetilde{x})}(\Pi(\tau^k(\widetilde{x})), \Pi(\tau^k(\widetilde{x}))) > C_0\epsilon_0,
$$

meanwhile we have

$$
d^u_{\tau^k(\widetilde{x})}(\Pi(\tau^k(\widetilde{x})), \Pi(\tau^k(\widetilde{x}))) \le \lambda_0 d^u_{\tau^{k-1}(\widetilde{x})}(\Pi(\tau^{k-1}(\widetilde{x})), \Pi(\tau^{k-1}(\widetilde{x}))) \le \lambda_0 C_0 \epsilon_0
$$

by the uniform expansion of W^u, while we have $\tau^k(\widetilde{y}) \in \alpha(\tau^k(\widetilde{x}))$, where a contradiction is obtained.

Pick up $\beta \in \mathcal{P}^u(M^f)$. Since $H_{\widetilde{\mu}}(\beta|\alpha_0^\infty \vee \eta) = 0$, for any $\rho > 0$, we can choose $k \in \mathbb{N}$ such that $H_{\widetilde{\mu}}(\beta|\alpha_0^{k-1} \vee \eta) < \rho$. Then we have

$$
\begin{aligned}
H_{\widetilde{\mu}}(\beta_0^{n-1}|\eta) &\le H_{\widetilde{\mu}}(\beta_0^{n-1}|(\alpha_0^{k-1} \vee \eta)_0^{n-1}) + H_{\widetilde{\mu}}((\alpha_0^{k-1} \vee \eta)_0^{n-1}|\eta) \\
&\le nH_{\widetilde{\mu}}(\beta|(\alpha_0^{k-1} \vee \eta)_0^{n-1}) + H_{\widetilde{\mu}}((\alpha_0^{k-1} \vee \eta)_0^{n-1}|\eta)
\end{aligned}
$$

$$\leq n\rho + H_{\widetilde{\mu}}((\alpha_0^{k-1} \vee \eta)_0^{n-1}|\eta).$$

On the other hand, we have

$$\begin{aligned} H_{\widetilde{\mu}}((\alpha_0^{k-1} \vee \eta)_0^{n-1}|\eta) &= H_{\widetilde{\mu}}(\alpha_0^{n+k-2} \vee \tau^{-(n-1)}\eta|\eta) \\ &\leq H_{\widetilde{\mu}}(\alpha_0^{n+k-2}|\eta) + H_{\widetilde{\mu}}(\tau^{-(n-1)}\eta|\alpha_0^{n+k-2} \vee \eta) \\ &\leq H_{\widetilde{\mu}}(\alpha_0^{n+k-2}|\eta) + H_{\widetilde{\mu}}(\eta|\alpha_{-(n-1)}^{k-1} \vee \tau^{n-1}\eta) \\ &\leq H_{\widetilde{\mu}}(\alpha_0^{n+k-2}|\eta) + H_{\widetilde{\mu}}(\eta|(\tau\alpha)^u), \end{aligned}$$

in the last inequality, Lemma 3.4 is applied. Then by Lemma 3.6 we have

$$\begin{aligned} h_\mu(f, \beta|\eta) &= \lim_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\beta_0^{n-1}|\eta) \\ &\leq \rho + \lim_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\alpha_0^{n+k-2}|\eta) + \lim_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\eta|(\tau\alpha)^u) \\ &= \rho + h_\mu(f, \alpha|\eta). \end{aligned}$$

Since $\rho > 0$ is arbitrary, we have

$$h_\mu(f, \beta|\eta) \leq h_\mu(f, \alpha|\eta),$$

then by the arbitrariness of β, we complete the proof.

The following lemma gives the relationship between $h_\mu(f, \beta|\eta)$ and $h_\mu(f, \beta|\xi)$, whose proof is similar to that in [19], so we omit its proof.

Lemma 3.8 *Let $\alpha \in \mathcal{P}(M^f)$, $\eta \in \mathcal{P}^u(M^f)$ and $\xi \in \mathcal{Q}^u(M^f)$. Then for μ-a.e. $\widetilde{x} \in M^f$, we have*

$$\liminf_{n\to\infty} \frac{1}{n} I_{\widetilde{\mu}}(\alpha_0^{n-1}|\xi)(\widetilde{x}) = \liminf_{n\to\infty} \frac{1}{n} I_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta)(\widetilde{x})$$

and

$$\limsup_{n\to\infty} \frac{1}{n} I_{\widetilde{\mu}}(\alpha_0^{n-1}|\xi)(\widetilde{x}) = \limsup_{n\to\infty} \frac{1}{n} I_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta)(\widetilde{x}).$$

We also need the following theorem from [17].

Theorem 3.1 (see [17, Theorem B] *Let f be a C^1 partially hyperbolic endomorphism. Suppose that μ is an ergodic measure of f. For any $\alpha \in \mathcal{P}(M^f), \eta \in \mathcal{P}^u(M^f)$, we have*

$$\lim_{n\to\infty} \frac{1}{n} I_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta)(\widetilde{x}) = h_\mu(f, \alpha|\eta).$$

Now we give the following proposition, which plays an important role in this paper.

Proposition 3.1 *For any $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$ and $\alpha \in \mathcal{P}(M^f)$ with* $\operatorname{diam}(\alpha) < \frac{\epsilon_0}{\lambda_0}$, *we have*

$$h_\mu^u(f) = h_\mu^u(f|\zeta) = h_\mu(f, \alpha|\zeta) = \lim_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\alpha_0^{n-1}|\zeta).$$

Proof We prove Proposition 3.1 in two cases.

Case 1 For $\zeta \in \mathcal{P}^u(M^f)$.

This is the results of Lemmas 3.6–3.7.

Case 2 For $\zeta \in \mathcal{Q}^u(M^f)$.

Given $\eta \in \mathcal{P}^u(M^f)$, and denote ζ by ξ. By Lemmas 3.1–3.2, we know that $\widetilde{\mu} \mapsto h_\mu(f,\alpha|\eta)$ is affine and upper semi-continuous, for any $\widetilde{\mu} \in \mathcal{M}(\tau)$, by the Ergodic Decomposition theorem, let $\widetilde{\mu} = \int_{\mathcal{M}^e(\tau)} \widetilde{\nu} \mathrm{d}m(\widetilde{\nu})$, where $\mathcal{M}^e(\tau)$ is the set of ergodic measures with respect to τ and m is the measure on $\mathcal{M}(\tau)$ such that $m(\mathcal{M}(\tau)) = 1$, then by Theorem 3.1, we have

$$h_\mu(f,\alpha|\eta) = \int_{\mathcal{M}^e(\tau)} h_\nu(f,\alpha|\eta)\mathrm{d}m(\widetilde{\nu}) = \int_{M^f} \lim_{n\to\infty} \frac{1}{n} I_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta)(\widetilde{x})\mathrm{d}\widetilde{\mu}.$$

Then by Fatou's lemma and Lemma 3.8, we have

$$\begin{aligned}
h_\mu(f,\alpha|\xi) &\geq \liminf_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\alpha_0^{n-1}|\xi) \\
&\geq \int_{M^f} \liminf_{n\to\infty} \frac{1}{n} I_{\widetilde{\mu}}(\alpha_0^{n-1}|\xi)(\widetilde{x})\mathrm{d}\widetilde{\mu} \\
&= \int_{M^f} \liminf_{n\to\infty} \frac{1}{n} I_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta)(\widetilde{x})\mathrm{d}\widetilde{\mu} \\
&= h_\mu(f,\alpha|\eta).
\end{aligned}$$

On the other hand, we have

$$H_{\widetilde{\mu}}(\alpha_0^{n-1}|\xi) \leq H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\eta|\xi),$$

by Lemma 3.5 we know that $H_{\widetilde{\mu}}(\eta|\xi) < \infty$, thus we have

$$h_\mu(f,\alpha|\xi) = \limsup_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\alpha_0^{n-1}|\xi) \leq \lim_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\alpha_0^{n-1}|\eta) = h_\mu(f,\alpha|\eta).$$

Combining Lemmas 3.3 and 3.7, we complete the proof of Proposition 3.1.

The following corollary can be obtained easily from Proposition 3.1.

Corollary 3.1 *If $\mathcal{U} \in \mathcal{C}_{M^f}$, has diameter smaller than $\frac{\epsilon_0}{\lambda_0}$, then*

$$h_\mu^{u,+}(f,\mathcal{U}|\zeta) = h_\mu^u(f|\zeta) = h_\mu^u(f)$$

for any $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$.

Now we begin to define another notation of local unstable metric entropy $h_\mu^u(f,\mathcal{U}|\zeta)$.

Definition 3.4 *For $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$, define*

$$h_\mu^u(f,\mathcal{U}|\zeta) := \limsup_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\mathcal{U}_0^{n-1}|\zeta),$$

where $H_{\widetilde{\mu}}(\mathcal{U}|\zeta) = \inf_{\alpha\in\mathcal{P}(M^f),\alpha\geq\mathcal{U}} H_{\widetilde{\mu}}(\alpha|\zeta)$.

The following proposition gives the relation between $h_\mu^u(f,\mathcal{U}|\zeta)$ and $h_\mu^{u,+}(f,\mathcal{U}|\zeta)$, whose proof is similar to that of [19, Proposition 3.18].

Proposition 3.2 $h_\mu(f,\mathcal{U}|\zeta) \leq h_\mu^{u+}(f,\mathcal{U}|\zeta)$ *for any $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$.*

In the definition of $h_\mu^u(f,\mathcal{U}|\zeta)$, we use "lim sup", in fact, we can show that for any $\eta \in \mathcal{P}^u(M^f)$, it can be replaced by "lim". To prove this, we need some lemmas.

Using the similar method to that for Lemma 3.4, we have the following lemma.

Lemma 3.9 *Let $\alpha \in \mathcal{P}(M^f)$ with $\alpha \geq \mathcal{U}_0^{n-1}$, and $\eta \in \mathcal{P}^u(M^f)$. Then $\tau^n\alpha \vee \tau^n\eta \geq (\tau^n\alpha)^u$ for any $n \in \mathbb{N}$.*

Lemma 3.10 *For any $\eta \in \mathcal{P}^u(M^f)$, $h_\mu^u(f,\mathcal{U}|\eta) := \lim\limits_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\mathcal{U}_0^{n-1}|\eta)$ exists.*

Proof Let $\beta \in \mathcal{P}(M^f)$ with $\eta = \beta^u$. Choose any α, $\gamma \in \mathcal{P}(M^f)$ such that $\alpha \geq \mathcal{U}_0^{n-1}$ and $\gamma \geq \mathcal{U}_0^{m-1}$, then we have

$$\begin{aligned}
H_{\widetilde{\mu}}(\mathcal{U}_0^{m+n-1}|\eta) &\leq H_{\widetilde{\mu}}(\alpha \vee \tau^{-n}\gamma|\eta) \\
&= H_{\widetilde{\mu}}(\alpha|\eta) + H_{\widetilde{\mu}}(\gamma|\tau^n\alpha \vee \tau^n\eta) \\
&\leq H_{\widetilde{\mu}}(\alpha|\eta) + H_{\widetilde{\mu}}(\gamma|(\tau^n\alpha)^u) \\
&\leq H_{\widetilde{\mu}}(\alpha|\eta) + H_{\widetilde{\mu}}(\gamma|\eta) + H_{\widetilde{\mu}}(\eta|(\tau^n\alpha)^u) \\
&\leq H_{\widetilde{\mu}}(\alpha|\eta) + H_{\widetilde{\mu}}(\gamma|\eta) + H_{\widetilde{\mu}}(\beta),
\end{aligned}$$

in the third and last inequality Lemmas 3.9 and 3.5 are applied respectively. Because of the arbitrariness of α and γ, we have

$$H_{\widetilde{\mu}}(\mathcal{U}_0^{m+n-1}|\eta) \leq H_{\widetilde{\mu}}(\mathcal{U}_0^{m-1}|\eta) + H_{\widetilde{\mu}}(\mathcal{U}_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\beta).$$

As in the proof of Lemma 3.6, we have shown that $\{H_{\widetilde{\mu}}(\mathcal{U}_0^{n-1}|\eta) + H_{\widetilde{\mu}}(\beta)\}$ is a subadditive sequence, which implies what we need.

The following lemmas are similar to [19, Proposition 3.17 and Lemma 2.5(ii)] respectively.

Lemma 3.11 *$h_\mu^u(f,\mathcal{U}|\eta)$ is independent of $\eta \in \mathcal{P}^u(M^f)$.*

Lemma 3.12 *Fix $N \in \mathbb{N}$, for any $k \geq 1$ and $\alpha \geq \mathcal{U}_0^{N-1}$, we have*

$$\tau^{Nk}\alpha \vee \cdots \vee \tau^N\alpha \vee \tau^{Nk}\eta \geq (\tau^N\alpha)^u.$$

Lemma 3.13 *For $\eta \in \mathcal{P}^u(M^f)$, we have*
(i) *$h_\mu^u(f,\mathcal{U}|\eta) = \frac{1}{n} h_\mu^u(f^n,\mathcal{U}_0^{n-1}|\eta)$ for any $n \in \mathbb{N}$,*
(ii) *$h_\mu^u(f,\mathcal{U}|\eta) = \lim\limits_{n\to\infty} \frac{1}{n} h_\mu^{u,+}(f^n,\mathcal{U}_0^{n-1}|\eta)$.*

Proof In the proof, Lemmas 3.5, 3.12 and Proposition 3.2 are used. The proof is completely parallel to that of [19, Lemma 3.19], so we omit it here.

Lemma 3.14 *$h_\mu^{u,+}(f^n,\mathcal{U}_0^{n-1}|\eta) = nh_\mu^u(f|\eta)$ for any $\eta \in \mathcal{P}^u(M^f)$ and $n \in \mathbb{N}$.*

Proof Choose arbitrary $\alpha \geq \mathcal{U}_0^{n-1}$, as in Lemma 3.7, we can show that

$$\eta \vee \bigvee_{i=0}^{\infty} \tau^{-ni}\alpha = \varepsilon.$$

Then following the line of the proof of [19, Lemma 3.13], for any $\beta \in \mathcal{P}(M^f)$ and $\rho > 0$ we can show that

$$h_\mu^u(f^n,\beta|\eta) \leq \rho + h_\mu(f^n,\alpha|\eta).$$

Then by the arbitrariness of β, ρ and α, we have

$$nh_\mu^u(f|\eta) = h_\mu^u(f^n|\eta) \le h_\mu^{u,+}(f^n,\mathcal{U}_0^{n-1}|\eta).$$

And it is clear that $nh_\mu^u(f|\eta) \ge h_\mu^{u,+}(f^n,\mathcal{U}_0^{n-1}|\eta)$.

4 Local Unstable Topological Entropy and Pressure

In this section, we give the definition of local unstable topological entropy of f with respect to a Borel cover $\mathcal{U} \in \mathcal{C}_{M^f}$.

Let $K \subset M^f$. For any $\mathcal{U} \in \mathcal{C}_{M^f}$, denote

$$N(K,\mathcal{U}) := \min\Big\{\text{the cardinality of } \mathcal{V}\colon \mathcal{V} \subset \mathcal{U}, \bigcup_{V\in\mathcal{V}} V \supset K\Big\},$$

and denote $\log N(K,\mathcal{U})$ by $H(\mathcal{U}|K)$.

Definition 4.1 *For any $\mathcal{U} \in \mathcal{C}_{M^f}$, we define*

$$h_{\text{top}}^u(f,\mathcal{U}) := \lim_{\delta\to 0} h_{\text{top}}^u(f,\mathcal{U},\delta),$$

where

$$h_{\text{top}}^u(f,\mathcal{U},\delta) = \sup_{\widetilde{x}\in M^f} h_{\text{top}}(f,\mathcal{U}|\overline{\widetilde{W}^u(\widetilde{x},\delta)}),$$

$$h_{\text{top}}(f,\mathcal{U}|\overline{\widetilde{W}^u(\widetilde{x},\delta)}) := \limsup_{n\to\infty} \frac{1}{n} H(\mathcal{U}_0^{n-1}|\overline{\widetilde{W}^u(\widetilde{x},\delta)})$$

and $\overline{\widetilde{W}^u(\widetilde{x},\delta)} = \{\widetilde{y} \in M^f\colon \Pi(\widetilde{y}) \in \overline{W^u(\widetilde{x},\delta)}$ *and satisfies* (2.1)$\}$.

Using the same method in [3, 17, 19], we can prove that $h_{\text{top}}^u(f,\mathcal{U})$ is independent of δ.

Lemma 4.1 $h_{\text{top}}^u(f,\mathcal{U}) = h_{\text{top}}^u(f,\mathcal{U},\delta)$ *for any* $\delta > 0$.

As a generalization of local unstable topological entropy, we can give the definition of local unstable pressure of f.

Definition 4.2 *Let $\phi \in C(M)$. Define*

$$\begin{aligned}P^u(f,\phi,\widetilde{x},\delta,n,\mathcal{U}) = \inf\Big\{ &\sum_{V\in\mathcal{V}} \sup_{\widetilde{y}\in V\cap\overline{\widetilde{W}^u(\widetilde{x},\delta)}} \exp(S_n\phi)(\Pi(\widetilde{y}))\colon \\ &\mathcal{V}\in\mathcal{P}(M^f),\ \mathcal{V}\ge\mathcal{U}_0^{n-1}\Big\},\end{aligned}$$

where $S_n\phi(x) := \sum_{i=0}^{n-1} \phi(f^i(x))$ *for* $x \in M$. *Then* $P^u(f,\phi,\mathcal{U}|\overline{\widetilde{W}^u(\widetilde{x},\delta)})$ *is defined as*

$$P^u(f,\phi,\mathcal{U}|\overline{\widetilde{W}^u(\widetilde{x},\delta)}) = \limsup_{n\to\infty} \frac{1}{n} \log P^u(f,\phi,\widetilde{x},\delta,n,\mathcal{U}).$$

Next, we define

$$P^u(f,\phi,\mathcal{U},\delta) = \sup_{\widetilde{x}\in M^f} P^u(f,\phi,\mathcal{U}|\overline{\widetilde{W}^u(\widetilde{x},\delta)}).$$

Then the local unstable pressure of f with respect to ϕ is defined as

$$P^u(f,\phi,\mathcal{U}) = \lim_{\delta\to 0} P^u(f,\phi,\mathcal{U},\delta).$$

Now we give the relation between local unstable metric entropy and local unstable topological entropy.

Proposition 4.1 *Let $\mu \in \mathcal{M}(f)$ and $\mathcal{U} \in \mathcal{C}_{M^f}$. Then for any $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$, we have*

$$h^u_\mu(f,\mathcal{U}|\zeta) \le h^u_{\rm top}(f,\mathcal{U}).$$

Proof Fix $\delta > 0$ such that $\zeta(\widetilde{x}) \subset \overline{\widetilde{W}^u(\widetilde{x},\delta)}$. In the next, for any $\mathcal{V} = \{V_j\}_{j=1}^k \in \mathcal{C}^o_{M^f}$, we will construct a finite partition $\alpha \in \mathcal{P}(M^f)$ with $\alpha \ge \mathcal{V}$ such that

$$H_{\widetilde{\mu}}(\alpha|\zeta) \le \int_{M^f} \log N(\overline{\widetilde{W}^u(\widetilde{x},\delta)}, \mathcal{V}) {\rm d}\widetilde{\mu}(\widetilde{x}).$$

For any $\widetilde{y} \in M^f$, we can find a subset $I_{\widetilde{y}}(\widetilde{x})$ of $\{1,2,\cdots,k\}$ with minimal cardinality no more than $N(\overline{\widetilde{W}^u(\widetilde{y},\delta)}, \mathcal{V})$ such that

$$\bigcup_{j\in I_{\widetilde{y}}} V_j \supset \zeta(\widetilde{y}).$$

Then following the way in [19], we can construct a partition $\alpha = \{A_j\}_{j=1}^N$ of M^f with $\alpha \ge \mathcal{V}$ such that

$$H_{\widetilde{\mu}}(\alpha|\zeta) \le \int_{M^f} \log N(\overline{\widetilde{W}^u(\widetilde{y},\delta)}, \mathcal{V}) {\rm d}\widetilde{\mu}(\widetilde{y}). \tag{4.1}$$

Then by Fatou's lemma, we have

$$\begin{aligned}
h^u_\mu(f,\mathcal{U}|\zeta) &= \limsup_{n\to\infty} \frac{1}{n} H_{\widetilde{\mu}}(\mathcal{U}_0^{n-1}|\zeta) \\
&\le \limsup_{n\to\infty} \int_{M^f} \frac{1}{n} \log N(\overline{\widetilde{W}^u(\widetilde{x},\delta)}, \mathcal{U}_0^{n-1}) {\rm d}\widetilde{\mu}(\widetilde{x}) \\
&\le \int_{M^f} \limsup_{n\to\infty} \frac{1}{n} \log N(\overline{\widetilde{W}^u(\widetilde{x},\delta)}, \mathcal{U}_0^{n-1}) {\rm d}\widetilde{\mu}(\widetilde{x}) \\
&= \int_{M^f} h_{\rm top}(f,\mathcal{U}|\overline{\widetilde{W}^u(\widetilde{x},\delta)}) {\rm d}\widetilde{\mu}(\widetilde{x}) \\
&\le \int_{M^f} \max_{\widetilde{x}\in M^f} h_{\rm top}(f,\mathcal{U}|\overline{\widetilde{W}^u(\widetilde{x},\delta)}) {\rm d}\widetilde{\mu}(\widetilde{x}) \\
&= h^u_{\rm top}(f,\mathcal{U}),
\end{aligned}$$

which completes the proof of Proposition 4.1.

5 Unstable Topological Conditional Entropy and Unstable Tail Entropy

In this section, we give the definitions of unstable topological conditional entropy and unstable tail entropy, which are useful in the proof of Theorem A, for the case when $\xi \in \mathcal{Q}^u(M^f)$.

Definition 5.1 *For $Y \in M^f$ and any two covers $\mathcal{U}$, $\mathcal{V} \in \mathcal{C}_{M^f}$, define*

$$N^u(Y,\mathcal{U}) = \sup_{\widetilde{y}\in Y} N(Y \cap \overline{\widetilde{W}^u(\widetilde{y},\delta)}, \mathcal{U})$$

and

$$H^u(Y,\mathcal{U}) = \log N^u(Y,\mathcal{U}),$$

and if $Y = \emptyset$, *we set* $H^u(\emptyset,\mathcal{U}) = 0$. *Define*

$$N^u(\mathcal{U}|\mathcal{V}) = \max_{V\in\mathcal{V}} N^u(V,\mathcal{U})$$

and

$$H^u(\mathcal{U}|\mathcal{V}) = \log N^u(\mathcal{U}|\mathcal{V}).$$

The following proposition is a collection of properties for H^u, whose proof is simple.

Proposition 5.1 (i) $H^u(Y,\mathcal{U}) \le H^u(Z,\mathcal{V})$ *if* $Y \subset Z$ *and* $\mathcal{V} \ge \mathcal{U}$.
(ii) $H^u(\mathcal{U}_1|\mathcal{V}_1) \le H^u(\mathcal{U}_2|\mathcal{V}_2)$ *if* $\mathcal{U}_2 \ge \mathcal{U}_1$ *and* $\mathcal{V}_1 \ge \mathcal{V}_2$.
(iii) $H^u(Y,\mathcal{U}) = \sup_{\widetilde{y}\in Y} \log N(\tau^{-1}(Y) \cap \tau^{-1}(\overline{\widetilde{W}^u(\widetilde{y},\delta)}), \tau^{-1}\mathcal{U})$.
(iv) $H^u(\mathcal{U}\vee\mathcal{V}|\mathcal{W}) \le H^u(\mathcal{U}|\mathcal{W}) + H^u(\mathcal{V}|\mathcal{U}\vee\mathcal{W})$.
(v) $H^u(\mathcal{U}_1\vee\mathcal{V}_1|\mathcal{U}_2\vee\mathcal{V}_2) \le H^u(\mathcal{U}_1|\mathcal{U}_2) + H^u(\mathcal{V}_1|\mathcal{V}_2)$.
(vi) $H^u(Y,\mathcal{U}\vee\mathcal{V}) \le H^u(Y,\mathcal{U}) + H^u(Y,\mathcal{V})$.
(vii) $H^u(M^f,\mathcal{U}) \le H^u(M^f,\mathcal{V}) + H^u(\mathcal{U}|\mathcal{V})$.
(viii) $H^u(\mathcal{U}|\mathcal{V}) \le H^u(\mathcal{U}|\mathcal{W}) + H^u(\mathcal{W}|\mathcal{V})$.

Lemma 5.1 *If* $\operatorname{diam}(\mathcal{V}) < \epsilon_0 \ll \delta$, *then* $\lim_{n\to\infty} \frac{1}{n} H^u(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{n-1})$ *exists.*

Proof Firstly, we show that $V \cap \tau^n(\overline{\widetilde{W}^u(\widetilde{y},\delta)}) = V \cap \overline{\widetilde{W}^u(\tau^n(\widetilde{y}),\delta)}$ for any $V \in \mathcal{V}_{-n}^{m-1}$ and $\widetilde{y} \in \tau^{-n}V$. Because $V \in \mathcal{V}_{-n}^{m-1}$, we know that if $\widetilde{z} \in V \cap \tau^n(\overline{\widetilde{W}^u(\widetilde{y},\delta)})$, then

$$d^u_{\tau^{n-j}(\widetilde{y})}(\Pi(\tau^{-j}(\widetilde{z})), \Pi(\tau^{-j}(\widetilde{y}))) \le C_0\epsilon_0$$

for $0 \le j \le n$, so we have

$$d^u_{\tau^n(\widetilde{y})}(\Pi(\widetilde{z}), \Pi\tau^n(\widetilde{y})) \le C_0\epsilon_0$$

which implies that $\widetilde{z} \in V \cap \overline{\widetilde{W}^u(\tau^n(\widetilde{y}),\delta)}$. Then by Proposition 5.1, we have

$$\begin{aligned}
&H^u(\mathcal{U}_0^{m+n-1}|\mathcal{V}_0^{m+n-1}) \\
&\le H^u(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{m+n-1}) + H^u(\tau^{-n}\mathcal{U}_0^{m-1}|\tau^{-n}\mathcal{V}_{-n}^{m-1}) \\
&\le H^u(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{n-1}) + \\
&\quad \log \max_{V\in\mathcal{V}_{-n}^{m-1}} \sup_{\widetilde{y}\in\tau^{-n}V} N(\tau^{-n}V \cap (\overline{\widetilde{W}^u(\widetilde{y},\delta)}), \tau^{-n}\mathcal{U}_0^{m-1}) \\
&= H^u(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{n-1}) + \log \max_{V\in\mathcal{V}_{-n}^{m-1}} \sup_{\widetilde{y}\in\tau^{-n}V} N(V \cap \tau^n\overline{\widetilde{W}^u(\widetilde{y},\delta)}, \mathcal{U}_0^{m-1}) \\
&= H^u(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{n-1}) + \log \max_{V\in\mathcal{V}_{-n}^{m-1}} \sup_{\tau^n(\widetilde{y})\in V} N(V \cap \overline{\widetilde{W}^u(\tau^n(\widetilde{y}),\delta)}, \mathcal{U}_0^{m-1}) \\
&= H^u(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{n-1}) + H^u(\mathcal{U}_0^{m-1}|\mathcal{V}_{-n}^{m-1}) \\
&\le H^u(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{n-1}) + H^u(\mathcal{U}_0^{m-1}|\mathcal{V}_0^{m-1}),
\end{aligned}$$

which means that $\{H^u(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{n-1})\}$ is subadditive, hence we complete the proof.

Because of Lemma 5.1, we have the following definition.

Definition 5.2 *The unstable conditional entropy of f on the cover $\mathcal{U}$ with respect to the cover $\mathcal{V}$ is defined as*

$$h^u(f,\mathcal{U}|\mathcal{V}) = \lim_{n\to\infty} \frac{1}{n} H^u(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{n-1}).$$

And define $h^u(f,Y,\mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} H^u(Y,\mathcal{U}_0^{n-1})$.

The unstable conditional entropy with respect to $\mathcal{V}$ is defined as

$$h^u(f|\mathcal{V}) = \sup_{\mathcal{U}\in\mathcal{C}^o_{M^f}} h^u(f,\mathcal{U}|\mathcal{V}).$$

And define $h^u(f,Y) = \sup_{\mathcal{U}\in\mathcal{C}^o_{M^f}} h^u(f,Y,\mathcal{U})$.

The unstable topological conditional entropy of f in the sense of Misiurewicz is defined as

$$h^{*u}(f) := \inf_{\mathcal{V}\in\mathcal{C}^o_{M^f}} h^u(f|\mathcal{V}).$$

The following proposition is a collection of properties of unstable conditional entropy.

Proposition 5.2 (i) $h^u(f,Y,\mathcal{U}) \le h^u(f,Z,\mathcal{V})$ *if* $Y \subset Z$ *and* $\mathcal{V} \ge \mathcal{U}$.
(ii) $h^u(f,\mathcal{U}_1|\mathcal{V}_1) \le h^u(f,\mathcal{U}_2|\mathcal{V}_2)$ *if* $\mathcal{U}_2 \ge \mathcal{U}_1$ *and* $\mathcal{V}_1 \ge \mathcal{V}_2$.
(iii) $h^u(f,M^f,\mathcal{U}) \le h^u(f,M^f,\mathcal{V}) + h^u(f,\mathcal{U}|\mathcal{V})$.
(iv) $h^u(f,\mathcal{U}|\mathcal{V}) \le h^u(f,\mathcal{U}|\mathcal{W}) + h^u(f,\mathcal{W}|\mathcal{V})$.
(v) $h^u_{\text{top}}(f,\mathcal{U}) \le h^u_{\text{top}}(f,\mathcal{V}) + h^u(f,\mathcal{U}|\mathcal{V})$.

Proof (i)–(iv) are simple. For (v), by Lebesgue Dominated Convergence theorem, we have

$$h^u(f,M^f,\mathcal{U}) = h^u_{\text{top}}(f,\mathcal{U}),$$

then by Proposition 5.1, we obtain what we need.

The following proposition is important.

Proposition 5.3

$$h^u(f,M^f) \le h^u(f,M^f,\mathcal{U}) + h^u(f|\mathcal{U})$$

and

$$h^u_{\text{top}}(f) \le h^u_{\text{top}}(f,\mathcal{U}) + h^u_{\text{top}}(f|\mathcal{U}).$$

Proof The two inequalities can be obtained by Proposition 5.2(iii) and (v), respectively.

In the next, we begin to define the unstable tail entropy of f in the sense of Bowen.

Fix $\delta > 0$, for $\epsilon > 0$, $\widetilde{x} \in M^f$ and $Y \subset M^f$, a subset $\widetilde{E}_n$ of $\overline{\widetilde{W}^u(\widetilde{x},\delta)}$ is called an (n,ϵ)-spanning set of $Y \cap \overline{\widetilde{W}^u(\widetilde{x},\delta)}$ if for any $\widetilde{y}_1,\widetilde{y}_2 \in \widetilde{E}_n$, we have $d^n(\widetilde{y}_1,\widetilde{y}_2) \le \epsilon$, which means

$$d^u_{\tau^j(\widetilde{x})}(\Pi(\tau^j(\widetilde{y}_1)),\Pi(\tau^j(\widetilde{y}_2))) \le \epsilon \quad \text{for } 0 \le j \le n-1.$$

Denote

$$R^u_n(\overline{\widetilde{W}^u(\widetilde{x},\delta)},\epsilon) := \text{the smallest cardinality of } (n,\epsilon)\text{-spanning sets of } Y \cap \overline{\widetilde{W}^u(\widetilde{x},\delta)}.$$

Then define

$$r_n^u(Y,\epsilon) = \sup_{\widetilde{y}\in Y} R_n^u(\overline{\widetilde{W}^u(\widetilde{y},\delta)},\epsilon),$$
$$\overline{r}^u(Y,\epsilon) = \limsup_{n\to\infty} \frac{1}{n}\log r_n^u(Y,\epsilon)$$

and

$$\overline{h}^u(f,Y) := \lim_{\epsilon\to 0} \overline{r}^u(Y,\epsilon).$$

For $\epsilon>0$, denote $\bigcap\limits_{n=1}^{\infty} \widetilde{B}_n(\widetilde{x},\epsilon)$ by $\Phi(\widetilde{x},\epsilon)$, where

$$\widetilde{B}_n(\widetilde{x},\epsilon) = \{\widetilde{y}\in M^f : d^n(\widetilde{x},\widetilde{y}) < \epsilon\}$$

and $d^n(\widetilde{x},\widetilde{y}) < \epsilon$ means $d(\Pi(\tau^k(\widetilde{x})),\Pi(\tau^k(\widetilde{x}))) < \epsilon$ for $0\le k\le n-1$. Now we can give the following definition.

Definition 5.3

$$h^{*u}(f,\epsilon) = \sup_{\widetilde{x}\in M^f} \overline{h}^u(f,\widetilde{x},\Phi(x,\epsilon)).$$

The following proposition gives the relation between unstable conditional entropy and unstable tail entropy, whose proof is completely similar to that of [19, Proposition 4.10], so we omit it here. For $\mathcal{U}\in\mathcal{C}^o_{M^f}$, let $\mathrm{Leb}(\mathcal{U})$ be the Lebesgue number of $\Pi(\mathcal{U})$.

Proposition 5.4 *For $Y\in M^f$ and $\mathcal{U},\mathcal{V}\in\mathcal{C}^o_{M^f}$ with $C_0\mathrm{diam}(\mathcal{U}) < \epsilon < \frac{\mathrm{Leb}(\mathcal{V})}{2}$.*

$$N(Y\cap\overline{\widetilde{W}^u(\widetilde{y},\delta)},\mathcal{V}_0^{n-1}) \le r_n^u(Y,\epsilon) \le N(Y\cap\overline{\widetilde{W}^u(\widetilde{y},\delta)},\mathcal{U}_0^{n-1}).$$

In fact, in our setting, both unstable conditional entropy and unstable tail entropy vanish, i.e., we have the following theorem.

Theorem 5.1

$$h^{*u}(f) = 0$$

and

$$h^{*u}(f,\epsilon) = 0$$

for any $\epsilon>0$ small enough.

Proof For $\mathcal{U}\in\mathcal{C}_{M^f}$ with $\mathrm{diam}(\mathcal{U})\ll\epsilon_0$, we show that $h^u(f|\mathcal{U})=0$. For any $\epsilon>0$, choose $\mathcal{W}\in\mathcal{C}^o_{M^f}$ with $\mathrm{Leb}(\mathcal{W})=3\epsilon$. Then by Proposition 5.4, we have

$$\begin{aligned}
&\max_{U\in\mathcal{U}_0^{n-1}} \sup_{\widetilde{y}\in U} \log N(U\cap\overline{\widetilde{W}^u(\widetilde{y},\delta)},\mathcal{W}_0^{n-1}) \\
&\le \max_{U\in\mathcal{U}_0^{n-1}} \log r_n^u(U,\epsilon) \\
&\le \log r_n^u(\widetilde{B}_n(\widetilde{x},\epsilon_0),\epsilon).
\end{aligned} \tag{5.1}$$

Fix $\widetilde{y}\in\widetilde{B}_n(\widetilde{x},\epsilon_0)$. Let $\widetilde{z}\in\widetilde{B}_n(\widetilde{x},\epsilon_0)\cap\overline{\widetilde{W}^u(\widetilde{y},\delta)}$, which implies that

$$d(\Pi(\tau^k(\widetilde{z})),\Pi(\tau^k(\widetilde{y}))) \le 2\epsilon_0$$

for any $0 \le k \le n-1$. Thus we have

$$d^u_{\tau^k(\widetilde{x})}(\Pi(\tau^k(\widetilde{z})), \Pi(\tau^k(\widetilde{y}))) \le 2C_0\epsilon_0$$

for any $0 \le k \le n-1$, which means

$$\widetilde{B}_n(\widetilde{x}, \epsilon_0) \cap \overline{\widetilde{W}^u(\widetilde{y}, \delta)} \subset B^u_n(\widetilde{y}, 2C_0\epsilon_0).$$

Noticing that

$$B^u_n(\widetilde{y}, 2C_0\epsilon_0) \subset \tau^{-n} B^u(\tau^n(\widetilde{y}), 2C_0\epsilon_0),$$

where $B^u(\widetilde{x}, \rho) := \{\widetilde{y} \in \widetilde{W}^u(\widetilde{x}) \colon d^u_{\widetilde{x}}(\Pi(\widetilde{x}), \Pi(\widetilde{y})) < \rho\}$, we can find $\widetilde{z}_j$, $1 \le j \le N$ such that

$$B^u(\tau^n(\widetilde{y}), 2C_0\epsilon_0) \subset \bigcup_{j=1}^{N} B^u(\tau^n(\widetilde{z}_j), \epsilon),$$

where $N = D\big(2C_0\frac{\epsilon_0}{\epsilon}\big)^{\dim E^u}$ for some $D > 1$. So we have

$$\widetilde{B}_n(\widetilde{x}, \epsilon_0) \cap \overline{\widetilde{W}^u(\widetilde{y}, \delta)} \subset \bigcup_{j=1}^{N} B^u_n(\widetilde{z}_j, \epsilon).$$

Thus

$$r^u_n(\widetilde{B}_n(\widetilde{x}, \epsilon_0), \epsilon) \le N.$$

By (5.1) we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} H^u(\mathcal{W}_0^{n-1} | \mathcal{U}_0^{n-1}) \\
&= \lim_{n\to\infty} \frac{1}{n} \max_{U \in \mathcal{U}_0^{n-1}} \sup_{\widetilde{y} \in U} \log N(U \cap \overline{\widetilde{W}^u(\widetilde{y}, \delta)}, \mathcal{W}_0^{n-1}) \\
&= \lim_{n\to\infty} \frac{1}{n} r^u_n(\widetilde{B}_n(\widetilde{x}, \epsilon_0), \epsilon) \\
&\le \lim_{n\to\infty} \frac{1}{n} \log D\Big(2C_0 \frac{\epsilon_0}{\epsilon}\Big)^{\dim E^u} \\
&= 0,
\end{aligned}$$

which implies that $h^u(\mathcal{W}|\mathcal{U}) = 0$, because of the arbitrariness of $\mathcal{W}$, we know that $h^u(f|\mathcal{U}) = 0$.

As in the proof of [19, Theorem 1.3], we have that $\overline{\widetilde{W}^u(\widetilde{y}, \delta)} \cap \Phi(\widetilde{x}, \epsilon)$ contains at most one point due to the expansion of f along W^u. Then following the line of the proof of [19, Theorem 1.3], we have

$$h^{*u}(f, \epsilon) = 0$$

for $\epsilon > 0$ small enough. Now we complete the proof of Theorem 5.1.

6 Variational Principles for Local Unstable Entropy and Unstable Pressure

In this section, we prove Theorem A, then variational principles for local unstable entropy and unstable pressure are obtained. First of all, we give two propositions as follows.

Proposition 6.1 *For any $\mu \in \mathcal{M}(f)$, $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$, and $\mathcal{U}$, $\mathcal{V} \in \mathcal{C}_{M^f}$ with small enough diameter, we have*

$$H^u_{\widetilde{\mu}}(\mathcal{V}|\zeta) \le H^u_{\widetilde{\mu}}(\mathcal{U}|\zeta) + H^u(\mathcal{V}|\mathcal{U}).$$

Proof Let $\mathcal{V} = \{V_1, V_2, \cdots, V_m\}$ and $\beta \in \mathcal{P}(M^f)$ such that $\beta \ge \mathcal{U}$. Let $B \in \beta$. For each $\widetilde{y} \in B$, there exists $I_{\widetilde{y}} \subset \{1, 2, \cdots, m\}$ with minimal cardinality no more than $N^u(\mathcal{V}|\beta)$ such that $\bigcup\limits_{j \in I_{\widetilde{y}}} V_i \supset B \cap \zeta(\widetilde{y})$. Thus we can choose $\widetilde{y}_1, \widetilde{y}_2, \cdots, \widetilde{y}_s \in B$ such that for each $\widetilde{y} \in B$, $I_{\widetilde{y}} = I_{\widetilde{y}_j}$ for some $1 \le i \le s$. Then as in the proof of Proposition 4.1, we can contruct a partition γ_B of B, then a partition $\gamma = \bigcup\limits_{B \in \beta} \gamma_B$ of M^f. According to the construct of γ, we know that

$$N(\gamma|\beta \vee \zeta) \le N^u(\mathcal{V}|\beta),$$

where

$$N(\gamma|\beta \vee \zeta) = \max_{B \in \beta} \sup_{\widetilde{y} \in B} N(B \cap \zeta(\widetilde{y}), \gamma).$$

Then we have

$$\begin{aligned} H_{\widetilde{\mu}}(\gamma|\zeta) &\le H_{\widetilde{\mu}}(\beta|\zeta) + H_{\widetilde{\mu}}(\gamma|\beta \vee \zeta) \\ &= H_{\widetilde{\mu}}(\beta|\zeta) + \int_{M^f} H_{\mu^{\beta\vee\zeta}_{\widetilde{x}}}(\gamma) \mathrm{d}\widetilde{\mu} \\ &\le H_{\widetilde{\mu}}(\beta|\zeta) + \log N^u(\gamma|\beta \vee \zeta) \\ &\le H_{\widetilde{\mu}}(\beta|\zeta) + \log N^u(\mathcal{V}|\mathcal{U}). \end{aligned}$$

Thus

$$\begin{aligned} H^u_{\widetilde{\mu}}(\mathcal{V}|\zeta) &\le H_{\widetilde{\mu}}(\gamma|\zeta) \\ &\le H_{\widetilde{\mu}}(\beta|\zeta) + H^u(\mathcal{V}|\mathcal{U}). \end{aligned}$$

Since $\beta \ge \mathcal{U}$ is arbitrary, we complete the proof.

Proposition 6.2 *For any $\mu \in \mathcal{M}(f)$, $\zeta \in \mathcal{P}^u(M^f) \cup \mathcal{Q}^u(M^f)$, $\mathcal{U} \in \mathcal{C}_{M^f}$ with sufficiently small diameter, we have*

$$h^u_\mu(f|\zeta) \le h^u_\mu(f, \mathcal{U}|\zeta) + h^u(f|\mathcal{U}).$$

Proof Proposition 6.2 can be obtained from Proposition 6.1 easily, so we omit the proof.

Proof of Theorem A We divide the proof into two cases.

Case 1 For $\eta \in \mathcal{P}^u(M^f)$.

Let $\mathcal{U} \in \mathcal{C}_{M^f}$ with $\mathrm{diam}(\mathcal{U}) \ll \epsilon_0$. By Corollary 3.1, we know that

$$h^{u,+}_\mu(f, \mathcal{U}|\zeta) = h^u_\mu(f|\zeta) = h^u_\mu(f). \tag{6.1}$$

By Lemmas 3.13–3.14, we have

$$h^u_\mu(\mathcal{U}|\eta) = \lim_{n\to\infty} \frac{1}{n} h^{u,+}_\mu(f^n, \mathcal{U}_0^{n-1}|\eta) = \lim_{n\to\infty} \frac{1}{n} n h^u_\mu(f|\eta) = h^u_\mu(f|\eta). \tag{6.2}$$

Then by (6.1), (6.2) and Proposition 4.1, we know that

$$h^u_\mu(f) = h^u_\mu(f,\mathcal{U}|\zeta) \le h^u_{\rm top}(f,\mathcal{U}).$$

By the variational principle for unstable entropy of f, we can obtain

$$h^u_{\rm top}(f) = \sup_{\mu\in\mathcal{M}(f)} h^u_\mu(f) \le h^u_{\rm top}(f,\mathcal{U}).$$

And it is easy to see that $h^u_{\rm top}(f,\mathcal{U}) \le h^u_{\rm top}(f)$, then we have

$$h^u_{\rm top}(f,\mathcal{U}) = h^u_{\rm top}(f).$$

This ends the proof of Theorem A for $\eta \in \mathcal{P}^u(M^f)$.

Case 2 For $\xi \in \mathcal{Q}^u(M^f)$.

Let $\mathcal{U} \in \mathcal{C}_{M^f}$. By Theorem 5.1 and Proposition 6.2, we have

$$h^u_\mu(f|\zeta) = h^u_\mu(f,\mathcal{U}|\zeta).$$

By Corollary 3.1, we know that

$$h^u_\mu(f|\zeta) = h^{u,+}_\mu(f,\mathcal{U}|\zeta).$$

By Proposition 5.3 and Theorem 5.1, we have

$$h^u_{\rm top}(f) \le h^u_{\rm top}(f,\mathcal{U}),$$

and it is clear that for $\mathcal{U} \in \mathcal{C}^o_{M^f}$, we have

$$h^u_{\rm top}(f) \ge h^u_{\rm top}(f,\mathcal{U}),$$

which completes the proof.

As an application of Theorem A, following the line of [19, Proposition 3.14], we have the following proposition.

Proposition 6.3 *For $\mathcal{U} \in \mathcal{C}^o_{M^f}$, the local unstable entropy map $\mu \mapsto h^+_\mu(f,\mathcal{U}|\eta)$ is upper semi-continuous for $\eta \in \mathcal{P}^u(M^f)$.*

Now we begin to discuss the variational principle for local pressure. Firstly, we need the following lemma from [10], which is adapted in our paper. For $\mathcal{V} \in \mathcal{C}_{M^f}$, let α be the Borel partition generated by $\mathcal{V}$, let

$$\begin{aligned}&\mathcal{P}^*(\mathcal{V})\\ &= \{\beta \in \mathcal{P}(M^f) : \beta \ge \mathcal{V} \text{ and each atom of } \beta \text{ is the union of some atoms of } \alpha\}.\end{aligned}$$

Denote $\widetilde{\phi}(\widetilde{x}) := \phi(\Pi(\widetilde{x}))$, it is clear that $\int_M \phi {\rm d}\mu = \int_{M^f} \widetilde{\phi} {\rm d}\widetilde{\mu}$.

Lemma 6.1 (see [10, Lemma 2.1]) *Let $\mathcal{U} \in \mathcal{C}_{M^f}$ and $\phi \in C(M)$, then we have*

$$\inf_{\mathcal{V}\in\mathcal{C}_{M^f},\mathcal{V}\ge\mathcal{U}} \Big\{\sum_{B\in\mathcal{V}} \sup_{\widetilde{y}\in B\cap\overline{\widetilde{W}^u(\widetilde{x},\delta)}} \widetilde{\phi}(\widetilde{y})\Big\} = \min_{\beta\in\mathcal{P}^*(\mathcal{U})} \Big\{\sum_{B\in\beta} \sup_{\widetilde{y}\in B\cap\overline{\widetilde{W}^u(\widetilde{x},\delta)}} \widetilde{\phi}(\widetilde{y})\Big\}.$$

Now we begin to prove Theorem C. Firstly, we prove the following proposition.

Proposition 6.4 *For any $\zeta \in \mathcal{P}^u(M^f)\cap\mathcal{Q}^u(M^f)$, we have*

$$h_\mu^u(f,\mathcal{U}|\zeta)+\int_M \phi \mathrm{d}\mu \le P^u(f,\phi,\mathcal{U}).$$

Proof Let $\zeta \in \mathcal{P}^u(M^f)\cap\mathcal{Q}^u(M^f)$, we have

$$\begin{aligned}
& h_\mu^u(f,\mathcal{U}|\zeta)+\int_M \phi \mathrm{d}\mu \\
=\ & h_\mu^u(f,\mathcal{U}|\zeta)+\int_{M^f} \widetilde{\phi}\mathrm{d}\widetilde{\mu} \\
=\ & \limsup_{n\to\infty}\frac{1}{n}H_{\widetilde{\mu}}(\mathcal{U}_0^{n-1}|\zeta)+\int_{M^f}\widetilde{\phi}\mathrm{d}\widetilde{\mu}(\widetilde{x}) \\
=\ & \limsup_{n\to\infty}\frac{1}{n}\int_{M^f}\Big(H_{\widetilde{\mu}_{\widetilde{x}}^{\zeta}}(\mathcal{U}_0^{n-1})+\int_{M^f}S_n\widetilde{\phi}\mathrm{d}\widetilde{\mu}_{\widetilde{x}}^{\zeta}\Big)\mathrm{d}\mu(\widetilde{x}), \qquad (6.3)
\end{aligned}$$

where $S_n\widetilde{\phi} := \sum\limits_{i=0}^{n-1}\widetilde{\phi}(\tau^i(\widetilde{x}))$. Choose $\delta>0$ such that $\zeta(\widetilde{x})\subset\overline{\widetilde{W}^u(\widetilde{x},\delta)}$ for every $\widetilde{x}\in\Lambda'\subset M^f$, where $\widetilde{\mu}(\Lambda')=1$. For any $\beta\in\mathcal{P}(M^f)$, and any $\widetilde{x}\in\Lambda'$, denote $\{C\colon C=B\cap\zeta(\widetilde{x})$ for some $B\in\beta\}$ by $\beta_{\widetilde{x}}$. Then by Lemma 6.1, we know that there exists a $\beta\in\mathcal{P}^*(\mathcal{U}_0^{n-1})$ such that

$$\begin{aligned}
\log P^u(f,\phi,\widetilde{x},\delta,n,\mathcal{U}) &= \log\Big(\sum_{B\in\beta}\sup_{\widetilde{y}\in B\cap\overline{\widetilde{W}^u(\widetilde{x},\delta)}}\exp((S_n\widetilde{\phi})(\widetilde{y}))\Big) \\
&\ge \log\Big(\sum_{C\in\beta_{\widetilde{x}}}\sup_{\widetilde{y}\in C}\exp((S_n\widetilde{\phi})(\widetilde{y}))\Big) \\
&\ge \sum_{C\in\beta_{\widetilde{x}}}\widetilde{\mu}_{\widetilde{x}}^{\zeta}(C)\Big(\sup_{\widetilde{y}\in C}(S_n\widetilde{\phi})(\widetilde{y})-\log\widetilde{\mu}_{\widetilde{x}}^{\zeta}(C)\Big) \\
&= H_{\widetilde{\mu}_{\widetilde{x}}^{\zeta}}(\beta_{\widetilde{x}})+\sum_{C\in\beta_{\widetilde{x}}}\widetilde{\mu}_{\widetilde{x}}^{\zeta}(C)\sup_{\widetilde{y}\in C}(S_n\widetilde{\phi})(\widetilde{y}) \\
&\ge H_{\widetilde{\mu}_{\widetilde{x}}^{\zeta}}(\beta_{\widetilde{x}})+\int_{M^f}S_n\widetilde{\phi}\mathrm{d}\widetilde{\mu}_{\widetilde{x}}^{\zeta} \\
&\ge H_{\widetilde{\mu}_{\widetilde{x}}^{\zeta}}(\mathcal{U}_0^{n-1})+\int_{M^f}S_n\widetilde{\phi}\mathrm{d}\widetilde{\mu}_{\widetilde{x}}^{\zeta}. \qquad (6.4)
\end{aligned}$$

By (6.3) and (6.4), we know that

$$\begin{aligned}
h_\mu^u(f,\mathcal{U}|\zeta)+\int_M\phi\mathrm{d}\mu &= \int_{M^f}\Big(H_{\widetilde{\mu}_{\widetilde{x}}^{\zeta}}(\mathcal{U}_0^{n-1})+\int_{M^f}S_n\widetilde{\phi}\mathrm{d}\widetilde{\mu}_{\widetilde{x}}^{\zeta}\Big)\mathrm{d}\mu(\widetilde{x}) \\
&\le \limsup_{n\to\infty}\frac{1}{n}\int_{M^f}\log P^u(f,\phi,\widetilde{x},\delta,n,\mathcal{U})\mathrm{d}\mu(\widetilde{x}) \\
&\le \int_{M^f}\limsup_{n\to\infty}\frac{1}{n}\log P^u(f,\phi,\widetilde{x},\delta,n,\mathcal{U})\mathrm{d}\mu(\widetilde{x}) \\
&= \int_{M^f}\log P^u(f,\phi,\delta,\mathcal{U})\mathrm{d}\mu(\widetilde{x}) \\
&\le P^u(f,\phi,\mathcal{U}), \qquad (6.5)
\end{aligned}$$

where in the third inequality, Fatou's lemma is applied. This completes the proof.

Proof of Theorem C By Proposition 6.4, Theorem A and the principle for unstable pressure for partially hyperbolic endomorphisms obtained in [17], we have

$$\begin{aligned}P^u(f,\phi) &= \sup\Big\{h_\mu^u(f|\zeta) + \int_M \phi \mathrm{d}\mu \colon \mu \in \mathcal{M}(f)\Big\} \\ &= \sup\Big\{h_\mu^u(f,\mathcal{U}|\zeta) + \int_M \phi \mathrm{d}\mu \colon \mu \in \mathcal{M}(f)\Big\} \\ &\le P^u(f,\phi,\mathcal{U}).\end{aligned}$$

On the other hand, it is clear that when $\mathcal{U} \in \mathcal{C}^o_{M^f}$, $P^u(f,\phi,\mathcal{U}) \le P^u(f,\phi)$, which completes the proof.

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