

Riemannian Geometry on Hom- ρ -commutative Algebras

Zahra BAGHERI¹ Esmail PEYGHAN¹

Abstract Recently, some concepts such as Hom-algebras, Hom-Lie algebras, Hom-Lie admissible algebras, Hom-coalgebras are studied and some classical properties of algebras and some geometric objects are extended on them. In this paper by recalling the concept of Hom- ρ -commutative algebras, the authors intend to develop some of the most classical results in Riemannian geometry such as metric, connection, torsion tensor, curvature tensor on it and also they discuss about differential operators and get some results of differential calculus by using them. The notions of symplectic structures and Poisson structures are included and an example of ρ -Poisson bracket is given.

Keywords Extended hyper-plan, Hom- ρ -commutative algebra, ρ -Poisson bracket

2000 MR Subject Classification 53B05, 53B21, 53C05

1 Introduction

One branch of differential geometry is Riemannian geometry that studies Riemannian manifolds (a smooth manifold with a Riemannian metric) (see [26] for more details). Riemannian geometry was first brought up by Bernhard Riemann in the nineteenth century. The concept of linear connection is one of the main concepts of the Riemannian geometry, which arose by the idea of parallel transport along a path in a Riemannian manifold at the end of 19th century (see [10, 27]). There is no direct and quick way to companion between distance points of a curve space, however the connection permits to contrast what is happening at these points. Earlier, in the 1910's, Albert Einstein discovered that the Riemannian geometry is substantial to general relativity theory. It is also the foundational revelation for gauge theories. This division into two branches has led to many representations tending to either the specific (e.g. presented in tensor notation assuming a coordinate frame and zero torsion) or the abstract (e.g. using the language of fiber bundles). By worth of its applications, the Riemannian geometry stands at the nucleus of modern mathematics.

Differential calculus is a branch of mathematics concerned with the determination, properties and applications of derivatives and differentials in study of functions. The development of differential calculus is closely dealing with the concept of integral calculus. In this approach, the differential calculus on the manifold is deduced from the properties of the manifold and it involves functions on the manifold, differential operators, differential forms and derivatives. If we denote by d the exterior derivative on differential forms, then the operator d satisfies $d^2 = 0$ that is an important property of this calculus.

Definition of a multiplication over a vector space was the original notion of Hom-algebra structure, where the structure is twisted by a homomorphism. The structure of Hom-Lie

Manuscript received April 23, 2021. Revised August 27, 2021.

¹Department of Mathematics, Faculty of Science, Arak University, Arak, 38156-8-8349, Iran.

E-mail: z.bagheri6848@gmail.com e-peyghan@araku.ac.ir

algebra appeared first as a generalization of Lie algebra in [13] by Hartwig et al. Physics and deformations of Lie algebras, in particular Lie algebra of vector fields were the stimulants to study Hom-Lie structures. Lie algebras are special cases of Hom-Lie algebras in which ϕ is the identity map. Also, q -deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra. Later Hom-Lie algebras were extended to Hom-associative algebras by Makhlof and Silverstrov in [18] and to quasi-Hom Lie and quasi-Lie algebras by Larsson and Silvestrov in [14–15]. Other interesting Hom-algebraic structures, such as Leibniz and Hom-Lie admissible algebras, Hom-Lie superalgebras, Hom-alternative algebras, Hom-Hopf algebras, Hom-coalgebras were studied in [2, 3, 12, 17, 19–20, 25, 28].

Non-commutative geometry is a branch of mathematics concerned with a geometric approach to non-commutative algebras and with the construction of spaces that are presented by non-commutative algebras of functions. Extension of the concept of differential forms on manifolds plays the basic role in non-commutative geometry (see [9, 11, 16, 23] for instance). Important examples of non-commutative geometry are ρ -commutative algebras. They have a great ability to generalize geometric objects. Accordingly, Riemannian geometry and its objects such as metric, connection, curvature, torsion, differential form and also differential calculus and application to hyperplane are discussed on ρ -commutative algebra by Bongaarts, Ciupala and Ngakeu in [6–8, 21]. In this paper we recall and study Hom- ρ -commutative algebra and develop some of the most classical results in Riemannian geometry and differential calculus on it.

This paper is arranged as follows. In Section 2, we recall some necessary background knowledge including ρ -commutative and Hom- ρ -commutative algebras, Hom-associative and Hom- ρ -commutative Lie algebras. In Section 3, we define p -forms and wedge product on Hom- ρ -algebras and the reader will get some important properties of ρ -tensor products, this section further develops the foundational topics for Riemannian manifolds, metric, connection, torsion tensor and curvature tensor are included. Also, we check some examples, properties and lemmas to obtain important results. Section 4 has been assigned to discuss about representations, cochain, Hom-cochain and some results will be derived of differential calculus. Also, symplectic structures and Poisson brackets are studied in this section.

2 Hom- ρ -Commutative Algebra

In this section, we summarize some definitions concerning ρ -commutative and Hom- ρ -commutative algebras and related results.

Let A be an associative and unital algebra over a field k ($k = \mathbb{R}$ or $k = \mathbb{C}$), grading by an abelian group $(G, +)$, i.e., the vector space A has a G -grading $A = \bigoplus_{a \in G} A_a$ such that $A_a A_b \subset A_{a+b}$. A map $\rho : G \times G \rightarrow k^*$ is called a two-cycle if the following conditions hold

$$\rho(a, b) = \rho(b, a)^{-1}, \quad a, b \in G, \quad (2.1)$$

$$\rho(a + b, c) = \rho(a, c)\rho(b, c), \quad a, b, c \in G. \quad (2.2)$$

The above conditions say that $\rho(a, b) \neq 0$, $\rho(0, b) = 1$ and $\rho(c, c) = \pm 1$ for all $a, b, c \in A$, $c \neq 0$.

The ρ -commutator of two homogeneous elements f and g of A is

$$[f, g]_\rho = fg - \rho(|f|, |g|)gf, \quad (2.3)$$

where $|f|$ is the G -degree of a (non-zero) homogeneous element $f \in A$ and the set of homogeneous elements in A is denoted by $Hg(A)$.

A ρ -commutative algebra is a G -graded algebra A with a given two-cycle ρ such that $fg = \rho(|f|, |g|)gf$ for all homogeneous elements f and g in A (i.e., $[f, g]_\rho = 0$).

In the following, we have some preliminary definitions from [1, 5].

Definition 2.1 A Hom- ρ -algebra is a quadruple (A, \cdot, ρ, ϕ) consisting of a G -graded vector space A , i.e., $A = \bigoplus_{a \in G} A_a$, an even bilinear map $\cdot : A \times A \rightarrow A$, i.e., $A_a \cdot A_b \subseteq A_{a+b}$ for all $a, b \in G$, a two-cycle $\rho : G \times G \rightarrow k^*$ and an even linear map $\phi : A \rightarrow A$. In addition, if $f \cdot g = \rho(|f|, |g|)g \cdot f$, for any $f, g \in \text{Hg}(A)$, we have a Hom- ρ -commutative algebra.

A Hom- ρ -algebra (A, \cdot, ρ, ϕ) is called a Hom-associative ρ -algebra if

$$\phi(f)(g \cdot h) = (f \cdot g)\phi(h).$$

Moreover if $fg = \rho(|f|, |g|)gf$, (A, \cdot, ρ, ϕ) is called Hom-associative ρ -commutative algebra.

A Hom- ρ -algebra (A, \cdot, ρ, ϕ) is said to be multiplicative if ϕ is a morphism for \cdot , regular if ϕ is an automorphism for \cdot , and involutive if $\phi^2 = Id_A$ (see [1, 5]).

Example 2.1 The quaternion algebra H is a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra in the following sense. Associate the ‘‘Triple degree’’ to the standard basis elements of H ,

$$\varepsilon = (0, 0, 0), \quad i = (0, 1, 1), \quad j = (1, 0, 1), \quad k = (1, 1, 0),$$

where ε denotes the unit and the following multiplication conditions are imposed

- i) $i^2 = j^2 = k^2 = -1$,
- ii) $ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j$.

Also the two-cocycle ρ is defined by $\rho(a, b) = (-1)^{\langle a, b \rangle}$, where $\langle a, b \rangle$ is the usual scalar product of 3-vectors. Indeed $\langle i, j \rangle = 1$ and similarly for k , so that i, j and k , ρ -commute with each other. But $\langle i, i \rangle = \langle j, j \rangle = \langle k, k \rangle = 2$, so that i, j and k commute with themselves. Thus, quaternion algebra H is a ρ -commutative algebra. If we set linear map $\phi_H(i) = ai, \phi_H(j) = bj, \phi_H(k) = ck$, $a, b, c \in \mathbb{C}$, then we have a Hom- ρ -commutative quaternion algebra. But, H is a Hom-associative ρ -commutative algebra if $a = b = c$.

Definition 2.2 A Hom- ρ -Lie algebra is a G -graded vector space A together with a bilinear map $[\cdot, \cdot]_\rho : A \times A \rightarrow A$, a two-cycle ρ and a linear map $\phi : A \rightarrow A$ satisfying the following relations

- $[A_g, A_{g'}]_\rho \subset A_{g+g'}$,
- $[f, g]_\rho = -\rho(|f|, |g|)[g, f]_\rho$,
- $\rho(|h|, |f|)[\phi(f), [g, h]_\rho]_\rho + \rho(|g|, |h|)[\phi(h), [f, g]_\rho]_\rho + \rho(|f|, |g|)[\phi(g), [h, f]_\rho]_\rho = 0$.

The second condition is called ρ -antisymmetry and the third condition is called ρ -Jacobi identity.

Proposition 2.1 (see [29]) The multiplex $(A, \cdot, \rho, \phi, [\cdot, \cdot]_\rho)$ consisting of a Hom-associative ρ -algebra (A, \cdot, ρ, ϕ) and ρ -commutator $[f, g]_\rho = fg - \rho(|f|, |g|)gf$ is a Hom- ρ -Lie algebra, called Hom- ρ -commutative Lie algebra.

A Hom- ρ -Lie algebra $(A, \rho, \phi, [\cdot, \cdot]_\rho)$ is called multiplicative if $\phi[f, g]_\rho = [\phi(f), \phi(g)]_\rho$, regular if ϕ is an automorphism and involutive if ϕ is an involution, that is $\phi^2 = Id_A$ (see [1]).

Definition 2.3 (see [5]) Let (A, \cdot, ρ, ϕ) be a Hom- ρ -algebra. A ρ -derivation of degree $|X|$ on A is a linear map $X : A \rightarrow A$ such that

$$X(f \cdot g) = X(f) \cdot g + \rho(|X|, |f|)f \cdot X(g). \tag{2.4}$$

If $\rho\text{-Der}A$ is denoted the space of all ρ -derivations of A , then for $X \in Hg(\rho\text{-Der}A)$ and $Y \in Hg(\rho\text{-Der}A)$, the ρ -commutator of X, Y , defined by $[X, Y]_\rho = X \circ Y - \rho(|X|, |Y|)Y \circ X$, is a ρ -derivation. Furthermore, when A is Hom- ρ -commutative, $\rho\text{-Der}A$ is also a A -bimodule with actions \triangleright and \triangleleft defined by

$$(f \triangleright X) \cdot g = f(X \cdot g), \quad X \triangleleft f = \rho(|X|, |f|)f \triangleright X, \tag{2.5}$$

respectively. In fact any G -graded left module M over a Hom- ρ -commutative algebra A is a A -bimodule with

$$f \triangleright (X \triangleleft g) = (f \triangleright X) \triangleleft g, \quad f, g \in A, X \in M. \tag{2.6}$$

Moreover, $\rho\text{-Der}A$ equipped with the ρ -commutator is a ρ -Lie algebra.

We will simply denote the left action $f \triangleright X$ as fX and $\rho(|X|, |Y|)$ by $\rho(X, Y)$.

3 p -Forms, ρ -Tensor Product

In this section, at first we define the p -forms and the wedge product, next we turn to a brief discussion of the concept of tensor products to develop Riemannian geometry on Hom- ρ -commutative algebras, then we concentrate on metrics to introduce the notion of linear connection and define the torsion and curvature associated to the connection. Later, we survey some properties and give to the reader some main points. Symbols $Hg(\Omega^1(A))$ and $Hg(\rho\text{-Der}A)$ apply respectively for homogeneous elements of $\Omega^1(A)$ and $\rho\text{-Der}A$.

Definition 3.1 *A p -form on Hom- ρ -algebra A is a p -linear map $\alpha_p : \times^p(\rho\text{-Der}A) \rightarrow A$, p -linear in sense of left A -modules*

- $\alpha_p(fX_1, \dots, X_p) = f\alpha_p(X_1, \dots, X_p)$,
- $\alpha_p(X_1, \dots, X_j f, X_{j+1}, \dots, X_p) = \alpha_p(X_1, \dots, X_j, fX_{j+1}, \dots, X_p)$,
- $\alpha_p(X_1, \dots, X_j, X_{j+1}, \dots, X_p) = -\rho(X_j, X_{j+1})\alpha_p(X_1, \dots, X_{j+1}, X_j, \dots, X_p)$,

where $j = 1, \dots, p-1$, $X_k \in Hg(\rho\text{-Der}A)$, $k = 1, \dots, p$, $f \in A$ and Xf is the right A -action on $\rho\text{-Der}A$ defined by (2.5). Let $\Omega^p(A)$ denote the set of p -forms. Then $\Omega^p(A)$ is a G -graded A -bimodule with

$$\begin{aligned} (\alpha_p f)(X_1, \dots, X_p) &= \alpha_p(X_1, \dots, X_p)f, \\ f\alpha_p &= \rho(f, \alpha_p)\alpha_p f, \\ |\alpha_p| &= |\alpha_p(X_1, \dots, X_p)| - (|X_1| + \dots + |X_p|). \end{aligned}$$

Then we have the exterior algebra $\Omega(A) = \bigoplus_{p \geq 0} \Omega^p(A)$ with $\Omega^0(A) = A$.

Definition 3.2 *The wedge product \wedge in $\Omega(A)$ is the map*

$$\wedge : \Omega^p(A) \times \Omega^q(A) \rightarrow \Omega^{p+q}(A),$$

defined by

$$\begin{aligned} &(\alpha \wedge \beta)(X_1, \dots, X_p, \dots, X_{p+q}) \\ &= \sum_{\sigma \in S_{p,q}} \text{sign}(\sigma) \times \rho\left(\sum_{j=p+1}^{p+q} X_{\sigma(j)}, \alpha\right) \rho(X_{\sigma(k)}, X_{\sigma(l)}) \\ &\quad \times \alpha(X_{\sigma(1)}, \dots, X_{\sigma(p)}) \beta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}) \end{aligned}$$

for $\alpha \in Hg(\Omega^p(A))$ and $\beta \in Hg(\Omega^q(A))$, where $S_{p,q}$ is the set of permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(p)$ and $\sigma(p+1) < \sigma(k+2) < \dots < \sigma(p+q)$, $l < k$ and $\sigma(l) > \sigma(k)$.

ρ -Tensor Product: Let (A, \cdot, ρ, ϕ) be a Hom- ρ -commutative algebra and $(\rho\text{-Der}A, [\cdot, \cdot]_\rho, \rho, \phi_A)$ be a Hom- ρ -Lie algebra, where $\rho\text{-Der}A$ is the set of all ρ -derivations on Hom- ρ -commutative algebra A .

For $\alpha_1, \dots, \alpha_p \in Hg(\Omega^1(A))$, we make a p -linear homogeneous map s of G -degree $|s| = \sum_{i=1}^p |\alpha_i|$ as $s = \alpha_1 \otimes_\rho \dots \otimes_\rho \alpha_p$ and define it by

$$\alpha_1 \otimes_\rho \dots \otimes_\rho \alpha_p(X_1, \dots, X_p) := \prod_{i=1}^p \alpha_i(X_i) \prod_{k=1}^{p-1} \rho\left(\sum_{j=k+1}^p X_j, \alpha_k\right),$$

where $X_1, \dots, X_p \in Hg(\rho\text{-Der}A)$, that satisfy the following relations

$$\begin{aligned} s(fX_1, X_2, \dots, X_p) &= fs(X_1, X_2, \dots, X_p), \\ s(X_1, \dots, X_i \triangleleft f, X_{i+1}, \dots, X_p) &= s(X_1, \dots, X_i, fX_{i+1}, \dots, X_p). \end{aligned}$$

Let $T_\rho^{\otimes p}$ denote the linear space generated by elements $s = \alpha_1 \otimes_\rho \dots \otimes_\rho \alpha_p$. $T_\rho^{\otimes p}$ is a A -bimodule with the following actions

$$\begin{aligned} (\alpha_1 \otimes_\rho \dots \otimes_\rho \alpha_p)f &= \alpha_1 \otimes_\rho \dots \otimes_\rho (\alpha_p f), \\ f(\alpha_1 \otimes_\rho \dots \otimes_\rho \alpha_p) &= (f\alpha_1) \otimes_\rho \dots \otimes_\rho \alpha_p = \rho\left(f, \sum_{i=1}^p \alpha_i\right) \alpha_1 \otimes_\rho \dots \otimes_\rho (\alpha_p f). \end{aligned}$$

We have the ρ -tensor algebra $T^\otimes = \bigoplus_{p \geq 0} T_\rho^{\otimes p}$ with $T_\rho^{\otimes 0} = A$ and natural algebra structure \otimes_ρ , which is defined on homogeneous elements in T^\otimes by

$$\begin{aligned} T_p \otimes_\rho T_q(X_1, X_2, \dots, X_p, X_{p+1}, \dots, X_{p+q}) \\ = T_p(X_1, X_2, \dots, X_p)T_q(X_{p+1}, \dots, X_{p+q}) \times \rho\left(\sum_{j=1}^q X_{p+j}, T_p\right), \end{aligned}$$

$\forall T_p \in T_\rho^{\otimes p}, T_q \in T_\rho^{\otimes q}, X_1, \dots, X_{p+q} \in Hg(\rho\text{-Der}A)$. Also, $T_p \otimes_\rho T_q$ has the degree $|T_p \otimes_\rho T_q| = |T_p| + |T_q|$. Notice that $T_\rho^{\otimes p}$ coincides with the space of all ρ - p -linear maps on $\times^p \rho\text{-Der}A$, if $\Omega^1(A)$ and $\rho\text{-Der}_\phi A$ are finitely generated. In this case, $T \in T_\rho^{\otimes p}$ is called covariant ρ -tensor.

Definition 3.3 A covariant 2-tensor $g \in T_\rho^{\otimes 2}$ is called non-degenerate tensor if

$$\forall \alpha \in Hg(\Omega^1(A)), \quad \exists X_\alpha \in \rho\text{-Der}A : g(Y, X_\alpha) = \alpha(Y), \quad \forall Y \in \rho\text{-Der}A.$$

A (homogeneous) metric g of degree $|g|$ on a ρ -commutative algebra A is an homogeneous symmetric and nondegenerate covariant 2-tensor on A .

Definition 3.4 Let (A, \cdot, ρ, ϕ) be a multiplicative Hom- ρ -commutative algebra and $(\rho\text{-Der}A, [\cdot, \cdot]_\rho, \rho, \phi_A)$ be a Hom- ρ -Lie algebra, where $\rho\text{-Der}A$ is the set of all ρ -derivations on A endowed with a bilinear map $g(\cdot, \cdot)$ such that for any $X, Y \in Hg(\rho\text{-Der}(A))$ the following equations are satisfied

- (i) $g(fX, Y) = fg(X, Y), \quad \forall X, Y \in \rho\text{-Der}(A),$
- (ii) $g(X, Y) = \rho(X, Y)g(Y, X), \quad \forall X, Y \in Hg(\rho\text{-Der}(A)),$
- (iii) $g(X, Y) = g(\phi_A(X), \phi_A(Y)), \quad \forall X, Y \in \rho\text{-Der}(A),$
- (iv) the map $\tilde{g} : \rho\text{-Der}A \rightarrow \Omega^1(A)$ defined by $\tilde{g}(X)Y = g(Y, X)$ is a left A -module isomorphism.

Then, we say that A admits a metric $g(\cdot, \cdot)$ (not necessary homogeneous). Also, homogeneous metric \tilde{g} satisfies $g(X, fY) = g(Xf, Y)$ and $\tilde{g}(aX) = a\tilde{g}(X)$, $\tilde{g}(X \triangleleft a) = \tilde{g}(X)a\rho(a, g)$, $a \in Hg(A)$.

Definition 3.5 A linear connection on ρ -Der A is a linear map

$$\begin{cases} \nabla : \rho\text{-Der}A \rightarrow \text{End}(\rho\text{-Der}A), \\ X \rightarrow \nabla_X, \end{cases}$$

such that

$$\begin{aligned} \nabla_{aX}Y &= a\nabla_XY, \quad a \in A, X, Y \in \rho\text{-Der}A, \\ \nabla_X(aY) &= (X \cdot a)Y + \rho(X, a)\phi(a)\nabla_XY, \quad a \in Hg(A), X \in Hg(\rho\text{-Der}A). \end{aligned}$$

Definition 3.6 Let $T \in T^{\otimes}$, $X \in Hg(\rho\text{-Der}A)$ and $\nabla_X T$ denote the covariant derivative of T , which is a p -linear map. This map is defined on $\rho\text{-Der}A$ by

$$\begin{aligned} \rho\left(X, \sum_{i=1}^p X_i\right) \nabla_X T(X_1, \dots, X_p) &= \phi_A(X) \cdot T(X_1, \dots, X_p) \\ &\quad - \sum_{i=1}^p \rho\left(X, \sum_{l=1}^{i-1} X_l\right) T(\phi_A(X_1), \dots, \nabla_X X_i, \dots, \phi_A(X_p)). \end{aligned}$$

ρ -tensor T is said to be parallel or compatible with respect to a linear connection ∇ if $\nabla T = 0$.

Connection ∇ is called compatible with metric g if

$$\phi_A(X) \cdot g(Y, Z) = g(\nabla_X Y, \phi_A(Z)) + \rho(X, Y)g(\phi_A(Y), \nabla_X Z), \quad \forall X, Y, Z \in Hg(\rho\text{-Der}A).$$

The curvature of a linear connection ∇ is the map

$$\begin{cases} R : \rho\text{-Der}A \times \rho\text{-Der}A \rightarrow \text{End}(\rho\text{-Der}A), \\ (X, Y) \rightarrow R(X, Y) := R_{XY}, \end{cases}$$

defined by

$$R(X, Y)(Z) = \nabla_{\phi_A(X)} \nabla_Y Z - \rho(X, Y) \nabla_{\phi_A(Y)} \nabla_X Z - \nabla_{[X, Y]_\rho} \phi_A(Z).$$

Lemma 3.1 By the definition of curvature R , we get the following equalities

- a) $R(X, Y) = -\rho(X, Y)R(Y, X)$,
- b) $R(aX, Y)Z = \phi(a)R(X, Y)Z - \rho(X + a, Y)(\phi_A(Y) \cdot a)\nabla_X Z$
 $+ \rho(X + a, Y)(Y \cdot a)\nabla_X \phi_A(Z) + a\nabla_{\phi_A(X)} \nabla_Y Z - \phi(a)\nabla_{\phi_A(X)} \nabla_Y Z$,
- c) $R(X, aY)(Z) = \rho(X, a)\phi(a)(R(X, Y)Z) + (\phi_A(X) \cdot a)\nabla_Y Z$
 $- (X \cdot a)\nabla_Y \phi_A(Z) - \rho(X, a + Y)a\nabla_{\phi_A(Y)} \nabla_X Z$
 $+ \rho(X, a + Y)\phi(a)\nabla_{\phi_A(Y)} \nabla_X Z$
 $- \phi(a)\nabla_{[X, Y]_\rho} \phi_A(Z) + \rho(X, a)\phi(a)\nabla_{[X, Y]_\rho} \phi_A(Z)$,
- d) $R(X, Y)(aZ) = \rho(X + Y, a)\phi^2(a)(R(X, Y)Z) + (\phi_A(X) \cdot Y \cdot a)Z$
 $- \rho(X, Y)(\phi_A(Y) \cdot X \cdot a)Z + \rho(Y, a)(\phi_A(X) \cdot \phi(a))\nabla_Y Z$

$$\begin{aligned}
 & -\rho(X, Y)\rho(X, a)(\phi_A(Y) \cdot \phi(a))\nabla_X Z \\
 & -\rho(X, Y)\rho(Y, X+a)\rho(X, a)\phi(a) \cdot X \\
 & -\rho(X, Y)\rho(Y, X+a)\phi(X \cdot a)\nabla_{\phi_A(Y)} Z \\
 & +\rho(X, Y+a)\phi(Y \cdot a)\nabla_{\phi_A(X)} Z \\
 & -(X \cdot Y \cdot a)\phi_A(Z) + \rho(X, Y)(Y \cdot X \cdot a)\phi_A(Z) \\
 & -\rho(X+Y, a)\phi(a)\nabla_{[X, Y]_\rho}\phi_A(Z) + \rho(X+Y, a)\phi^2(a)\nabla_{[X, Y]_\rho}\phi_A(Z).
 \end{aligned}$$

Proof For sake of the brevity, we proof the relation (b). By the definitin of curvature R , we have

$$R(aX, Y)Z = \nabla_{\phi_A(aX)}\nabla_Y Z - \rho(X+a, Y)\nabla_{\phi_A(Y)}\nabla_{aX} Z - \nabla_{[aX, Y]_\rho}\phi_A(Z).$$

Now, by the properties of connection ∇ and the relation $[aX, Y]_\rho = -\rho(X+a, Y)(Y \cdot a)X + \phi(a)[X, Y]_\rho$, we get

$$\begin{aligned}
 R(aX, Y)Z &= a\nabla_{\phi_A(X)}\nabla_Y Z - \rho(X+a, Y)(\phi_A(Y) \cdot a)\nabla_X Z \\
 &\quad - \rho(X+a, Y)\rho(Y, a)\phi(a)\nabla_{\phi_A(Y)}\nabla_X Z \\
 &\quad - (Y \cdot a)\nabla_X \phi_A(Z) - \phi(a)\nabla_{[X, Y]_\rho}\phi_A(Z) \\
 &= \phi(a)R(X, Y)Z - \rho(X+a, Y)(\phi_A(Y) \cdot a)\nabla_X Z \\
 &\quad + \rho(X+a, Y)(Y \cdot a)\nabla_X \phi_A(Z) + a\nabla_{\phi_A(X)}\nabla_Y Z - \phi(a)\nabla_{\phi_A(X)}\nabla_Y Z.
 \end{aligned}$$

Also, the torsion of the connection ∇ is defined by

$$\begin{cases} T_\nabla : \rho\text{-Der}A \times \rho\text{-Der}A \rightarrow \rho\text{-Der}A, \\ T_\nabla(X, Y) = \nabla_X Y - \rho(X, Y)\nabla_Y X - [X, Y]_\rho. \end{cases}$$

Connection ∇ is called torsion-free if

$$T_\nabla = 0, \quad \text{i.e.,} \quad [X, Y]_\rho = \nabla_X Y - \rho(X, Y)\nabla_Y X, \quad \forall X, Y \in Hg(\rho\text{-Der}A).$$

In the similar way of the curvature, it is easy to see that the torsion has the following properties

$$\begin{aligned}
 T(aX, Y) &= T(X, Y) + a\nabla_X Y - \phi(a)\nabla_X Y, \\
 T(X, aY) &= \rho(X, a)\phi(a)T(X, Y) - \rho(X, a+Y)a\nabla_Y X \\
 &\quad + \rho(X+a, Y)\phi(a)\nabla_Y X - \rho(X, a)\phi(a)[X, Y]_\rho + \phi(a)[X, Y]_\rho.
 \end{aligned}$$

Theorem 3.1 *There exists a unique linear connection on every Hom- ρ -commutative algebra with homogeneous metric g , which is torsion-free and compatible with metric g . This connection is called Levi-Civita connection associated to g .*

Proof By the compatibility condition of the connection ∇ with the metric g , we can easily show that the Koszul equation is given by

$$\begin{aligned}
 2\rho(Z, Y)g(\phi_A(X), \nabla_Y Z) &= \rho(X, Z)\phi_A(Z) \cdot g(X, Y) + \rho(X, Z)g(\phi_A(Z), [X, Y]_\rho) \\
 &\quad - \phi_A(X) \cdot g(Z, Y) - \rho(X, Z)g([Z, X]_\rho, \phi_A(Y)) \\
 &\quad + \rho(Z, Y)\rho(X, Y+Z)\phi_A(Y) \cdot g(Z, X) \\
 &\quad + \rho(Z, Y)\rho(X, Y+Z)g([Y, Z]_\rho, \phi_A(X)),
 \end{aligned}$$

where $X, Y, Z \in Hg(\rho\text{-Der}A)$. So, the proof is clear.

Christoffel coefficients: Let us assume that A is generated as algebra by n homogeneous coordinates x_1, x_2, \dots, x_n and $\rho\text{-Der}A$ is generated by $\partial_1, \dots, \partial_n$, where $\partial_i = \frac{\partial}{\partial x_i}$ such that $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$, $|\partial_i| = -|x_i|$. Also, we assume that $\Omega^1(A) = \langle dx^1, \dots, dx^n \rangle$ such that $|dx^i| = |x_i|$. Let (dx^i) be the dual basis of (∂_i) and set

$$g = dx^m \otimes dx^n g_{mn} \quad (\text{summation}).$$

Proposition 3.1 *Assuming that $[\partial_i, \partial_j] = 0$ and $\nabla_{\partial_i} \partial_j = \sum_s \phi_A(\partial_s) \triangleleft \Gamma_{ij}^s$, the ρ -Christoffel coefficients Γ_{ij}^t of G -degree $|\Gamma_{ij}^t| = x_t - x_i - x_j$ of the Levi-Civita connection ∇ are as follows*

$$\begin{aligned} \Gamma_{ij}^t &= \frac{1}{2} \rho(g, x_t - x_i - x_j) \times \sum_k \tilde{g}^{tk} \{ -\phi_A(\partial_k) \cdot \tilde{g}_{ij} + \rho(x_i + x_k, x_j) \phi_A(\partial_j) \cdot \tilde{g}_{ki} \\ &\quad + \rho(x_k, x_i + x_j) \phi_A(\partial_i) \cdot \tilde{g}_{jk} \}. \end{aligned} \quad (3.1)$$

Proof Setting $X = \partial_k$, $Y = \partial_i$ and $Z = \partial_j$ in the Koszul formula, we get

$$\begin{aligned} &2g\left(\phi_A(\partial_k), \sum_s \phi_A(\partial_s) \triangleleft \Gamma_{ij}^s\right) \\ &= \rho(\partial_k, \partial_i + \partial_j) \phi_A(\partial_i) \cdot \tilde{g}_{jk} + \rho(\partial_k + \partial_i, \partial_j) \phi_A(\partial_j) \cdot \tilde{g}_{ki} - \phi_A(\partial_k) \cdot \tilde{g}_{ij}, \end{aligned}$$

thus

$$\begin{aligned} &2 \sum_s \rho(\partial_s, \Gamma_{ij}^s) \rho(\partial_k, \Gamma_{ij}^s) \Gamma_{ij}^s \tilde{g}_{ks} \\ &= \rho(\partial_k, \partial_i + \partial_j) \phi_A(\partial_i) \cdot \tilde{g}_{jk} + \rho(\partial_k + \partial_i, \partial_j) \phi_A(\partial_j) \cdot \tilde{g}_{ki} - \phi_A(\partial_k) \cdot \tilde{g}_{ij}. \end{aligned}$$

Multiplying both sides of the above relation by \tilde{g}^{tk} , we get

$$\begin{aligned} &2 \sum_s \rho(-x_t - x_k, \Gamma_{ij}^t) \rho(\tilde{g}^{tk}, \Gamma_{ij}^t) \Gamma_{ij}^t \\ &= \rho(x_k, x_i + x_j) \phi_A(\partial_i) \cdot \tilde{g}_{jk} + \rho(x_k + x_i, x_j) \phi_A(\partial_j) \cdot \tilde{g}_{ki} - \phi_A(\partial_k) \cdot \tilde{g}_{ij}. \end{aligned}$$

Since $g = dx^t \otimes dx^k g_{tk}$, so $|g| = x_t + x_k + g_{tk}$ and then $-x_t - x_k = g_{tk} - |g|$. Therefore

$$\begin{aligned} &2 \sum_s \rho(g_{tk}, \Gamma_{ij}^t) \rho(-g, \Gamma_{ij}^t) \rho(-g_{tk}, \Gamma_{ij}^t) \Gamma_{ij}^t \\ &= \rho(x_k, x_i + x_j) \phi_A(\partial_i) \cdot \tilde{g}_{jk} \\ &\quad + \rho(x_k + x_i, x_j) \phi_A(\partial_j) \cdot \tilde{g}_{ki} - \phi_A(\partial_k) \cdot \tilde{g}_{ij}. \end{aligned}$$

Thus, in the last step, the result will be obtained, that is

$$\Gamma_{ij}^t = \frac{1}{2} \rho(g, \Gamma_{ij}^t) \sum_k \tilde{g}^{tk} \{ \rho(x_k, x_i + x_j) \phi_A(\partial_i) \cdot \tilde{g}_{jk} + \rho(x_k + x_i, x_j) \phi_A(\partial_j) \cdot \tilde{g}_{ki} - \phi_A(\partial_k) \cdot \tilde{g}_{ij} \}.$$

Example 3.1 Extended-hyperplane $A_q^2 = \langle 1, x, y, x^{-1}, y^{-1} : x \cdot y = qy \cdot x \rangle$ is a $\mathbb{Z} \times \mathbb{Z}$ -graded algebra. Considering the two-cycle

$$\rho(n, n') = q^{\sum_{j,k=1}^n n_j n'_k \alpha_{jk}},$$

where $\alpha_{jk} = 1$ if $j < k$, 0 if $j = k$ and -1 if $j > k$, A_q^2 is a ρ -commutative algebra. In this example, we intend to define morphisms ϕ and $\phi_{A_q^2}$ on the A_q^2 and $\rho\text{-Der}A_q^2$, respectively, to make Hom- ρ -commutative algebra and Hom- ρ -commutative Lie algebra, respectively. In the next, by considering the metric g and its components that is defined in [21], we obtain the ρ -Christoffel coefficients Γ_{ij}^t corresponding to $\phi_{A_q^2}$ by (3.1). If we define the linear map $\phi : A_q^2 \rightarrow A_q^2$ by

$$\phi(x) = ax, \quad \phi(y) = ay, \quad \phi(x^{-1}) = bx^{-1}, \quad \phi(y^{-1}) = by^{-1}, \quad a, b \in k,$$

then we have the Hom- ρ -commutative algebra A_q^2 . Note that $(A_q^2, \cdot, \rho, \phi)$ is a Hom-associative ρ -commutative algebra if $a = b$. The set $\rho\text{-Der}A_q^2$ of all ϕ - ρ -derivations on A_q^2 is a A_q^2 -bimodule generated by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, and $\Omega^1(A)$ is generated by dx, dy such that $dx_j(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$, $|\frac{\partial}{\partial x_i}| = -|x_i|$ and $|dx_i| = |x_i|$, where $x_1 = x$ and $x_2 = y$. We define the linear map $\phi_{A_q^2} : \rho\text{-Der}A_q^2 \rightarrow \rho\text{-Der}A_q^2$ by

$$\phi_{A_q^2}\left(\frac{\partial}{\partial x}\right) = \lambda\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y}, \quad \phi_{A_q^2}\left(\frac{\partial}{\partial y}\right) = (\lambda + \gamma)\frac{\partial}{\partial y},$$

where $\lambda, \gamma \in k$. All homogeneous metrics on $\rho\text{-Der}A_q^2$ were defined by [21],

$$g = g_{11}dx \otimes_{\rho} dx + g_{12}dx \otimes_{\rho} dy + qg_{21}dy \otimes_{\rho} dx + g_{22}dy \otimes_{\rho} dy.$$

Now, by using metric g , we try to find the following relation

$$g(X, Y) = g(\phi_{A_q^2}(X), \phi_{A_q^2}(Y)), \quad \forall X, Y \in Hg(\rho\text{-Der}A_q^2).$$

Case 1 If $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial x}$, we have $g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = g_{11}$. Also, we get

$$\begin{aligned} g\left(\phi_{A_q^2}\left(\frac{\partial}{\partial x}\right), \phi_{A_q^2}\left(\frac{\partial}{\partial x}\right)\right) &= g\left(\lambda\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y}, \lambda\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y}\right) \\ &= \lambda^2g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) + \lambda\gamma g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) + \gamma\lambda g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) \\ &\quad + \gamma^2g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \lambda^2g_{11} + \lambda\gamma(1+q)g_{12} + \gamma^2g_{22}. \end{aligned}$$

In this case, we find $\lambda^2 = 1$, $\lambda\gamma = 0$, $\gamma^2 = 0$, so $\lambda = \pm 1$, $\gamma = 0$.

Case 2 If $X = \frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial y}$, we have

$$\begin{cases} g\left(\phi_{A_q^2}\left(\frac{\partial}{\partial y}\right), \phi_{A_q^2}\left(\frac{\partial}{\partial y}\right)\right) = g\left((\lambda + \gamma)\frac{\partial}{\partial y}, (\lambda + \gamma)\frac{\partial}{\partial y}\right) = (\lambda + \gamma)^2g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = (\lambda + \gamma)^2g_{22}, \\ g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = g_{22}. \end{cases}$$

In this case, we find $(\lambda + \gamma)^2 = 1$.

Case 3 If $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$, we get $g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = g_{21} = qg_{12}$ and

$$\begin{aligned} g\left(\phi_{A_q^2}\left(\frac{\partial}{\partial x}\right), \phi_{A_q^2}\left(\frac{\partial}{\partial y}\right)\right) &= g\left(\lambda\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y}, (\lambda + \gamma)\frac{\partial}{\partial y}\right) = \lambda(\lambda + \gamma)g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \\ &\quad + \gamma(\lambda + \gamma)g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = q\lambda(\lambda + \gamma)g_{12} + \gamma(\lambda + \gamma)g_{22}. \end{aligned}$$

In this case, we find $\lambda(\lambda + \gamma) = 1, \gamma(\lambda + \gamma) = 0$.

Case 4 If $X = \frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial x}$, we have

$$\begin{cases} g\left(\phi_{A_q^2}\left(\frac{\partial}{\partial y}\right), \phi_{A_q^2}\left(\frac{\partial}{\partial x}\right)\right) = g\left((\lambda + \gamma)\frac{\partial}{\partial y}, \lambda\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y}\right) = \lambda(\lambda + \gamma)g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) \\ \qquad \qquad \qquad + \gamma(\lambda + \gamma)g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \lambda(\lambda + \gamma)g_{12} + \gamma(\lambda + \gamma)g_{22}, \\ g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) = g_{12}. \end{cases}$$

In this case, we also find $\lambda(\lambda + \gamma) = 1, \gamma(\lambda + \gamma) = 0$.

By the above cases, if $\gamma = 0$ and $\lambda = \pm 1$, we obtain

$$\phi_{A_q^2}\left(\frac{\partial}{\partial x}\right) = \pm \frac{\partial}{\partial x}, \quad \phi_{A_q^2}\left(\frac{\partial}{\partial y}\right) = \pm \frac{\partial}{\partial y}.$$

Therefore $\phi_{A_q^2}$ can be written in the following two cases

$$\begin{cases} \phi_{A_q^2}\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x}, & \phi_{A_q^2}\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}, \\ \phi_{A_q^2}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}, & \phi_{A_q^2}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}. \end{cases}$$

Also, we know that $\rho\text{-Der}A_q^2$ with ρ -commutator $[X, Y] = XY - \rho(X, Y)YX$ and the linear map $\phi_{A_q^2}$ is a Hom- ρ -commutative Lie algebra. But, note that for the elements of the basis of $\rho\text{-Der}A_q^2$, we have

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]_\rho = \frac{\partial}{\partial x}\frac{\partial}{\partial y} - \rho\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}\frac{\partial}{\partial x}. \tag{3.2}$$

On the other hand, since $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]_\rho \in \rho\text{-Der}A_q^2$, then we can write

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]_\rho = p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y},$$

where $p, q \in A$, so $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]_\rho(x) = p$ and $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]_\rho(y) = q$. But, (3.2) gives us $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]_\rho(x) = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]_\rho(y) = 0$.

Let us set $\tilde{g}_{mk} := \rho(\partial_m, \partial_k)g_{mk} = \rho(x_m, x_k)g_{mk}$ and continue with $g_{11} = x^{-2}, g_{12} = x^{-1}y^{-1}, g_{22} = y^{-2}$. In this case, g is a homogeneous metric on A_q^2 (of degree $(0, 0)$) if and only if

$$D = \frac{1 - q^2}{q^2}x^{-2}y^{-2}$$

is invertible, in other words if and only if $q \neq 0, 1, -1$. So, by the definition of \tilde{g} , we have

$$(\tilde{g}_{mk}) = \begin{pmatrix} x^{-2} & qx^{-1}y^{-1} \\ x^{-1}y^{-1} & y^{-2} \end{pmatrix}$$

and

$$(\tilde{g}^{mk}) = (\tilde{g}_{mk})^{-1} = \frac{1}{1 - q^2} \begin{pmatrix} x^2 & -qxy \\ -xy & y^2 \end{pmatrix}.$$

Here, it seems that, we are ready to find the ρ -Christoffel coefficients Γ_{ij}^t corresponding to each $\phi_{A_q^2}$. So, we need to consider the following cases:

If $\phi_{A_q^2}\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x}$, $\phi_{A_q^2}\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}$, we have

$$\Gamma_{11}^1 = x^{-1}, \quad \Gamma_{22}^2 = y^{-1}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{11}^2 = \Gamma_{22}^1 = 0.$$

If $\phi_{A_q^2}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}$, $\phi_{A_q^2}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}$, we can find the following ρ -Christoffel coefficients

$$\Gamma_{11}^1 = -x^{-1}, \quad \Gamma_{22}^2 = -y^{-1}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{11}^2 = \Gamma_{22}^1 = 0.$$

Definition 3.7 Let (A, \cdot, ρ, ϕ) be a multiplicative Hom- ρ -commutative algebra and $(\rho\text{-Der}A, [\cdot, \cdot]_\rho, \rho, \phi_A)$ be a multiplicative Hom- ρ -Lie algebra, where $\rho\text{-Der}A$ is the set of all ρ -derivation on Hom- ρ -commutative algebra A . For a Levi-Civita connection ∇ , we define the covariant derivation of R as follows

$$\begin{aligned} (\nabla_Z R)(X, Y) &= \nabla_{\phi_A^2(Z)} R(X, Y)(\cdot) - R(\nabla_Z X, \phi_A(Y))\phi_A(\cdot) \\ &\quad - \rho(Z, X)R(\phi_A(X), \nabla_Z Y)\phi_A(\cdot) \\ &\quad - \rho(Z, X + Y)R(\phi_A(X), \phi_A(Y))\nabla_Z(\cdot). \end{aligned} \quad (3.3)$$

Lemma 3.2 For $X, Y, Z, V, W \in \rho\text{-Der}A$, the curvature R satisfies the following equalities

- (a) $\rho(X, Y)R_{YZ}X + \rho(Y, Z)R_{ZX}Y + \rho(Z, X)R_{XY}Z = 0$ (Bianchi identity 1).
 (b) Second Bianchi identity for a torsion-free connection

$$\rho(V, X)R(X, Y, V, W) + \rho(Y, V)R(V, X, Y, W) + \rho(X, Y)R(Y, V, X, W) = 0.$$

- (c) If $\nabla_{\phi_A(X)}\phi_A(Y) = \phi_A(\nabla_X Y)$, then

$$\rho(Y, Z)(\nabla_Z R)(X, Y) + \rho(X, Y)(\nabla_Y R)(Z, X) + \rho(Z, X)(\nabla_X R)(Y, Z) = 0,$$

where $R(X, Y, V, W) := g(R_{XY}V, W)$.

Proof (a) By the definition of curvature R , we can find the following relations

$$\begin{aligned} \rho(Y, Z)R_{ZX}Y &= \rho(Y, Z)\{\nabla_{\phi_A(Z)}\nabla_X Y - \rho(Z, X)\nabla_{\phi_A(X)}\nabla_Z Y - \nabla_{[Z, X]_\rho}\phi_A(Y)\} \\ &= \rho(Y, Z)\nabla_{\phi_A(Z)}\nabla_X Y - \rho(Y, Z)\rho(Z, X)\rho(Z, Y)\nabla_{\phi_A(X)}\nabla_Y Z \\ &\quad - \rho(Y, Z)\rho(Z, X)\nabla_{\phi_A(X)}[Z, Y]_\rho - \rho(Y, Z)\rho(Z, Y)\rho(X, Y)\nabla_{\phi(Y)}[Z, X]_\rho \\ &\quad - \rho(Y, Z)[[Z, X]_\rho, \phi_A(Y)]_\rho, \end{aligned} \quad (3.4)$$

$$\begin{aligned} \rho(X, Y)R_{YZ}X &= \rho(X, Y)\{\nabla_{\phi_A(Y)}\nabla_Z X - \rho(Y, Z)\nabla_{\phi_A(Z)}\nabla_Y X - \nabla_{[Y, Z]_\rho}\phi_A(X)\} \\ &= \rho(X, Y)\nabla_{\phi_A(Y)}\nabla_Z X - \rho(X, Y)\rho(Y, Z)\rho(Y, X)\nabla_{\phi_A(Z)}\nabla_X Y \\ &\quad - \rho(X, Y)\rho(Y, Z)\nabla_{\phi_A(Z)}[Y, X]_\rho - \rho(X, Y)\rho(Y, X)\rho(Z, X)\nabla_{\phi_A(X)}[Y, Z]_\rho \\ &\quad - \rho(X, Y)[[Y, Z]_\rho, \phi_A(X)]_\rho \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \rho(Z, X)R_{XY}Z &= \rho(Z, X)\{\nabla_{\phi_A(X)}\nabla_Y Z - \rho(X, Y)\nabla_{\phi_A(Y)}\nabla_X Z - \nabla_{[X, Y]_\rho}\phi_A(Z)\} \\ &= \rho(Z, X)\nabla_{\phi_A(X)}\nabla_Y Z - \rho(Z, X)\rho(X, Y)\rho(X, Z)\nabla_{\phi_A(Y)}\nabla_Z X \\ &\quad - \rho(Z, X)\rho(X, Y)\nabla_{\phi_A(Y)}[X, Z]_\rho - \rho(Z, X)\rho(X, Z)\rho(Y, Z)\nabla_{\phi_A(Z)}[X, Y]_\rho \\ &\quad - \rho(Z, X)[[X, Y]_\rho, \phi_A(Z)]_\rho. \end{aligned} \quad (3.6)$$

Summing the relations (3.4)–(3.6), implies

$$\rho(X, Y)R_{YZ}X + \rho(Y, Z)R_{ZX}Y + \rho(Z, X)R_{XY}Z = 0.$$

The second Bianchi identity follows immediately from the first and the relation (c) follows from a direct calculation by applying the relation $\nabla_{\phi_A(X)}\phi_A(Y) = \phi_A(\nabla_X Y)$.

Based on our knowledge of Riemannian geometry, the following properties hold for Riemannian curvature tensor $R(X, Y, Z, W)$:

- (1) R is skew-symmetric in the first two and last two entries

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = -R(X, Y, W, Z).$$

- (2) R is symmetric between the first two and last two entries

$$R(X, Y, Z, W) = R(Z, W, X, Y).$$

Also, in [21], Ngakeu presented that the curvature tensor $R(X, Y, Z, W)$ on ρ -commutative algebras has the same properties of Riemannian geometry, that is

- (i) $R(X, Y, V, W) = -\rho(X, Y)R(Y, X, V, W) = -\rho(V, W)R(X, Y, W, V)$,
- (ii) $R(X, Y, V, W) = \rho(X + Y, V + W)R(V, W, X, Y)$.

For the curvature tensor $R(X, Y, V, W)$ on Hom- ρ -commutative algebras, we can not have the following properties

$$\begin{aligned} R(X, Y, V, W) &= \rho(X + Y, V + W)R(V, W, X, Y), \\ R(X, Y, V, W) &= -\rho(X, Y)R(Y, X, V, W) = -\rho(V, W)R(X, Y, W, V). \end{aligned}$$

4 Differential Calculus, Poisson Bracket

In this section, we recall the notions of representation, cochains and Hom-cochains on Hom- ρ -Lie algebras. These notions are defined analogously of the classical case by Abdaoui et al. in [1]. Then, we try to develop differential calculus by using them and recall the notion of Poisson bracket on Hom- ρ -Lie algebras and investigate some examples (for the classical case see [22, 24]).

Definition 4.1 (see [1]) *Let $(A, [\cdot, \cdot]_\rho, \rho, \phi)$ be a Hom- ρ -Lie algebra. For any non-negative integer k , a ϕ^k - ρ -derivation of degree $|X|$ on A is a linear map $X : A \rightarrow A$ such that*

$$X \circ \phi = \phi \circ X, \quad \text{i.e.,} \quad [X, \phi]_\rho = 0$$

and

$$X[f, g]_\rho = [X(f), \phi^k(g)]_\rho + \rho(X, f)[\phi^k(f), X(g)]_\rho. \tag{4.1}$$

We denote by $\rho\text{-Der}_{\phi^k} A$ the space of all ϕ^k - ρ -derivations of A . It is easy to see that for $X \in \rho\text{-Der}_{\phi^k} A$ and $Y \in \rho\text{-Der}_{\phi^s} A$ the ρ -commutator of X, Y , defined by $[X, Y]_\rho = X \circ Y - \rho(X, Y)Y \circ X$, is a ϕ^{k+s} - ρ -derivation and

$$\rho\text{-Der}_{\phi} A = \bigoplus_{k \geq 0} \rho\text{-Der}_{\phi^k} A$$

is a ρ -Lie algebra with above bracket.

Proposition 4.1 *Let $(A, [\cdot, \cdot]_\rho, \rho, \phi)$ be a Hom- ρ -Lie algebra. The quadruple $(\rho\text{-Der}_{\phi} A, [\cdot, \cdot]_\rho, \rho, \phi_A)$ consisting of the space $\rho\text{-Der}_{\phi} A$, two-cycle ρ , ρ -commutator $[X, Y]_\rho = X \circ Y - \rho(X, Y)Y \circ X$ and even linear map $\phi_A : \rho\text{-Der}_{\phi} A \rightarrow \rho\text{-Der}_{\phi} A$ given by $\phi_A(X) = X \circ \phi$ is a Hom- ρ -Lie algebra.*

Proof We know that the composition of functions is always associative, that is if h_1 , h_2 and h_3 are three functions with suitably chosen domains and codomains, then $h_1 \circ (h_2 \circ h_3) = (h_1 \circ h_2) \circ h_3$. Let us use this to prove our proposition. So, we have

$$\begin{aligned}\phi_A(Y) \circ (Z \circ X) &= (\phi_A(Y) \circ Z) \circ X = ((Y \circ \phi) \circ Z) \circ X = (Y \circ (\phi \circ Z)) \circ X \\ &= Y \circ ((\phi \circ Z) \circ X) = Y \circ ((Z \circ \phi) \circ X) = Y \circ (Z \circ (\phi \circ X)) \\ &= Y \circ (Z \circ (X \circ \phi)) = Y \circ (Z \circ \phi_A(X)) = (Y \circ Z) \circ \phi_A(X).\end{aligned}$$

Now, by using the relation $\phi_A(Y) \circ (Z \circ X) = (Y \circ Z) \circ \phi_A(X)$ for $X, Y, Z \in Hg(\rho\text{-Der } \phi A)$, we study the Jacobi-identity. Direct calculations give us

$$\begin{aligned}\rho(Z, X)[\phi_A(X), [Y, Z]_\rho]_\rho &= \rho(Z, X)\phi_A(X) \circ (Y \circ Z) - \rho(Z, X)\rho(X, Y + Z)(Y \circ Z) \circ \phi_A(X) \\ &\quad - \rho(Z, X)\rho(Y, Z)\phi_A(X) \circ (Z \circ Y) \\ &\quad + \rho(Z, X)\rho(X, Y + Z)\rho(Y, Z)(Z \circ Y) \circ \phi_A(X), \\ \rho(X, Y)[\phi_A(Y), [Z, X]_\rho]_\rho &= \rho(X, Y)\phi_A(Y) \circ (Z \circ X) - \rho(X, Y)\rho(Y, X + Z)(Z \circ X) \circ \phi_A(Y) \\ &\quad - \rho(X, Y)\rho(Z, X)\phi_A(Y) \circ (X \circ Z) \\ &\quad + \rho(X, Y)\rho(Y, X + Z)\rho(Z, X)(X \circ Z) \circ \phi_A(Y), \\ \rho(Y, Z)[\phi_A(Z), [X, Y]_\rho]_\rho &= \rho(Y, Z)\phi_A(Z) \circ (X \circ Y) - \rho(Y, Z)\rho(Z, Y + X)(X \circ Y) \circ \phi_A(Z) \\ &\quad - \rho(Y, Z)\rho(X, Y)\phi_A(Z) \circ (Y \circ X) \\ &\quad + \rho(Y, Z)\rho(Z, Y + X)\rho(X, Y)(Y \circ X) \circ \phi_A(Z).\end{aligned}$$

In the end, summing three above equations implies the Jacobi identity.

Definition 4.2 (see [1]) *Let V be a vector space. A linear map $\mu : A \rightarrow \text{End}(V)$ is called a representation of the Hom- ρ -Lie algebra $(A, [\cdot, \cdot]_\rho, \rho, \phi)$ on V with respect to $B \in \text{End}(V)$ if the following equality is satisfied*

$$\mu[f, g]_\rho \circ B = \mu(\phi(f)) \circ \mu(g) - \rho(f, g)\mu(\phi(g)) \circ \mu(f).$$

Moreover, a representation (V, μ) is said to be graded if $V = \bigoplus_{a \in G} V_a$ is a G -graded space such that

$$\mu(f)(V_a) \subseteq V_{|f|+a}$$

for all the homogeneous elements $f \in A$ and $a \in G$.

Let V be a G -graded vector space and $B : V \rightarrow V$ be an even homomorphism.

Definition 4.3 (see [24]) *A k -cochain on a Hom- ρ -Lie algebra $(A, [\cdot, \cdot]_\rho, \rho, \phi)$ is a ρ -skew-symmetric and k -linear map $\alpha : A \times \cdots \times A \rightarrow V$ of G -degree $|\alpha|$, in the sense of*

$$\alpha(f_1, \dots, f_k) \in V_{|f_1|+\dots+|f_k|+|\alpha|},$$

where $f_1, \dots, f_k \in Hg(A)$. We denote by $C^k(A; V)$ the set of k -cochains on A .

$\alpha \in C^k(A; V)$ is called a k -Hom-cochain on A if for $f_1, \dots, f_k \in Hg(A)$, the following relation holds

$$B(\alpha(f_1, \dots, f_k)) = \alpha(\phi(f_1), \dots, \phi(f_k)).$$

Let $C_\phi^k(A, V)$ denote the set of k -Hom-cochains on A . Then $C_\phi^k(A, V)$ is a graded algebra with $C_\phi^0(A, V) = V$ and we have

$$C_\phi(A, V) = \bigoplus_{k \geq 0} C_\phi^k(A, V).$$

In the next, let $(A, [\cdot, \cdot]_\rho, \rho, \phi)$ be a Hom- ρ -Lie algebra and $(\rho\text{-Der}_\phi A, [\cdot, \cdot]_\rho, \rho, \phi_A)$ be Hom- ρ -Lie algebra of all ϕ^k - ρ -derivations on Hom- ρ -Lie algebra A , where $\rho\text{-Der}_\phi A$ is equipped with the representation μ_A on A ($\mu_A : \rho\text{-Der}_\phi A \rightarrow \text{End}(A)$) with respect to $B = Id_A : A \rightarrow A$.

We intend to define some operators on the set of k -Hom-cochains

$$C_{\phi_A}^k(\rho\text{-Der}_\phi A, A) = \{\alpha \in C^k(\rho\text{-Der}_\phi A, A) : \alpha \circ \phi_A = \alpha\}$$

on $\rho\text{-Der}_\phi A$.

Now, we define the co-boundary operator $d_A : C_{\phi_A}^k(\rho\text{-Der}_\phi A, A) \rightarrow C_{\phi_A}^{k+1}(\rho\text{-Der}_\phi A, A)$ by

$$d_A f(X) = \mu_A(\phi_A^{-1}(X)) \cdot f, \quad f \in A$$

and

$$\begin{aligned} d_A \alpha(X_1, \dots, X_{k+1}) &= \sum_{j=1}^{k+1} (-1)^{j-1} \rho \left(\sum_{i=1}^{j-1} X_i, X_j \right) \mu_A(\phi_A^{k-1}(X_j)) \cdot \alpha(X_1, \dots, \widehat{X_j}, \dots, X_{k+1}) \\ &+ \sum_{1 \leq j < l \leq k+1} (-1)^{j+l} \rho \left(\sum_{i=1}^{j-1} X_i, X_j \right) \rho \left(\sum_{i=1}^{j-1} X_i, X_l \right) \\ &\times \rho \left(\sum_{i=j+1}^{l-1} X_i, X_l \right) \alpha([X_j, X_l]_\rho, \phi_A(X_1), \dots, \widehat{\phi_A(X_j)}, \dots, \\ &\quad \widehat{\phi_A(X_l)}, \dots, \phi_A(X_{k+1})) \end{aligned} \quad (4.2)$$

for $k \geq 1$, $\alpha \in C_{\phi_A}^k(\rho\text{-Der}_\phi A, A)$ and $X_l \in Hg(\rho\text{-Der}_\phi A)$, $l \in \{1, \dots, k+1\}$, where $\widehat{X_j}$ means that X_j is omitted. Note that $|d_A \alpha| = |\alpha|$ and $d_A^2 = 0$ (the condition $d_A^2 = 0$ does not follow if the condition $\alpha \circ \phi_A = \alpha$ is omitted, so it is necessary to define the differential operators on k -Hom-cochains).

The inner and Lie derivations also are defined on $C_{\phi_A}(\rho\text{-Der}_\phi A, A)$ by

$$\begin{aligned} i_X \alpha(X_1, \dots, X_{k-1}) &:= \rho \left(\sum_{i=1}^{k-1} X_i, X \right) \alpha(X, X_1, \dots, X_{k-1}), \quad i_X(f) = 0, \\ L_X &= i_X \circ d_A + d_A \circ i_X, \end{aligned}$$

where $X_l \in Hg(\rho\text{-Der}_\phi A)$, $l \in \{1, \dots, k\}$. Note that $|i_X| = |L_X| = |X|$.

Remark 4.1 For the inner and Lie derivations i_X and L_X , we have $i_X \circ i_Y + \rho(X, Y) i_Y \circ i_X = 0$ and $d \circ L_X = L_X \circ d$, where $X, Y \in Hg(\rho\text{-Der}_\phi A)$, But the property $[L_X, i_Y] = i_{[X, Y]}$ not necessarily holds. For instance, if we consider the extended-hyperplane A_q^2 introduced in Example 3.1, if we set $a = b = 1$, then $(A_q^2, \cdot, \rho, \phi = Id, [\cdot, \cdot]_\rho = 0)$ is a Hom- ρ -Lie algebra. For $\phi = Id$ and $k = 0$, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are ϕ^0 - ρ -derivation. Let us set $\mu_{A_q^2} = ad$ and consider $\phi_{A_q^2}(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial x}$ and $\phi_{A_q^2}(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial y}$. Thus for $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$, $Z = \frac{\partial}{\partial x}$, we have

$$\begin{aligned} [i_X \circ d + d \circ i_X, i_Y](\alpha)(Z) &= ([i_X \circ d, i_Y] + [d \circ i_X, i_Y])(\alpha)(Z) \\ &= (1 - q) \left[\frac{\partial}{\partial x}, \alpha \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right) \right] + 2 \left[\frac{\partial}{\partial x}, \alpha \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \right] \\ &\neq i_{[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]} = i_{[X, Y]} = 0. \end{aligned}$$

Definition 4.4 We say that Hom- ρ -Lie algebra $(A, [\cdot, \cdot]_\rho, \rho, \phi)$ satisfies in the Cartan identity if the following condition holds

$$[L_X, i_Y] = i_{[X, Y]}.$$

In this case, one can easily show that

$$L_{[X, Y]_\rho} = [L_X, L_Y]_\rho.$$

Definition 4.5 Let $\Omega \in C_{\phi_A}^2(\rho\text{-Der } \phi A, A)$ be called a symplectic structure on A if Ω is non-degenerate and closed ($d_A\Omega = 0$).

Let us define $\tilde{\Omega} : \rho\text{-Der } \phi A \rightarrow C_{\phi_A}^1(\rho\text{-Der } \phi A, A)$ by $\tilde{\Omega}(X) = \Omega(\cdot, \phi_A(X))$. Then the non-degenerate property of Ω is equivalent to the assertion that $\tilde{\Omega}$ is isomorphism (note that, the definition of non degeneracy creates a one-to-one correspondence between $\rho\text{-Der } \phi A$ and $C_{\phi_A}^1(\rho\text{-Der } \phi A, A)$ and they have the same cardinality. Therefore, if they are even infinite dimensional, then this isometric is meaningful). It is remarkable that Ω is homogeneous if and only if $\tilde{\Omega}$ is homogeneous and we have $|\Omega| = |\tilde{\Omega}|$.

Definition 4.6 $X \in \rho\text{-Der } \phi A$ is called a locally Hamiltonian ϕ^k - ρ -derivation if $L_{\phi_A(X)}\Omega = 0$.

Lemma 4.1 Let $(\rho\text{-Der } \phi A, [\cdot, \cdot]_\rho, \rho, \phi_A)$ satisfy the Cartan identity. $X \in \rho\text{-Der } \phi A$ is locally Hamiltonian if and only if $d_A(i_{\phi_A(X)}\Omega) = 0$.

Proof By $L_{\phi_A(X)} = d_A \circ i_{\phi_A(X)} + i_{\phi_A(X)} \circ d_A$, the proof is clear. By the relation $L_{\phi_A[X, Y]_\rho} = [L_{\phi_A(X)}, L_{\phi_A(Y)}]_\rho$, we can show that if $X, Y \in \rho\text{-Der } \phi A$ are locally Hamiltonian, then $[X, Y]_\rho$ is also locally Hamiltonian.

Definition 4.7 For any $f \in A$, the vector $X := \tilde{\Omega}^{-1}(d_A f)$ is called the Hamiltonian ϕ^k - ρ -derivation associated to f .

Let X_f denote the Hamiltonian ϕ^k - ρ -derivation associated to f , i.e., $X = X_f$. So X_f is of G -degree $|X_f| = |f| - |\Omega|$ and

$$d_A f = \Omega(\cdot, \phi_A(X_f)) = -i_{\phi_A(X_f)}\Omega.$$

In other words

$$\Omega(\phi_A(X_g), \phi_A(X_f)) = \mu_A(X_g) \cdot f.$$

Note that, since Ω is a 2-Hom-cochain, we have

$$\Omega(\phi_A(X), \phi_A(Y)) = \Omega(X, Y),$$

and so

$$\Omega(X_g, X_f) = \mu_A(X_g) \cdot f.$$

Let the following two relations exist between the maps ϕ and ϕ_A and $\mu_A(X) \in \text{End}(A)$,

$$\mu_A(\phi_A(X_f)) = \mu_A(X_{\phi(f)}), \tag{4.3}$$

$$\mu_A(\phi_A(X_f)) \cdot [g, h]_\rho = [\mu_A(X_f) \cdot g, \phi(h)]_\rho + \rho(X_f, g)[\phi(g), \mu_A(X_f) \cdot h]_\rho. \tag{4.4}$$

Lemma 4.2 Let $(\rho\text{-Der } \phi A, [\cdot, \cdot]_\rho, \rho, \phi_A)$ satisfy the Cartan identity. If $X, Y \in \rho\text{-Der } \phi A$ are locally Hamiltonians $\phi^k\text{-}\rho\text{-derivation}$ and $\phi^s\text{-}\rho\text{-derivation}$, respectively, then the following relation holds

$$[X, Y]_\rho = X_{\Omega(X, Y)} = -\rho(X, Y)X_{\Omega(Y, X)},$$

i.e., $[X, Y]_\rho$ is the Hamiltonian $\phi^{k+s}\text{-}\rho\text{-derivation}$ associated to $\Omega(X, Y)$.

Proof By the equality $i_{[X, Y]_\rho} = [L_X, i_Y]_\rho$ and equivalently $i_{\phi_A[X, Y]_\rho} = [L_{\phi_A(X)}, i_{\phi_A(Y)}]_\rho$, we have

$$i_{\phi_A[X, Y]_\rho}\Omega = [L_{\phi_A(X)}, i_{\phi_A(Y)}]_\rho\Omega = L_{\phi_A(X)}(i_{\phi_A(Y)}\Omega) - \rho(X, Y)i_{\phi_A(Y)}(L_{\phi_A(X)}\Omega).$$

Since X is a locally Hamiltonian $\phi^k\text{-}\rho\text{-derivation}$, we get

$$i_{\phi_A[X, Y]_\rho}\Omega = [L_{\phi_A(X)}, i_{\phi_A(Y)}]_\rho\Omega = L_{\phi_A(X)}(i_{\phi_A(Y)}\Omega).$$

In the next, by the Cartan identity and given that the $\phi^s\text{-}\rho\text{-derivation}$ Y is locally Hamiltonian, we easily obtain the following relation

$$i_{\phi_A[X, Y]_\rho}\Omega = \rho(X, Y)d_A(\Omega(\phi_A(Y), \phi_A(X))) = -d_A(\Omega(X, Y)),$$

and so

$$-i_{\phi_A[X, Y]_\rho}\Omega = d_A(\Omega(X, Y)).$$

Thus, the conclusion holds, that is

$$[X, Y]_\rho = X_{\Omega(X, Y)} = -\rho(X, Y)X_{\Omega(Y, X)}.$$

Definition 4.8 (see [4]) A Poisson Hom- ρ -algebra consists of a G -graded vector space A , bilinear maps $\cdot : A \times A \rightarrow A$ and $\{\cdot, \cdot\}_\rho : A \times A \rightarrow A$ of G -degree $|\{\cdot, \cdot\}_\rho| = P$, an even linear map $\phi : A \rightarrow A$ and a two-cycle $\rho : G \times G \rightarrow k^*$ such that

- (1) (A, \cdot, ρ, ϕ) is a Hom-associative ρ -algebra.
- (2) $(A, \{\cdot, \cdot\}_\rho, \rho, \phi)$ is a Hom- ρ -Lie algebra, i.e.,
 - (i) $|\{f, g\}_\rho| = P + |f| + |g|$,
 - (ii) $\{f, g\}_\rho = -\rho(f, g)\{g, f\}_\rho$,
 - (iii) $\rho(h, f)\{\phi(f), \{g, h\}_\rho\}_\rho + \rho(g, h)\{\phi(h), \{f, g\}_\rho\}_\rho + \rho(f, g)\{\phi(g), \{h, f\}_\rho\}_\rho = 0$.
- (3) For all $f, g, h \in A$, $\{f \cdot g, \phi(h)\}_\rho = \rho(g, h + P)\{f, h\}_\rho \cdot \phi(g) + \phi(f) \cdot \{g, h\}_\rho$.

Furthermore, if $fg = \rho(f, g)gf$ for all $f, g \in Hg(A)$, then we have a Poisson Hom- ρ -commutative algebra.

Equivalently, Poisson Hom- ρ -Lie algebra can be defined in the following expression.

Definition 4.9 A Poisson Hom- ρ -Lie algebra is a multiplex $(A, [\cdot, \cdot]_\rho, \{\cdot, \cdot\}_\rho, \rho, \phi)$ consisting of a G -graded vector space A , bilinear maps $[\cdot, \cdot]_\rho : A \times A \rightarrow A$ and $\{\cdot, \cdot\}_\rho : A \times A \rightarrow A$ of G -degree $|\{\cdot, \cdot\}_\rho| = P$, an even linear map $\phi : A \rightarrow A$ and a two-cycle $\rho : G \times G \rightarrow k^*$ satisfying

- (1) $(A, [\cdot, \cdot]_\rho, \rho, \phi)$ is a Hom- ρ -Lie algebra.
- (2) $(A, \{\cdot, \cdot\}_\rho, \rho, \phi)$ is a Hom- ρ -Lie algebra.
- (3) For all $f, g, h \in A$, $\{[f, g]_\rho, \phi(h)\}_\rho = \rho(g, h + P)[\{f, h\}_\rho, \phi(g)]_\rho + [\phi(f), \{g, h\}_\rho]_\rho$.

Example 4.1 Let $(A, \cdot, \{\cdot, \cdot\}_\rho, \rho, \phi)$ be a Poisson Hom- ρ -algebra. We define the bracket $[\cdot, \cdot] : A \times A \rightarrow A$ by $[f, g] = f \cdot g - \rho(f, g)g \cdot f$. In this case $(A, [\cdot, \cdot]_\rho, \cdot, \{\cdot, \cdot\}_\rho, \rho, \phi)$ is a Poisson Hom- ρ -Lie algebra.

It is enough to check that $\{[f, g]_\rho, \phi(h)\}_\rho = \rho(g, h + P)[\{f, h\}_\rho, \phi(g)]_\rho + [\phi(f), \{g, h\}_\rho]_\rho$. For this, we have

$$\{[f, g]_\rho, \phi(h)\}_\rho = \{f \cdot g, \phi(h)\}_\rho - \rho(f, g)\{g \cdot f, \phi(h)\}_\rho.$$

On the other hand, since $(A, \cdot, \{\cdot, \cdot\}_\rho, \rho, \phi)$ is a Poisson Hom- ρ -algebra, then

$$\{f \cdot g, \phi(h)\}_\rho = \rho(g, h + P)\{f, h\}_\rho \cdot \phi(g) + \phi(f) \cdot \{g, h\}_\rho$$

and

$$\{g \cdot f, \phi(h)\}_\rho = \rho(f, h + P)\{g, h\}_\rho \cdot \phi(f) + \phi(g) \cdot \{f, h\}_\rho.$$

Therefore

$$\begin{aligned} \{[f, g]_\rho, \phi(h)\}_\rho &= \rho(g, h + P)\{f, h\}_\rho \cdot \phi(g) + \phi(f) \cdot \{g, h\}_\rho \\ &\quad - \rho(f, g)\{\rho(f, h + P)\{g, h\}_\rho \cdot \phi(f) + \phi(g) \cdot \{f, h\}_\rho\} \\ &= \rho(g, h + P)[\{f, h\}_\rho, \phi(g)]_\rho + [\phi(f), \{g, h\}_\rho]_\rho. \end{aligned}$$

Theorem 4.1 *Let $(A, [\cdot, \cdot]_\rho, \rho, \phi)$ be a Hom- ρ -Lie algebra and Ω be the homogeneous symplectic structure. Defining the ρ -Poisson bracket $\{\cdot, \cdot\}_\rho$ associated to Ω as*

$$\{f, g\}_\rho := -\rho(\Omega, g)\mu_A(X_f) \cdot g = -\rho(\Omega, g)\Omega(\phi_A(X_f), \phi_A(X_g)), \quad f, g \in Hg(A),$$

$(A, [\cdot, \cdot]_\rho, \{\cdot, \cdot\}_\rho, \phi)$ is a Poisson Hom- ρ -Lie algebra.

Proof Since Ω is a 2-Hom-cochain, we have

$$\{f, g\}_\rho := -\rho(\Omega, g)\mu_A(X_f) \cdot g = -\rho(\Omega, g)\Omega(X_f, X_g), \quad f, g \in Hg(A). \quad (4.5)$$

At first, we show that $(A, \{\cdot, \cdot\}_\rho, \rho, \phi)$ is a Hom- ρ -Lie algebra. We have

$$|\{f, g\}_\rho| = |\{\cdot, \cdot\}_\rho| + |f| + |g|.$$

On the other hand, we have

$$|\Omega(X_f, X_g)| = |\Omega| + |X_f| + |X_g| = P + |f| + |g|.$$

By the definition of ρ -Poisson bracket, since $|\{f, g\}_\rho| = |\Omega(X_f, X_g)|$, we can find $|\{\cdot, \cdot\}_\rho| = P = -|\Omega|$.

Now, we investigate the following relation

$$\{f, g\}_\rho = -\rho(f, g)\{g, f\}_\rho.$$

By the relation (4.5), we have

$$\begin{aligned} \{f, g\}_\rho &= -\rho(\Omega, g)\Omega(X_f, X_g) = \rho(\Omega, g)\rho(X_f, X_g)\Omega(X_g, X_f) \\ &= \rho(\Omega, g)\rho(f, g)\rho(f, -\Omega)\rho(-\Omega, g)\Omega(X_g, X_f) \\ &= \rho(f, g)\rho(\Omega, f)\Omega(X_g, X_f) = -\rho(f, g)\{g, f\}_\rho. \end{aligned}$$

Now, it is time to complete the proof by showing that the ρ -Hom-Jacobi identity holds. At first, note that by Lemma 4.2, we have

$$\Omega([X_f, X_g]_\rho, \phi_A(X_h)) = \Omega(X_{\Omega(X_f, X_g)}, \phi_A(X_h)) = -\rho(X_f + X_g, X_h)\Omega(\phi_A(X_h), X_{\Omega(X_f, X_g)})$$

$$\begin{aligned}
&= -\rho(X_f + X_g, X_h)\mu_A(\phi_A(X_h)) \cdot \Omega(X_f, X_g) \\
&= -\rho(X_f + X_g, X_h)\mu_A(\phi_A(X_h))\mu_A(X_f) \cdot g.
\end{aligned} \tag{4.6}$$

Since Ω is a closed form, by (4.2), we have

$$\begin{aligned}
0 = d_A\Omega(X_f, X_g, X_h) &= \mu_A(\phi_A(X_f))\Omega(X_g, X_h) - \rho(X_f, X_g)\mu_A(\phi_A(X_g))\Omega(X_f, X_h) \\
&\quad + \rho(X_f + X_g, X_h)\mu_A(\phi_A(X_h))\Omega(X_f, X_g) - \Omega([X_f, X_g]_\rho, \phi_A(X_h)) \\
&\quad + \rho(X_g, X_h)\Omega([X_f, X_h]_\rho, \phi_A(X_g)) \\
&\quad - \rho(X_f, X_g + X_h)\Omega([X_g, X_h]_\rho, \phi_A(X_f)).
\end{aligned}$$

This case will complete by invoking (4.6) and again (4.5), as

$$\begin{aligned}
0 = d_A\Omega(X_f, X_g, X_h) &= -2\rho(X_g, X_h)\mu_A(\phi_A(X_f))\mu_A(X_h) \cdot g \\
&\quad + 2\rho(X_f, X_g + X_h)\mu_A(\phi_A(X_g))\mu_A(X_h) \cdot f \\
&\quad - 2\rho(X_f + X_g, X_h)\rho(X_f, X_g)\mu_A(\phi_A(X_h))\mu_A(X_g) \cdot f.
\end{aligned}$$

By (4.5) again, this time vice versa, we obtain

$$\begin{aligned}
0 = d_A\Omega(X_f, X_g, X_h) &= \rho(g, \Omega)\rho(h, 2\Omega)\rho(f, h)\rho(h, f)\{\phi(f), \{g, h\}_\rho\}_\rho \\
&\quad + \rho(f + g, h)\rho(g, \Omega)\rho(h, 2\Omega)\{\phi(h), \{f, g\}_\rho\}_\rho \\
&\quad + \rho(f, g + h)\rho(g, \Omega)\rho(h, 2\Omega)\{\phi(g), \{h, f\}_\rho\}_\rho.
\end{aligned}$$

With this result, we can write

$$\rho(h, f)\{\phi(f), \{g, h\}_\rho\}_\rho + \rho(g, h)\{\phi(h), \{f, g\}_\rho\}_\rho + \rho(f, g)\{\phi(g), \{h, f\}_\rho\}_\rho = 0.$$

We continue with the checking of the following relation

$$\{[f, g]_\rho, \phi(h)\}_\rho = \rho(g, h + P)\{[f, h]_\rho, \phi(g)\}_\rho + [\phi(f), \{h, g\}_\rho]_\rho.$$

We have

$$\begin{aligned}
\{[f, g]_\rho, \phi(h)\}_\rho &= -\rho(\Omega, h)\Omega(X_{[f, g]_\rho}, X_{\phi(h)}) \\
&= \rho(\Omega, h)\rho(f + g - \Omega, h - \Omega)\Omega(X_{\phi(h)}, X_{[f, g]_\rho}).
\end{aligned}$$

Now, by using relation (4.5), we have

$$\{[f, g]_\rho, \phi(h)\}_\rho = \rho(\Omega, h)\rho(f + g - \Omega, h - \Omega)\mu_A(X_{\phi(h)}) \cdot [f, g]_\rho.$$

In the next, the relations (4.3)–(4.4) give us

$$\begin{aligned}
\{[f, g]_\rho, \phi(h)\}_\rho &= \rho(f + g, h)\rho(f + g, -\Omega)[\mu_A(X_h) \cdot f, \phi(g)]_\rho \\
&\quad + \rho(f + g, h)\rho(f + g, -\Omega)\rho(X_h, f)[\phi(f), \mu_A(X_h) \cdot g]_\rho \\
&= \rho(g, h - \Omega)\{[f, h]_\rho, \phi(g)\}_\rho + [\phi(f), \{g, h\}_\rho]_\rho.
\end{aligned}$$

Lemma 4.2 implies the following corollary.

Corollary 4.1 *We have*

$$[X_f, X_g]_\rho = X_{\Omega(X_f, X_g)} = -\rho(g, \Omega)X_{\{f, g\}_\rho}.$$

The above theorem is the key to construct Hamiltonian derivation and Poisson bracket on the specific Hom- ρ -commutative algebras. To illustrate an application of this lemma, let us state the following example.

Example 4.2 Let us go back to Example 3.1. In this example, for the extended hyperplane A_q^2 and space $\rho\text{-Der}A_q^2$, assuming that $\Omega = dy \wedge dx$ and $|f| = (f_1, f_2)$, the Hamiltonian ϕ - ρ -derivation associated to $f \in A_q^2$ has the following expression

$$X_f = q^{1-f_1} \mu_{A_q^2} \left(\frac{\partial}{\partial y} \right) \cdot f \frac{\partial}{\partial x} - q^{f_2} \mu_{A_q^2} \left(\frac{\partial}{\partial x} \right) \cdot f \frac{\partial}{\partial y}.$$

So, the Poisson bracket corresponding to $\phi_{A_q^2}$ gives as follows

$$\begin{aligned} \{f, g\}_\rho &= -\rho(\Omega, g) \mu_{A_q^2}(X_f) \cdot g = -\rho(x + y, g) \left\{ q^{1-f_1} \left(\mu_{A_q^2} \left(\frac{\partial}{\partial y} \right) \cdot f \right) \left(\mu_{A_q^2} \left(\frac{\partial}{\partial x} \right) \cdot g \right) \right. \\ &\quad \left. - q^{f_2} \left(\mu_{A_q^2} \left(\frac{\partial}{\partial x} \right) \cdot f \right) \left(\mu_{A_q^2} \left(\frac{\partial}{\partial y} \right) \cdot g \right) \right\}. \end{aligned}$$

Example 4.3 This example is intended to give us a Poisson structure on quaternion algebra H (Example 2.1). Let us define the Poisson bracket $\{\cdot, \cdot\}_\rho$ on H by the following structure

$$\begin{aligned} \{i, i\}_\rho &= 0, \quad \{j, j\}_\rho = 0, \quad \{k, k\}_\rho = 0, \quad \{i, j\}_\rho = -\{j, i\}_\rho = k, \\ \{k, i\}_\rho &= -\{i, k\}_\rho = j, \quad \{j, k\}_\rho = -\{k, j\}_\rho = i. \end{aligned}$$

So, $(H, \{\cdot, \cdot\}_\rho, \phi_H)$ is a Poisson Hom- ρ -commutative algebra.

References

- [1] Abdaoui, K., Ammar, F. and Makhlouf, A., Construction and cohomology of color Hom-Lie algebras, *Commu. Alg.*, **43**(11), 2015, 4581–4612.
- [2] Ammar, F. and Makhlouf, A., Hom-Lie superalgebras and Hom-Lie admissible superalgebras, *J. Alg.*, **324**(7), 2010, 1513–1528.
- [3] Armakan, A. R., Silvestrov, S. and Farhangdoost, M. R., Extensions of hom-Lie color algebras, *Georgian Math. J.*, **28**(1), 2021, 15–27.
- [4] Bakayoko, I., Modules over color Hom-Poisson algebras, *J. Gen. Lie Theo. Appl.*, **8**(1), 2014, 6 pages.
- [5] Bakayoko, I., Hom-Novikov color algebras, 2016, arXiv:1609.07813v1, 16 pages.
- [6] Bongaarts, P. J. and Pijls, H. G. J., Almost commutative algebra and differential calculus on the quantum hyperplane, *J. Math. Phys.*, **35**(2), 1994, 959–970.
- [7] Ciupala, C., Linear connections on almost commutative algebras, *Acta. Math. Univ. Comenianiae*, **72**(2), 2003, 197–207.
- [8] Ciupala, C., Differential calculus on almost commutative algebras and application to the quantum hyperplane, *Arch. Math. (BRNO) Tomus*, **41**, 2005, 359–377.
- [9] Ciupala, C., 2- ρ -Derivations on a ρ -algebra and applications to the quaternionic algebra, *Int. J. Geom. Meth. Mod. Phys.*, **4**(3), 2007, 457–469.
- [10] Darling, R. W. R., *Differential Forms and Connections*, Cambridge University Press, Cambridge, 1994.
- [11] Dubois-Violette, M., Dérivations et calcul différentiel non commutatif, *C. R. Acad. Sci. Paris, Série I*, **307**, 1988, 403–408.
- [12] Fregier, Y. and Gohr, A., On Hom-type algebras, *J. Gen. Lie Theo. Appl.*, **4**, 2010, 16 pages.
- [13] Hartwig, J. T., Larsson, D. and Silvestrov, S. D., Deformations of Lie algebras using σ -derivations, *J. Alg.*, **295**, 2006, 314–361.
- [14] Larsson, D. and Silvestrov, S. D., Quasi-Hom-Lie algebras, central extensions and 2-cocycle-like identities, *J. Alg.*, **288**, 2005, 321–344.

- [15] Larsson, D. and Silvestrov, S. D., Quasi-Lie algebras, in Noncommutative Geometry and Representation Theory in Mathematical Physics, *Contemp. Math.*, **391**, Amer. Math. Soc., Providence, RI, 2005, 241–248.
- [16] Majid, S., Riemannian geometry of quantum groups and finite groups with nonuniversal differentials, *Commun. Math. Phys.*, **225**, 2002, 131–170.
- [17] Makhlouf, A., Hom-alternative algebras and Hom-Jordan algebras, *Int. Elect. J. Alg.*, **8**, 2010, 177–190.
- [18] Makhlouf, A. and Silvestrov, S., Hom-algebra structures, *J. Gen. Lie Theo. Appl.*, **2**, 2008, 51–64.
- [19] Makhlouf, A. and Silvestrov, S., Hom-algebras and Hom-coalgebras, *J. Alg. Appl.*, **9**(4), 2010, 553–589.
- [20] Makhlouf, A. and Silvestrov, S., Notes on formal deformations of Hom-associative algebras and Hom-Lie algebras, *Forum Math.*, **22**(4), 2010, 715–739.
- [21] Ngakeu, F., Levi-Civita connection on almost commutative algebras, *Int. J. Geom. Meth. Mod. Phys.*, **4**(7), 2007, 1075–1085.
- [22] Ngakeu, F., Graded Poisson structures and Schouten-Nijenhuis bracket on almost commutative algebras, *Int. J. Geom. Meth. Mod. Phys.*, **9**(5), 2012, 12500421–125004220.
- [23] Ngakeu, F., Majid, S. and Lambert, D., Noncommutative Riemannian geometry of the alternating group A_4 , *J. Geom. Phys.*, **42**, 2002, 259–282.
- [24] Sheng, Y., Representations of Hom-Lie algebras, *Alg. Rep. Theo.*, **15**, 2012, 1081–1098.
- [25] Sheng, Y. and Chen, D., Hom-Lie 2 algebras, *J. Alg.*, **376**, 2013, 174–195.
- [26] Sternberg, S., *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.
- [27] Weintraub, S. H., *Differential Forms, A Complement to Vector Calculus*, Academic Press, Inc., San Diego, CA, 1997.
- [28] Yau, D., Hom-Malcev, Hom-alternative and Hom-Jordan algebras, *Int. Elect. J. Alg.*, **11**, 2012, 177–217.
- [29] Yuan, L., Hom-Lie color algebra structures, *Commu. Alg.*, **40**(2), 2012, 575–592.