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Limits of One-dimensional Interacting Particle Systems with Two-scale Interaction

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Abstract This paper characterizes the limits of a large system of interacting particles distributed on the real line. The interaction occurring among neighbors involves two kinds of independent actions with different rates. This system is a generalization of the voter process, of which each particle is of type **A** or **a**. Under suitable scaling, the local proportion functions of **A** particles converge to continuous functions which solve a class of stochastic partial differential equations driven by Fisher-Wright white noise. To obtain the convergence, the tightness of these functions is derived from the moment estimate method.

Keywords Interacting particle systems, Stochastic partial differential equations, Two-scale interaction, Tightness
2000 MR Subject Classification 60H15, 35R60

1 Introduction

This paper studies the dynamics of a large system of interacting particles distributed on the real line. In our model, the particles are placed at grids $\rho^{-1}\mathbb{Z}$, where the parameter ρ denotes the numbers of grids in a unit interval. Each grid is occupied by one and only one particle, and each particle is of type **A** or **a**. Each particle interacts with its neighbors in a certain way that will be specified later; here we say two particles x and y are neighbors (denoted by $x \sim y$) if their distance is less than a given bound D. Our model can be regarded as a generalization of the voter process studied in [11].

The central problem in this paper is how to characterize the limit behavior of the system when the grids are more and more dense, say, $\rho \to \infty$. To be more specified, let ρ_n be a sequence tending to infinity, and D_n be the corresponding bounds for neighborhood. Define the local proportion of type **A** around a grid $x \in \rho_n^{-1}\mathbb{Z}$ at time t as $u_n(t,x) = N_n^{-1} \sum\limits_{y\sim x} \xi_t^n(y)$, where $N_n \approx 2D_n\rho_n$ is the number of neighbors of the particle at x, and $\xi_t^n(x)$ indicates the type of the particle at (t,x) (1 for **A** and 0 for **a**). We extend $u_n(t,x)$ to the entire space $\mathbb{R}^+ \times \mathbb{R}$ by linear interpolation. Our goal is to investigate the convergence of u_n as n tends to infinity, and to characterize the limit if it exists.

The dynamics of u_n depends on the interacting manner of the particle system. In our model, each particle interacts with its neighbors independently according to a Poisson process. We further assume that the interaction (i.e., two-scale interaction) involves two types of independent actions: The regular one-to-one interaction with higher-rate H_n and the rare interaction with

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lower-rate L_n. In the first type, the particle chooses one neighbor randomly and duplicates its type. In contrast, the other type of action is much more flexible: We assume that the particle at grid x and time t updates its type to $i \in \{\mathbf{A}, \mathbf{a}\}$ with probability $p_i(u_n(t,x))$ and with rate $F_i(u_n(t,x))$, where $p_i : [0,1] \to [0,1]$ and $F_i : [0,1] \to [0,+\infty)$ are given bounded and measurable functions; in this case, $L_n(t,x) = F_{\mathbf{A}}(u_n(t,x)) + F_{\mathbf{a}}(u_n(t,x))$. In other words, the rare interaction can be state-dependent (see [6, Theorem I.3.9]), which endows the model with the ability to capture various features in specific applications. For example, the voter process studied in [11] is associated with the setting that $p_{\mathbf{A}}(u) = u$, $F_{\mathbf{a}} \equiv 0$, and $F_{\mathbf{A}}$ is a positive constant. When applying to population genetics, it can model various effects in gene frequency diffusion, such as mutation (e.g., p_i and F_i are all constants), selection (e.g., $p_{\mathbf{A}}(u) = u$, $p_{\mathbf{a}}(u) = 1-u$, and $F_{\mathbf{A}} > F_{\mathbf{a}}$ if $\mathbf{A}$ is advantaged), Allee's effect (e.g., $p_{\mathbf{A}}(u) = u$, $p_{\mathbf{a}}(u) = (1-u)$, and $F_{\mathbf{A}}(u) = u, F_{\mathbf{a}}(u) = (1-u)$), and so on (see [4, 14, 17]); all these effects can be overlaid.

The main result of this paper is to obtain the convergence of u_n in a proper way. Define the λ-norm $\|f\|_\lambda := \sup_x |f(x)\mathrm{e}^{\lambda|x|}|$ and the following topological vector space of continuous functions

$$\mathscr{C} := \Big\{ f \in C(\mathbb{R}, [0,\infty)) \mid \lim_{x\to\infty} f(x)\mathrm{e}^{\lambda|x|} = 0,\ \forall\, \lambda < 0 \Big\}$$

with norm $\|\cdot\|_\lambda$, where $\lambda < 0$.

Theorem 1.1 *Let $\rho_n = n, D_n = n^{-\frac{1}{2}}, H_n = 2n$, and $u_n(0,\cdot)$ converges to f_0 in $\mathscr{C}$ as $n \to \infty$. Then u_n converges in distribution to a $\mathscr{C}$-valued continuous process u which satisfies the following equation with initial condition $u(0,x) = f_0$:*

$$\partial_t u = \frac{1}{3}\partial_x^2 u + (1-u)p_{\mathbf{A}}(u)F_{\mathbf{A}}(u) - u\, p_{\mathbf{a}}(u)F_{\mathbf{a}}(u) + 2\sqrt{u(1-u)}\dot{W}, \tag{1.1}$$

where $\dot{W}$ is a space-time white noise.

Remark 1.1 The higher-rate interaction contributes to the second-order term (i.e., $\frac{1}{3}\partial_x^2 u$) by random walk with generator (see [5])

$$\Delta_n(f)(x) := \frac{H_n}{N_n}\sum_{y\sim x}(f(y) - f(x)),$$

and the noise term (i.e., $2\sqrt{u(1-u)}\dot{W}$) by Fisher-Wright model (see [2, 4, 17]), while the lower-rate interaction generates the other terms on the right-hand side of (1.1) (we call them the "reaction terms" in what follows) through the increment on local proportion in unit time and space.

Remark 1.2 This conclusion is able to be carried over to a case on ring if measures, functions and noise are periodic. However, the corresponding result for high-dimension case does not come true. Because super Brownian motion in higher dimensions exists as singular measure-valued process rather than a density value process in one dimension case which can be expressed as in Theorem 1.1 (see [12, Theorem III.4.2]).

Remark 1.3 The coefficients are a little different from those in [11], which results from the choice of parameters.

(1.1) can be given rigorous meaning in terms of an integral equation as explained in [16, Chapter 3]. Our theorem gives existence of solutions to the associate martingale problem (see [13]), while uniqueness in law also holds in a general condition, due to the following result from [13].

Lemma 1.1 *If p_i and F_i with $i \in \{\mathbf{A}, \mathbf{a}\}$ are Lipschitz continuous, the solution to the martingale problem associated with* (1.1) *is unique.*

The proof strategy of Theorem 1.1 is adopted from [11], and the key idea is to show that the model satisfies a martingale problem that approximates the martingale problem for the limiting processes. Tightness is proved through estimating moments of small increments for the local proportion and arguing as in the Kolmogorov tightness criterion; to this end, we establish an approximate Green's function representation (2.12) for the local proportion $u_n(t,x)$, which is analogous to the one for the solution to (1.1) but with certain error terms. The introduction of probabilities $p_{\mathbf{A}}(u)$ and $p_{\mathbf{a}}(u)$ not only generalizes the model, but also helps us simplify the proof of Green's function representation, comparing to that for the voter process in [11].

It is worth noting that various properties of solutions to SPDEs like (1.1) have been studied in the literature; for instance, the compact support property for solutions was discovered by [9]; as a stochastic version of reaction-diffusion equations, random traveling waves were introduced and investigated in [10, 15], and further analysis of the traveling speed was carried out in [7–8], etc.

The rest of the paper is all devoted to the proof of Theorem 1.1. After figuring out the dynamics of $\xi^n_t(x)$, we decompose each term in the expansion of a functional of $\xi^n_t(x)$ into the sum of a fluctuation term and an average term in Subsection 2.1. Green's function representation is derived in Subsection 2.2, and tightness of u_n is proved in Subsection 2.3. The limit is characterized in Subsection 2.4, which concludes the proof.

2 Proof of Theorem 1.1

Let us introduce three independent Poisson processes associated with $x, y \in \rho_n^{-1}\mathbb{Z}$, i.e., $P_t(x,y)$ with rate H_n/N_n, $P^{\mathbf{A}}_t(x)$ with rate $F_{\mathbf{A}}(u_{t-}(x))$, and $P^{\mathbf{a}}_t(x)$ with rate $F_{\mathbf{a}}(u_{t-}(x))$, characterizing the events of x interviewing y, x updating its type to $\mathbf{A}$, and x updating its type to $\mathbf{a}$, respectively.

In what follows, we simply write

$$\sum_x := \sum_{x \in \rho_n^{-1}\mathbb{Z}} \quad \text{and} \quad \sum_{y\sim x} := \sum_{y \in \{y | y \sim x\}}.$$

For $f, g : \rho_n^{-1}\mathbb{Z} \to \mathbb{R}$ and v a measure on $\rho_n^{-1}\mathbb{Z}$, we denote

$$\langle\langle f, g\rangle\rangle := \rho_n^{-1} \sum_x f(x)g(x) \quad \text{and} \quad \langle\langle v, f\rangle\rangle := \int f \mathrm{d}v.$$

Moreover, we denote $e_\lambda(x) := \mathrm{e}^{\lambda|x|}$.

According to the setting of our model, the process $\xi^n_t(x)$ satisfies

$$\xi^n_t(x) = \xi^n_0(x) + \sum_{y\sim x} \int_0^t (\xi^n_{s-}(y) - \xi^n_{s-}(x))\mathrm{d}P_s(x,y)$$

$$+\int_0^t (1-\xi^n_{s-}(x))p_{\mathbf{A}}\Big(\frac{\sum_{y\sim x}\xi^n_{s-}(y)}{N_n}\Big)\mathrm{d}P^{\mathbf{A}}_s(x)$$
$$-\int_0^t \xi^n_{s-}(x)p_{\mathbf{a}}\Big(\frac{\sum_{y\sim x}\xi^n_{s-}(y)}{N_n}\Big)\mathrm{d}P^{\mathbf{a}}_s(x).$$

Take a test function $\phi : [0,\infty)\times\rho_n^{-1}\mathbb{Z}\to\mathbb{R}$ with $t\to\phi_t(x)$ being continuously differentiable and satisfying

$$\int_0^T \langle\!\langle |\phi_s|+\phi_s^2+|\partial_s\phi_s|, 1\rangle\!\rangle \mathrm{d}s<\infty.$$

Define the measure valued process

$$v^n_t := \rho_n^{-1}\sum_x \delta(\cdot-\xi^n_t(x)),$$

and then using the integration by parts, for $t\le T$, we have

$$\begin{aligned}
&\langle\!\langle v_t,\phi_t\rangle\!\rangle-\langle\!\langle v_0,\phi_0\rangle\!\rangle-\int_0^t\langle\!\langle v_s,\partial_s\phi_s\rangle\!\rangle\mathrm{d}s\\
&=\rho_n^{-1}\sum_x\sum_{y\sim x}\int_0^t(\xi^n_{s-}(y)-\xi^n_{s-}(x))\phi_s(x)\mathrm{d}P_s(x,y)\\
&\quad+\rho_n^{-1}\sum_x\int_0^t(1-\xi^n_{s-}(x))p_{\mathbf{A}}\Big(\frac{\sum_{y\sim x}\xi^n_{s-}(y)}{N_n}\Big)\phi_s(x)\mathrm{d}P^{\mathbf{A}}_s(x)\\
&\quad-\rho_n^{-1}\sum_x\int_0^t\xi^n_{s-}(x)p_{\mathbf{a}}\Big(\frac{\sum_{y\sim x}\xi^n_{s-}(y)}{N_n}\Big)\phi_s(x)\mathrm{d}P^{\mathbf{a}}_s(x).
\end{aligned}$$

By symmetry, one can see that

$$\rho_n^{-1}\sum_x\sum_{y\sim x}\int_0^t\xi^n_{s-}(y)\phi_s(y)\mathrm{d}P_s(x,y)=\rho_n^{-1}\sum_x\sum_{y\sim x}\int_0^t\xi^n_{s-}(x)\phi_s(x)\mathrm{d}P_s(y,x),$$

so the previous formula can be written as

$$\begin{aligned}
&\langle\!\langle v_t,\phi_t\rangle\!\rangle-\langle\!\langle v_0,\phi_0\rangle\!\rangle-\int_0^t\langle\!\langle v_s,\partial_s\phi_s\rangle\!\rangle\mathrm{d}s\\
&=\rho_n^{-1}\sum_x\sum_{y\sim x}\int_0^t\xi^n_{s-}(y)(\phi_s(x)-\phi_s(y))\mathrm{d}P_s(x,y)\\
&\quad+\rho_n^{-1}\sum_x\int_0^t(1-\xi^n_{s-}(x))p_{\mathbf{A}}\Big(\frac{\sum_{y\sim x}\xi^n_{s-}(y)}{N_n}\Big)\phi_s(x)\mathrm{d}P^{\mathbf{A}}_s(x)\\
&\quad-\rho_n^{-1}\sum_x\int_0^t\xi^n_{s-}(x)p_{\mathbf{a}}\Big(\frac{\sum_{y\sim x}\xi^n_{s-}(y)}{N_n}\Big)\phi_s(x)\mathrm{d}P^{\mathbf{a}}_s(x)\\
&\quad+\rho_n^{-1}\sum_x\sum_{y\sim x}\int_0^t\xi^n_{s-}(x)\phi_s(x)(\mathrm{d}P_s(y,x)-\mathrm{d}P_s(x,y))\\
&=: J_1+J_2+J_3+J_4.
\end{aligned}\tag{2.1}$$

2.1 The Doob-Meyer decomposition

We need to decompose each of the four terms (2.1), denoted by J_1, J_2, J_3 and J_4, respectively, into the sum of a fluctuation term and an average term. In the following argument, we will omit superscript n without ambiguity and denote $u_n(t,x)$ as $u_t(x)$. Define

$$\mathcal{D}(f,\delta)(x) := \sup\{|f(y)-f(x)| : |y-x| \le \delta, y \in \rho_n^{-1}\mathbb{Z}\}$$

for $f : \rho_n^{-1}\mathbb{Z} \to \mathbb{R}$, $x \in \rho_n^{-1}\mathbb{Z}$ and $\delta > 0$.

For J_1, one computes that

$$\begin{aligned} J_1 &= \rho_n^{-1} \sum_x \sum_{y\sim x} \int_0^t \xi_{s-}^n(y)(\phi_s(x)-\phi_s(y))\Big(\mathrm{d}P_s(x,y) - \frac{H_n}{N_n}\mathrm{d}s\Big) \\ &\quad + \rho_n^{-1} \sum_x \sum_{y\sim x} \int_0^t \xi_s^n(y)(\phi_s(x)-\phi_s(y))\frac{H_n}{N_n}\mathrm{d}s \\ &=: E_t^{(1)}(\phi) + \int_0^t \langle\!\langle v_s, \Delta_n(\phi_s)\rangle\!\rangle \mathrm{d}s, \end{aligned} \tag{2.2}$$

where $E_t^{(1)}(\phi)$ is a martingale with brackets process given by

$$\begin{aligned} \mathrm{d}\langle E_n^{(1)}(\phi)\rangle_t &= \rho_n^{-2} \sum_x \sum_{y\sim x} \xi_t^n(y)(\phi_t(x)-\phi_t(y))^2 \frac{H_n}{N_n}\mathrm{d}t \\ &\le \rho_n^{-2} \sum_y \xi_t^n(y)[\mathcal{D}(\phi_t, D_n)(y)]^2 H_n \mathrm{d}t \\ &= \rho_n^{-1} H_n \langle\!\langle v_t, [\mathcal{D}(\phi_t, D_n)]^2\rangle\!\rangle \mathrm{d}t \\ &\le \rho_n^{-1} H_n \|\mathcal{D}(\phi_t, D_n)\|_\lambda^2 \langle\!\langle e_{-2\lambda}, v_t\rangle\!\rangle \mathrm{d}t \\ &\le \rho_n^{-1} H_n \|\mathcal{D}(\phi_t, D_n)\|_\lambda^2 \langle\!\langle e_{-2\lambda}, 1\rangle\!\rangle \mathrm{d}t. \end{aligned} \tag{2.3}$$

Alternatively, we bound it by

$$\begin{aligned} \mathrm{d}\langle E_n^{(1)}(\phi)\rangle_t &\le \rho_n^{-2} \sum_x \sum_{y\sim x} 2\xi_t^n(y)\|\phi_t\|_0[|\phi_t(x)| + |\phi_t(y)|]\frac{H_n}{N_n}\mathrm{d}t \\ &= 2\|\phi_t\|_0 \rho_n^{-1} H_n [\langle\!\langle |\phi_t|, u_t\rangle\!\rangle + \langle\!\langle v_t, |\phi_t|\rangle\!\rangle]\mathrm{d}t \\ &\le 4\|\phi_t\|_0 \rho_n^{-1} H_n \langle\!\langle |\phi_t(x)|, 1\rangle\!\rangle \mathrm{d}t. \end{aligned} \tag{2.4}$$

Hereafter, C denotes a generic positive constant that may change from line to line.

For J_2, one has that

$$\begin{aligned} J_2 &= \rho_n^{-1} \sum_x \int_0^t (1-\xi_{s-}^n(x)) p_{\mathbf{A}}\Big(\frac{\sum_{y\sim x} \xi_{s-}^n(y)}{N_n}\Big)\phi_s(x)(\mathrm{d}P_s^{\mathbf{A}}(x) - F_{\mathbf{A}}(u_s)\mathrm{d}s) \\ &\quad + \rho_n^{-1} \sum_x \int_0^t (1-\xi_s^n(x)) p_{\mathbf{A}}\Big(\frac{\sum_{y\sim x} \xi_s^n(y)}{N_n}\Big)\phi_s(x) F_{\mathbf{A}}(u_s)\mathrm{d}s \\ &=: E_t^{(2)}(\phi) + \int_0^t [\langle\!\langle 1, F_{\mathbf{A}}(u_s)p_{\mathbf{A}}(u_s)\phi_s\rangle\!\rangle - \langle\!\langle v_s, F_{\mathbf{A}}(u_s)p_{\mathbf{A}}(u_s)\phi_s\rangle\!\rangle]\mathrm{d}s, \end{aligned} \tag{2.5}$$

where $E_t^{(2)}(\phi)$ is a martingale with brackets process given by

$$
\begin{aligned}
\mathrm{d}\langle E_n^{(2)}(\phi)\rangle_t &= \rho_n^{-2}\sum_x (1-\xi_t^n(x))p_{\mathbf{A}}^2(u_t)\phi_t^2(x)F_{\mathbf{A}}(u_t)\mathrm{d}t \\
&= \rho_n^{-1}[\langle\!\langle 1, p_{\mathbf{A}}^2(u_t)F_{\mathbf{A}}(u_t)\phi_t^2\rangle\!\rangle - \langle\!\langle v_t, p_{\mathbf{A}}^2(u_t)F_{\mathbf{A}}(u_t)\phi_t^2\rangle\!\rangle]\mathrm{d}t \\
&\le C(p_{\mathbf{A}}, F_{\mathbf{A}})\rho_n^{-1}\langle\!\langle 1, \phi_t^2\rangle\!\rangle \mathrm{d}t \\
&\le C(p_{\mathbf{A}}, F_{\mathbf{A}})\rho_n^{-1}\|\phi_t\|_\lambda^2\langle\!\langle e_{-2\lambda}, 1\rangle\!\rangle \mathrm{d}t.
\end{aligned}
\tag{2.6}
$$

For J_3, one obtains that

$$
\begin{aligned}
J_3 &= -\rho_n^{-1}\sum_x\int_0^t \xi_{s-}^n(x)p_{\mathbf{a}}\Big(\frac{\sum_{y\sim x}\xi_{s-}^n(y)}{N_n}\Big)\phi_s(x)(\mathrm{d}P_s^{\mathbf{a}}(x) - F_{\mathbf{a}}(u_{s-})\mathrm{d}s) \\
&\quad - \rho_n^{-1}\sum_x\int_0^t \xi_s^n(x)p_{\mathbf{a}}\Big(\frac{\sum_{y\sim x}\xi_s^n(y)}{N_n}\Big)\phi_s(x)F_{\mathbf{a}}(u_s)\mathrm{d}s \\
&=: E_t^{(3)}(\phi) - \int_0^t \langle\!\langle v_s, p_{\mathbf{a}}(u_s)F_{\mathbf{a}}(u_s)\phi_s\rangle\!\rangle \mathrm{d}s,
\end{aligned}
\tag{2.7}
$$

where $E_t^{(3)}(\phi)$ is a martingale with brackets process given by

$$
\begin{aligned}
\mathrm{d}\langle E_n^{(3)}(\phi)\rangle_t &= \rho_n^{-2}\sum_x \xi_t^n(x)p_{\mathbf{a}}^2(u_t)\phi_t^2(x)F_{\mathbf{a}}(u_t)\mathrm{d}t = \rho_n^{-1}\langle\!\langle v_t, \phi_t^2 p_{\mathbf{a}}^2(u_t)F_{\mathbf{a}}(u_t)\rangle\!\rangle \mathrm{d}t \\
&\le \rho_n^{-1}C(p_{\mathbf{a}}, F_{\mathbf{a}})\langle\!\langle 1, \phi_t^2\rangle\!\rangle \mathrm{d}t \le \rho_n^{-1}C(p_{\mathbf{a}}, F_{\mathbf{a}})\|\phi_t\|_\lambda^2\langle\!\langle e_{-2\lambda}, 1\rangle\!\rangle \mathrm{d}t.
\end{aligned}
\tag{2.8}
$$

For J_4, one obtains that

$$
\begin{aligned}
J_4 &= \rho_n^{-1}\sum_x\sum_{y\sim x}\int_0^t \xi_{s-}^n(x)\phi_s(x)(\mathrm{d}P_s(y,x) - \mathrm{d}P_s(x,y)) \\
&= \rho_n^{-1}\sum_x\sum_{y\sim x}\int_0^t \xi_{s-}^n(x)\phi_s(x)\Big[\Big(\mathrm{d}P_s(y,x) - \frac{H_n}{N_n}\mathrm{d}s\Big) - \Big(\mathrm{d}P_s(x,y) - \frac{H_n}{N_n}\mathrm{d}s\Big)\Big] \\
&=: Z_t(\phi),
\end{aligned}
\tag{2.9}
$$

where $Z_t(\phi)$ is a martingale with brackets process given by

$$
\begin{aligned}
\mathrm{d}\langle Z(\phi)\rangle_t &= \rho_n^{-2}\sum_x\sum_{y\sim x}\sum_{x'}\sum_{y'\sim x'}\xi_t^n(x)\phi_t(x)\xi_t^n(x')\phi_t(x')2I(x=x', y=y')\frac{H_n}{N_n}\mathrm{d}t \\
&\quad - \rho_n^{-2}\sum_x\sum_{y\sim x}\sum_{x'}\sum_{y'\sim x'}\xi_t^n(x)\phi_t(x)\xi_t^n(x')\phi_t(x')2I(x=y', y=x')\frac{H_n}{N_n}\mathrm{d}t \\
&= 2\rho_n^{-2}\sum_x\sum_{y\sim x}(\xi_t^n(x)\phi_t(x)^2 - \xi_t^n(x)\phi_t(x)\xi_t^n(y)\phi_t(y))\frac{H_n}{N_n}\mathrm{d}t \\
&= 2\rho_n^{-1}H_n\Big[\langle\!\langle v_t, \phi_t^2\rangle\!\rangle - \Big\langle\!\!\Big\langle v_t, \phi_t\frac{\sum_{y\sim\cdot}(\xi_t^n\phi_t)(y)}{N_n}\Big\rangle\!\!\Big\rangle\Big]\mathrm{d}t \\
&\le 4\rho_n^{-1}H_n\langle\!\langle v_t, \phi_t^2\rangle\!\rangle \mathrm{d}t \le 4\rho_n^{-1}H_n\|\phi_t\|_\lambda^2\langle\!\langle e_{-2\lambda}, 1\rangle\!\rangle \mathrm{d}t.
\end{aligned}
\tag{2.10}
$$

Combining (2.2), (2.5), (2.7) and (2.9), one gains that

$$
\langle\!\langle v_t, \phi_t\rangle\!\rangle = \langle\!\langle v_0, \phi_0\rangle\!\rangle + \int_0^t \langle\!\langle v_s, \partial_s\phi_s + \Delta_n(\phi_s)\rangle\!\rangle \mathrm{d}s
$$

$$
\begin{aligned}
&- \int_0^t \langle\!\langle v_s, [p_{\mathbf{a}}(u_s)F_{\mathbf{a}}(u_s) + F_{\mathbf{A}}(u_s)p_{\mathbf{A}}(u_s)]\phi_s\rangle\!\rangle \mathrm{d}s \\
&+ \int_0^t \langle\!\langle 1, F_{\mathbf{A}}(u_s)p_{\mathbf{A}}(u_s)\phi_s\rangle\!\rangle \mathrm{d}s + \sum_{i=1}^{3} E_t^{(i)}(\phi) + Z_t(\phi). \qquad (2.11)
\end{aligned}
$$

2.2 Green's function representation

For each $z \in \rho_n^{-1}\mathbb{Z}$, let ψ_t^z be the unique solution of

$$
\partial_t \psi_t^z = \Delta_n(\psi_t^z), \quad \psi_0^z(x) = \rho_n N_n^{-1} I(x \sim z).
$$

Linearly interpolating ψ_t^z and letting $n \to \infty$, one obtains that $\psi_t^z(x)$ converges to $p\big(\frac{D_n^2 H_n t}{3}, x - z\big)$, where $p(\sigma^2, \cdot)$ is the density function of centered normal distribution with variance σ^2.

Remark 2.1 A continuous-time random walk with generator Δ_n is of variance $\frac{D_n^2}{3}$.

2.2.1 Property of ψ_t^z

Set $\phi_s = \psi_{t-s}^x$ for $s \le t$ and substitute it into (2.11), then the second term on the right-hand side vanishes, and $\langle\!\langle v_t, \psi_0^x\rangle\!\rangle = u_t(x)$. So

$$
\begin{aligned}
u_t(x) = \langle\!\langle v_0, \psi_t^x\rangle\!\rangle &- \int_0^t \langle\!\langle v_s, [p_{\mathbf{a}}(u_s)F_{\mathbf{a}}(u_s) + F_{\mathbf{A}}(u_s)p_{\mathbf{A}}(u_s)]\psi_{t-s}^x\rangle\!\rangle \mathrm{d}s \\
&+ \int_0^t \langle\!\langle 1, F_{\mathbf{A}}(u_s)p_{\mathbf{A}}(u_s)\psi_{t-s}^x\rangle\!\rangle \mathrm{d}s + \sum_{i=1}^{3} E_t^{(i)}(\psi_{t-\cdot}^x) + Z_t(\psi_{t-\cdot}^x). \qquad (2.12)
\end{aligned}
$$

Lemma 2.1 *For $T \ge 0$, $p \ge 2, \lambda > 0$, we have*

$$
\mathbb{E}(|E_t^{(i)}(\psi_{t-\cdot}^z)|^p) \le \widehat{C} n^{-\frac{p}{16}} e_{\lambda p}(z) \quad \textit{for all}\, t \le T,
$$

where $\widehat{C} = C(\lambda, p, T, p_{\mathbf{A}}, F_{\mathbf{A}}, p_{\mathbf{a}}, F_{\mathbf{a}})$, $1 \le i \le 3$.

Proof From [11, Lemma 3(a)], the greatest jumps of the martingales $E_t^{(i)}(\phi)$ are bounded by $\rho_n^{-1} \sup\limits_{s \ge 0} \|\psi_s^z\|_0 \le C\rho_n^{-1} n^{\frac{1}{2}}$ a.s., where C is a constant.

For $i = 1$, using Burkholder's inequality, (2.3) and (2.4), we have

$$
\begin{aligned}
&\mathbb{E}(|E_t^{(1)}(\psi_{t-\cdot}^z)|^p) \\
&\le C(p)(\rho_n^{-1}H_n)^{\frac{p}{2}} \Big(\int_{(t-n^{-\frac{3}{8}})_+}^t \|\psi_{t-s}^z\|_0 \langle\!\langle \psi_{t-s}^z, 1\rangle\!\rangle \mathrm{d}s \Big)^{\frac{p}{2}} \\
&\quad + C(p)(\rho_n^{-1}H_n)^{\frac{p}{2}} \mathbb{E}\Big[\Big(\int_0^{(t-n^{-\frac{3}{8}})_+} \|\mathcal{D}(\psi_{t-s}^z, D_n)\|_\lambda^2 \langle\!\langle e_{-2\lambda}, v_s\rangle\!\rangle \mathrm{d}s \Big)^{\frac{p}{2}}\Big] \\
&\quad + C(p)\rho_n^{-p} n^{\frac{p}{2}}.
\end{aligned}
$$

By [11, Lemma 3(a,c)], one has

$$
\begin{aligned}
\Big(\int_{(t-n^{-\frac{3}{8}})_+}^t \|\psi_{t-s}^z\|_0 \langle\!\langle \psi_{t-s}^z, 1\rangle\!\rangle \mathrm{d}s \Big)^{\frac{p}{2}} &\le C(T) \Big(\int_{(t-n^{-\frac{3}{8}})_+}^t (t-s)^{-\frac{2}{3}} \langle\!\langle 1, \psi_{t-s}^z\rangle\!\rangle \mathrm{d}s \Big)^{\frac{p}{2}} \\
&\le C(p, T) n^{-\frac{p}{16}}
\end{aligned}
$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^{(t-n^{-\frac{3}{8}})_+}\|\mathcal{D}(\psi_{t-s}^z,D_n)\|_\lambda^2\langle\!\langle e_{-2\lambda},v_s\rangle\!\rangle\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\
&\le C(\lambda,p,T)\Big(\int_0^{(t-n^{-\frac{3}{8}})_+}\|\mathcal{D}(\psi_{t-s}^z,D_n)\|_\lambda^2\mathrm{d}s\Big)^{\frac{p}{2}}\\
&\le C(\lambda,p,T)e_{\lambda p}(z)n^{-\frac{p}{4}}\Big(\int_0^{(t-n^{-\frac{3}{8}})_+}(t-s)^{-2}\mathrm{d}s\Big)^{\frac{p}{2}}\\
&\le C(\lambda,p,T)n^{-\frac{p}{16}}e_{\lambda p}(z).
\end{aligned}$$

Finally,

$$\mathbb{E}(|E_t^{(1)}(\psi_{t-\cdot}^z)|^p)\le C(\lambda,p,T)e_{\lambda p}(z)\rho_n^{-\frac{p}{2}}((H_nn^{-\frac{1}{8}})^{\frac{p}{2}}+(\rho_n^{-1}n)^{\frac{p}{2}}).\tag{2.13}$$

Similarly, for $i=2,3$, by (2.6), (2.8) and [11, Lemma 3(c)], we get

$$\begin{aligned}
\mathbb{E}(|E_t^{(i)}(\psi_{t-\cdot}^z)|^p)&\le \widehat{C}\rho_n^{-\frac{p}{2}}\Big[\Big(\int_0^t\|\psi_{t-s}^z\|_\lambda^2\mathrm{d}s\Big)^{\frac{p}{2}}+(\rho_n^{-1}n)^{\frac{p}{2}}\Big]\\
&\le \widehat{C}\rho_n^{-\frac{p}{2}}[(n^{\frac{1}{4}})^{\frac{p}{2}}+(\rho_n^{-1}n)^{\frac{p}{2}}]e_{\lambda p}(z).
\end{aligned}\tag{2.14}$$

According to (2.13) and (2.14), one has

$$\mathbb{E}(|E_t^{(i)}(\psi_{t-\cdot}^z)|^p)\le \widehat{C}n^{-\frac{p}{16}}e_{\lambda p}(z),$$

where $1\le i\le 3$. The proof is complete.

2.3 Tightness

Our objective is to prove the tightness of u_n. Define

$$\widehat{u}_n(t,x):=u_t(x)-\langle\!\langle v_0,\psi_t^x\rangle\!\rangle.$$

Lemma 2.2 *For $0\le s\le t\le T,\ y,z\in\rho_n^{-1}\mathbb{Z},\ |t-s|,|y-z|\le 1,\ \lambda>0,\ p\ge 2,$ we have*

$$\mathbb{E}(|\widehat{u}_t(z)-\widehat{u}_s(y)|^p)\le \widehat{C}e_{\lambda p}(z)(|t-s|^{\frac{p}{24}}+|z-y|^{\frac{p}{24}}+n^{-\frac{p}{12}}),$$

where $\widehat{C}=C(\lambda,p,T,p_{\mathbf{A}},F_{\mathbf{A}},p_{\mathbf{a}},F_{\mathbf{a}})$.

Proof Set

$$\delta:=(|z-y|^{\frac{1}{4}}\vee n^{-\frac{1}{2}})\wedge t,\quad \overline{\delta}:=(|t-s|^{\frac{1}{4}}\vee n^{-\frac{1}{2}})\wedge s.$$

From (2.12), we have

$$\begin{aligned}
\widehat{u}_t(x)=&-\int_0^t\langle\!\langle v_s,(p_{\mathbf{a}}(u_s)F_{\mathbf{a}}(u_s)+F_{\mathbf{A}}(u_s)p_{\mathbf{A}}(u_s))\psi_{t-s}^x\rangle\!\rangle\mathrm{d}s\\
&+\int_0^t\langle\!\langle 1,F_{\mathbf{A}}(u_s)p_{\mathbf{A}}(u_s)\psi_{t-s}^x\rangle\!\rangle\mathrm{d}s+\sum_{i=1}^3E_t^{(i)}(\psi_{t-\cdot}^x)+Z_t(\psi_{t-\cdot}^x).
\end{aligned}$$

By Lemma 2.1 and using Hölder's inequality, we estimate that

$$\begin{aligned}
&\mathbb{E}(|\widehat{u}_t(z)-\widehat{u}_t(y)|^p)-\widehat{C}n^{-\frac{p}{16}}e_{\lambda p}(z)\\
&\le C(p)\mathbb{E}\Big|\int_0^t \langle\!\langle v_s,(p_{\mathbf{a}}(u_s)F_{\mathbf{a}}(u_s)+F_{\mathbf{A}}(u_s)p_{\mathbf{A}}(u_s))(\psi^z_{t-s}-\psi^y_{t-s})\rangle\!\rangle \mathrm{d}s\Big|^p\\
&\quad + C(p)\mathbb{E}\Big|\int_0^t \langle\!\langle 1,F_{\mathbf{A}}(u_s)p_{\mathbf{A}}(u_s)(\psi^z_{t-s}-\psi^y_{t-s})\rangle\!\rangle \mathrm{d}s\Big|^p\\
&\quad + C(p)\mathbb{E}|Z_t(\psi^z_{t-\cdot}-\psi^y_{t-\cdot})|^p\\
&=: T_1+T_2+T_3.
\end{aligned}$$

For the term T_1, using [11, Lemma 3(c, e)] one has that

$$\begin{aligned}
T_1 &\le \widehat{C}\Big(\mathbb{E}\Big|\int_0^{t-\delta}\langle\!\langle v_s,(\psi^z_{t-s}-\psi^y_{t-s})\rangle\!\rangle \mathrm{d}s\Big|^p+\mathbb{E}\Big|\int_{t-\delta}^t\langle\!\langle v_s,(\psi^z_{t-s}-\psi^y_{t-s})\rangle\!\rangle \mathrm{d}s\Big|^p\Big)\\
&\le \widehat{C}\Big[\Big(\int_0^{t-\delta}\langle\!\langle 1,|\psi^z_{t-s}-\psi^y_{t-s}|\rangle\!\rangle \mathrm{d}s\Big)^p+\Big(\int_{t-\delta}^t\langle\!\langle 1,|\psi^z_{t-s}-\psi^y_{t-s}|\rangle\!\rangle \mathrm{d}s\Big)^p\Big]\\
&\le \widehat{C}\Big[\Big(\int_0^{t-\delta}\langle\!\langle 1,e_{-\lambda}\rangle\!\rangle \mathrm{d}s\Big)^p\Big(\sup_{s\in(\delta,t]}\|\psi^z_s-\psi^y_s\|_\lambda\Big)^p+\Big(\int_{t-\delta}^t\|\psi^z_{t-s}\|_\lambda\langle\!\langle 1,e_{-\lambda}\rangle\!\rangle \mathrm{d}s\Big)^p\Big]\\
&\le \widehat{C}\Big[\Big(\sup_{s\in(\delta,t]}\|\psi^z_s-\psi^y_s\|_\lambda\Big)^p+e_{\lambda p}(z)\Big(\int_{t-\delta}^t|t-s|^{-\frac{2}{3}}\mathrm{d}s\Big)^p\Big]\\
&\le \widehat{C}e_{\lambda p}(z)(|z-y|^{\frac{p}{2}}\delta^{-p}+n^{-\frac{p}{2}}\delta^{-\frac{3p}{4}})+\widehat{C}e_{\lambda p}(z)\delta^{\frac{p}{3}}.
\end{aligned}$$

Similarly, for the term T_2, one obtains

$$\begin{aligned}
T_2 &\le \widehat{C}\Big(\Big|\int_0^{t-\delta}\langle\!\langle 1,(\psi^z_{t-s}-\psi^y_{t-s})\rangle\!\rangle \mathrm{d}s\Big|^p+\Big|\int_{t-\delta}^t\langle\!\langle 1,(\psi^z_{t-s}-\psi^y_{t-s})\rangle\!\rangle \mathrm{d}s\Big|^p\Big)\\
&\le \widehat{C}\Big(\Big(\int_0^{t-\delta}\langle\!\langle 1,|\psi^z_{t-s}-\psi^y_{t-s}|\rangle\!\rangle \mathrm{d}s\Big)^p+\Big(\int_{t-\delta}^t\langle\!\langle 1,|\psi^z_{t-s}-\psi^y_{t-s}|\rangle\!\rangle \mathrm{d}s\Big)^p\Big)\\
&\le \widehat{C}e_{\lambda p}(z)(|z-y|^{\frac{p}{2}}\delta^{-p}+n^{-\frac{p}{2}}\delta^{-\frac{3p}{4}})+\widehat{C}e_{\lambda p}(z)\delta^{\frac{p}{3}}.
\end{aligned}$$

For the term T_3, use [11, Lemma 3(c, e)] and (2.10), we find

$$\begin{aligned}
T_3 &\le C(p)(\rho_n^{-1}H_n)^{\frac{p}{2}}\Big(\mathbb{E}\Big(\int_0^{t-\delta}\langle\!\langle v_s,(\psi^z_{t-s}-\psi^y_{t-s})^2\rangle\!\rangle \mathrm{d}s\Big)^{\frac{p}{2}}+\mathbb{E}\Big(\int_{t-\delta}^t\langle\!\langle v_s,(\psi^z_{t-s}-\psi^y_{t-s})^2\rangle\!\rangle \mathrm{d}s\Big)^{\frac{p}{2}}\Big)\\
&\le C(p)(\rho_n^{-1}H_n)^{\frac{p}{2}}\Big(\Big(\int_0^{t-\delta}\langle\!\langle 1,(\psi^z_{t-s}-\psi^y_{t-s})^2\rangle\!\rangle \mathrm{d}s\Big)^{\frac{p}{2}}+\Big(\int_{t-\delta}^t\langle\!\langle 1,(\psi^z_{t-s}-\psi^y_{t-s})^2\rangle\!\rangle \mathrm{d}s\Big)^{\frac{p}{2}}\Big)\\
&\le C(p)(\rho_n^{-1}H_n)^{\frac{p}{2}}\Big(\Big(\int_0^{t-\delta}\langle\!\langle 1,e_{-2\lambda}\rangle\!\rangle \mathrm{d}s\Big)^{\frac{p}{2}}\Big(\sup_{s\in(\delta,t]}\|\psi^z_s-\psi^y_s\|_\lambda\Big)^p\\
&\quad +\Big(\int_{t-\delta}^t\|\psi^z_{t-s}+\psi^y_{t-s}\|_0\langle\!\langle 1,\psi^z_{t-s}+\psi^y_{t-s}\rangle\!\rangle \mathrm{d}s\Big)^{\frac{p}{2}}\Big)\\
&\le C(\lambda,p,T)(\rho_n^{-1}H_n)^{\frac{p}{2}}\Big(\Big(\sup_{s\in(\delta,t]}\|\psi^z_s-\psi^y_s\|_\lambda\Big)^p+\Big(\int_{t-\delta}^t|t-s|^{-\frac{2}{3}}\mathrm{d}s\Big)^{\frac{p}{2}}\Big)\\
&\le \widehat{C}e_{\lambda p}(z)(|z-y|^{\frac{p}{2}}\delta^{-p}+n^{-\frac{p}{2}}\delta^{-\frac{3p}{4}})+\widehat{C}e_{\lambda p}(z)\delta^{\frac{p}{6}}.
\end{aligned}$$

Combining the estimates for T_1, T_2 and T_3 and noticing the definition of δ, we have

$$\begin{aligned}\mathbb{E}(|\widehat{u}_t(z) - \widehat{u}_t(y)|^p) &\le \widehat{C}e_{\lambda p}(z)(n^{-\frac{p}{16}} + |z-y|^{\frac{p}{2}}\delta^{-p} + n^{-\frac{p}{2}}\delta^{-\frac{3p}{4}} + \delta^{\frac{p}{3}} + \delta^{\frac{p}{6}}) \\ &\le \widehat{C}e_{\lambda p}(z)(|z-y|^{\frac{p}{24}} + n^{-\frac{p}{12}}).\end{aligned} \tag{2.15}$$

Also using Lemma 2.1, (2.10) and Hölder's inequality, we can similarly estimate that

$$\begin{aligned}&\mathbb{E}(|\widehat{u}_t(y) - \widehat{u}_s(y)|)^p - \widehat{C}n^{-\frac{p}{16}}e_{\lambda p}(y) \\ &\le C(p)\mathbb{E}\Big(\int_s^t \langle\!\langle v_r, (p_{\mathbf{a}}(u_r)F_{\mathbf{a}}(u_r) + F_{\mathbf{A}}(u_r)p_{\mathbf{A}}(u_r))\psi^y_{t-r}\rangle\!\rangle \mathrm{d}r\Big)^p \\ &\quad + C(p)\mathbb{E}\Big(\int_0^s \langle\!\langle v_r, (p_{\mathbf{a}}(u_r)F_{\mathbf{a}}(u_r) + F_{\mathbf{A}}(u_r)p_{\mathbf{A}}(u_r))|\psi^y_{t-r} - \psi^y_{s-r}|\rangle\!\rangle \mathrm{d}r\Big)^p \\ &\quad + C(p)\mathbb{E}\Big(\int_s^t \langle\!\langle 1, F_{\mathbf{A}}(u_r)p_{\mathbf{A}}(u_r)\psi^y_{t-r}\rangle\!\rangle \mathrm{d}r\Big)^p \\ &\quad + C(p)\mathbb{E}\Big(\int_0^s \langle\!\langle 1, F_{\mathbf{A}}(u_r)p_{\mathbf{A}}(u_r)|\psi^y_{t-r} - \psi^y_{s-r}|\rangle\!\rangle \mathrm{d}r\Big)^p \\ &\quad + C(p)(\rho_n^{-1}H_n)^{\frac{p}{2}}\mathbb{E}\Big(\int_s^t \langle\!\langle v_r, (\psi^y_{t-r})^2\rangle\!\rangle \mathrm{d}r\Big)^{\frac{p}{2}} \\ &\quad + C(p)(\rho_n^{-1}H_n)^{\frac{p}{2}}\mathbb{E}\Big(\int_0^s \langle\!\langle v_r, (\psi^y_{t-r} - \psi^y_{s-r})^2\rangle\!\rangle \mathrm{d}r\Big)^{\frac{p}{2}} \\ &=: T_4 + T_5 + T_6 + T_7 + T_8 + T_9.\end{aligned}$$

For terms T_4 and T_6, we use [11, Lemma 3(a)] and get

$$T_4 + T_6 \le \widehat{C}\Big(\int_s^t \langle\!\langle 1, \psi^y_{t-r}\rangle\!\rangle \mathrm{d}r\Big)^p \le \widehat{C}|t-s|^p.$$

For terms T_5 and T_7, we use [11, Lemma 3(c, f)] and obtain

$$\begin{aligned}T_5 + T_7 &\le \widehat{C}\Big(\Big(\int_0^{s-\overline{\delta}} \langle\!\langle 1, |\psi^y_{t-r} - \psi^y_{s-r}|\rangle\!\rangle \mathrm{d}r\Big)^p + \Big(\int_{s-\overline{\delta}}^s \langle\!\langle 1, |\psi^y_{t-r} - \psi^y_{s-r}|\rangle\!\rangle \mathrm{d}r\Big)^p\Big) \\ &\le \widehat{C}\Big(\Big(\sup_{r\in[0,s-\overline{\delta})} \|\psi^y_{t-r} - \psi^y_{s-r}\|_\lambda\Big)^p + \Big(\int_{s-\overline{\delta}}^s \|\psi^y_{s-r}\|_\lambda \mathrm{d}r\Big)^p\Big) \\ &\le \widehat{C}e_{\lambda p}(y)(|t-s|^{\frac{p}{2}}\overline{\delta}^{-\frac{3p}{2}} + n^{-\frac{p}{2}}\overline{\delta}^{-\frac{3p}{4}}) + \widehat{C}e_{\lambda p}(y)\overline{\delta}^{\frac{p}{3}}.\end{aligned}$$

For term T_8, using [11, Lemma 3(c)], we have

$$T_8 \le \widehat{C}\Big(\int_s^t \|\psi^y_{t-r}\|_0 \langle\!\langle 1, \psi^y_{t-r}\rangle\!\rangle \mathrm{d}r\Big)^{\frac{p}{2}} \le \widehat{C}\Big(\int_s^t (t-r)^{-\frac{2}{3}}\mathrm{d}r\Big)^{\frac{p}{2}} \le \widehat{C}|t-s|^{\frac{p}{6}}.$$

For term T_9, we use [11, Lemma 3(c, f)] and obtain

$$\begin{aligned}T_9 &\le \widehat{C}\Big(\Big(\int_0^{s-\overline{\delta}} \langle\!\langle 1, (\psi^y_{t-r} - \psi^y_{s-r})^2\rangle\!\rangle \mathrm{d}r\Big)^{\frac{p}{2}} + \Big(\int_{s-\overline{\delta}}^s \langle\!\langle 1, (\psi^y_{t-r} - \psi^y_{s-r})^2\rangle\!\rangle \mathrm{d}r\Big)^{\frac{p}{2}}\Big) \\ &\le \widehat{C}\Big(\Big(\sup_{r\in[0,s-\overline{\delta})} \|\psi^y_{t-r} - \psi^y_{s-r}\|_\lambda\Big)^p + \Big(\int_{s-\overline{\delta}}^s \|\psi^y_{s-r} + \psi^y_{t-r}\|_0 \langle\!\langle 1, \psi^y_{t-r} + \psi^y_{s-r}\rangle\!\rangle \mathrm{d}r\Big)^{\frac{p}{2}}\Big) \\ &\le \widehat{C}e_{\lambda p}(y)(|t-s|^{\frac{p}{2}}\overline{\delta}^{-\frac{3p}{2}} + n^{-\frac{p}{2}}\overline{\delta}^{-\frac{3p}{4}}) + \widehat{C}e_{\lambda p}(y)\overline{\delta}^{\frac{p}{6}}.\end{aligned}$$

Combining the estimates for T_i $(4 \le i \le 9)$ and noticing the definition of $\overline{\delta}$, we have

$$\begin{aligned}
&\mathbb{E}(|\widehat{u}_t(y) - \widehat{u}_s(y)|)^p \\
&\le \widehat{C}e_{\lambda p}(y)(n^{-\frac{p}{16}} + |t-s|^p + |t-s|^{\frac{p}{6}} + |t-s|^{\frac{p}{2}}\overline{\delta}^{-\frac{3p}{2}} + n^{-\frac{p}{2}}\overline{\delta}^{-\frac{3p}{4}} + \overline{\delta}^{\frac{p}{3}} + \overline{\delta}^{\frac{p}{6}}) \\
&\le \widehat{C}e_{\lambda p}(y)(|t-s|^{\frac{p}{24}} + n^{-\frac{p}{12}}) \\
&\le \widehat{C}e_{\lambda p}(z)(|t-s|^{\frac{p}{24}} + n^{-\frac{p}{12}}).
\end{aligned} \tag{2.16}$$

Put (2.15)–(2.16) together and get

$$\begin{aligned}
\mathbb{E}(|\widehat{u}_t(z) - \widehat{u}_s(y)|^p) &\le C(p)\mathbb{E}(|\widehat{u}_t(z) - \widehat{u}_t(y)|^p) + C(p)\mathbb{E}(|\widehat{u}_t(y) - \widehat{u}_s(y)|^p) \\
&\le \widehat{C}e_{\lambda p}(z)(|t-s|^{\frac{p}{24}} + |z-y|^{\frac{p}{24}} + n^{-\frac{p}{12}}).
\end{aligned}$$

The proof is complete.

To get the tightness of $u_n(\cdot,\cdot)$, define $\widetilde{u}_n(t,x) := \widehat{u}_n(t,x)$ on the grid $z \in \rho_n^{-1}\mathbb{Z},\ t \in \mathbb{N}/(n\rho_n)$, then linearly interpolate it first in x and then in t to obtain a continuous $\mathscr{C}$ valued process. Using Lemma 2.2, it is easy to find that

$$\mathbb{E}(|\widetilde{u}_t(z) - \widetilde{u}_s(y)|^p) \le \widehat{C}e_{\lambda p}(z)(|t-s|^{\frac{p}{24}} + |z-y|^{\frac{p}{24}}),$$

because that means to find an m such that $\big(\frac{1}{n\rho_n}\big)^{\frac{p}{m}} \ge \big(\frac{1}{n\rho_n}\big)^{\frac{p}{24}} + \big(\frac{1}{n}\big)^{\frac{p}{12}}$ and $\big(\frac{1}{\rho_n}\big)^{\frac{p}{m}} \ge \big(\frac{1}{\rho_n}\big)^{\frac{p}{24}} + \big(\frac{1}{n}\big)^{\frac{p}{12}}$. If $\rho_n = n$, m should be 24.

The following result is taken from [11, Lemma 7].

Lemma 2.3 *For any $\lambda > 0, T < \infty$,*
(i) $\mathbb{P}\big(\sup\limits_{t\le T}\|\widehat{u}_t(z) - \widetilde{u}_t(z)\|_{-\lambda} \ge 7n^{-\frac{1}{4}}\big) \to 0$ *as* $n \to \infty$,
(ii) $\sup\limits_{t\le T}\|\langle\langle v_0, \psi_t^{\cdot}\rangle\rangle - P_{\frac{t}{3}}f_0\|_{-\lambda} \to 0$ *as* $n \to \infty$.

On any given compact subset $K \subset \mathbb{R}^+\times\mathbb{R}$, $\widetilde{u}_n(t,x)$ is uniformly bounded and equicontinuous by using Kolmogorov's continuity criterion (see [16, Corollary 1.2(ii)]) almost surely. Therefore, we get the tightness of $\widetilde{u}_n(t,x)$ as continuous $\mathscr{C}$-valued process. Then the tightness of $u_n(t,x)$ follows from the above lemma. Also, the continuity of all limit points follows.

2.4 Characterizing limit points

Taking a continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with compact support, we define

$$\begin{aligned}
|E_t^{(4)}(\phi)| &:= |\langle\langle v_t^n, \phi\rangle\rangle - \langle\langle u_n(t), \phi\rangle\rangle| = \rho_n^{-1}N_n^{-1}\Big|\sum_x\sum_{y\sim x}(\xi_t^n(x) - \xi_t^n(y))\phi(x)\Big| \\
&\le \rho_n^{-1}N_n^{-1}\sum_x\sum_{y\sim x}\xi_t^n(x)|\phi(y) - \phi(x)| \le \langle\langle 1, \mathcal{D}(\phi, D_n)\rangle\rangle,
\end{aligned} \tag{2.17}$$

and get $\mathbb{E}\big(\sup\limits_{t\le T}|E_t^{(4)}(\phi)|\big) \le \langle\langle 1, \mathcal{D}(\phi, D_n)\rangle\rangle$. From the tightness of $u_n(t)$, we can get the tightness of $(\langle\langle v_t^n, \phi\rangle\rangle, t \ge 0)$. This in turn implies the tightness of $(v_t^n, t \ge 0)$ as cadlag Radon measure valued process with the vague topology (see [1, Theorem 3.6.4]). $\langle\langle v_t^n, e_{-\lambda}\rangle\rangle \le \langle\langle 1, e_{-\lambda}\rangle\rangle \le C(\lambda)$ assures the compact containment condition is satisfied, and then all limit points are continuous.

Because of simultaneous convergence of subsequence of the pairs $(u_n(t), v_t^n)$, by Skorokhod's theorem (see [3, Theorem 2.1.8]), we can find random variables with the same distribution as ξ_t^n, which converges almost surely. We can still label it as $(u_n(t), v_t^n)$ since our interest is to identify the distribution of the limit. Since the limits are continuous, the almost sure convergence holds not only in Skorokhod sense but also in uniform sense on compact sets. Thus, with probability one, for any $T < \infty$, $\lambda > 0$ and function ϕ with compact support, we have

$$\sup_{t\le T} \|u_n(t) - u_t\|_{-\lambda} \to 0, \quad \sup_{t\le T} \Big| \int \phi(x) v_t^n(\mathrm{d}x) - \int \phi(x) v_t(\mathrm{d}x) \Big| \to 0,$$

where $v_t(\mathrm{d}x) = u_t(x)\mathrm{d}x$ for all $t \ge 0$, from (2.17).

Taking a $\phi \in C_c^3(\mathbb{R})$, and substituting it into (2.11), we have

$$\begin{aligned} Z_t(\phi) = & \int \phi(x) v_t^n(\mathrm{d}x) - \int \phi(x) v_0^n(\mathrm{d}x) - \int_0^t \int \Delta_n(\phi)(x) v_s^n(\mathrm{d}x)\mathrm{d}s \\ & + \int_0^t \int (p_{\mathbf{a}}(u_s) F_{\mathbf{a}}(u_s) + F_{\mathbf{A}}(u_s) p_{\mathbf{A}}(u_s)) \phi(x) v_s^n(\mathrm{d}x)\mathrm{d}s \\ & - \int_0^t \int F_{\mathbf{A}}(u_s) p_{\mathbf{A}}(u_s) \phi(x)\mathrm{d}x\mathrm{d}s - \sum_{i=1}^3 E_t^{(i)}(\phi). \end{aligned}$$

When n goes to infinity, we know that $E_t^{(i)}(\phi)$ $(1 \le i \le 3)$ tend to zero almost surely for all t by (2.3), (2.6) and (2.8), and also know $\Delta_n(\phi)(x)$ tends to $\frac{1}{3}\partial_x^2$ uniformly by Taylor's expansion. Therefore, $Z_t(\phi)$ tends to a continuous local martingale $z_t(\phi)$, where

$$\begin{aligned} z_t(\phi) = & \int \phi(x) u_t(x)\mathrm{d}x - \int \phi(x) u_0(x)\mathrm{d}x - \int_0^t \int \frac{1}{3} u_s(x) \partial_x^2 \phi(x)\mathrm{d}x\mathrm{d}s \\ & - \int_0^t \int ((1 - u_s(x)) F_{\mathbf{A}}(u_s(x)) p_{\mathbf{A}}(u_s(x)) \\ & - u_s(x) p_{\mathbf{a}}(u_s(x)) F_{\mathbf{a}}(u_s(x)))\phi(x)\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{2.18}$$

From (2.10), we know the following process

$$\begin{aligned} & Z_t^2(\phi) - 2\int_0^t \rho_n^{-1} H_n \Big[\langle\!\langle v_s, \phi^2 \rangle\!\rangle - \Big\langle\!\!\Big\langle v_s, \phi \frac{\sum_{y\sim\cdot} (\xi_s^n \phi_s)(y)}{N_n} \Big\rangle\!\!\Big\rangle \Big] \mathrm{d}s \\ = & Z_t^2(\phi) - 2\int_0^t \rho_n^{-1} H_n \langle\!\langle v_s, (1 - u_s)\phi^2 \rangle\!\rangle \mathrm{d}s \\ & + 2\int_0^t \rho_n^{-1} H_n \Big\langle\!\!\Big\langle v_s \phi, \frac{\sum_{y\sim\cdot} (\xi_s^n \phi_s)(y)}{N_n} - u_s \phi \Big\rangle\!\!\Big\rangle \mathrm{d}s \\ = & Z_t^2(\phi) - 2\int_0^t \rho_n^{-1} H_n \langle\!\langle v_s, (1 - u_s)\phi^2 \rangle\!\rangle \mathrm{d}s + E_t^{(5)}(\phi) \end{aligned}$$

is a martingale, where

$$E_t^{(5)}(\phi) := 2\int_0^t \rho_n^{-1} H_n \Big\langle\!\!\Big\langle v_s \phi, \frac{\sum_{y\sim\cdot} (\xi_s^n \phi_s)(y)}{N_n} - u_s \phi \Big\rangle\!\!\Big\rangle \mathrm{d}s.$$

Fortunately, we have a bound

$$
\begin{aligned}
E_t^{(5)}(\phi) &\leq 2\int_0^t \rho_n^{-2} H_n \sum_x \xi_s^n(x)\phi(x) \frac{\sum\limits_{y\sim x} \xi_s^n(y)|\phi(y)-\phi(x)|}{N_n} \mathrm{d}s \\
&\leq 2\int_0^t \rho_n^{-1} H_n \langle\!\langle v_s\phi, u_s\mathcal{D}(\phi, D_n)\rangle\!\rangle \mathrm{d}s \\
&\leq 2t\rho_n^{-1} H_n \langle\!\langle \phi, \mathcal{D}(\phi, D_n)\rangle\!\rangle \to 0.
\end{aligned}
$$

So

$$
z_t^2(\phi) - 4\int_0^t \int (1-u_s(x))\phi^2(x)u_s(x)\mathrm{d}x\mathrm{d}s \tag{2.19}
$$

is a continuous local martingale. Since $C_c^3(\mathbb{R})$ is dense in $C_c^2(\mathbb{R})$, (2.18)–(2.19) hold for any $\phi \in C_c^2(\mathbb{R})$. Hence, the solution $u(t,x)$ to the martingale problem associated with the following SPDE

$$
\partial_t u = \frac{1}{3}\partial_x^2 u + (1-u)p_{\mathbf{A}}(u)F_{\mathbf{A}}(u) - up_{\mathbf{a}}(u)F_{\mathbf{a}}(u) + 2\sqrt{u(1-u)}\dot{W}
$$

comes.

Remark 2.2 Instead of taking $\phi \in C_c^2(\mathbb{R})$ directly, we use $\phi \in C_c^3(\mathbb{R})$. The reason is when we conclude $\Delta_n(\phi)(x)$ tends to $\frac{1}{3}\partial_x^2$, Taylor's expansion is required.

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