Chin. Ann. Math. Ser. B
43(2), 2022, 209–222
DOI: 10.1007/s11401-022-0312-y

Chinese Annals of
Mathematics, Series B

Sufficiency of Kalman's Rank Condition for the Approximate Boundary Controllability on an Annular Domain

Chengxia ZU[1]

Abstract In this paper the author establishes the sufficiency of Kalman's rank condition on the approximate boundary controllability at a finite time for diagonalizable systems on an annular domain in higher dimensional case.

Keywords Kalman's rank condition, Approximate boundary controllability, Diagonalizable systems of wave equations, Annular domain

2000 MR Subject Classification 93B05, 93B07, 35L20

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset$ and $\mathrm{mes}(\Gamma_1) > 0$. Let $\mathcal{H}_0 = L^2(\Omega)$, $\mathcal{H}_1 = H_0^1(\Omega)$, $\mathcal{L} = L^2_{\mathrm{loc}}(0, +\infty; L^2(\Gamma_1))$ and $\mathcal{H}_{-1} = H^{-1}(\Omega)$ denotes the dual of $\mathcal{H}_1$.

Let $U = (u^{(1)}, \cdots, u^{(N)})^T$. Consider the following coupled system of wave equations with Dirichlet boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ U = DH & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \tag{1.1}$$

with the initial condition

$$t = 0: \ U = \widehat{U}_0, \ U' = \widehat{U}_1 \quad \text{in } \Omega, \tag{1.2}$$

where " $'$" stands for the time derivative; $\Delta = \sum\limits_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ is the Laplacian operator; the coupling matrix $A = (a_{ij})$ is of order N and the boundary control matrix $D = (d_{pq})$ is a full column-rank matrix of order $N \times M$ ($M \leq N$), both with constant elements; $H = (h^{(1)}, \cdots, h^{(M)})^T$ denotes the boundary controls.

From the approximate boundary null controllability of system (1.1) introduced by Li and Rao in [5–6], we have the following definition.

Definition 1.1 *System* (1.1) *is approximately boundary null controllable at the time* $T > 0$, *if for any given initial data* $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$, *there exists a sequence* $\{H_n\}$ *of*

Manuscript received January 20, 2021. Revised August 30, 2021.
[1]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: chengxia@fudan.edu.cn

boundary controls in $\mathcal{L}^M$ *with compact support in* $[0,T]$, *such that the corresponding sequence* $\{U_n\}$ *of solutions to problem* (1.1)–(1.2) *satisfies*

$$(U_n, U_n') \to (0,0) \quad \textit{in } C^0_{\rm loc}([T,+\infty);(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N) \quad \textit{as } n \to +\infty. \tag{1.3}$$

Let $\Phi = (\phi^{(1)}, \cdots, \phi^{(N)})^T$. The adjoint system of system (1.1) is given by

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in } (0,+\infty)\times\Omega, \\ \Phi = 0 & \text{on } (0,+\infty)\times\Gamma \end{cases} \tag{1.4}$$

with the initial data

$$t = 0: \ \Phi = \Phi_0, \ \Phi' = \Phi_1 \quad \text{in } \Omega. \tag{1.5}$$

Definition 1.2 (see [8–9]) *The adjoint system* (1.4) *is D-observable on the interval* $[0,T]$, *if the following partial Neumann observation*

$$D\partial_\nu\Phi \equiv 0 \quad \textit{on } [0,T]\times\Gamma_1, \tag{1.6}$$

∂_ν *being the outward normal derivative, implies that* $(\Phi_0,\Phi_1) = (0,0)$, *then* $\Phi \equiv 0$.

The relationship between the approximate boundary null controllability of system (1.1) and the D-observability of the adjoint system (1.4) was also given by Li and Rao in [5–6] as follows.

Theorem 1.1 *System* (1.1) *is approximately null controllable at the time* $T > 0$ *if and only if the adjoint system* (1.4) *is D-observable on the interval* $[0,T]$.

The necessity of Kalman's rank condition to the D-observability of the adjoint system (1.4), proved by Li and Rao in [5–6], can be written as the following theorem.

Theorem 1.2 *If the adjoint system* (1.4) *is D-observable, then we necessarily have the following Kalman's rank condition*:

$$\text{rank}(D, AD, \cdots, A^{N-1}D) = N. \tag{1.7}$$

Kalman's rank condition (1.7) is not sufficient for the approximate boundary null controllability of system (1.1) in general. Otherwise, noting that Kalman's rank condition (1.7) is independent of the control (and observation) time $T > 0$, if system (1.1) is approximately null controllable at the time $T > 0$, then the approximate boundary null controllability can be realized almost immediately, which contradicts the finite speed of wave propagation. However, in some special cases, Kalman's rank condition (1.7) is sufficient for the approximate boundary null controllability of system (1.1) on a finite time interval $[0,T]$, when $T > 0$ is large enough (see [5–6, 9]). This paper as a continuation of [9] is to investigate the sufficiency of Kalman's rank condition (1.7) for diagonalizable systems on an annular domain $\Omega = \{x : a < |x| < 1\} \subset \mathbb{R}^d$ with $\Gamma_0 = \{x : |x| = a\}$ and $\Gamma_1 = \{x : |x| = 1\}$, where a is a positive constant with $a < 1$.

In Section 2, we will investigate the eigenfunctions and eigenvalues of $-\Delta$ on the annular domain $\Omega = \{x : a < |x| < 1\}$ based on the coordinate transformation, and give some properties of the eigenvalues. The uniqueness result for non-harmonic series on this annular domain will be established in Section 3. The sufficiency of Kalman's rank condition (1.7) for diagonalizable systems on an annular domain will be given in Section 4 by a way similar to the one-space-dimensional case and to [9].

2 Preliminaries

In this section, we will give the eigenfunctions and eigenvalues of $-\Delta$ on an annular domain $\Omega = \{x : a < |x| < 1\}$ with $0 < a < 1$, which will be used in Section 4 to show the sufficiency of Kalman's rank condition (1.7) to the D-observability for $T > 0$ large enough for diagonalizable systems. For this purpose, we consider the eigenvalue problem

$$\begin{cases} -\Delta e(x) = \mu^2 e(x) & \text{in } \Omega, \\ e(x) = 0 & \text{on } \Gamma \end{cases} \tag{2.1}$$

in spherical coordinates.

Let $e(x) = R(r)Y(\theta)$. Similarly to the spherical domain in [9], we get the corresponding eigenvalue problems for $Y(\theta)$ and $R(r)$, respectively.

For $Y(\theta)$, we have

$$-\Delta_\theta Y(\theta) = m(m + d - 2)Y(\theta) \quad \text{on } S^{d-1}, \tag{2.2}$$

where $m \in \mathbb{N}$ and Δ_θ is the Laplacian operator on the unit sphere S^{d-1} with $d \geq 2$ (see [1]).

For $R(r)$, we have

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}(r^{d-1}R'(r)) - m(m + d - 2)r^{d-3}R(r) + \mu^2 r^{d-1}R(r) = 0, \quad a < r < 1, \\ R(a) = 0, \quad R(1) = 0. \end{cases} \tag{2.3}$$

Atkinson and Han introduced the eigenfunctions and eigenvalues of problem (2.2) in [1]. In what follows, $\mathbb{N}$ and $\mathbb{N}^+$ denote the set of natural numbers and the set of positive integers, respectively.

Lemma 2.1 (see [1]) *Let Δ_θ be the Laplacian operator on the unit sphere S^{d-1} with $d \geq 2$. Then, we have*

(i) *for any given $m \in \mathbb{N}$, $\{Y_{m,j}\}_{1\leq j\leq j_m}$ are the eigenfunctions of $-\Delta_\theta$, corresponding to the eigenvalue $m(m + d - 2)$, i.e., we have*

$$-\Delta_\theta Y_{m,j} = m(m + d - 2)Y_{m,j} \quad \text{on } S^{d-1}, \tag{2.4}$$

where j_m, the multiplicity of the eigenvalue $m(m + d - 2)$, is given by

$$j_m = \begin{cases} 1, & \text{when } m = 0, \\ \dfrac{(2m + d - 2)(m + d - 3)!}{m!(d - 2)!}, & \text{when } m \in \mathbb{N}^+; \end{cases} \tag{2.5}$$

(ii) $\{Y_{m,j}\}_{m\in\mathbb{N},1\leq j\leq j_m}$ *are orthonormal in $L^2(S^{d-1})$, i.e., we have*

$$\int_{S^{d-1}} Y_{m,j}Y_{m',j'}\mathrm{d}\Gamma = \delta_{m,m'}\delta_{j,j'} \tag{2.6}$$

with $m, m' \in \mathbb{N}$, $1 \leq j \leq j_m$ and $1 \leq j' \leq j_{m'}$, where $\delta_{m,m'}$ stands for the Kronecker symbol;

(iii) $\{Y_{m,j}\}_{m\in\mathbb{N},1\leq j\leq j_m}$ *are complete in $L^2(S^{d-1})$.*

Next, we consider problem (2.3) as a Sturm-Liouville problem (2.7) below. Some properties of the eigenfunctions of the Sturm-Liouville problem was introduced by Bagrov and Belov in [2] as follows.

Lemma 2.2 *Consider the following Sturm-Liouville problem*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}[\phi(r)x'(r)] - q(r)x(r) + \lambda\rho(r)x(r) = 0, \quad a < r < 1, \\ x(a) = 0, \quad x(1) = 0. \end{cases} \tag{2.7}$$

Assume that $\phi'(r)$, $q(r)$ and $\rho(r)$ are continuous, and $\phi(r) > 0$, $\rho(r) > 0$ and $q(r) \geq 0$ on the interval $[a, 1]$. Then, we have

(i) *there exists a sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}^+}$ and the corresponding sequence of eigenfunctions $\{x_k(r)\}_{k\in\mathbb{N}^+}$ for Sturm-Liouville problem* (2.7)*, and all eigenvalues can be ordered so that*

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_k| \leq \cdots; \tag{2.8}$$

(ii) *every eigenvalue corresponds to, up to a multiplier constant, only one eigenfunction*;

(iii) *the eigenfunctions of Sturm-Liouville problem* (2.7) *corresponding to different eigenvalues are pairwisely orthogonal on the interval $(a, 1)$ with the weight function $\rho(t)$, i.e., for $k, k' \in \mathbb{N}^+$, we have*

$$\int_a^1 x_k(r)x_{k'}(r)\rho(r)\mathrm{d}r = 0, \quad k \neq k', \tag{2.9}$$

where $x_k(r)$ and $x_{k'}(r)$ are eigenfunctions of (2.7) *corresponding to the eigenvalues λ_k and $\lambda_{k'}$ with $\lambda_k \neq \lambda_{k'}$, respectively*;

(iv) (*Steklov's expansion theorem*) *if a function $f(r)$ is twice continuously differentiable on $[a, 1]$ and satisfies the boundary conditions in* (2.7)*, it can be expanded in a series of the eigenfunctions $x_k(r)$ of Sturm-Liouville problem* (2.7) *absolutely and uniformly converging on $[a, 1]$.*

Remark 2.1 Lemma 2.2 is also valid when the interval $[a, 1]$ is replaced by $[r_1, r_2]$.

By [7–8], setting $R(r) = r^{1-\frac{d}{2}}x(r)$, for any fixed $m \in \mathbb{N}$, problem (2.3) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}[rx_m'(r)] - \dfrac{(m + \frac{d}{2} - 1)^2}{r}x_m(r) + \mu^2 r x_m(r) = 0, \quad a < r < 1, \\ x_m(a) = 0, \quad x_m(1) = 0. \end{cases} \tag{2.10}$$

By Wu [8], we have the following lemma.

Lemma 2.3 *For any fixed $m \in \mathbb{N}$, let*

$$\begin{aligned} x_{m,k}(r) = [&N_{m+\frac{d}{2}-1}(\mu_{m,k})J_{m+\frac{d}{2}-1}(\mu_{m,k}r) - J_{m+\frac{d}{2}-1}(\mu_{m,k})N_{m+\frac{d}{2}-1}(\mu_{m,k}r)], \\ &k \in \mathbb{N}^+, \end{aligned} \tag{2.11}$$

where $J_{m+\frac{d}{2}-1}(r)$ and $N_{m+\frac{d}{2}-1}(r)$ are the $\big(m+\frac{d}{2}-1\big)$-th Bessel function and Neumann function, respectively, and $\mu_{m,k}$ is the k-th positive root of

$$N_{m+\frac{d}{2}-1}(\mu)J_{m+\frac{d}{2}-1}(\mu a) - J_{m+\frac{d}{2}-1}(\mu)N_{m+\frac{d}{2}-1}(\mu a) = 0. \tag{2.12}$$

We have

(i) $\mu_{m,k}^2$ $(k \in \mathbb{N}^+)$ *are all the eigenvalues of problem* (2.10);

(ii) $x_{m,k}(r)$ *is the eigenfunction of problem* (2.10)*, corresponding to the eigenvalue $\mu_{m,k}^2$.*

By Lemma 2.2(iii) and Lemma 2.3, for

$$R_{m,k}(r) = r^{1-\frac{d}{2}} x_{m,k}(r), \tag{2.13}$$

we have the following proposition.

Proposition 2.1 (i) *For any given $m \in \mathbb{N}$, $\{R_{m,k}(r)\}_{k\in\mathbb{N}^+}$ is a sequence of orthogonal functions with the weight r^{d-1} in $L^2(a,1)$.*

(ii) (see [5]) *If $d = 2$ and $m \in \mathbb{N}^+$, or if $d \geq 3$ and $m \in \mathbb{N}$, we have*

$$\int_a^1 r^{d-1} R_{m,k}^2(r)\mathrm{d}r \geq \frac{2(1-a)}{\pi^2 \mu_{m,k}^2}, \quad k \in \mathbb{N}^+; \tag{2.14}$$

while, for $d = 2$ and $m = 0$, we have

$$\int_a^1 r^{d-1} R_{0,k}^2(r)\mathrm{d}r \geq \frac{2}{\pi^2 \mu_{0,k}^2}\Big(1 - \frac{J_0^2(\mu_{0,1}) + N_0^2(\mu_{0,1})}{J_0^2(\mu_{0,1}a) + N_0^2(\mu_{0,1}a)}\Big), \quad k \in \mathbb{N}^+. \tag{2.15}$$

Proof (i) Since (2.10) is a Sturm-Liouville problem (2.7) with $\rho(r) = r$, by Lemma 2.2(iii), for any fixed $m \in \mathbb{N}$, we have

$$\int_a^1 x_{m,k}(r) x_{m,k'}(r) r\mathrm{d}r = 0, \quad k \neq k',$$

where $k, k' \in \mathbb{N}^+$. Then, noting (2.13), for any fixed $m \in \mathbb{N}$, we have

$$\int_a^1 R_{m,k}(r) R_{m,k'}(r) r^{d-1}\mathrm{d}r = \int_a^1 x_{m,k}(r) x_{m,k'}(r) r\mathrm{d}r = 0, \quad k, k' \in \mathbb{N}^+, \ k \neq k'.$$

Namely, for any given $m \in \mathbb{N}$, $\{R_{m,k}(r)\}_{k\in\mathbb{N}^+}$ is a sequence of orthogonal functions with the weight r^{d-1} in $L^2(a,1)$.

(ii) By [4, Lemma 4(iv)], a direct computation gives (2.14) and (2.15).

Remark 2.2 (i) Let

$$c_{m,k} = \Big(\int_a^1 r^{d-1} R_{m,k}^2(r)\mathrm{d}r\Big)^{-\frac{1}{2}}. \tag{2.16}$$

Then, for any given $m \in \mathbb{N}$, $\{c_{m,k} R_{m,k}(r)\}_{k\in\mathbb{N}^+}$ is a sequence of orthonormal functions with the weight r^{d-1} in $L^2(a,1)$.

(ii) Furthermore, let

$$c_0 = \frac{\sqrt{2}}{2} \max\Big\{\Big(1 - \frac{J_0^2(\mu_{0,1}) + N_0^2(\mu_{0,1})}{J_0^2(\mu_{0,1}a) + N_0^2(\mu_{0,1}a)}\Big)^{-\frac{1}{2}}, (1-a)^{-\frac{1}{2}}\Big\}. \tag{2.17}$$

Then, by Proposition 2.1(ii), we have

$$|c_{m,k}| \leq c_0 \pi \mu_{m,k}. \tag{2.18}$$

The inequality (2.18) will be useful to guarantee the convergence of the infinite series given in Section 4. We now give the eigenfunctions and eigenvalues of $-\Delta$ on $\Omega = \{x : a < |x| < 1\}$ as follows.

Lemma 2.4 *Let $\Omega = \{x : a < |x| < 1\}$ with $0 < a < 1$, and*

$$e_{m,k,j}(x) = c_{m,k} R_{m,k}(r) Y_{m,j}(\theta), \quad m \in \mathbb{N};\ k \in \mathbb{N}^+;\ 1 \le j \le j_m, \tag{2.19}$$

in which $R_{m,k}(r)$ is given by (2.13), $c_{m,k}$ is given by (2.16), and $Y_{m,j}(\theta)$ is given by Lemma 2.1(ii). We have

(i) *for any given $m \in \mathbb{N}$ and $k \in \mathbb{N}^+$, $e_{m,k,j}(x)$ $(1 \le j \le j_m)$ are all the eigenfunctions of $-\Delta$, corresponding to the eigenvalue $\mu^2_{m,k}$;*

(ii) *$\{e_{m,k,j}(x)\}_{m\in\mathbb{N};k\in\mathbb{N}^+;1\le j\le j_m}$ is an orthonormal sequence in $L^2(\Omega)$;*

(iii) *$\partial_\nu e_{m,k,j}(x)|_{\Gamma_1} = -2\pi^{-1} c_{m,k} Y_{m,j}(\theta)$, where ∂_ν denotes the outward normal derivative on the boundary.*

Proof (i) By the above discussion, it is easy to get (i).

(ii) Let $m, m' \in \mathbb{N}$, $k, k' \in \mathbb{N}^+$, $1 \le j \le j_m$ and $1 \le j' \le j_{m'}$. By Lemma 2.1(ii)–(iii) and Proposition 2.1, we have

$$\begin{aligned}
&\int_\Omega e_{m,k,j}(x) e_{m',k',j'}(x) \mathrm{d}x \\
&= \int_a^1 \int_{S^{d-1}} c_{m,k} R_{m,k}(r) c_{m',k'} R_{m',k'}(r) Y_{m,j} Y_{m',j'} r^{d-1} \mathrm{d}r \mathrm{d}\Gamma \\
&= c_{m,k} c_{m',k'} \int_a^1 R_{m,k}(r) R_{m',k'}(r) r^{d-1} \mathrm{d}r \int_{S^{d-1}} Y_{m,j} Y_{m',j'} \mathrm{d}\Gamma \\
&= \delta_{m,m'} \delta_{k,k'} \delta_{j,j'},
\end{aligned}$$

then we get (ii).

(iii) Since

$$\begin{aligned}
\frac{\mathrm{d}R_{m,k}(r)}{\mathrm{d}r} &= \Big(1 - \frac{d}{2}\Big) r^{-\frac{d}{2}} [N_{m+\frac{d}{2}-1}(\mu_{m,k}) J_{m+\frac{d}{2}-1}(\mu_{m,k} r) - J_{m+\frac{d}{2}-1}(\mu_{m,k}) N_{m+\frac{d}{2}-1}(\mu_{m,k} r)] \\
&\quad + r^{1-\frac{d}{2}} \mu_{m,k} [N_{m+\frac{d}{2}-1}(\mu_{m,k}) J'_{m+\frac{d}{2}-1}(\mu_{m,k} r) - J_{m+\frac{d}{2}-1}(\mu_{m,k}) N'_{m+\frac{d}{2}-1}(\mu_{m,k} r)],
\end{aligned}$$

by the boundary condition in (2.10), we have

$$\begin{aligned}
\frac{\mathrm{d}R_{m,k}(r)}{\mathrm{d}r}\Big|_{r=1} &= \mu_{m,k} [N_{m+\frac{d}{2}-1}(\mu_{m,k}) J'_{m+\frac{d}{2}-1}(\mu_{m,k}) - J_{m+\frac{d}{2}-1}(\mu_{m,k}) N'_{m+\frac{d}{2}-1}(\mu_{m,k})] \\
&= -\frac{2}{\pi},
\end{aligned} \tag{2.20}$$

in which we used the fact that $J_\sigma(x) N'_\sigma(x) - N_\sigma(x) J'_\sigma(x) = \frac{2}{\pi x}$ (see [2]). By (2.20), we have

$$\partial_\nu e_{m,k,j}(x)|_{\Gamma_1} = c_{m,k} \frac{\mathrm{d}R_{m,k}(r)}{\mathrm{d}r}\Big|_{r=1} Y_{m,j}(\theta) = -2\pi^{-1} c_{m,k} Y_{m,j}(\theta). \tag{2.21}$$

Now, we introduce some properties of the eigenvalues $\mu^2_{m,k}$ of $-\Delta$ on $\Omega = \{x : a < |x| < 1\}$. Let $\alpha_{\sigma,k}$ denote the k-th zero point of the cross-product of the σ-th order Bessel function and Neumann function:

$$f_\sigma(x) = N_\sigma(Kx) J_\sigma(x) - J_\sigma(Kx) N_\sigma(x), \tag{2.22}$$

where K is a positive constant with $K > 1$. The property of $\alpha_{\sigma,k}$ is given in [3–4]. By (2.12) and taking $K = a^{-1}$, we get $a \cdot \mu_{m,k} = \alpha_{\sigma,k}$ with $\sigma = m + \frac{d}{2} - 1$. Then, we have the following peroposition.

Proposition 2.2 *Let $\mu_{m,k}$ be the k-th positive root of* (2.12) *with $m \in \mathbb{N}$ and $k \in \mathbb{N}^+$.*

(i) *For $d = 2$, we have*

$$\mu_{m,k+1} - \mu_{m,k} > \frac{\pi}{2-a}, \quad m \in \mathbb{N}^+;\ k \geq 2;$$

while, for any fixed $d > 2$, we have

$$\mu_{m,k+1} - \mu_{m,k} > \frac{\pi}{2-a}, \quad m \in \mathbb{N};\ k \geq 2.$$

(ii) *For any fixed $d \geq 2$, when $k \to +\infty$, we have*

$$\mu_{m,k} = \frac{k\pi}{1-a} + O_m\Big(\frac{1}{k}\Big), \quad m \in \mathbb{N}.$$

(iii) *For any fixed $d \geq 2$, we have*

$$\mu_{m+1,k} - \mu_{m,k} > 0, \quad m \in \mathbb{N};\ k \in \mathbb{N}^+.$$

(iv) *For any fixed $d \geq 2$, we have*

$$\frac{2m+d-2}{2} < \mu_{m,k} < \frac{\pi k}{1-a} + \frac{\pi(2m+d-2)}{4}.$$

3 A Uniqess Result

Let $\mathbb{Z}^*$ denote the set of all nonzero integers. We now give the following uniqueness result introduced by Zu, Li and Rao in [9], which will be useful for proving the sufficiency of Kalman's rank condition (1.7) on the annular domain $\Omega = \{x : a < |x| < 1\}$.

Lemma 3.1 *Assume that*

$$\cdots < \beta^{(1)}_{m,-1} < \cdots < \beta^{(s)}_{m,-1} < \beta^{(1)}_{m,1} < \cdots < \beta^{(s)}_{m,1} < \beta^{(1)}_{m,2} < \cdots \tag{3.1}$$

for any fixed $m \in \mathbb{N}$. Assume furthermore that for any given $m \in \mathbb{N}$, there exist positive constants γ_m, c_m and τ_m such that

$$\beta^{(l)}_{m,k+1} - \beta^{(l)}_{m,k} \geq s\gamma_m \tag{3.2}$$

and

$$\frac{c_m}{|k|^{\tau_m}} \leq \beta^{(l+1)}_{m,k} - \beta^{(l)}_{m,k} \leq \gamma_m \tag{3.3}$$

for all $1 \leq l \leq s$ and all $k \in \mathbb{Z}^*$ with $|k|$ large enough.

Assume finally that

$$\sum_{m\in\mathbb{N}} \sum_{k\in\mathbb{Z}^*} \sum_{l=1}^{s} \sum_{j=1}^{j_m} a^{(l)}_{m,k,j} \mathrm{e}^{\mathrm{i}\beta^{(l)}_{m,k} t} Y_{m,j}(\theta) \equiv 0 \quad \text{on } S^{d-1} \times [0,T] \tag{3.4}$$

with

$$\sum_{m\in\mathbb{N}}\sum_{k\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_m}|a_{m,k,j}^{(l)}|^2<+\infty \tag{3.5}$$

and $T>2\pi D^+$, *where*

$$D^+=\sup_{m\in\mathbb{N}} D_m^+<+\infty, \tag{3.6}$$

in which D_m^+ *is the upper density of the sequence* $\{\beta_{m,k}^{(l)}\}_{k\in\mathbb{Z}^*;1\le l\le s}$. *Then, we have*

$$a_{m,k,j}^{(l)}=0,\quad m\in\mathbb{N};\ k\in\mathbb{Z}^*;\ 1\le l\le s;\ 1\le j\le j_m. \tag{3.7}$$

When Ω is an annular domain, by Proposition 2.2(i), for $d>2$ and $d=2$, the uniform gap condition of sequence $\{\mu_{m,k}\}_{k\in\mathbb{N}^+}$ starts from $k=2$ for any given $m\in\mathbb{N}$ and $m\in\mathbb{N}^+$, respectively, which is different from the case that Ω is a spherical domain. Hence, in order to use Lemma 3.1, we should add condition (3.10) below and rearrange $2s$ elements of sequence $\{\beta_{m,k}^{(l)}\}_{k\in\mathbb{Z}^*;1\le l\le s}$ to guarantee condition (3.1).

Corollary 3.1 *Assume that*

$$\delta_1<\delta_2<\cdots<\delta_s. \tag{3.8}$$

For any fixed $m\in\mathbb{N}$, *we define*

$$\begin{cases}\beta_{m,k}^{(l)}=\sqrt{\mu_{m,k}^2+\epsilon\delta_l}, & l=1,2,\cdots,s,\ k\ge1,\\ \beta_{m,-k}^{(l)}=-\beta_{m,k}^{(s-l+1)}, & l=1,2,\cdots,s,\ k\ge1,\end{cases} \tag{3.9}$$

where $\mu_{m,k}$ *is the* k*-th positive root of* (2.12) *for any fixed* $m\in\mathbb{N}$. *Then, for* $\epsilon>0$ *small enough and*

$$\epsilon\notin\Big\{\frac{\mu_{m,2}^2-\mu_{m,1}^2}{\delta_l-\delta_{l'}}, m\in\mathbb{N}, 1\le l,l'\le s \text{ and } l\ne l'\Big\}, \tag{3.10}$$

the sequence $\{\beta_{m,k}^{(l)}\}_{k\in\mathbb{Z}^*;1\le l\le s}$ *satisfies* (3.1)–(3.3) *and* (3.6) *for any given* $m\in\mathbb{N}$.

Proof For any given $m\in\mathbb{N}$ and $k\in\mathbb{N}^+$, by the definition of $\mu_{m,k}$ and Proposition 2.2(ii), we have

$$\min\{\mu_{m,k}\mid m\in\mathbb{N},k\in\mathbb{N}^+\}=\mu_{0,1}. \tag{3.11}$$

For $d=2$, by Proposition 2.2(ii), we have

$$\mu_{0,k+1}-\mu_{0,k}\to\frac{\pi}{1-a}\quad\text{as } k\to+\infty.$$

Then there exists $\widetilde{\gamma}>0$, such that for $d=2$, we have

$$\mu_{0,k+1}-\mu_{0,k}\ge\widetilde{\gamma},\quad k\ge2.$$

Let

$$\gamma_0=\min\Big\{\frac{\pi}{2-a},\widetilde{\gamma}\Big\}.$$

For any fixed $d \geq 2$, by Proposition 2.2(i), we have

$$\mu_{m,k+1} - \mu_{m,k} \geq \gamma_0, \quad k \geq 2;\ m \in \mathbb{N}. \tag{3.12}$$

Let $0 < \epsilon < \frac{2\mu_{0,1}\gamma_0}{(\delta_s - \delta_1)}$. Similarly to the proof of that on a spherical domain, by (3.11)–(3.12), we have $\beta^{(s)}_{m,k} < \beta^{(1)}_{m,k+1}$ for $|k| \geq 2$. By (3.10), $\beta^{(l)}_{m,k}$ ($k = 1, 2; 1 \leq l \leq s$) are distinct for any fixed $m \in \mathbb{N}$, we can rearrange them in an increasing order. The rearranged sequence, still denoted by $\{\beta^{(l)}_{m,k}\}_{k\in\mathbb{Z}^*;1\leq l\leq s}$, satisfies (3.1).

On the other hand, for any fixed $m \in \mathbb{N}$, by Proposition 2.2(ii), a direct computation similar to that on a spherical domain gives that

$$\beta^{(l)}_{m,k+1} - \beta^{(l)}_{m,k} \to \frac{\pi}{1-a} \quad \text{as } k \to +\infty$$

and

$$(\beta^{(l+1)}_{m,k} - \beta^{(l)}_{m,k})k \to \frac{(\delta_{l+1} - \delta_l)(1-a)\epsilon}{2\pi} \quad \text{as } k \to +\infty.$$

Then it is easy to see that the sequence $\{\beta^{(l)}_{m,k}\}_{m\in\mathbb{N};k\in\mathbb{Z}^*;1\leq l\leq s}$ satisfies (3.2)–(3.3) with

$$\tau_m = 1; \quad c_m = \frac{(1-a)\epsilon}{4\pi} \min_{1\leq l\leq s-1}\{\delta_{l+1} - \delta_l\}; \quad \gamma_m = \frac{\pi}{2s(1-a)}.$$

Next, we will prove that the sequence $\{\beta^{(l)}_{m,k}\}_{m\in\mathbb{N};k\in\mathbb{Z}^*;1\leq l\leq s}$ satisfies (3.6) in a way similar to the proof of that on a spherical domain.

For any fixed $m \in \mathbb{N}$ and $k \in \mathbb{N}^+$, if $\beta^{(l)}_{m,k} < R < \beta^{(l+1)}_{m,k}$ for $1 \leq l \leq s-1$, then we have

$$\frac{N(\beta^{(l)}_{m,k})}{2\beta^{(l+1)}_{m,k}} < \frac{N(R)}{2R} < \frac{N(\beta^{(l)}_{m,k})}{2\beta^{(l)}_{m,k}}; \tag{3.13}$$

while, if $\beta^{(s)}_{m,k} < R < \beta^{(1)}_{m,k+1}$, then we have

$$\frac{N(\beta^{(s)}_{m,k})}{2\beta^{(1)}_{m,k+1}} < \frac{N(R)}{2R} < \frac{N(\beta^{(s)}_{m,k})}{2\beta^{(s)}_{m,k}}. \tag{3.14}$$

By Proposition 2.2(ii), for each $m \in \mathbb{N}$, we have

$$\lim_{k\to+\infty} \frac{N(\beta^{(l)}_{m,k})}{2\beta^{(l)}_{m,k}} = \frac{s(1-a)}{\pi}, \quad 1 \leq l \leq s, \tag{3.15}$$

$$\lim_{k\to+\infty} \frac{N(\beta^{(l)}_{m,k})}{2\beta^{(l+1)}_{m,k}} = \frac{s(1-a)}{\pi}, \quad 1 \leq l \leq s-1 \quad \text{and} \quad \lim_{k\to+\infty} \frac{N(\beta^{(s)}_{m,k})}{2\beta^{(1)}_{m,k+1}} = \frac{s(1-a)}{\pi}. \tag{3.16}$$

Thus, by (3.13)–(3.16), we get

$$D^+_m = \limsup_{R\to+\infty} \frac{N(R)}{2R} = \frac{s(1-a)}{\pi}.$$

Since

$$D^+ = \sup_{m\in\mathbb{N}} D^+_m = \frac{s(1-a)}{\pi},$$

the sequence $\{\beta^{(l)}_{m,k}\}_{k\in\mathbb{Z}^*;1\leq l\leq s}$ satisfies all the requirements of Lemma 3.1.

4 The Sufficiency of Kalman's Rank Condition on an Annular Domain

On the annular domain $\Omega = \{x : a < |x| < 1\}$, we consider the following system:

$$\begin{cases} U'' - \Delta U + \epsilon AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ U = DH & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \tag{4.1}$$

with the initial condition

$$t = 0 : \ U = \widehat{U}_0, \ U' = \widehat{U}_1 \quad \text{in } \Omega. \tag{4.2}$$

The adjoint system of (4.1) is given by

$$\begin{cases} \Phi'' - \Delta\Phi + \epsilon A^T \Phi = 0 & \text{in } (0, +\infty) \times \Omega, \\ \Phi = 0 & \text{on } (0, +\infty) \times \Gamma \end{cases} \tag{4.3}$$

with the initial data

$$t = 0 : \Phi = \Phi_0, \ \Phi' = \Phi_1 \quad \text{in } \Omega. \tag{4.4}$$

In this section, we will prove the sufficiency of Kalman's rank condition (1.7) for $T > 0$ large enough to the approximate boundary null controllability of system (4.1) on the annular domain Ω. By Theorem 1.1, it is sufficient to prove the sufficiency of Kalman's rank condition (1.7) to the D-observability of the corresponding adjoint system (4.3) on Ω. The following necessary and sufficient condition of Kalman's rank condition (1.7) given by Li and Rao in [5–6] is very useful to prove the sufficiency of Kalman's rank condition (1.7) in this case.

Lemma 4.1 *Assume that $k \geq 0$ is an integer, A is a matrix of order N and D is a full column-rank matrix of order $N \times M$ with $M \leq N$. Then Kalman's rank condition*

$$\operatorname{rank}(D, AD, \cdots, A^{N-1}D) = N - k \tag{4.5}$$

holds if and only if the largest dimension of invariant subspaces of A^T, contained in $\operatorname{Ker}(D^T)$, is equal to k.

Theorem 4.1 *Let*

$$\Omega = \{x : a < |x| < 1\}$$

with $0 < a < 1$ and let the sequence $\{\beta_{m,k}^{(l)}\}_{k \in \mathbb{Z}^*; 1 \leq l \leq s}$ be defined by (3.9). Assume that the coupling matrix A is diagonalizable with the real eigenvalues given by

$$\delta_1 < \delta_2 < \cdots < \delta_s. \tag{4.6}$$

Assume furthermore that $\epsilon > 0$ is small enough and (3.10) holds so that for each $m \in \mathbb{N}$, the sequence $\{\beta_{m,k}^{(l)}\}_{k \in \mathbb{Z}^*; 1 \leq l \leq s}$ satisfies (3.1)–(3.3) and (3.6).

Then Kalman's rank condition (1.7) is sufficient for the approximate boundary null controllability of system (4.1), provided that $T > 2s(1 - a)$.

Proof By Theorem 1.1, it is sufficient to prove the sufficiency of Kalman's rank condition (1.7) to the D-observability of the corresponding adjoint system (4.3). The proof is similar to that on a spherical domain in higher dimension case (see [9]). In this paper, we just give the essential differences.

Let $\omega^{(l,\mu)} = (\omega_1^{(l,\mu)}, \cdots, \omega_N^{(l,\mu)})^T$ be the eigenvectors of A^T, corresponding to the eigenvalue δ_l:

$$A^T \omega^{(l,\mu)} = \delta_l \omega^{(l,\mu)}, \quad 1 \le l \le s;\ 1 \le \mu \le \mu_l \tag{4.7}$$

with

$$\sum_{l=1}^{s} \mu_l = N \quad \text{and} \quad |\omega^{(l,\mu)}| = 1. \tag{4.8}$$

Furthermore, let

$$E_{m,k,j}^{(l,\mu)} = \begin{pmatrix} \frac{e_{m,k,j}\omega^{(l,\mu)}}{\mathrm{i}\beta_{m,k}^{(l)}} \\ e_{m,k,j}\omega^{(l,\mu)} \end{pmatrix}, \quad m \in \mathbb{N};\ k \in \mathbb{Z}^*;\ 1 \le l \le s;\ 1 \le \mu \le \mu_l;\ 1 \le j \le j_m,$$

in which we define $e_{m,-k,j} = e_{m,k,j}$ for all $m \in \mathbb{N}, k \in \mathbb{N}^+$ and $1 \le j \le j_m$. Then, by [9, Theorem 3], $\{E_{m,k,j}^{(l,\mu)}\}$ forms a Riesz basis of $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$.

Thus, for any given initial data $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (H_0^1(\Omega))^N \times (L^2(\Omega))^N$, there exists $\{a_{m,k,j}^{(l,\mu)}\}$ such that

$$\begin{pmatrix} \widehat{\Phi}_0 \\ \widehat{\Phi}_1 \end{pmatrix} = \sum_{m\in\mathbb{N}} \sum_{k\in\mathbb{Z}^*} \sum_{l=1}^{s} \sum_{\mu=1}^{\mu_l} \sum_{j=1}^{j_m} a_{m,k,j}^{(l,\mu)} E_{m,k,j}^{(l,\mu)}$$

with

$$\sum_{m\in\mathbb{N}} \sum_{k\in\mathbb{Z}^*} \sum_{l=1}^{s} \sum_{\mu=1}^{\mu_l} \sum_{j=1}^{j_m} |a_{m,k,j}^{(l,\mu)}|^2 < +\infty.$$

Then the corresponding solution to problem (4.3)–(4.4) is given by

$$\Phi = \sum_{m\in\mathbb{N}} \sum_{k\in\mathbb{Z}^*} \sum_{l=1}^{s} \sum_{\mu=1}^{\mu_l} \sum_{j=1}^{j_m} \frac{a_{m,k,j}^{(l,\mu)}}{\mathrm{i}\beta_{m,k}^{(l)}} \mathrm{e}^{\mathrm{i}\beta_{m,k}^{(l)} t} e_{m,k,j}(x) \omega^{(l,\mu)},$$

where $e_{m,k,j}(x)$ are given by (2.19).

Using Lemma 2.4(iii), the observation (1.6) becomes

$$0 \equiv \sum_{m\in\mathbb{N}} \sum_{k\in\mathbb{Z}^*} \sum_{l=1}^{s} D^T \sum_{\mu=1}^{\mu_l} \sum_{j=1}^{j_m} \frac{2c_{m,k} a_{m,k,j}^{(l,\mu)}}{\mathrm{i}\pi\beta_{m,k}^{(l)}} \omega^{(l,\mu)} Y_{m,j}(\theta) \mathrm{e}^{\mathrm{i}\beta_{m,k}^{(l)} t} \quad \text{on} \quad [0,T] \times \Gamma_1. \tag{4.9}$$

Noting $D = (d_{pq})$, we define

$$b_{(m,k,j,q)}^{l} = \sum_{p=1}^{N} \sum_{\mu=1}^{\mu_l} \frac{2d_{pq} c_{m,k} a_{m,k,j}^{(l,\mu)} \omega_p^{(l,\mu)}}{\mathrm{i}\pi\beta_{m,k}^{(l)}},$$

and $\mu_{m,-k}=\mu_{m,k}$ for all $m\in\mathbb{N}$ and $k\in\mathbb{N}^+$.

Then, for any fixed q with $1\le q\le M$, the observation (1.6) can be rewritten as

$$\sum_{m\in\mathbb{N}}\sum_{k\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_m} b^l_{(m,k,j,q)}Y_{m,j}(\theta)e^{i\beta^{(l)}_{m,k}t}\equiv 0 \quad \text{on} \quad [0,T]\times\Gamma_1.$$

The difference from the proof of that on a spherical domain is the verification of

$$\sum_{m\in\mathbb{N}}\sum_{k\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_m}|b^l_{(m,k,j,q)}|^2<+\infty$$

for any fixed q with $1\le q\le M$.

Using (2.18) and (4.8), we have

$$\begin{aligned}|b^l_{(m,k,j,q)}|^2 &= \Big|\sum_{p=1}^{N}\sum_{\mu=1}^{\mu_l}\frac{2c_{m,k}d_{pq}a^{(l,\mu)}_{m,k,j}\omega^{(l,\mu)}_p}{i\pi\beta^{(l)}_{m,k}}\Big|^2\\ &\le c_0^2\Big(N\max_{p,q}\{|d_{pq}|\}\max_{p,l,\mu}\{|\omega^{(l,\mu)}_p|\}\Big)^2\Big|\sum_{\mu=1}^{\mu_l}\frac{|a^{(l,\mu)}_{m,k,j}||\mu_{m,k}|}{|\beta^{(l)}_{m,k}|}\Big|^2\\ &\le c_0^2\Big(N\max_{p,q}\{|d_{pq}|\}\Big)^2\Big|\sum_{\mu=1}^{\mu_l}\frac{|a^{(l,\mu)}_{m,k,j}||\mu_{m,k}|}{|\beta^{(l)}_{m,k}|}\Big|^2. \end{aligned} \tag{4.10}$$

In the present situation, by Proposition 2.2(iv), for any fixed l with $1\le l\le s$, we have

$$\Big|\frac{\mu_{m,k}}{\beta^{(l)}_{m,k}}\Big|=\frac{\mu_{m,k}}{\sqrt{\mu^2_{m,k}+\epsilon\delta_l}}\to 1 \quad \text{as } m \text{ and } |k|\to+\infty. \tag{4.11}$$

Hence, there exists a positive constant c_1 such that

$$\Big|\frac{\mu_{m,k}}{\beta^{(l)}_{m,k}}\Big|\le c_1, \quad m\in\mathbb{N};\ k\in\mathbb{Z}^*. \tag{4.12}$$

By (4.12) and Cauchy-Schwartz inequality, we have

$$\Big|\sum_{\mu=1}^{\mu_l}\frac{|a^{(l,\mu)}_{m,k,j}||\mu_{m,k}|}{|\beta^{(l)}_{m,k}|}\Big|^2\le c_1^2\Big|\sum_{\mu=1}^{\mu_l}|a^{(l,\mu)}_{m,k,j}|\Big|^2\le c_1^2\mu_l\sum_{\mu=1}^{\mu_l}|a^{(l,\mu)}_{m,k,j}|^2. \tag{4.13}$$

Let

$$C=Nc_0^2\Big(Nc_1\max_{p,q}\{|d_{pq}|\}\Big)^2.$$

By (4.10) and (4.13), for $q=1,2,\cdots,M$, we have

$$\sum_{m\in\mathbb{N}}\sum_{k\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_m}|b^l_{(m,k,j,q)}|^2\le C\sum_{m\in\mathbb{N}}\sum_{k\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{\mu=1}^{\mu_l}\sum_{j=1}^{j_m}|a^{(l,\mu)}_{m,k,j}|^2<+\infty.$$

Applying Lemma 3.1 to each line of (4.9), we get

$$D^T\sum_{\mu=1}^{\mu_l}\frac{2c_{m,k}a^{(l,\mu)}_{m,k,j}}{i\pi\beta^{(l)}_{m,k}}\omega^{(l,\mu)}=0, \quad m\in\mathbb{N};\ k\in\mathbb{Z}^*;\ 1\le j\le j_m;\ 1\le l\le s.$$

By Lemma 4.1, it follows from Kalman's rank condition (1.7) that $\mathrm{Ker}(D^T)$ does not contain any non-trivial invariant subspace of A^T, then we have

$$\sum_{\mu=1}^{\mu_l} \frac{2c_{m,k}a_{m,k,j}^{(l,\mu)}}{\mathrm{i}\pi\beta_{m,k}^{(l)}}\omega^{(l,\mu)} = 0, \quad m \in \mathbb{N};\ k \in \mathbb{Z}^*;\ 1 \le j \le j_m;\ 1 \le l \le s.$$

Since $\{\omega^{(l,\mu)}\}_{1\le\mu\le\mu_l}$ are linearly independent, noting that by (2.16), $c_{m,k} \neq 0$ for any given $m \in \mathbb{N}$ and any given $k \in \mathbb{N}^+$, we have

$$a_{m,k,j}^{(l,\mu)} = 0, \quad m \in \mathbb{N};\ k \in \mathbb{Z}^*;\ 1 \le l \le s;\ 1 \le \mu \le \mu_l;\ 1 \le j \le j_m, \tag{4.14}$$

namely, $\Phi \equiv 0$. The proof is complete.

Example 4.1 Let δ_1, δ_2 be positive constants with $\delta_1 < \delta_2$, and k_1, k_2 be positive integers. Consider the adjoint system (4.3) with $A = \mathrm{diag}(\delta_1, \delta_2)$ and $D = (1, -1)^T$. We will show that Kalman's rank condition is not sufficient for the D-observability of adjoint system (4.3) at the infinite horizon for $\epsilon \in \mathcal{N}$, where

$$\mathcal{N} = \Big\{\frac{\mu_{m,k_1}^2 - \mu_{m,k_2}^2}{\delta_1 - \delta_2}, m \in \mathbb{N}, 1 \le k_1 < k_2\Big\}.$$

For $\epsilon > 0$ with $\epsilon \in \mathcal{N}$, there exist m and k_1, k_2 with $1 \le k_1 < k_2$, such that

$$\mu_{m,k_1}^2 + \epsilon\delta_2 = \mu_{m,k_2}^2 + \epsilon\delta_1 = \alpha^2,$$

where $\alpha > 0$. Let

$$\Phi = \mathrm{e}^{\mathrm{i}\alpha t}\Big(\frac{R_{m,k_2}(r)}{R'_{m,k_2}(1)}Y_{m,1}(\theta), \frac{R_{m,k_1}(r)}{R'_{m,k_1}(1)}Y_{m,1}(\theta)\Big)^T,$$

where $Y_{m,1}$ is given in Lemma 2.1 with $j = 1$; $R_{m,k_1}(r)$ and $R_{m,k_2}(r)$ are given by (2.13) with $k = k_1$ and $k = k_2$, respectively. Then by Lemma 2.4, Φ is a non-trivial solution of system (4.3) and satisfies the observation.

Example 4.1 shows that Kalman's rank condition (1.7) is not sufficient in general for the approximate boundary null controllability of system (1.2) even at the infinite horizon. Hence, it is essential to add condition (3.10) to guarantee the sufficiency of Kalman's rank condition (1.7).

We now indicate the relationship between the controllability time T and the rank of D.

Theorem 4.2 *Let* $\Omega = \{x : a < |x| < 1\}$ *with* $0 < a < 1$. *Assume that* $\mathrm{rank}(D) = N - k$ *with* $0 \le k \le N - 1$ *and the coupling matrix A is diagonalizable with the real eigenvalues given by* (4.6). *Then, Kalman's rank condition* (1.7) *is sufficient for the approximate boundary null controllability of system* (4.1) *on the interval* $[0, T]$, *provided that* $T > 2(k+1)(1-a)$ *and* $\epsilon > 0$ *is small enough.*

Proof The proof is same as that of Theorem 4 given by Zu, Li and Rao in [9].

Remark 4.1 Let $\Omega = \{x : r_1 < |x| < r_2\}$ with $0 < r_1 < r_2$. Assume that the coupling matrix A is diagonalizable with the real eigenvalues given by (4.6). Assume furthermore that

$\epsilon > 0$ is so small that for each $m \in \mathbb{N}$, the sequence $\{\beta_{m,k}^{(l)}\}_{k\in\mathbb{Z}^*;1\le l\le s}$ defined by (3.9) satisfies (3.1)–(3.3) and (3.6).

Then, Kalman's rank condition (1.7) is sufficient for the approximate boundary null controllability of system (4.1) on the interval $[0,T]$, provided that $T > 2s(r_2 - r_1)$.

Proof When $\Omega = \{x : a < |x| < 1\}$ is changed to $\{x : r_1 < |x| < r_2\}$, a, $\mu_{m,k}$ and D^+ are replaced by $\frac{r_1}{r_2}$, $\frac{\mu_{m,k}}{r_2}$ and $r_2 D^+$, respectively. Thus, by Lemma 3.1, the controllability time T is replaced by $r_2 T$, i.e., $2s(r_2 - r_1)$.

Similarly, we have the following remark.

Remark 4.2 Let $\Omega = \{x : r_1 < |x| < r_2\}$ with $0 < r_1 < r_2$. Assume that $\mathrm{rank}(D) = N - k$ with $0 \le k \le N-1$ and the coupling matrix A is diagonalizable with the real eigenvalues given by (4.6). Assume furthermore that $\epsilon > 0$ is small enough and condition (3.10) holds. Then, Kalman's rank condition (1.7) is sufficient for the approximate boundary null controllability of system (4.1) on the interval $[0,T]$, provided that $T > 2(k+1)(r_2 - r_1)$.

Remark 4.3 The controllability time T given by Theorems 4.1 or 4.2 is not optimal.

Acknowledgement The author would like to thank Professors Tatsien Li and Bopeng Rao for their valuable advices.

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