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Dual-holomorphic Functions and Problems of Lifts

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Abstract The main purpose of this paper is to study the differential geometrical objects on tangent bundle corresponding to dual-holomorphic objects of dual-holomorphic manifold. As a result of this approach, the authors find a new class of lifts (deformed complete lifts) in the tangent bundle.

Keywords Dual numbers, Tangent bundle, Complete lift, Dual-holomorphic functions, Anti-Kähler manifold
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1 Introduction

We can define the following classical numbers of order two: Dual numbers (or parabolic numbers): i.e., $a+\varepsilon b,\ a,b\in \mathrm{R},\ \varepsilon^2=0$; where R is the field of real numbers. Let M_n be a differentiable manifold and $T(M_n)$ be its tangent bundle. Two types of lift (extension) problems have been studied in the previous works (see for example [1–2, 8]): a) The lift of various objects (functions, vector fields, forms, tensor fields, linear connections, etc.) from the base manifold to the tangent bundle. b) The lift on the total manifold $T(M_n)$ by means of a specific geometric structure on $T(M_n)$. In the present paper we continue such a study by considering the structure given by the dual numbers on the tangent bundle and defining new lifts of functions, vector fields, forms, tensor fields and linear connections.

1.1 We consider a two-dimensional dual algebra $\mathrm{R}(\varepsilon)$, $\varepsilon^2=0$ (ε is nilpotent) with a standard basis $\{e_1,e_2\}=\{1,\varepsilon\}$ and structural constants $C_{\alpha\beta}^{\gamma}$: $e_\alpha e_\beta=C_{\alpha\beta}^{\gamma}e_\gamma,\ \alpha,\beta,\gamma=1,2$, where $C_{11}^1=C_{12}^2=C_{21}^2=1,\ C_{12}^1=C_{21}^1=C_{22}^1=C_{11}^2=C_{22}^2=0$ are components of the (1,2)-tensor $C:\mathrm{R}(\varepsilon)\times\mathrm{R}(\varepsilon)\to\mathrm{R}(\varepsilon)$.

Let $Z=x^\alpha e_\alpha$ be a variable in $\mathrm{R}(\varepsilon)$, where $x^\alpha\ (\alpha=1,2)$ are real variables. Using real-valued C^∞-functions $f^\beta(x)=f^\beta(x^1,x^2),\ \beta=1,2$, we introduce a dual function $F=f^\beta(x)e_\beta$ of variable $Z\in\mathrm{R}(\varepsilon)$. Let $\mathrm{d}Z=\mathrm{d}x^\alpha e_\alpha$ and $\mathrm{d}F=\mathrm{d}f^\alpha e_\alpha$ be respectively the differentials of Z and $F(Z)$. We shall say that the function $F=F(Z)$ is a dual-holomorphic function if there exists a new dual function $F'(Z)$ such that $\mathrm{d}F=F'(Z)\mathrm{d}Z$. The function $F'(Z)$ is called the derivative of $F(Z)$. It is well known that the dual function $F=F(Z)$ is holomorphic if and only if the following Scheffers condition holds (see [3]):

$$C_2D=DC_2, \tag{1.1}$$

where $D=\left(\frac{\partial f^\alpha}{\partial x^\beta}\right)$ is the Jacobian matrix of $f^\alpha(x)$, $C_2=(C_{2\beta}^{\gamma})=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, γ and β denote the row and column numbers of matrix C_2, respectively. The condition (1.1) reduces to the following

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equations:

$$\frac{\partial f^1}{\partial x^2}=0, \quad \frac{\partial f^2}{\partial x^2}=\frac{\partial f^1}{\partial x^1}.$$

From here follows that the dual-holomorphic function $F=F(Z)$ has the following explicit form:

$$F(Z)=f(x^1)+\varepsilon(x^2 f'(x^1)+g(x^1)),$$

where $f(x^1)=f^1(x^1)$, $f'(x^1)=\frac{\mathrm{d}f}{\mathrm{d}x^1}$ and $g=g(x^1)$ is any real C^∞-function.

By similar devices, we see that the dual-holomorphic multi-variable function $F=F(Z^1,\cdots,Z^n)$, $Z^i=x^i+\varepsilon x^{n+i}$, $i=1,\cdots,n$, has the form:

$$F(Z^1,\cdots,Z^n)=f(x^1,\cdots,x^n)+\varepsilon(x^{n+s}\partial_s f+g(x^1,\cdots,x^n)), \tag{1.2}$$

where $g=g(x^1,\cdots,x^n)$ is any real multi-variable C^∞-function, $\partial_s f=\frac{\partial f}{\partial x^s}$.

A dual-holomorphic manifold (see [6]) $X_n(\mathrm{R}(\varepsilon))$ of dimension n is a Hausdorff space with a fixed atlas compatible with a group of $\mathrm{R}(\varepsilon)$-holomorphic transformations of space $\mathrm{R}^n(\varepsilon)$, where $\mathrm{R}^n(\varepsilon)=\mathrm{R}(\varepsilon)\times\cdots\times\mathrm{R}(\varepsilon)$ is the space of n-tupes of dual numbers $(z^1,z^2,\cdots,z^n)$ with $z^i=x^i+\varepsilon y^i\in\mathrm{R}(\varepsilon)$, $x^i,y^i\in\mathrm{R}$, $i=1,\cdots,n$. We shall identify $\mathrm{R}^n(\varepsilon)$ with R^{2n}, when necessary, by mapping $(z^1,z^2,\cdots,z^n)\in\mathrm{R}^n(\varepsilon)$ into $(x^1,\cdots,x^n,y^1,\cdots,y^n)\in\mathrm{R}^{2n}$ and therefore the $\mathrm{R}(\varepsilon)$-holomorphic manifold $X_n(\mathrm{R}(\varepsilon))$ is a real manifold M_{2n} of dimension $2n$.

1.2 Let now M_n be a differentiable manifold and $T(M_n)$ be its tangent bundle, and π be the projection $T(M_n)\to M_n$. The tangent bundle $T(M_n)$ consists of pair (x,v), where $x\in M_n$ and $v\in T_x(M_n)$ ($T_x(M_n)$ is a tangent vector space at $x\in M_n$). Let $(U,\ x=(x^1,\cdots,x^n))$ be a coordinate chart in M_n. Then it induces local coordinates $(x^1,\cdots,x^n,x^{n+1},\cdots,x^{2n})$ in $\pi^{-1}(U)$, where $x^{n+1},\cdots,x^{2n}$ represent the components of $v\in T_x(M_n)$ with respect to local frame $\{\partial_i\}$. In the following we use the notation $\overline{i}=i+n$ for all $i=1,\cdots,n$.

If $(U',\ x'=(x^{1'},\cdots,x^{n'}))$ is another coordinate chart in M_n, then the induced coordinates $(x^{1'},\cdots,x^{n'},x^{\overline{1}'},\cdots,x^{\overline{n}'})$ in $\pi^{-1}(U')$, will be given by

$$\begin{cases} x^{i'}=x^{i'}(x^i), & i=1,\cdots,n,\\ x^{\overline{i}'}=\dfrac{\partial x^{i'}}{\partial x^i}x^{\overline{i}}, & \overline{i}=n+1,\cdots,2n. \end{cases} \tag{1.3}$$

The Jacobian of (1.3) is given by matrix

$$S=\Big(\frac{\partial x^{\alpha'}}{\partial x^\alpha}\Big)=\begin{pmatrix} \frac{\partial x^{i'}}{\partial x^i} & 0\\ x^{\overline{s}}\frac{\partial^2 x^{i'}}{\partial x^i\partial x^s} & \frac{\partial x^{i'}}{\partial x^i}\end{pmatrix}, \quad \alpha=1,\cdots,2n.$$

From here follows that there exists a tensor field of type (1,1),

$$\varphi=(\varphi^\alpha_\beta)=\begin{pmatrix}\varphi^i_j & \varphi^i_{\overline{j}}\\ \varphi^{\overline{i}}_j & \varphi^{\overline{i}}_{\overline{j}}\end{pmatrix}=\begin{pmatrix}0 & 0\\ I & 0\end{pmatrix}\ (I=(\delta^i_j)-\text{identity matrix of degree } n) \tag{1.4}$$

with properties $\varphi^2=0$ and $S\varphi=\varphi S$, i.e., the transformation $S:\{\partial_\alpha\}\to\{\partial_{\alpha'}\}$ preserving φ is an admissible dual transformation. Thus, the tangent bundle $T(M_n)$ of a manifold M_n carries a natural dual structure φ, which is integrable ($\partial_k\varphi^i_j=0$). Therefore, with each induced coordinates $(x^i,x^{\overline{i}})$ in $\pi^{-1}(U)\subset T(M_n)$, we associate the local dual coordinates $X^i=x^i+\varepsilon x^{\overline{i}}$, $\varepsilon^2=0$. Using (1.3) we see that the local dual coordinates $X^i=x^i+\varepsilon x^{\overline{i}}$ is transformed by

$$X^{i'}=x^{i'}(x^i)+\varepsilon x^{\overline{s}}\partial_s(x^{i'}(x^i)). \tag{1.5}$$

The equation (1.5) shows that the quantities $X^{i'}$ are dual-holomorphic functions of $X^i = x^i + \varepsilon x^{\overline{i}}$ (see (1.2) with $g(x^1, \cdots, x^n) = 0$). Thus the tangent bundle $T(M_n)$ with a natural integrable φ-structure is a real image of dual-holomorphic manifold $X_n(\mathrm{R}(\varepsilon))$ ($\dim X_n(\mathrm{R}(\varepsilon)) = n$) (see [5]). In such interpretation there exists a one-to-one correspondence between dual tensor fields on $X_n(\mathrm{R}(\varepsilon))$ and pure tensor fields with respect to φ-structure on $T(M_n)$ (see [6]). A real C^∞-tensor field t of type (1,q) or ω of type (0,q) on $T(M_n)$ is called pure with respect to φ-structure if

$$\begin{aligned}\varphi t(X_1, X_2, \cdots, X_q) &= t(\varphi X_1, X_2, \cdots, X_q)\\ &= t(X_1, \varphi X_2, \cdots, X_q) = \cdots = t(X_1, X_2, \cdots, \varphi X_q)\end{aligned}$$

or

$$\omega(\varphi X_1, X_2, \cdots, X_q) = \omega(X_1, \varphi X_2, \cdots, X_q) = \cdots = \omega(X_1, \cdots, \varphi X_q).$$

In particular, vector and covector fields will be considered to be pure for convenience sake.

It is important that the dual tensor field on $X_n(\mathrm{R}(\varepsilon))$ corresponding to a pure C^∞-tensor field is not necessarily dual-holomorphic. This tensor field is dual-holomorphic on $X_n(\mathrm{R}(\varepsilon))$ if and only if Φ-operator associated with φ and applied to a pure tensor field t of type (1,q) or ω of type (0,q) satisfies the following conditions (see [4, 7]),

$$(\Phi_\varphi t)(Y, X_1, \cdots, X_q) = -(L_{t(X_1, X_2, \cdots, X_q)}\varphi)Y + \sum_{\lambda=1}^{q} t(X_1, X_2, \cdots, (L_{X_\lambda}\varphi)Y, \cdots, X_q) = 0$$

or

$$\begin{aligned}(\Phi_\varphi \omega)(Y, X_1, \cdots, X_q) &= (\varphi Y)(\omega(X_1, X_2, \cdots, X_q)) - Y(\omega(\varphi X_1, X_2, \cdots, X_q))\\ &\quad + \sum_{\lambda=1}^{q} \omega(X_1, X_2, \cdots, \varphi(L_Y X_\lambda), \cdots, X_q) = 0,\end{aligned}$$

where L_Y is the Lie derivation with respect to Y.

2 Deformed Complete Lifts of Functions

From (1.2) we immediately have

$$F = {}^V f + \varepsilon({}^C f + {}^V g),$$

where g is any function on M_n, ${}^V f = f \circ \pi$, ${}^V g = g \circ \pi$ are vertical lifts of f, g, respectively, and ${}^C f = x^{n+s}\partial_s f$ is complete lift of f from M_n to its tangent bundle $T(M_n)$ (see [8]). We call ${}^D f = {}^C f + {}^V g$ the deformed complete lift of function f to tangent bundle $T(M_n)$.

Thus we have the following theorem.

Theorem 2.1 *Let $T(M_n)$ be a tangent bundle of M_n, which is a real image of dual-holomorphic manifold $X_n(\mathrm{R}(\varepsilon))$. Then the vertical and the deformed complete lifts to $T(M_n)$ of any function on M_n are a real and dual part of corresponding dual-holomorphic function on $X_n(\mathrm{R}(\varepsilon))$, respectively.*

3 Deformed Complete Lifts of Vector Fields

In a tangent bundle $T(M_n)$ with dual structure φ, a vector field $\widetilde{V} = (\widetilde{v}^\alpha) = (\widetilde{v}^i, \widetilde{v}^{n+i}) = (\widetilde{v}^i, \widetilde{v}^{\overline{i}})$ is called a dual-holomorphic vector field if $L_{\widetilde{V}}\varphi = 0$ (see [3]). Such vector field is a real image of corresponding dual-holomorphic vector field $V = (V^i)$ on $X_n(\mathrm{R}(\varepsilon))$, where $V^i = \widetilde{v}^i + \widetilde{v}^{\bar{i}}\varepsilon$. The condition of dual-holomorphy of a vector field $\widetilde{V}$ on $T(M_n)$ may be now locally written as follows:

$$L_{\widetilde{V}}\varphi_\beta^\alpha = \widetilde{v}^\sigma\partial_\sigma\varphi_\beta^\alpha - (\partial_\sigma\widetilde{v}^\alpha)\varphi_\beta^\sigma + (\partial_\beta\widetilde{v}^\sigma)\varphi_\sigma^\alpha = 0. \tag{3.1}$$

By (1.4), we have:

a) The case where $\alpha = i,\ \beta = j$, (3.1) reduces to

$$\begin{aligned} L_{\widetilde{V}}\varphi_j^i &= \widetilde{v}^\sigma\partial_\sigma\varphi_j^i - (\partial_\sigma\widetilde{v}^i)\varphi_j^\sigma + (\partial_j\widetilde{v}^\sigma)\varphi_\sigma^i = \widetilde{v}^m\partial_m\varphi_j^i + \widetilde{v}^{\overline{m}}\partial_{\overline{m}}\varphi_j^i - (\partial_m\widetilde{v}^i)\varphi_j^m \\ &- (\partial_{\overline{m}}\widetilde{v}^i)\varphi_j^{\overline{m}} + (\partial_j\widetilde{v}^m)\varphi_m^i + (\partial_j\widetilde{v}^{\overline{m}})\varphi_{\overline{m}}^i = -(\partial_{\overline{m}}\widetilde{v}^i)\delta_j^m = -(\partial_{\bar{j}}\widetilde{v}^i) = 0, \end{aligned}$$

from which follows

$$\widetilde{v}^i = v^i(x^1, \cdots, x^n). \tag{3.2}$$

b) The cases where $\alpha = i,\ \beta = \bar{j}$ and $\alpha = \bar{i},\ \beta = \bar{j}$, (3.1) reduces to $0 = 0$.

c) The case where $\alpha = \bar{i},\ \beta = j$, (3.1) reduces to

$$\begin{aligned} L_{\widetilde{V}}\varphi_j^{\bar{i}} &= \widetilde{v}^\sigma\partial_\sigma\varphi_j^{\bar{i}} - (\partial_\sigma\widetilde{v}^{\bar{i}})\varphi_j^\sigma + (\partial_j\widetilde{v}^\sigma)\varphi_\sigma^{\bar{i}} = \widetilde{v}^m\partial_m\varphi_j^{\bar{i}} + \widetilde{v}^{\overline{m}}\partial_{\overline{m}}\varphi_j^{\bar{i}} - (\partial_m\widetilde{v}^{\bar{i}})\varphi_j^m \\ &-(\partial_{\overline{m}}\widetilde{v}^{\bar{i}})\varphi_j^{\overline{m}} + (\partial_j\widetilde{v}^m)\varphi_m^{\bar{i}} + (\partial_j\widetilde{v}^{\overline{m}})\varphi_{\overline{m}}^{\bar{i}} = -(\partial_{\overline{m}}\widetilde{v}^{\bar{i}})\varphi_j^{\overline{m}} + (\partial_j\widetilde{v}^m)\varphi_m^{\bar{i}} \\ &= -(\partial_{\overline{m}}\widetilde{v}^{\bar{i}})\delta_j^m + (\partial_j\widetilde{v}^m)\delta_m^i = 0, \end{aligned}$$

from which follows

$$\partial_{\bar{j}}\widetilde{v}^{\bar{i}} = \partial_j v^i,$$

and after integrating, we find

$$\widetilde{v}^{\bar{i}} = x^{\bar{j}}\partial_j v^i + w^i(x^1, \cdots, x^n), \tag{3.3}$$

where $w^i = w^i(x^1, \cdots, x^n)$ are any real multi-variable C^∞-functions.

Remark 3.1 Using (1.3), (3.2)–(3.3) and $\widetilde{v}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\widetilde{v}^\alpha$, we easily see that $v = (v^i(x^1, \cdots, x^n))$ and $w = (w^i(x^1, \cdots, x^n))$ are vector fields on M_n.

Thus a real dual-holomorphic vector field $\widetilde{V}$ on tangent bundle can be written in the form

$$\begin{aligned} \widetilde{V} = (\widetilde{v}^\alpha) &= \begin{pmatrix} \widetilde{v}^i \\ \widetilde{v}^{\bar{i}} \end{pmatrix} = \begin{pmatrix} v^i(x^1, \cdots, x^n) \\ x^{\bar{j}}\partial_j v^i + w^i(x^1, \cdots, x^n) \end{pmatrix} = \begin{pmatrix} v^i \\ x^{\bar{j}}\partial_j v^i \end{pmatrix} + \begin{pmatrix} 0 \\ w^i \end{pmatrix} \\ &= {}^C v + {}^V w, \end{aligned}$$

where ${}^C v$ and ${}^V w$ are the complete and vertical lifts of vector fields $v = (v^i)$ and $w = (w^i)$ from M_n to tangent bundle $T(M_n)$, respectively (see [8]). Thus we have the following theorem.

Theorem 3.1 *Let $T(M_n)$ be a tangent bundle of M_n, which is a real image of dual-holomorphic manifold $X_n(\mathrm{R}(\varepsilon))$. Then a real image of corresponding dual-holomorphic vector field $V = (V^i) = (\widetilde{v}^i + \widetilde{v}^{\bar{i}}\varepsilon)$ is a deformed complete lift in the form ${}^D V = {}^C v + {}^V w$, where ${}^C v$ and ${}^V w$ are the complete and vertical lifts of vector fields $v = (v^i)$ and $w = (w^i)$ from M_n to $T(M_n)$, respectively.*

4 Deformed Complete Lifts of Tensor Fields of Type (1,1)

A tensor field $\widetilde{t}$ of type (1,1) on tangent bundle $T(M_n)$ is called pure tensor field with respect to the dual structure φ if

$$\widetilde{t}(\varphi X) = \varphi(\widetilde{t}X)$$

for any vector fields X on $T(M_n)$. From here we see that, the condition of pure tensor fields may be expressed in terms of the local induced coordinates as follows:

$$\widetilde{t}_\sigma^\beta \varphi_\alpha^\sigma = \widetilde{t}_\alpha^\sigma \varphi_\sigma^\beta.$$

Using (1.4), from the last conditon we have

$$\widetilde{t} = (\widetilde{t}_\beta^\alpha) = \begin{pmatrix} \widetilde{t}_j^i & 0 \\ \widetilde{t}_j^{\overline{i}} & \widetilde{t}_j^i \end{pmatrix}. \tag{4.1}$$

A pure tensor field $\widetilde{t}$ is called a dual-holomorphic tensor field if $\Phi_\varphi \widetilde{t} = 0$, where Φ_φ is the Tachibana operator defined by [4, 7],

$$(\Phi_\varphi \widetilde{t})(X, Y) = [\varphi X, \widetilde{t}Y] - \varphi[X, \widetilde{t}Y] - \widetilde{t}[\varphi X, Y] + \varphi\widetilde{t}[X, Y].$$

We note that, such tensor field is a real image of corresponding dual-holomorphic tensor field from $X_n(\mathrm{R}(\varepsilon))$ (see [3]). Sometimes the tensor $\Phi_\varphi \widetilde{t}$ of type (1,2) is called the Nijenhuis-Shirokov tensor field. It is clear that, if $\varphi = \widetilde{t}$, then $\Phi_\varphi \widetilde{t}$ is the Nijenhuis tensor N_φ, i.e., $\Phi_\varphi \varphi = N_\varphi$.

The condition of dual-holomorphy of a pure tensor field $\widetilde{t}$ on $T(M_n)$ may be now locally written as follows:

$$(\Phi_\varphi \widetilde{t})_{\gamma\beta}^\alpha = \varphi_\gamma^\sigma \partial_\sigma \widetilde{t}_\beta^\alpha - \varphi_\sigma^\alpha \partial_\gamma \widetilde{t}_\beta^\sigma - \widetilde{t}_\beta^\sigma \partial_\sigma \varphi_\gamma^\alpha + \widetilde{t}_\sigma^\alpha \partial_\beta \varphi_\gamma^\sigma = 0. \tag{4.2}$$

By virtue of (1.4) and (4.1), (4.2) after some calculations reduces to

$$\partial_{\overline{k}} \widetilde{t}_j^i = 0, \quad \partial_{\overline{k}} \widetilde{t}_j^{\overline{i}} - \partial_k \widetilde{t}_j^i = 0.$$

From here follows

$$\widetilde{t}_j^i = t_j^i(x^1, \cdots, x^n), \quad \widetilde{t}_j^{\overline{i}} = x^{\overline{k}} \partial_k t_j^i + g_j^i, \tag{4.3}$$

where $g_j^i = g_j^i(x^1, \cdots, x^n)$.

Remark 4.1 Using (1.3), (4.3) and $t_{\beta'}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^{\beta'}} t_\beta^\alpha$, we easily see that $t_j^i(x^1, \cdots, x^n)$ and $g_j^i(x^1, \cdots, x^n)$ are components of any tensor fields t and g of type (1,1) on M_n.

Thus a dual-holomorphic tensor field $\widetilde{t}$ on tangent bundle can be written in the form

$$\widetilde{t} = (\widetilde{t}_\beta^\alpha) = \begin{pmatrix} t_j^i & 0 \\ x^{\overline{k}} \partial_k t_j^i + g_j^i & t_j^i \end{pmatrix} = \begin{pmatrix} t_j^i & 0 \\ x^{\overline{k}} \partial_k t_j^i & t_j^i \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ g_j^i & 0 \end{pmatrix} = {}^C t + {}^V g\,,$$

where ${}^C t$ and ${}^V g$ are the complete and vertical lifts of (1,1)-tensor fields t and g from M_n to tangent bundle $T(M_n)$, respectively (see [8]). Thus we have the following theorem.

Theorem 4.1 *Let $T(M_n)$ be a tangent bundle of M_n, which is a real image of dual-holomorphic manifold $X_n(\mathrm{R}(\varepsilon))$. Then a real image of corresponding dual-holomorphic tensor field T of type (1,1) from $X_n(\mathrm{R}(\varepsilon))$ is a deformed complete lift in the form ${}^D t = {}^C t + {}^V g$, where ${}^C t$ and ${}^V g$ are the complete and vertical lifts of (1,1)-tensor fields t and g from M_n to $T(M_n)$, respectively.*

Let (M_{4n}, F, G, H) be an almost quaternion manifold, i.e.,

$$F^2 = -I, \quad G^2 = -I, \quad H^2 = -I,$$
$$F = GH = -HG, \quad G = HF = -FH, \quad H = FG = -GF.$$

Then for three fields F, G and H of type (1,1), we now consider the following deformed complete lifts:

$${}^DF = {}^CF + {}^VG, \quad {}^DG = {}^CG + {}^VH, \quad {}^DH = {}^CH + {}^VF.$$

From here, we find

$$\begin{aligned}({}^DF)^2 &= \begin{pmatrix} F^i_m & 0 \\ x^{\overline{s}}\partial_s F^i_m + G^i_m & F^i_m \end{pmatrix}\begin{pmatrix} F^m_j & 0 \\ x^{\overline{s}}\partial_s F^m_j + G^m_j & F^m_j \end{pmatrix} \\ &= \begin{pmatrix} F^i_m F^m_j & 0 \\ x^{\overline{s}}(\partial_s F^i_m)F^m_j + G^i_m F^m_j + F^i_m x^{\overline{s}}\partial_s F^m_j + F^i_m G^m_j & F^i_m F^m_j \end{pmatrix} \\ &= \begin{pmatrix} F^2 & 0 \\ x^{\overline{s}}\partial_s F^2 + GF + FG & F^2 \end{pmatrix} = \begin{pmatrix} -I_{M_n} & 0 \\ 0 & -I_{M_n} \end{pmatrix} = -I_{T(M_n)}.\end{aligned}$$

Similarly

$$({}^DG)^2 = -I_{T(M_n)}, \quad ({}^DH)^2 = -I_{T(M_n)}.$$

Thus we have the following theorem.

Theorem 4.2 *Let (M_{4n}, F, G, H) be an almost quaternion manifold. Then the deformed complete lifts of each structure F, G and H are almost complex structures on the tangent bundle.*

5 Deformed Complete Lifts of 1-Forms

An 1-form $\widetilde{\omega}$ on the tangent bundle $T(M_n)$ is called a dual-holomorphic 1-form, if $\Phi_\varphi\widetilde{\omega} = 0$, where Φ_φ is the Tachibana operator defined by [4, 7],

$$(\Phi_\varphi\widetilde{\omega})(X, Y) = (\varphi X)(\widetilde{\omega}(Y)) - X(\widetilde{\omega}(\varphi Y)) + \widetilde{\omega}((L_Y\varphi)X).$$

Such 1-form is a real image of corresponding dual-holomorphic 1-form from $X_n(\mathrm{R}(\varepsilon))$ (see [3]). The tensor field $\Phi_\varphi\widetilde{\omega}$ of type $(0, 2)$ has components

$$(\Phi_\varphi\widetilde{\omega})_{\alpha\beta} = \varphi^\sigma_\alpha\partial_\sigma\widetilde{\omega}_\beta - \varphi^\sigma_\beta\partial_\alpha\widetilde{\omega}_\sigma - \widetilde{\omega}_\sigma(\partial_\alpha\varphi^\sigma_\beta - \partial_\beta\varphi^\sigma_\alpha)$$

with respect to the natural frame $\{\partial_\alpha\} = \{\partial_i, \partial_{\overline{i}}\}$.

By virtue of (1.4), $(\Phi_\varphi\widetilde{\omega})_{\alpha\beta} = 0$ reduces to

$$\partial_{\overline{i}}\widetilde{\omega}_j - \partial_i\widetilde{\omega}_{\overline{j}} = 0, \quad \partial_{\overline{i}}\widetilde{\omega}_{\overline{j}} = 0.$$

From here we have

$$\widetilde{\omega}_{\overline{j}} = \omega_j(x^1, \cdots, x^n), \quad \widetilde{\omega}_j = x^{\overline{i}}\partial_i\omega_j + \theta_j\,(x^1, \cdots, x^n). \tag{5.1}$$

Remark 5.1 Using (1.3), (5.1) and $\widetilde{\omega}_{\beta'} = \frac{\partial x^\beta}{\partial x^{\beta'}}\widetilde{\omega}_\beta$, we easily see that $\omega_j(x^1, \cdots, x^n)$ and $\theta_j\,(x^1, \cdots, x^n)$ are components of any 1-forms ω and θ on M_n, respectively.

Thus a real dual-holomorphic 1-form $\widetilde{\omega}$ on tangent bundle can be rewritten in the form

$$\widetilde{\omega}=(\widetilde{\omega}_j,\,\widetilde{\omega}_{\overline{j}})=(x^{\overline{i}}\partial_i\omega_j+\theta_j,\ \omega_j\,)=(x^{\overline{i}}\partial_i\omega_j,\ \omega_j\,)+(\,\theta_j,\,0\,)={}^C\omega+{}^V\theta,$$

where ${}^C\omega$ and ${}^V\theta$ are the complete and vertical lifts of 1-forms $\omega=(\omega_j)$ and $\theta=(\theta_j)$ from M_n to tangent bundle $T(M_n)$, respectively (see [8]). Thus we have the following theorem.

Theorem 5.1 *Let $T(M_n)$ be a tangent bundle of M_n, which is a real image of dual-holomorphic manifold $X_n(\mathrm{R}(\varepsilon))$. Then a real image of corresponding dual-holomorphic 1-form from $X_n(\mathrm{R}(\varepsilon))$ is a deformed complete lift in the form ${}^D\omega={}^C\omega+{}^V\theta$, where ${}^C\omega$ and ${}^V\theta$ are the complete and vertical lifts of 1-forms $\omega=(\omega_j)$ and $\theta=(\theta_j)$ from M_n to $T(M_n)$, respectively.*

6 Deformed Complete Lifts of Riemannian Metrics

A tensor field $\widetilde{g}$ of type (0,2) on the tangent bundle $T(M_n)$ is called a pure tensor field with respect to the dual structure φ if

$$\widetilde{g}(\varphi X,Y)=\widetilde{g}(X,\varphi Y)$$

for any vector fields X and Y on $T(M_n)$. From here we see that, the condition of purity of $\widetilde{g}$ may be expressed in terms of the local induced coordinates as follows:

$$\widetilde{g}_{\sigma\beta}\varphi^{\sigma}_{\alpha}=\widetilde{g}_{\alpha\sigma}\varphi^{\sigma}_{\beta}.$$

Using (1.4), from the last conditon we have

$$\widetilde{g}=(\widetilde{g}_{\alpha\beta})=\begin{pmatrix}\widetilde{g}_{ij} & \widetilde{g}_{\overline{i}j}\\ \widetilde{g}_{\overline{i}j} & 0\end{pmatrix},\quad \widetilde{g}_{\overline{i}\,\overline{j}}=0,\ \widetilde{g}_{\overline{i}j}=\widetilde{g}_{i\overline{j}}.$$

A pure tensor field $\widetilde{g}$ of type (0,2) on tangent bundle $T(M_n)$ is called a dual-holomorphic with respect to φ, if $\Phi_\varphi\widetilde{g}=0$, where Φ_φ is the Tachibana operator defined by [4, 7],

$$(\Phi_\varphi\widetilde{g})(X,Y,Z)=(\varphi X)(\widetilde{g}(Y,Z))-X(\widetilde{g}(\varphi Y,Z))+\widetilde{g}((L_Y\varphi)X,Z)+\widetilde{g}(Y,(L_Z\varphi)X).$$

Such tensor field is a real image of corresponding dual-holomorphic tensor field of type (0,2) from $X_n(\mathrm{R}(\varepsilon))$. It is well known that, if $\widetilde{g}$ is a Riemannian metric and $\nabla^{\widetilde{g}}$ is its Levi-Civita connection, then the condition $\Phi_\varphi\widetilde{g}=0$ is equivalent to the condition $\nabla^{\widetilde{g}}\varphi=0$ (see [3]), i.e., the triple $(T(M_n),\widetilde{g},\varphi)$ is a dual anti-Kähler (or Kähler-Norden) manifold.

The tensor field $\Phi_\varphi\widetilde{g}$ of type (0,3) has components

$$(\Phi_\varphi\widetilde{g})_{\alpha\beta\gamma}=\varphi^{\sigma}_{\alpha}\partial_\sigma\widetilde{g}_{\beta\gamma}-\varphi^{\sigma}_{\beta}\partial_\alpha\widetilde{g}_{\sigma\gamma}-\widetilde{g}_{\sigma\gamma}(\partial_\alpha\varphi^{\sigma}_{\beta}-\partial_\beta\varphi^{\sigma}_{\alpha})+\widetilde{g}_{\beta\sigma}\partial_\gamma\varphi^{\sigma}_{\alpha}$$

with respect to the natural frame $\{\partial_\alpha\}=\{\partial_i,\partial_{\overline{i}}\}$.

By virtue of (1.4), after some calculations, $(\Phi_\varphi\widetilde{g})_{\alpha\beta\gamma}=0$ reduces to

$$\partial_{\overline{i}}\widetilde{g}_{jk}-\partial_i\widetilde{g}_{\overline{j}k}=0,\quad \partial_{\overline{i}}\widetilde{g}_{\overline{j}k}=0,$$

from which we have

$$\widetilde{g}_{\overline{j}k}=g_{jk}(x^1,\cdots,x^n),\quad \widetilde{g}_{jk}=x^{\overline{i}}\partial_i g_{jk}+h_{jk}(x^1,\cdots,x^n).\tag{6.1}$$

Remark 6.1 Using (1.3), (6.1) and $\widetilde{g}_{\alpha'\beta'}=\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}\widetilde{g}_{\alpha\beta}$, we easily see that $g_{jk}(x^1,\cdots,x^n)$ and $h_{jk}(x^1,\cdots,x^n)$ are components of any tensor fields g and h of type $(0,2)$ on M_n, respectively.

Thus a real dual-holomorphic tensor field $\widetilde{g}$ of type $(0,2)$ on tangent bundle can be rewritten in the form

$$\widetilde{g} = (\widetilde{g}_{\beta\gamma}) = \begin{pmatrix} x^{\bar{i}}\partial_i g_{jk} + h_{jk} & g_{jk} \\ g_{jk} & 0 \end{pmatrix} = \begin{pmatrix} x^{\bar{i}}\partial_i g_{jk} & g_{jk} \\ g_{jk} & 0 \end{pmatrix} + \begin{pmatrix} h_{jk} & 0 \\ 0 & 0 \end{pmatrix} = {}^{C}g + {}^{V}h\,,$$

where ${}^{C}g$ and ${}^{V}h$ are the complete and vertical lifts of tesor fields $g = (g_{jk})$ and $h = (h_{jk})$ of type $(0,2)$ from M_n to tangent bundle $T(M_n)$, respectively (see [8]). Therefore we have the following theorem.

Theorem 6.1 *Let $T(M_n)$ be a tangent bundle of M_n, which is a real image of dual-holomorphic manifold $X_n(\mathrm{R}(\varepsilon))$. Then a real image of corresponding dual-holomorphic tensor field of type $(0,2)$ from $X_n(\mathrm{R}(\varepsilon))$ is a deformed complete lift in the form ${}^{D}g = {}^{C}g + {}^{V}h$, where ${}^{C}g$ and ${}^{V}h$ are the complete and vertical lifts of $g = (g_{jk})$ and $h = (h_{jk})$ from M_n to $T(M_n)$, respectively.*

Remark 6.2 Now let g be a Riemannian metric, and h be any symmetric $(0,2)$-tensor field on M_n. It is clear that in such case the tensor ${}^{D}g = {}^{C}g + {}^{V}h$ is a Riemannian metric on $T(M_n)$. We note that lifts of this kind have been also studied under the names: The metric I+II (see [8]) if $g = h$ and the synectic lift (see [5]).

7 Deformed Complete Lifts of Connections

Let $\widetilde{\nabla}$ be a connection with components $\widetilde{\Gamma}^{\gamma}_{\alpha\beta}$ on the tangent bundle $T(M_n)$ preserving the structure φ. That connection is called a pure connection by definition if

$$\widetilde{\Gamma}^{\sigma}_{\alpha\beta}\varphi^{\gamma}_{\sigma} = \widetilde{\Gamma}^{\gamma}_{\sigma\beta}\varphi^{\sigma}_{\alpha} = \widetilde{\Gamma}^{\gamma}_{\alpha\sigma}\varphi^{\sigma}_{\beta}.$$

Using (1.4), from the purity conditon we have

$$\widetilde{\Gamma}^{k}_{\bar{i}j} = \widetilde{\Gamma}^{k}_{i\bar{j}} = \widetilde{\Gamma}^{k}_{\bar{i}\,\bar{j}} = \widetilde{\Gamma}^{\bar{k}}_{\bar{i}\,\bar{j}} = 0. \tag{7.1}$$

The pure connection $\widetilde{\nabla}$ with components $\widetilde{\Gamma}^{\gamma}_{\alpha\beta}$ is called a dual-holomorphic connection, if (see [6])

$$(\Phi_{\varphi}\Gamma)^{\gamma}_{\tau\alpha\beta} = \varphi^{\sigma}_{\tau}\partial_{\sigma}\widetilde{\Gamma}^{\gamma}_{\alpha\beta} - \varphi^{\sigma}_{\alpha}\partial_{\tau}\widetilde{\Gamma}^{\gamma}_{\sigma\beta} = 0.$$

It is well known that, such connection is a real image of corresponding dual-holomorphic connection from $X_n(\mathrm{R}(\varepsilon))$.

From here, by virtue of (1.4) and (7.1), we have

$$\begin{aligned}
&(\Phi_{\varphi}\Gamma)^{k}_{tij} = \varphi^{m}_{t}\partial_{m}\widetilde{\Gamma}^{k}_{ij} + \varphi^{\overline{m}}_{t}\partial_{\overline{m}}\widetilde{\Gamma}^{k}_{ij} - \varphi^{m}_{i}\partial_{t}\widetilde{\Gamma}^{k}_{mj} - \varphi^{\overline{m}}_{i}\partial_{t}\widetilde{\Gamma}^{k}_{\overline{m}j} = 0 \\
&\;\Leftrightarrow \widetilde{\Gamma}^{k}_{ij} = \Gamma^{k}_{ij}(x^{1},\cdots,x^{n}), \\
&(\Phi_{\varphi}\Gamma)^{k}_{\bar{t}ij} = \varphi^{m}_{\bar{t}}\partial_{m}\widetilde{\Gamma}^{k}_{ij} + \varphi^{\overline{m}}_{\bar{t}}\partial_{\overline{m}}\widetilde{\Gamma}^{k}_{ij} - \varphi^{m}_{i}\partial_{\bar{t}}\widetilde{\Gamma}^{k}_{mj} - \varphi^{\overline{m}}_{i}\partial_{\bar{t}}\widetilde{\Gamma}^{k}_{\overline{m}j} = 0 \Leftrightarrow 0 = 0, \\
&(\Phi_{\varphi}\Gamma)^{k}_{t\bar{i}j} = \varphi^{m}_{t}\partial_{m}\widetilde{\Gamma}^{k}_{\bar{i}j} + \varphi^{\overline{m}}_{t}\partial_{\overline{m}}\widetilde{\Gamma}^{k}_{\bar{i}j} - \varphi^{m}_{\bar{i}}\partial_{t}\widetilde{\Gamma}^{k}_{mj} - \varphi^{\overline{m}}_{\bar{i}}\partial_{t}\widetilde{\Gamma}^{k}_{\overline{m}j} = 0 \Leftrightarrow 0 = 0, \\
&(\Phi_{\varphi}\Gamma)^{k}_{ti\bar{j}} = \varphi^{m}_{t}\partial_{m}\widetilde{\Gamma}^{k}_{i\bar{j}} + \varphi^{\overline{m}}_{t}\partial_{\overline{m}}\widetilde{\Gamma}^{k}_{i\bar{j}} - \varphi^{m}_{i}\partial_{t}\widetilde{\Gamma}^{k}_{m\bar{j}} - \varphi^{\overline{m}}_{i}\partial_{t}\widetilde{\Gamma}^{k}_{\overline{m}\bar{j}} = 0 \Leftrightarrow 0 = 0, \\
&(\Phi_{\varphi}\Gamma)^{k}_{\bar{t}i\bar{j}} = \varphi^{m}_{\bar{t}}\partial_{m}\widetilde{\Gamma}^{k}_{i\bar{j}} + \varphi^{\overline{m}}_{\bar{t}}\partial_{\overline{m}}\widetilde{\Gamma}^{k}_{i\bar{j}} - \varphi^{m}_{i}\partial_{\bar{t}}\widetilde{\Gamma}^{k}_{m\bar{j}} - \varphi^{\overline{m}}_{i}\partial_{\bar{t}}\widetilde{\Gamma}^{k}_{\overline{m}\bar{j}} = 0 \Leftrightarrow 0 = 0, \\
&(\Phi_{\varphi}\Gamma)^{k}_{\bar{t}\,\bar{i}j} = \varphi^{m}_{\bar{t}}\partial_{m}\widetilde{\Gamma}^{k}_{\bar{i}j} + \varphi^{\overline{m}}_{\bar{t}}\partial_{\overline{m}}\widetilde{\Gamma}^{k}_{\bar{i}j} - \varphi^{m}_{\bar{i}}\partial_{\bar{t}}\widetilde{\Gamma}^{k}_{mj} - \varphi^{\overline{m}}_{\bar{i}}\partial_{\bar{t}}\widetilde{\Gamma}^{k}_{\overline{m}j} = 0 \Leftrightarrow 0 = 0, \\
&(\Phi_{\varphi}\Gamma)^{k}_{t\bar{i}\,\bar{j}} = \varphi^{m}_{t}\partial_{m}\widetilde{\Gamma}^{k}_{\bar{i}\,\bar{j}} + \varphi^{\overline{m}}_{t}\partial_{\overline{m}}\widetilde{\Gamma}^{k}_{\bar{i}\,\bar{j}} - \varphi^{m}_{\bar{i}}\partial_{t}\widetilde{\Gamma}^{k}_{m\bar{j}} - \varphi^{\overline{m}}_{\bar{i}}\partial_{t}\widetilde{\Gamma}^{k}_{\overline{m}\bar{j}} = 0 \Leftrightarrow 0 = 0,
\end{aligned}$$

$$
\begin{aligned}
&(\Phi_\varphi\Gamma)^{k}_{t\bar i\bar j} = \varphi^m_t\partial_m\widetilde\Gamma^k_{\bar i\bar j} + \varphi^{\bar m}_t\partial_{\bar m}\widetilde\Gamma^k_{\bar i\bar j} - \varphi^m_{\bar i}\partial_t\widetilde\Gamma^k_{m\bar j} - \varphi^{\bar m}_{\bar i}\partial_t\widetilde\Gamma^k_{\bar m\bar j} = 0 \Leftrightarrow 0 = 0,\\
&(\Phi_\varphi\Gamma)^{\bar k}_{\bar t ij} = \varphi^m_{\bar t}\partial_m\widetilde\Gamma^{\bar k}_{ij} + \varphi^{\bar m}_{\bar t}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{ij} - \varphi^m_i\partial_{\bar t}\widetilde\Gamma^{\bar k}_{mj} - \varphi^{\bar m}_i\partial_{\bar t}\widetilde\Gamma^{\bar k}_{\bar m j} = 0\\
&\Leftrightarrow \varphi^{\bar m}_i\partial_{\bar t}\widetilde\Gamma^{\bar k}_{\bar m j} = 0 \Leftrightarrow \widetilde\Gamma^{\bar k}_{\bar i j} = \Gamma^k_{ij}(x^1,\cdots,x^n),\\
&(\Phi_\varphi\Gamma)^{\bar k}_{t\bar i j} = \varphi^m_t\partial_m\widetilde\Gamma^{\bar k}_{\bar i j} + \varphi^{\bar m}_t\partial_{\bar m}\widetilde\Gamma^{\bar k}_{\bar i j} - \varphi^m_{\bar i}\partial_t\widetilde\Gamma^{\bar k}_{mj} - \varphi^{\bar m}_{\bar i}\partial_t\widetilde\Gamma^{\bar k}_{\bar m j} = 0 \Leftrightarrow 0 = 0,\\
&(\Phi_\varphi\Gamma)^{\bar k}_{ti\bar j} = \varphi^m_t\partial_m\widetilde\Gamma^{\bar k}_{i\bar j} + \varphi^{\bar m}_t\partial_{\bar m}\widetilde\Gamma^{\bar k}_{i\bar j} - \varphi^m_i\partial_t\widetilde\Gamma^{\bar k}_{m\bar j} - \varphi^{\bar m}_i\partial_t\widetilde\Gamma^{\bar k}_{\bar m\bar j} = 0\\
&\Leftrightarrow \varphi^{\bar m}_t\partial_{\bar m}\widetilde\Gamma^{\bar k}_{i\bar j} = 0 \Leftrightarrow \widetilde\Gamma^{\bar k}_{i\bar j} = \Gamma^k_{ij}(x^1,\cdots,x^n),\\
&(\Phi_\varphi\Gamma)^{\bar k}_{tij} = \varphi^m_t\partial_m\widetilde\Gamma^{\bar k}_{ij} + \varphi^{\bar m}_t\partial_{\bar m}\widetilde\Gamma^{\bar k}_{ij} - \varphi^m_i\partial_t\widetilde\Gamma^{\bar k}_{mj} - \varphi^{\bar m}_i\partial_t\widetilde\Gamma^{\bar k}_{\bar m j} = 0\\
&\Leftrightarrow \varphi^{\bar m}_t\partial_{\bar m}\widetilde\Gamma^{\bar k}_{ij} - \varphi^{\bar m}_i\partial_t\widetilde\Gamma^{\bar k}_{\bar m j} = 0 \Leftrightarrow \partial_{\bar t}\widetilde\Gamma^{\bar k}_{ij} - \partial_t\widetilde\Gamma^{\bar k}_{\bar i j} = 0\\
&\Leftrightarrow \widetilde\Gamma^{\bar k}_{ij} = x^{\bar t}\partial_t\Gamma^k_{ij} + H^k_{ij}(x^1,\cdots,x^n),\\
&(\Phi_\varphi\Gamma)^{\bar k}_{\bar t\bar i j} = \varphi^m_{\bar t}\partial_m\widetilde\Gamma^{\bar k}_{\bar i j} + \varphi^{\bar m}_{\bar t}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{\bar i j} - \varphi^m_{\bar i}\partial_{\bar t}\widetilde\Gamma^{\bar k}_{mj} - \varphi^{\bar m}_{\bar i}\partial_{\bar t}\widetilde\Gamma^{\bar k}_{\bar m j} = 0 \Leftrightarrow 0 = 0,\\
&(\Phi_\varphi\Gamma)^{\bar k}_{t\bar i\bar j} = \varphi^m_t\partial_m\widetilde\Gamma^{\bar k}_{\bar i\bar j} + \varphi^{\bar m}_t\partial_{\bar m}\widetilde\Gamma^{\bar k}_{\bar i\bar j} - \varphi^m_{\bar i}\partial_t\widetilde\Gamma^{\bar k}_{\bar m j} - \varphi^{\bar m}_{\bar i}\partial_t\widetilde\Gamma^{\bar k}_{\bar m\bar j} = 0 \Leftrightarrow 0 = 0,\\
&(\Phi_\varphi\Gamma)^{\bar k}_{\bar t i\bar j} = \varphi^m_{\bar t}\partial_m\widetilde\Gamma^{\bar k}_{i\bar j} + \varphi^{\bar m}_{\bar t}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{i\bar j} - \varphi^m_i\partial_{\bar t}\widetilde\Gamma^{\bar k}_{m\bar j} - \varphi^{\bar m}_i\partial_{\bar t}\widetilde\Gamma^{\bar k}_{\bar m\bar j} = 0 \Leftrightarrow 0 = 0,\\
&(\Phi_\varphi\Gamma)^{\bar k}_{\bar t\bar i\bar j} = \varphi^m_{\bar t}\partial_m\widetilde\Gamma^{\bar k}_{\bar i\bar j} + \varphi^{\bar m}_{\bar t}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{\bar i\bar j} - \varphi^m_{\bar i}\partial_{\bar t}\widetilde\Gamma^{\bar k}_{m\bar j} - \varphi^{\bar m}_{\bar i}\partial_{\bar t}\widetilde\Gamma^{\bar k}_{\bar m\bar j} = 0 \Leftrightarrow 0 = 0.
\end{aligned}
$$

Thus $(\Phi_\varphi\Gamma)^\gamma_{\tau\alpha\beta} = 0$ reduces to

$$
\widetilde\Gamma^k_{ij} = \Gamma^{\bar k}_{\bar i j} = \widetilde\Gamma^{\bar k}_{i\bar j} = \Gamma^k_{ij}(x^1,\cdots,x^n), \quad \widetilde\Gamma^{\bar k}_{ij} = x^{\bar t}\partial_t\Gamma^k_{ij} + H^k_{ij}(x^1,\cdots,x^n). \tag{7.2}
$$

Remark 7.1 Using (1.3), (7.1)–(7.2) and

$$
\widetilde\Gamma^{\gamma'}_{\alpha'\beta'} = \frac{\partial x^{\gamma'}}{\partial x^\gamma}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}\widetilde\Gamma^\gamma_{\alpha\beta} + \frac{\partial x^{\gamma'}}{\partial x^\gamma}\frac{\partial^2 x^\gamma}{\partial x^{\alpha'}\partial x^{\beta'}},
$$

after straightforward calculations we see that $\Gamma^k_{ij}(x^1,\cdots,x^n)$ and $H^k_{ij}(x^1,\cdots,x^n)$ are components of any connection ∇ and tensor field H of type (1,2) on M_n, respectively.

Taking account of the definition of the complete lift ${}^C\nabla$ of connection ∇ (see [8]), we see that a real dual-holomorphic connection $\widetilde\nabla$ on tangent bundle can be rewritten in the form

$$
\widetilde{\nabla} = {}^C\nabla + {}^VH,
$$

where ${}^VH$ is the vertical lift of tensor field $H = (H^k_{ij})$ of type (1,2) from M_n to tangent bundle $T(M_n)$. Thus we have following theorem.

Theorem 7.1 *Let $T(M_n)$ be a tangent bundle of M_n, which is a real image of dual-holomorphic manifold $X_n(\mathrm{R}(\varepsilon))$. Then a real image of corresponding dual-holomorphic connection from $X_n(\mathrm{R}(\varepsilon))$ is a deformed complete lift in the form ${}^D\nabla = {}^C\nabla + {}^VH$, where ${}^C\nabla$ and ${}^VH$ are the complete and vertical lifts of $\nabla = (\Gamma^k_{ij})$ and $H = (H^k_{ij})$ from M_n to $T(M_n)$, respectively.*

Example 7.1 Let (M, g) be a Riemannian manifold, and $(T(M_n), \varphi)$ be its tangent bundle with natural dual φ-structure: $\varphi = \left(\begin{smallmatrix}0&0\\I&0\end{smallmatrix}\right)$.

The complete and vertical lifts of vector and tensor fields from M_n to $T(M_n)$ have the following properties

$$
\varphi^C X = {}^VX, \quad {}^VX^Vf = 0, \quad {}^VX^Cf = {}^CX^Vf = {}^V(Xf),
$$

$$
\begin{aligned}
&{}^{V}h({}^{V}X,{}^{C}Y)=0, \quad {}^{C}g({}^{V}X,{}^{C}Y)={}^{V}(g(X,Y)),\\
&{}^{V}h({}^{C}X,{}^{C}Y)={}^{V}(h(X,Y)), \quad {}^{C}g({}^{C}X,{}^{C}Y)={}^{C}(g(X,Y)), \quad {}^{V}h({}^{V}X,{}^{C}Y)=0,\\
&[{}^{C}Y,{}^{V}X]={}^{V}[Y,X], \quad [{}^{C}Y,{}^{C}X]={}^{C}[Y,X]
\end{aligned}
$$

for any function f on M_n (see [8]). Using these formulas, we find

$$
\begin{aligned}
{}^{D}g(\varphi{}^{C}X,{}^{C}Y) &= ({}^{C}g+{}^{V}h)(\varphi{}^{C}X,{}^{C}Y)=({}^{C}g+{}^{V}h)({}^{V}X,{}^{C}Y)\\
&={}^{C}g({}^{V}X,{}^{C}Y)+{}^{V}h({}^{V}X,{}^{C}Y)={}^{C}g({}^{V}X,{}^{C}Y)={}^{V}(g(X,Y))={}^{C}g({}^{C}X,{}^{V}Y)\\
&={}^{C}g({}^{C}X,{}^{V}Y)+{}^{V}h({}^{C}X,{}^{V}Y)=({}^{C}g+{}^{V}h)({}^{C}X,{}^{V}Y)\\
&=({}^{C}g+{}^{V}h)({}^{C}X,\varphi{}^{C}Y)={}^{D}g({}^{C}X,\varphi{}^{C}Y)
\end{aligned}
$$

and

$$
\begin{aligned}
(\Phi_{\varphi}{}^{D}g)({}^{C}X,{}^{C}Y,{}^{C}Z) &= (\varphi{}^{C}X)({}^{D}g({}^{C}Y,{}^{C}Z))-{}^{C}X({}^{D}g(\varphi{}^{C}Y,{}^{C}Z))\\
&\quad +{}^{D}g((L_{{}^{C}Y}\varphi){}^{C}X,{}^{C}Z)+{}^{D}g({}^{C}Y,(L_{{}^{C}Z}\varphi){}^{C}X)={}^{V}X^{C}(g(Y,Z))\\
&\quad +{}^{V}X^{V}(h(Y,Z))-{}^{C}X^{V}(g(Y,Z))+{}^{D}g(L_{{}^{C}Y}(\varphi{}^{C}X)-\varphi(L_{{}^{C}Y}{}^{C}X),{}^{C}Z)\\
&\quad +{}^{D}g({}^{C}Y,L_{{}^{C}Z}(\varphi{}^{C}X)-\varphi(L_{{}^{C}Z}{}^{C}X))={}^{V}(X(g(Y,Z)))-{}^{V}(X(g(Y,Z)))\\
&\quad +{}^{D}g(L_{{}^{C}Y}{}^{V}X-\varphi[{}^{C}Y,{}^{C}X],{}^{C}Z)+{}^{D}g({}^{C}Y,L_{{}^{C}Z}{}^{V}X-\varphi[{}^{C}Z,{}^{C}X])\\
&={}^{D}g([{}^{C}Y,{}^{V}X]-\varphi^{C}[Y,X],{}^{C}Z)+{}^{D}g({}^{C}Y,[{}^{C}Z,{}^{V}X]-\varphi[{}^{C}Z,{}^{C}X])\\
&={}^{D}g({}^{V}[Y,X]-{}^{V}[Y,X],{}^{C}Z)+{}^{D}g({}^{C}Y,{}^{V}[Z,X]-{}^{V}[Z,X])=0.
\end{aligned}
$$

From here we see that the triple $(T(M_n),{}^{D}g,\varphi)$ is a dual anti-Kähler manifold ($\nabla^{{}^{D}g}\varphi=0$) (see Section 6). In such manifolds, the Levi-Civita connection $\nabla^{{}^{D}g}$ of ${}^{D}g$ also is dual-holomorphic (see [3]). Thus the Levi-Civita connection $\nabla^{{}^{D}g}$ is a simplest example of deformed complete lift of connection.

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