Chin. Ann. Math. Ser. B
43(2), 2022, 233–252
DOI: 10.1007/s11401-022-0314-9

Chinese Annals of Mathematics, Series B

Stability of the Rarefaction Wave for a Non-isentropic Navier-Stokes/Allen-Cahn System*

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Abstract This paper is concerned with the large time behavior of solutions to the Cauchy problem for a one-dimensional compressible non-isentropic Navier-Stokes/Allen-Cahn system which is a combination of the classical Navier-Stokes system with an Allen-Cahn phase field description. Motivated by the relationship between Navier-Stokes/Allen-Cahn and Navier-Stokes, the author can prove that the solutions to the one dimensional compressible non-isentropic Navier-Stokes/Allen-Cahn system tend time-asymptotically to the rarefaction wave, where the strength of the rarefaction wave is not required to be small. The proof is mainly based on a basic energy method.

Keywords Navier-Stokes/Allen-Cahn system, Rarefaction wave, Stability
2000 MR Subject Classification 35M10, 35B40, 35B35

1 Introduction

1.1 The problem

In this paper, we deal with the so-called Navier-Stokes/Allen-Cahn equations, a system of balance laws for two-phase mixtures of fluids undergoing phase transitions. In this model the interfaces between the phases are assumed to be of "diffuse" nature, that is, sharp interfaces are replaced by narrow transitions layers. These regions, as well as the two species, are located by a phase field variable χ governed by the Allen-Cahn equation, while the dynamics are described by the Navier-Stokes equations. The model was proposed by Blesgen [1], who developed a thermodynamically and mechanically consistent set of partial differential equations extending the Navier-Stokes equations to a compressible binary Allen-Cahn mixture. Recently, Freistühler and Kotschote have shown the thermodynamical consistency of a carefully chosen general form of Navier-Stokes/Allen-Cahn in [10]. Before formulating the non-isentropic Navier-Stokes/Allen-Cahn system, we refer to [10, 17] for a complete derivation. We are interested in

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*This work was supported by the National Natural Science Foundation of China (No. 12001249), the Natural Science Foundation of Jiangxi Province of China (No. GJJ190280) and the Scientific Research Funds of Jiangxi University of Finance and Economics (No. 012270624).

the form of the following compressible Navier-Stokes/Allen-Cahn system (see [17]) in $\mathbb{R}^n$,

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho u) = 0, \\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) - \nabla\cdot\mathbb{T} = \rho f_{ext}, \\ \partial_t(\rho\overline{E}) + \nabla\cdot(\rho\overline{E}u) - \nabla\cdot(\beta\nabla\theta) - \nabla\cdot(\mathbb{T}\cdot u) = \rho f_{ext}u, \\ \partial_t(\rho\chi) + \nabla\cdot(\rho\chi u) + \dfrac{\theta}{a}\Big\{ -\nabla\cdot\Big(\partial_{\nabla\chi}\Big(\dfrac{\rho}{\theta}F\Big)\Big) + \partial_\chi\Big(\dfrac{\rho}{\theta}F\Big)\Big\} = 0 \end{cases} \tag{1.1}$$

for $(t,x)\in(0,+\infty)\times\mathbb{R}^n$. Here ρ, u and θ denote the total density, the mean velocity, and the temperature, of the fluids mixture, respectively. χ is the concentration difference of the two components. $\overline{E}$ denotes the total energy density of the mixture,

$$\overline{E} := E + \frac{1}{2}|u|^2. \tag{1.2}$$

The thermal conductivity $\beta > 0$ may depend on $\xi = (\rho,\chi,\theta)$. The parameter $a > 0$ represents a relaxation time. Luo et al. [22] obtained a special form of the Navier-Stokes/Allen-Cahn system by taking the special free energy F, i.e.,

$$F(\rho,\theta,\chi,\phi) = R\theta\ln\rho - \frac{R\theta}{\gamma-1}\ln\theta + \frac{\theta}{\delta}Q(\chi) + \frac{\theta\delta}{2\rho}\phi, \quad \phi = |\nabla\chi|^2, \tag{1.3}$$

where $\gamma > 1$ and the constant $\delta > 0$, then the Navier-Stokes/Allen-Cahn system in three-dimensional under assumpitions becomes

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \nabla p = \mu\triangle u + \widetilde{\mu}\nabla\mathrm{div}u - \dfrac{\delta}{2}\mathrm{div}(\theta\nabla\chi\otimes\nabla\chi) + \rho f_{ext}, \\ \partial_t(\rho E) + \mathrm{div}(\rho E u) = \beta\triangle\theta + \mathbb{T}:\nabla u, \\ \partial_t(\rho\chi) + \mathrm{div}(\rho\chi u) = \dfrac{\delta\theta}{a}\triangle\chi - \dfrac{\theta\rho}{a\delta}\partial_\chi Q(\chi), \end{cases} \tag{1.4}$$

where $\mathbb{T}:\nabla u = \sum\limits_{i,j} T_{ij}\partial_i u_j$ and $\widetilde{\mu} = \frac{1}{3}\mu + \lambda$.

Furthermore, we assume that $Q(\chi) = \frac{\chi^4}{4} - \frac{\chi^2}{2}$, $f_{ext} = 0$ and $\lambda = -\frac{1}{3}\mu$. Then the Navier-Stokes/Allen-Cahn system in one-dimensional under assumpitions becomes

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \rho\partial_t u + \rho u\partial_x u + \partial_x p(\rho,\theta) = \mu\partial_x^2 u - \dfrac{\delta}{2}\partial_x(\theta(\partial_x\chi)^2), \\ \dfrac{R}{\gamma-1}\rho\partial_t\theta + \dfrac{R}{\gamma-1}\rho u\partial_x\theta + p(\rho,\theta)\partial_x u = \beta\partial_x^2\theta + \mu(\partial_x u)^2 - \dfrac{\delta}{2}\theta\partial_x u(\partial_x\chi)^2, \\ \rho\partial_t\chi + \rho u\partial_x\chi = \dfrac{\delta\theta}{a}\partial_x^2\chi - \dfrac{\theta\rho}{a\delta}(\chi^3-\chi) \end{cases} \tag{1.5}$$

for $(x,t)\in\mathbb{R}\times(0,+\infty)$. Here $\rho > 0$ denotes the total density, u represents the mean velocity of the fluid mixture, θ corresponds to temperature of the fluid mixture, χ is the concentration difference of the two components. The positive constants μ and β denote the viscosity and heat conduction coefficients, respectively. The parameter $a > 0$ represents a relaxation time and the constant $\delta > 0$ represents the thickness of the interfacial region. The Navier-Stokes/Allen-Cahn system describes two-phase patterns in a flowing liquid including phase transformations.

A phase field variable χ is introduced and a mixing energy is defined in terms of χ and its spatial gradient.

Initial data for system (1.5) are given by

$$[\rho, u, \theta, \chi](0, x) = [\rho_0, u_0, \theta_0, \chi_0](x), \quad x \in \mathbb{R}. \tag{1.6}$$

We assume that those initial data at both far fields $x = \pm\infty$ are constants, namely

$$\lim_{x\to\pm\infty} [\rho_0, u_0, \theta_0, \chi_0](x) = [\rho_\pm, u_\pm, \theta_\pm, \chi_\pm], \tag{1.7}$$

where $\rho_\pm > 0$, $u_\pm$, $\theta_\pm$and $\chi_\pm$ can be distinct.

1.2 Euler system and rarefaction wave

In order to study the large time behavior of solutions to Cauchy problem (1.5)–(1.7), we notice that in the setting of the concentration $\chi = 1$ for the large time behavior, Navier-Stokes/Allen-Cahn system (1.5) can be reduced to the following Navier-Stokes system in the form

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p(\rho, \theta) = \mu \partial_x^2 u, \\ \dfrac{R}{\gamma - 1}\rho \partial_t \theta + \dfrac{R}{\gamma - 1}\rho u \partial_x \theta + p(\rho, \theta)\partial_x u = \beta \partial_x^2 \theta + \mu(\partial_x u)^2. \end{cases} \tag{1.8}$$

We define its entropy in terms of its density and temperature by

$$S = \frac{R}{\gamma - 1}\ln\Big(\frac{R\theta}{A}\Big) - R\ln\rho + \frac{R}{\gamma - 1}, \tag{1.9}$$

where A is a positive constant. It is straightforward to verify from (1.5) that

$$\partial_t S + u\partial_x S = \frac{\mu}{\rho\theta}(\partial_x u)^2 + \frac{\beta}{\rho\theta}\partial_x^2\theta. \tag{1.10}$$

For later application, under the assumptions (1.7), we set $S_\pm$ to be the values of S at both far fields in terms of $\rho_\pm$ and $\theta_\pm$, i.e.,

$$S_\pm = \frac{R}{\gamma - 1}\ln\Big(\frac{R\theta_\pm}{A}\Big) - R\ln\rho_\pm + \frac{R}{\gamma - 1} \tag{1.11}$$

at both far fields. Also note that in terms of S and ρ,

$$p(\rho, \theta) = \frac{A}{\mathrm{e}}\exp\Big(\frac{\gamma - 1}{R}S\Big)\rho^\gamma.$$

To construct the rarefaction wave of Navier-Stokes/Allen-Cahn system in the non-isentropic case, we assume $S_+ = S_- = S_\infty$, equivalently, by (1.11),

$$\frac{\theta_+}{\theta_-} = \Big(\frac{\rho_+}{\rho_-}\Big)^{\gamma - 1}. \tag{1.12}$$

Therefore, as in [19], in the rarefaction wave θ can be determined in terms of ρ by (1.9) in the way that S takes the constant value S_∞. Furthermore, skipping terms of the second-order derivatives, we may expect that system (1.8) tends in large time to the following Euler equations:

$$\begin{cases} \partial_t\rho + u\partial_x\rho + \rho\partial_x u = 0, \\ \rho\partial_t u + \rho u\partial_x u + B\partial_x(\rho^\gamma) = 0, \end{cases} \tag{1.13}$$

where B is constant given by

$$B = \frac{A}{\mathrm{e}}\exp\Big(\frac{\gamma-1}{R}S_\infty\Big) > 0. \tag{1.14}$$

Motivated by the above formal computations, the large time behavior of solutions to Cauchy problem (1.5)–(1.6) under the condition (1.7) is expected to be determined by the Riemann problem for the above Euler system (1.13) with initial data given by

$$[\rho, u](0, x) = \begin{cases} [\rho_-, u_-], & x < 0, \\ [\rho+, u_+], & x > 0. \end{cases} \tag{1.15}$$

To study the rarefaction wave, one can see that system (1.13) has two characteristics (see [4, 25])

$$\begin{cases} \lambda_1(\rho, u) = u - \sqrt{B\gamma\rho^{\gamma-1}}, \\ \lambda_2(\rho, u) = u + \sqrt{B\gamma\rho^{\gamma-1}}, \end{cases}$$

which are genuinely nonlinear and give rise to the rarefaction wave curves

$$R_1[\rho_-, u_-] \equiv \Big\{[\rho, u] \in \mathbb{R}_+ \times \mathbb{R} \;\Big|\; u = u_- - \int_{\rho_-}^{\rho} \sqrt{B\gamma z^{\gamma-3}}\mathrm{d}z,\ \rho < \rho_-,\ u > u_-\Big\}$$

and

$$R_2[\rho_-, u_-] \equiv \Big\{[\rho, u] \in \mathbb{R}_+ \times \mathbb{R} \;\Big|\; u = u_- + \int_{\rho_-}^{\rho} \sqrt{B\gamma z^{\gamma-3}}\mathrm{d}z,\ \rho > \rho_-,\ u > u_-\Big\},$$

respectively. For the simplicity of presentation, we consider only the 1-rarefaction wave, as the case for 2-rarefaction wave can be treated in the completely same way. Thus two constant states $[\rho_\pm, u_\pm]$ satisfy

$$u_+ = u_- - \int_{\rho_-}^{\rho_+} \sqrt{B\gamma z^{\gamma-3}}\mathrm{d}z, \quad 0 < \rho_+ < \rho_-, \tag{1.16}$$

and the Riemann problem (1.13) and (1.15) admits a self-similar solution, the centered 1-rarefaction wave $[\rho^R, u^R](\frac{x}{t})$, which is defined by

$$\begin{cases} u^R = u_- - \displaystyle\int_{\rho_-}^{\rho^R} \sqrt{B\gamma z^{\gamma-3}}\mathrm{d}z, \\ \lambda_1\Big(\rho^R\Big(\dfrac{x}{t}\Big), u^R\Big(\dfrac{x}{t}\Big)\Big) = \begin{cases} \lambda_1(\rho_-, u_-), & \dfrac{x}{t} < \lambda_1(\rho_-, u_-), \\ \dfrac{x}{t}, & \lambda_1(\rho_-, u_-) \le \dfrac{x}{t} < \lambda_1(\rho_+, u_+), \\ \lambda_1(\rho_+, u_+), & \dfrac{x}{t} \ge \lambda_1(\rho_+, u_+). \end{cases} \end{cases}$$

In light of (1.9) and (1.14), one can define the corresponding profiles of θ as follows:

$$\theta^R = \frac{B}{R}(\rho^R)^{\gamma-1} \tag{1.17}$$

as long as the conditions for the far-field data hold, namely, (1.12).

1.3 Approximate rarefaction wave

As in [23], to study the asymptotic stability of the rarefaction wave $[\rho^R, u^R, \theta^R](\frac{x}{t})$ for Euler system (1.13), one has to use its smooth approximation $[\rho^r, u^r, \theta^r](t,x)$. To construct it, one can start with the Riemann problem on the Burgers' equation

$$\begin{cases} \partial_t w + w\partial_x w = 0, \\ w(x,0) = w_0 = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0 \end{cases} \end{cases} \tag{1.18}$$

for $w_- < w_+$. Note that the solution to (1.18) is given by

$$w^R\Big(\frac{x}{t}\Big) = \begin{cases} w_-, & x \le w_- t, \\ \dfrac{x}{t}, & w_- t < x < w_+ t, \\ w_+, & w_+ t \le x. \end{cases}$$

The solution to the Burger's equation turns out to be smooth if the initial data is smooth and increasing. Here we refer to the construction introduced in [23] where the initial data is chosen so that its spatial derivative is proportional to a parameter $\epsilon > 0$. Hence, $w^R(\frac{x}{t})$ can be approximated by the smooth function $\overline{w}(t,x)$, which is a solution to

$$\begin{cases} \partial_t w + w\partial_x w = 0, \\ w(0,x) = w_0(x) = \dfrac{1}{2}(w_+ + w_-) + \dfrac{1}{2}(w_+ - w_-)\tanh(\epsilon x), \end{cases} \tag{1.19}$$

where the constant $\epsilon > 0$ is to be chosen later on.

Lemma 1.1 *Let $\overline{\delta} = w_+ - w_-$ be the wave strength of the* 1*-rarefaction wave. Then problem* (1.19) *has a unique smooth solution $\overline{w}(t,x)$ which satisfies the following properties*:

(i) $w_- < \overline{w}(t,x) < w_+$, $\partial_x \overline{w} > 0$ *for $x \in \mathbb{R}$ and $t \ge 0$.*

(ii) *For any $1 \le p \le +\infty$, there exists a constant C_p such that for $t > 0$,*

$$\begin{aligned} \|\partial_x \overline{w}\|_{L^p} &\le C_p \min\{\overline{\delta}\epsilon^{1-\frac{1}{p}}, \overline{\delta}^{\frac{1}{p}} t^{-1+\frac{1}{p}}\}, \\ \|\partial_x^2 \overline{w}\|_{L^p} &\le C_p \min\{\overline{\delta}\epsilon^{2-\frac{1}{p}}, \epsilon^{1-\frac{1}{p}} t^{-1}\}, \\ \|\partial_x^3 \overline{w}\|_{L^p} &\le C_p \min\{\overline{\delta}\epsilon^{3-\frac{1}{p}}, \epsilon^{2-\frac{1}{p}} t^{-1}\}. \end{aligned}$$

(iii) $\lim\limits_{t\to+\infty} \sup\limits_{x\in\mathbb{R}} \big|\overline{w}(t,x) - w^R\big(\frac{x}{t}\big)\big| = 0.$

Now one can define the smooth approximate profile $[\rho^r, u^r, \theta^r](t,x)$ in terms of

$$\begin{cases} \lambda_1(\rho^r, u^r) = \overline{w}(1+t,x), \quad \lambda_1(\rho_\pm, u_\pm) = w_\pm, \\ u^r = u_- - \displaystyle\int_{\rho_-}^{\rho^r} \sqrt{B\gamma z^{\gamma-3}}\mathrm{d}z, \\ \theta^r = \dfrac{B}{R}(\rho^r)^{\gamma-1}. \end{cases} \tag{1.20}$$

Note that (ρ^r, u^r) satisfies Euler system (1.13). With Lemma 1.1 in hand, one has the following lemma.

Lemma 1.2 *Let $\delta_r = |\rho_+ - \rho_-| + |u_+ - u_-|$ be the wave strength. The approximate rarefaction profile $[\rho^r, u^r, \theta^r](t,x)$ given by* (1.20) *satisfies*

(i) $\partial_x u^r > 0$ *and* $\rho_+ < \rho^r(t,x) < \rho_-$, $u_- < u^r(t,x) < u_+$ *for* $x \in \mathbb{R}$ *and* $t \geq 0$.

(ii) *For any* $1 \leq p \leq +\infty$, *there exists a constant* C_p *such that for* $t > 0$,

$$\begin{aligned}
\|\partial_x[\rho^r, u^r, \theta^r]\|_{L^p} &\leq C_p \min\{\delta_r \epsilon^{1-\frac{1}{p}}, \delta_r^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\},\\
\|\partial_x^2[\rho^r, u^r, \theta^r]\|_{L^p} &\leq C_p \min\{\delta_r \epsilon^{2-\frac{1}{p}}, \epsilon^{1-\frac{1}{p}}(1+t)^{-1}\},\\
\|\partial_x^3[\rho^r, u^r, \theta^r]\|_{L^p} &\leq C_p \min\{\delta_r \epsilon^{3-\frac{1}{p}}, \epsilon^{2-\frac{1}{p}}(1+t)^{-1}\}.
\end{aligned}$$

(iii) $\lim\limits_{t\to+\infty} \sup\limits_{x\in\mathbb{R}} \left|[\rho^r, u^r, \theta^r](t,x) - [\rho^R, u^R, \theta^R]\left(\frac{x}{t}\right)\right| = 0.$

1.4 Main results

With the above preparation, the main result of this paper is stated as follows.

Theorem 1.1 *Let the far-field constant states* $[\rho_\pm, u_\pm, \theta_\pm, \chi_\pm]$ *satisfy* (1.16). *There are constants* $\varepsilon_0 > 0$ *and* $C_0 > 0$ *such that if*

$$\|[\rho_0(x) - \rho^r(0,x), u_0(x) - u^r(0,x), \theta_0(x) - \theta^r(0,x)]\|_{H^1} + \|\chi_0(x) - 1\|_{H^2} + \epsilon \leq \varepsilon_0,$$

where $\epsilon > 0$ *is the parameter appearing in* (1.19), *then Cauchy problem* (1.5)–(1.6) *admits a unique global solution* $[\rho, u, \theta, \chi]$ *satisfying*

$$\sup_{t\geq 0}(\|[\rho - \rho^r, u - u^r, \theta - \theta^r]\|_{H^1} + \|\chi - 1\|_{H^2}) \leq C_0 \varepsilon_0. \tag{1.21}$$

Moreover, the solution $[\rho, u, \theta, \chi]$ *tends time-asymptotically to the rarefaction wave in the sense that*

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left|[\rho, u, \theta, \chi](t,x) - [\rho^R, u^R, \theta^R, 1]\left(\frac{x}{t}\right)\right| = 0. \tag{1.22}$$

Let us recall some known results about the Navier-Stokes/Allen-Cahn system. For the compressible isentropic model, Feireisl et al. [9] proved the existence of global-in-time weak solutions in $\mathbb{R}^3$ without any restriction on the size of initial data. Ding et al. [5] proved the existence and uniqueness of global classical solution, the existence of weak solutions and the existence of unique strong solution of the Navier-Stokes/Allen-Cahn system in the interval $[0,1]$ for initial data without vacuum states. Chen and Guo [3] established the global existence and uniqueness of strong and classical solutions of Navier-Stokes/Allen-Cahn system in the interval $[0,1]$ with initial vacuum. Luo et al. in [21] proved the stability of rarefaction waves to the isentropic Navier-Stokes/Allen-Cahn system. Kotschote [17] investigated the stability of travelling wave solutions to the so-called Navier-Stokes/Allen-Cahn system. Kotschote [16] proved the existence and uniqueness of local strong solutions for the Navier-Stokes/Allen-Cahn system with arbitrary initial data. For the incompressible isentropic model, Gal and Grasselli [11] showed the existence of the trajectory attractor for both incompressible Navier-Stokes/Allen-Cahn and Navier-Stokes/Cahn-Hilliard systems and also obtained a convergence rate estimate in the phase-space metric. Xu et al. [26] discussed the global regularity of axisymmetric solutions in both large viscosity and small initial data cases in $\mathbb{R}^3$. For the coupled Navier-Stokes/Allen-Cahn system with $\frac{\partial f}{\partial \chi} = \frac{1}{\delta}(\chi^3 - \chi)$, where δ is a positive constant and $\sqrt{\delta}$ represents the thickness of the interface, Zhao et al. [28] investigated the vanishing viscosity limit and proved that the solutions of the Navier-Stokes/Allen-Cahn system converged to that of the Euler/Allen-Cahn system in a proper small time interval. Moreover, for numerical simulations, such as jet pinching-off and drop formation, we refer the readers to [2, 20, 27]. We also emphasize a different two-phase model, for which Evje et al. obtained a series results in [6–8] and references therein.

According to some special assumptions, we derive the one dimensional non-isentropic Navier-Stokes/Allen-Cahn system (1.5) in this paper. Notice that the Navier-Stokes/Allen-Cahn system is a combination of the compressible Navier-Stokes system with an Allen-Cahn phase field description. Actually, if we assume the concentration $\chi \equiv 1$, then the Navier-Stokes/Allen-Cahn system reduces to the classical Navier-Stokes system. It is well known that the large-time behavior of solutions to the Cauchy problem on Navier-Stokes system are basically the same as that of Riemann problem to the corresponding Euler equation. And the Riemann solutions for Euler equation are shock waves, rarefaction waves, contact discontinuities and the linear combinations of these basic waves. Here we only mention several results related to Navier-Stokes system: [12, 18] for the asymptotic stability of shock wave; [19, 23–24] for the asymptotic stability of rarefaction wave; [14–15] for the asymptotic stability of contact discontinuity wave; [13] for the asymptotic stability of combination of viscous contact wave with rarefaction waves.

This paper is mainly concerned with the large time behavior of solutions to Cauchy problem (1.5)–(1.7) for the compressible non-isentropic Navier-Stokes/Allen-Cahn system in $\mathbb{R}$. Motivated by the relationship between Navier-Stokes/Allen-Cahn system and Navier-Stokes system, we temporarily assume that $\chi_\pm = 1$, and we can consider the stability of the rarefaction wave to the Riemann problem on Euler system in the setting of $\chi(x,t) = 1$. Moreover, the case for $\chi_+ \neq \chi_-$ which leads to more complex structure is left for study in future. Compared with the classical Navier-Stokes system, the concentration χ in this model (1.5) brings both benefit and trouble. The benefit lies in the fact that the term $\delta\partial_x^2\chi$ in $(1.5)_4$ is a viscous dissipation term which provides extra regularity of $\partial_x\chi$, while the trouble is brought by the term $\frac{\delta}{2}\partial_x(\theta\partial_x\chi)^2$ in $(1.5)_2$ which increases the nonlinearity of the system, and it is also this term that requires $\|\partial_x m\|_{L^\infty}$ to be small when we deal with the high order nonlinear term J_{19} in Lemma 2.3, which just demands the initial perturbation to be small and also requires the smallness of $\|\zeta\|_{H^2}$ to close the a priori assumption in (2.3).

The rest of the paper is arranged as follows. After formulating the non-isentropic Navier-Stokes/Allen-Cahn system in $\mathbb{R}$, we deal with the rarefaction wave problem. In the main part Section 2, we give the a priori estimates on the solutions of the perturbation equations. The strength of the rarefaction wave δ_r is not necessarily small in the proof. The proof of Theorem 1.1 is concluded in Section 3.

Notations: Throughout this paper, C denotes some positive constant (generally large) and c denotes some positive constant (generally small), where both C and c may take different values in different places. $L^p = L^p(\mathbb{R})$ $(1 \le p \le +\infty)$ denotes the usual Lebesgue space on $\mathbb{R}$ with its norm $\|\cdot\|_{L^p}$, and when $p = 2, +\infty$, we write $\|\cdot\|_{L^2(\mathbb{R})} = \|\cdot\|$ and $\|\cdot\|_{L^\infty(\mathbb{R})} = \|\cdot\|_\infty$. We use $H^s = H^s(\mathbb{R})$ $(s \ge 0)$ to denote the usual Sobolev space with respect to x variable.

2 The a Priori Estimates

To prove Theorem 1.1, we use the energy method. Define the perturbation as

$$[\varphi, \psi, \zeta, m] = [\rho - \rho^r, u - u^r, \theta - \theta^r, \chi - 1],$$

then $[\varphi, \psi, \zeta, m]$ satisfies

$$\begin{cases} \partial_t \varphi + u\partial_x \varphi + \rho \partial_x \psi + \partial_x u^r \varphi + \partial_x \rho^r \psi = 0, \\ \rho(\partial_t \psi + u\partial_x \psi) + \partial_x [p(\rho, \theta) - p(\rho^r, \theta^r)] \\ = \mu \partial_x^2 \psi + \mu \partial_x^2 u^r - \rho \partial_x u^r \psi + \dfrac{\varphi}{\rho^r} \partial_x p(\rho^r, \theta^r) - \dfrac{\delta}{2} \partial_x (\theta (\partial_x m)^2), \\ \dfrac{R}{\gamma - 1} \rho(\partial_t \zeta + u\partial_x \zeta) + p(\rho, \theta) \partial_x \psi \\ = \beta \partial_x^2 \zeta + \beta \partial_x^2 \theta^r + \mu(\partial_x u)^2 - R\rho\zeta \partial_x u^r - \dfrac{R}{\gamma - 1} \rho \psi \partial_x \theta^r - \dfrac{\delta}{2} \theta \partial_x u (\partial_x m)^2, \\ \rho(\partial_t m + u\partial_x m) = \dfrac{\delta}{a} \theta \partial_x^2 m - \dfrac{\rho\theta}{a\delta}(m^3 + 3m^2 + 2m) \end{cases} \tag{2.1}$$

with initial data

$$\begin{aligned} [\varphi, \psi, \zeta, m](0, x) &= [\varphi_0, \psi_0, \zeta_0, m_0](x) \\ &= [\rho_0(x) - \rho^r(0, x), u_0(x) - u^r(0, x), \theta_0(x) - \theta^r(0, x), \chi_0(x) - 1] \end{aligned} \tag{2.2}$$

for $x \in \mathbb{R}$.

We define the solution space $X(0, T)$ by

$$\begin{aligned} X(0, T) := \{&[\varphi, \psi, \zeta] \in C([0, T]; H^1), m \in C([0, T]; H^2), \partial_x \varphi \in L^2([0, T]; L^2), \\ &[\partial_x \psi, \partial_x \zeta] \in L^2([0, T]; H^1), \partial_x m \in L^2([0, T]; H^2), \forall (x, t) \in (-\infty, \infty) \times [0, T]\}. \end{aligned}$$

The local existence of solutions to the reformulated Cauchy problem (2.1)–(2.2) can be established by the standard iteration argument. In the paper, to prove Theorem 1.1, for brevity we only devote ourselves to obtaining the global-in-time a priori estimates in the following.

Proposition 2.1 *Assume that all the conditions in Theorem* 1.1 *hold true. Let* $[\varphi, \psi, \zeta, m]$ *be a smooth solution to Cauchy problem* (2.1)–(2.2) *on* $0 \le t \le T$ *for* $T > 0$. *There are constants* $\varepsilon_0 > 0$, $C > 0$ *such that if*

$$\sup_{0 \le t \le T} (\|[\varphi, \psi, \zeta](t)\|_{H^1} + \|m(t)\|_{H^2}) + \epsilon \le \varepsilon_0, \tag{2.3}$$

where $\epsilon > 0$ *is the parameter appearing in* (1.19), *then one has*

$$\sup_{0 \le t \le T} (\|[\varphi, \psi, \zeta](t)\|_{H^1}^2 + \|m(t)\|_{H^2}^2) + \int_0^T (\|\partial_x \varphi\|^2 + \|[\partial_x \psi, \partial_x \zeta]\|_{H^1}^2 + \|\partial_x m\|_{H^2}^2) \mathrm{d}t$$

$$
+\int_0^T \|\sqrt{\partial_x u^r}[\varphi,\psi,\zeta]\|^2 \mathrm{d}t
$$
$$
\le C\|[\varphi_0,\psi_0,\zeta_0]\|_{H^1}^2 + C\|m_0\|_{H^2}^2 + C\epsilon^{\frac{1}{5}}. \tag{2.4}
$$

It is easy to get

$$
\|[\varphi,\psi,\zeta,m]\|_{L^\infty} \le \sqrt{2}\varepsilon_0, \quad \|\partial_x m\|_{L^\infty} \le \sqrt{2}\varepsilon_0, \tag{2.5}
$$

where the following Sobolev inequality

$$
|h(x)| \le \sqrt{2}\|h\|^{\frac{1}{2}}\|\partial_x h\|^{\frac{1}{2}} \quad \text{for } h(x)\in H^1(\mathbb{R}) \tag{2.6}
$$

is used.

Proposition 2.1 is a consequence of a series of lemmas as follows. The first one is concerned with the zero-order energy estimate.

Lemma 2.1 *Assume the conditions in Proposition* 2.1 *hold, then we have the following energy estimate for* $t\in[0,T]$,

$$
\|[\varphi,\psi,\zeta,m]\|^2 + \int_0^t (\|\partial_x\psi\|^2 + \|\partial_x\zeta\|^2 + \|\partial_x m\|^2 + \|m\|^2)\mathrm{d}\tau + \int_0^t \|\sqrt{\partial_x u^r}[\varphi,\psi,\zeta]\|^2 \mathrm{d}\tau
$$
$$
\le C\|[\varphi_0,\psi_0,\zeta_0,m_0]\|^2 + C\varepsilon_0\int_0^t \|\partial_x^2 m\|^2\mathrm{d}\tau + C\epsilon^{\frac{1}{5}}. \tag{2.7}
$$

Proof Set $\Phi(s)=s-1-\ln s$, and define $G=\frac{\rho}{2}\psi^2 + R\rho\theta^r\Phi\big(\frac{\rho^r}{\rho}\big) + \frac{R}{\gamma-1}\rho\theta^r\Phi\big(\frac{\theta}{\theta^r}\big) + \frac{\rho}{2}m^2$. Direct calculations give rise to

$$
\partial_t G + \mu(\partial_x\psi)^2 + \frac{\beta}{\theta}(\partial_x\zeta)^2 + \frac{\delta}{a}\theta(\partial_x m)^2 + 2\frac{\rho\theta}{a\delta}m^2 + \partial_x H + Q_1 = Q_2 + Q_3 + Q_4, \tag{2.8}
$$

where

$$
\begin{aligned}
&H = uG + [p(\rho,\theta)-p(\rho^r,\theta^r)]\psi - \mu\psi\partial_x\psi - \frac{\beta}{\theta}\zeta\partial_x\zeta + \frac{\rho u}{2}m^2 - \frac{\delta}{a}\theta m\partial_x m,\\
&Q_1 = \partial_x u^r\Big[\rho\psi^2 + R(\gamma-1)\rho\theta^r\Phi\Big(\frac{\rho^r}{\rho}\Big) + R\rho\theta^r\Phi\Big(\frac{\theta}{\theta^r}\Big)\Big] - R\varphi\psi\partial_x\theta^r + \frac{R}{(\gamma-1)\theta^r}\rho\psi\zeta\partial_x\theta^r,\\
&Q_2 = R\rho\psi\partial_x\theta^r\Phi\Big(\frac{\rho^r}{\rho}\Big) + \frac{R}{\gamma-1}\rho\psi\partial_x\theta^r\Phi\Big(\frac{\theta}{\theta^r}\Big),\\
&Q_3 = \mu\psi\partial_x^2 u^r + \beta\frac{\zeta}{\theta^2}\partial_x\zeta\partial_x\theta + \beta\frac{\zeta}{\theta}\partial_x^2\theta^r + \mu\frac{\zeta}{\theta}(\partial_x u)^2,\\
&Q_4 = -\frac{\delta}{2}\psi\partial_x(\theta(\partial_x m)^2) - \frac{\delta}{2}\zeta\partial_x u(\partial_x m)^2 - \frac{\delta}{a}m\partial_x\theta\partial_x m - \frac{\rho\theta}{a\delta}m(m^3+3m^2).
\end{aligned} \tag{2.9}
$$

Integrating the resulting equation with respect to x over $\mathbb{R}$, we arrive at

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} G\mathrm{d}x + \mu\int_{\mathbb{R}}(\partial_x\psi)^2\mathrm{d}x + \beta\int_{\mathbb{R}}\frac{1}{\theta}(\partial_x\zeta)^2\mathrm{d}x + \frac{\delta}{a}\int_{\mathbb{R}}\theta(\partial_x m)^2\mathrm{d}x + \frac{2}{a\delta}\int_{\mathbb{R}}\rho\theta m^2\mathrm{d}x + \int_{\mathbb{R}} Q_1\mathrm{d}x
$$
$$
= \int_{\mathbb{R}} Q_2\mathrm{d}x + \int_{\mathbb{R}} Q_3\mathrm{d}x + \int_{\mathbb{R}} Q_4\mathrm{d}x. \tag{2.10}
$$

In what follows, we first treat Q_1. The estimates for higher order terms Q_2, Q_3 and Q_4 are to be given later on. Recall $\partial_x u^r(t,x) > 0$ for all $t \geq 0$ and $x \in \mathbb{R}$. One can claim that

$$Q_1 = Q_5 + Q_6 \tag{2.11}$$

with

$$\begin{aligned} Q_5 &= \partial_x u^r \Big\{ \rho\psi^2 + \frac{B(\gamma-1)}{\rho}(\rho^r)^{\gamma-1}\varphi^2 + \frac{R^2}{B}\rho(\rho^r)^{1-\gamma}\zeta^2 + B(\gamma-1)\sqrt{\frac{(\rho^r)^{\gamma-1}}{B\gamma}}\varphi\psi \\ &\quad - R\rho\sqrt{\frac{1}{B\gamma(\rho^r)^{\gamma-1}}}\psi\zeta \Big\}, \\ Q_6 &= O(1)\partial_x u^r \varphi^3 + O(1)\partial_x u^r \zeta^3. \end{aligned} \tag{2.12}$$

To prove (2.11), we observe from $\Phi(1) = \Phi'(1) = 0$ and $\Phi''(1) = 1$ that

$$\begin{aligned} \Phi\Big(\frac{\rho^r}{\rho}\Big) &= \frac{\varphi^2}{\rho^2} + O(1)\varphi^3, \\ \Phi\Big(\frac{\theta}{\theta^r}\Big) &= \frac{\zeta^2}{(\theta^r)^2} + O(1)\zeta^3, \end{aligned} \tag{2.13}$$

as $|\varphi|$ and $|\zeta|$ can be sufficiently small. On the other hand, we get from (1.20) that

$$\partial_x\theta^r = -\frac{B(\gamma-1)}{R}\sqrt{\frac{(\rho^r)^{\gamma-1}}{B\gamma}}\partial_x u^r. \tag{2.14}$$

Then (2.11) follows from the third equation of (1.20), (2.13) and (2.14).

Next, we prove that there exists a constant $\alpha > 0$ such that

$$\int_{\mathbb{R}} Q_5 \mathrm{d}x \geq \alpha \|\sqrt{\partial_x u^r}[\varphi,\psi,\zeta]\|^2. \tag{2.15}$$

Elementary computations give

$$Q_5 = \partial_x u^r (\varphi,\psi,\zeta) M (\varphi,\psi,\zeta)^T, \tag{2.16}$$

where $()^T$ denotes the transpose of a row vector, and the 3×3 real symmetric matrix M is given by

$$\begin{pmatrix} \frac{B(\gamma-1)}{\rho}(\rho^r)^{\gamma-1} & \frac{B(\gamma-1)}{2}\sqrt{\frac{(\rho^r)^{\gamma-1}}{B\gamma}} & 0 \\ \frac{B(\gamma-1)}{2}\sqrt{\frac{(\rho^r)^{\gamma-1}}{B\gamma}} & \rho & -\frac{R\rho}{2}\sqrt{\frac{1}{B\gamma(\rho^r)^{\gamma-1}}} \\ 0 & -\frac{R\rho}{2}\sqrt{\frac{1}{B\gamma(\rho^r)^{\gamma-1}}} & \frac{R^2\rho}{B}(\rho^r)^{1-\gamma} \end{pmatrix}.$$

One can compute all the leading principal minors $\triangle_{ll}$ $(1 \leq l \leq 3)$ of M as follows:

$$\triangle_{11} = \frac{B(\gamma-1)}{\rho}(\rho^r)^{\gamma-1} > 0, \tag{2.17}$$

$$
\begin{aligned}
\triangle_{22} &= B(\gamma-1)(\rho^r)^{\gamma-1} - \frac{B(\gamma-1)^2}{4\gamma}(\rho^r)^{\gamma-1} \\
&= \frac{B(\gamma-1)(3\gamma+1)}{4\gamma}(\rho^r)^{\gamma-1} > 0, \qquad (2.18)\\
\triangle_{33} &= \frac{B(\gamma-1)}{\rho}(\rho^r)^{\gamma-1}\Big[\frac{R^2}{B}\rho^2(\rho^r)^{1-\gamma} - \frac{R^2\rho^2}{4B\gamma}(\rho^r)^{1-\gamma}\Big] - \Big[\frac{B(\gamma-1)}{2}\sqrt{\frac{(\rho^r)^{\gamma-1}}{B\gamma}}\Big]^2\frac{R^2\rho}{B}(\rho^r)^{1-\gamma} \\
&= \frac{3}{4}R^2(\gamma-1)\rho > 0. \qquad (2.19)
\end{aligned}
$$

We then conclude that M is positive definite, and (2.15) follows.

We now have by substituting (2.15) into (2.10) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} G\mathrm{d}x + \mu\int_{\mathbb{R}}(\partial_x\psi)^2\mathrm{d}x + \beta\int_{\mathbb{R}}\frac{1}{\theta}(\partial_x\zeta)^2\mathrm{d}x + \frac{\delta}{a}\int_{\mathbb{R}}\theta(\partial_x m)^2\mathrm{d}x + \frac{2}{a\delta}\int_{\mathbb{R}}\rho\theta m^2\mathrm{d}x \\
&+ \alpha\int_{\mathbb{R}}|\sqrt{\partial_x u^r}[\varphi,\psi,\zeta]|^2\mathrm{d}x \le \Big|\int_{\mathbb{R}} Q_2\mathrm{d}x\Big| + \Big|\int_{\mathbb{R}} Q_3\mathrm{d}x\Big| + \Big|\int_{\mathbb{R}} Q_4\mathrm{d}x\Big| + \Big|\int_{\mathbb{R}} Q_6\mathrm{d}x\Big|. \qquad (2.20)
\end{aligned}
$$

To complete the proof of Lemma 2.1, it remains to estimate all the terms on the right-hand side of (2.20). By applying Hölder's inequality, Cauchy-Schwarz inequality with $0<\eta<1$, Sobolev inequality (2.6), Lemma 1.2, (2.3), (2.5) and integrating by parts, it is direct to derive the following estimates:

$$
\int_{\mathbb{R}} |Q_2| + |Q_6|\mathrm{d}x \le C\|[\varphi,\psi,\zeta]\|_{L^\infty}\|\sqrt{\partial_x u^r}[\varphi,\psi,\zeta]\|^2 \le C\varepsilon_0\|\sqrt{\partial_x u^r}[\varphi,\psi,\zeta]\|^2, \qquad (2.21)
$$

$$
\begin{aligned}
\int_{\mathbb{R}} |Q_3|\mathrm{d}x &\le C\int_{\mathbb{R}} |\psi\partial_x^2 u^r| + |\zeta\partial_x\zeta\partial_x\theta| + |\zeta\partial_x^2\theta^r| + |\zeta(\partial_x u^r)^2| + |\zeta(\partial_x\psi)^2|\mathrm{d}x \\
&\le C\|[\psi,\zeta]\|_{L^\infty}\|\partial_x^2[u^r,\theta^r]\|_{L^1} + C\|\zeta\|_{L^\infty}\|\partial_x[\psi,\zeta]\|^2 + C\|\partial_x\theta^r\|_{L^\infty}\|\zeta\|\|\partial_x\zeta\| \\
&\quad + C\|\zeta\|_{L^\infty}\|\partial_x u^r\|^2 \\
&\le C\|[\psi,\zeta]\|^{\frac12}\|\partial_x[\psi,\zeta]\|^{\frac12}\epsilon^{\frac15}(1+t)^{-\frac45} + C\varepsilon_0(\|\partial_x\psi\|^2 + \|\partial_x\zeta\|^2) \\
&\quad + C\epsilon^{\frac14}(1+t)^{-\frac34}\|\zeta\|\|\partial_x\zeta\| + C\|\zeta\|^{\frac12}\|\partial_x\zeta\|^{\frac12}(1+t)^{-1} \\
&\le C(\varepsilon_0 + \epsilon^{\frac15} + \eta)(\|\partial_x\psi\|^2 + \|\partial_x\zeta\|^2) + \epsilon^{\frac15}(1+t)^{-\frac{16}{15}} + C\epsilon^{\frac12}(1+t)^{-\frac32}, \qquad (2.22)
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{R}} |Q_4|\mathrm{d}x &\le C\int_{\mathbb{R}} |\psi\partial_x\theta^r(\partial_x m)^2| + |\psi\partial_x\zeta(\partial_x m)^2| + |\psi\partial_x m\partial_x^2 m| \\
&\quad + |\partial_x u^r(\partial_x m)^2| + |\partial_x\psi(\partial_x m)^2| + |m\partial_x\theta^r\partial_x m| + |m\partial_x\zeta\partial_x m| + |m^3|\mathrm{d}x \\
&\le C\|\psi\|_{L^\infty}\|\partial_x\theta^r\|_{L^\infty}\|\partial_x m\|^2 + C\|\psi\|_{L^\infty}\|\partial_x m\|_{L^\infty}\|\partial_x m\|\|\partial_x\zeta\| \\
&\quad + C\|\psi\|_{L^\infty}\|\partial_x m\|\|\partial_x^2 m\| + C\|\partial_x u^r\|_{L^\infty}\|\partial_x m\|^2 \\
&\quad + C\|\partial_x m\|_{L^\infty}\|\partial_x\psi\|\|\partial_x m\| + C\|\partial_x\theta^r\|_{L^\infty}\|m\|\|\partial_x m\| \\
&\quad + C\|m\|_{L^\infty}\|\partial_x m\|\|\partial_x\zeta\| + C\|m\|_{L^\infty}\|m\|^2 \\
&\le C(\varepsilon_0 + \epsilon)(\|m\|^2 + \|\partial_x m\|^2) + C\varepsilon_0(\|\partial_x^2 m\|^2 + \|\partial_x\zeta\|^2). \qquad (2.23)
\end{aligned}
$$

Therefore, (2.7) follows by plugging (2.21)–(2.23) into (2.20), integrating the resulting inequality with respect to t and applying (2.13), where we recall that $\epsilon>0$, $1>\eta>0$ and $\varepsilon_0>0$ can be small enough. This completes the proof of lemma 2.1.

In the following lemma, we control the term $\|\partial_x\varphi(t)\|^2$.

Lemma 2.2 *Assume the conditions in Proposition* 2.1 *hold, then we have the following energy estimate for* $t \in [0,T]$,

$$\begin{aligned}&\|\partial_x\varphi\|^2+\int_0^t(\|\sqrt{\partial_x u^r}\partial_x\varphi\|^2+\|\partial_x\varphi\|^2)\mathrm{d}\tau\\&\le C(\|[\psi_0,\zeta_0,m_0]\|^2+\|\varphi_0\|_{H^1}^2)+C\varepsilon_0\int_0^t\|\partial_x^2 m\|^2\mathrm{d}\tau+C\epsilon^{\frac{1}{5}}.\end{aligned}\tag{2.24}$$

Proof We first differentiate $(2.1)_1$ with respect to x to obtain

$$\begin{aligned}&\partial_t\partial_x\varphi+\partial_x u\partial_x\varphi+u\partial_x^2\varphi+\partial_x\rho\partial_x\psi+\rho\partial_x^2\psi+\partial_x^2u^r\varphi+\partial_x u^r\partial_x\varphi\\&+\partial_x\rho^r\partial_x\psi+\partial_x^2\rho^r\psi=0.\end{aligned}\tag{2.25}$$

Then multiplying $(2.1)_2$ and (2.25) by $\frac{\partial_x\varphi}{\rho}$ and $\mu\frac{\partial_x\varphi}{\rho^2}$, respectively, and integrating the resulting equalities over $\mathbb{R}$, one has

$$\begin{aligned}&\int_{\mathbb{R}}\partial_t\psi\partial_x\varphi\mathrm{d}x+\int_{\mathbb{R}}u\partial_x\psi\partial_x\varphi\mathrm{d}x+\int_{\mathbb{R}}\frac{\partial_x\varphi}{\rho}\partial_x[p(\rho,\theta)-p(\rho^r,\theta^r)]\mathrm{d}x\\&=\int_{\mathbb{R}}\mu\partial_x^2\psi\frac{\partial_x\varphi}{\rho}\mathrm{d}x+\int_{\mathbb{R}}\mu\partial_x^2u^r\frac{\partial_x\varphi}{\rho}\mathrm{d}x-\int_{\mathbb{R}}\partial_x u^r\psi\partial_x\varphi\mathrm{d}x\\&\quad+\int_{\mathbb{R}}\frac{\varphi}{\rho\rho^r}\partial_x\varphi\partial_x p(\rho^r,\theta^r)\mathrm{d}x-\int_{\mathbb{R}}\frac{\delta}{2\rho}\partial_x\varphi\partial_x(\theta(\partial_x m)^2)\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}&\mu\int_{\mathbb{R}}\frac{\partial_x\varphi}{\rho^2}\partial_t\partial_x\varphi\mathrm{d}x+\mu\int_{\mathbb{R}}\partial_x u\frac{(\partial_x\varphi)^2}{\rho^2}\mathrm{d}x+\mu\int_{\mathbb{R}}u\frac{\partial_x\varphi\partial_x^2\varphi}{\rho^2}\mathrm{d}x+\mu\int_{\mathbb{R}}\frac{\partial_x\varphi}{\rho^2}\partial_x\rho\partial_x\psi\mathrm{d}x\\&+\mu\int_{\mathbb{R}}\partial_x u^r\frac{(\partial_x\varphi)^2}{\rho^2}\mathrm{d}x\\&=-\mu\int_{\mathbb{R}}\partial_x^2\psi\frac{\partial_x\varphi}{\rho}\mathrm{d}x-\mu\int_{\mathbb{R}}\partial_x^2u^r\varphi\frac{\partial_x\varphi}{\rho^2}\mathrm{d}x-\mu\int_{\mathbb{R}}\partial_x\rho^r\partial_x\psi\frac{\partial_x\varphi}{\rho^2}\mathrm{d}x-\mu\int_{\mathbb{R}}\partial_x^2\rho^r\psi\frac{\partial_x\varphi}{\rho^2}\mathrm{d}x.\end{aligned}$$

The summation of the equalities above further implies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\Big(\psi\partial_x\varphi+\frac{\mu}{2\rho^2}(\partial_x\varphi)^2\Big)\mathrm{d}x+\int_{\mathbb{R}}\Big[\mu\partial_x u^r\frac{(\partial_x\varphi)^2}{\rho^2}+\frac{R\theta}{\rho}(\partial_x\varphi)^2\Big]\mathrm{d}x\\&=\int_{\mathbb{R}}\psi\partial_t\partial_x\varphi\mathrm{d}x-\int_{\mathbb{R}}\mu\partial_x\psi\partial_x\varphi\mathrm{d}x-\int_{\mathbb{R}}\frac{R}{\rho}\zeta\partial_x\varphi\partial_x\rho^r\mathrm{d}x+\int_{\mathbb{R}}\frac{R\theta^r}{\rho\rho^r}\varphi\partial_x\varphi\partial_x\rho^r\mathrm{d}x\\&\quad-\int_{\mathbb{R}}R\partial_x\varphi\partial_x\zeta\mathrm{d}x+\int_{\mathbb{R}}\mu\frac{\partial_x\varphi}{\rho}\partial_x^2u^r\mathrm{d}x-\int_{\mathbb{R}}\psi\partial_x\varphi\partial_x u^r\mathrm{d}x+\int_{\mathbb{R}}\frac{\mu}{2\rho^2}\partial_x u(\partial_x\varphi)^2\mathrm{d}x\\&\quad-\int_{\mathbb{R}}\frac{\mu}{\rho^2}\partial_x\varphi\partial_x\rho\partial_x\psi\mathrm{d}x-\int_{\mathbb{R}}\frac{\mu}{\rho^2}\varphi\partial_x\varphi\partial_x^2u^r\mathrm{d}x-\int_{\mathbb{R}}\frac{\mu}{\rho^2}\psi\partial_x\varphi\partial_x^2\rho^r\mathrm{d}x\\&\quad-\int_{\mathbb{R}}\frac{\mu}{\rho^2}\partial_x\varphi\partial_x\rho^r\partial_x\psi\mathrm{d}x-\int_{\mathbb{R}}\frac{\delta}{2\rho}\partial_x\varphi\partial_x(\theta(\partial_x m)^2)\mathrm{d}x\\&=\sum_{l=1}^{13}J_l,\end{aligned}\tag{2.26}$$

where J_l $(1\le l\le 13)$ denote the corresponding terms on the left-hand side of (2.26).

We now turn to estimate J_l $(1 \le l \le 13)$ term by term. By applying Hölder's inequality, Cauchy-Schwarz inequality with $0 < \eta < 1$, $(2.1)_1$, (2.3), (2.5), (1.20), Lemma 1.2, Sobolev inequality (2.6) and integrating by parts, it is direct to derive the following estimates:

$$
\begin{aligned}
J_1 &= -\int_{\mathbb{R}} \partial_x\psi\partial_t\varphi \mathrm{d}x \\
&= \int_{\mathbb{R}} \partial_x\psi(u\partial_x\varphi + \rho\partial_x\psi + \partial_x u^r\varphi + \partial_x\rho^r\psi)\mathrm{d}x \\
&\le \|\psi\|_{L^\infty}\|\partial_x\psi\|\|\partial_x\varphi\| + C\|\partial_x\psi\|^2 + \|\partial_x u^r\|_{L^\infty}\|\varphi\|\|\partial_x\psi\| \\
&\quad + \|\partial_x\rho^r\|_{L^\infty}\|\psi\|\|\partial_x\psi\| + \|u^r\|_{L^\infty}\|\partial_x\psi\|\|\partial_x\varphi\| \\
&\le C\varepsilon_0(\|\partial_x\psi\|^2 + \|\partial_x\varphi\|^2) + \eta\|\partial_x\varphi\|^2 + (C + C_\eta)\|\partial_x\psi\|^2 \\
&\quad + C\epsilon^{\frac{1}{4}}(1+t)^{-\frac{3}{4}}\|[\varphi,\psi]\|\|\partial_x\psi\| \\
&\le (C\varepsilon_0 + \eta)\|\partial_x[\varphi,\psi]\|^2 + (C + C_\eta)\|\partial_x\psi\|^2 + C\epsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}, \\
J_2 + J_5 &\le C\int_{\mathbb{R}} |\partial_x\psi\partial_x\varphi| + |\partial_x\varphi\partial_x\zeta|\mathrm{d}x \\
&\le \eta\|\partial_x\varphi\|^2 + C_\eta(\|\partial_x\psi\|^2 + \|\partial_x\zeta\|^2), \\
J_3 + J_4 + J_7 &\le \int_{\mathbb{R}} |\zeta\partial_x\varphi\partial_x\rho^r| + |\varphi\partial_x\varphi\partial_x\rho^r| + |\psi\partial_x\varphi\partial_x u^r|\mathrm{d}x \\
&\le C\|\partial_x[\rho^r, u^r]\|_{L^\infty}\|\partial_x\varphi\|\|[\varphi,\psi,\zeta]\| \\
&\le C\epsilon^{\frac{1}{4}}(1+t)^{-\frac{3}{4}}\|\partial_x\varphi\|\|[\varphi,\psi,\zeta]\| \\
&\le \eta\|\partial_x\varphi\|^2 + C\epsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}, \\
J_6 + J_{10} + J_{11} &\le \int_{\mathbb{R}} |\partial_x\varphi\partial_x^2 u^r| + |\varphi\partial_x\varphi\partial_x^2 u^r| + |\psi\partial_x\varphi\partial_x^2\rho^r|\mathrm{d}x \\
&\le C\eta\|\partial_x\varphi\|^2 + C_\eta\|\partial_x^2 u^r\|^2 + C\|[\varphi,\psi]\|_{L^\infty}\|\partial_x\varphi\|\|\partial_{xx}[u^r,\rho^r]\| \\
&\le C(\eta + \varepsilon_0)\|\partial_x\varphi\|^2 + C\epsilon(1+t)^{-2}, \\
J_8 + J_9 + J_{12} &\le \int_{\mathbb{R}} |\partial_x\psi(\partial_x\varphi)^2| + |\partial_x u^r(\partial_x\varphi)^2| + |\partial_x\rho^r\partial_x\varphi\partial_x\psi|\mathrm{d}x \\
&\le C\|\partial_x\psi\|_{L^\infty}\|\partial_x\varphi\|^2 + C\|\partial_x u^r\|_{L^\infty}\|\partial_x\varphi\|^2 + C\|\partial_x\rho^r\|_{L^\infty}\|\partial_x\varphi\|\|\partial_x\psi\| \\
&\le C(\|\partial_x\psi\| + \|\partial_x^2\psi\|)\|\partial_x\varphi\|^2 + C\epsilon(\|\partial_x\varphi\|^2 + \|\partial_x\psi\|^2) \\
&\le C(\varepsilon_0 + \epsilon)\|\partial_x\varphi\|^2 + C\epsilon\|\partial_x\psi\|^2 + C\varepsilon_0\|\partial_x^2\psi\|^2, \\
J_{13} &\le C\int_{\mathbb{R}} |\partial_x\varphi\partial_x m\partial_x^2 m| + |\partial_x\varphi\partial_x\zeta(\partial_x m)^2| + |\partial_x\varphi\partial_x\theta^r(\partial_x m)^2|\mathrm{d}x \\
&\le C\|\partial_x m\|_{L^\infty}\|\partial_x\varphi\|\|\partial_x^2 m\| + C\|\partial_x m\|_{L^\infty}^2\|\partial_x\varphi\|\|\partial_x\zeta\| \\
&\quad + C\|\partial_x\theta^r\|_{L^\infty}\|\partial_x m\|_{L^\infty}\|\partial_x\varphi\|\|\partial_x m\| \\
&\le C\varepsilon_0(\|\partial_x\varphi\|^2 + \|\partial_x\zeta\|^2 + \|\partial_x m\|^2 + \|\partial_x^2 m\|^2).
\end{aligned}
$$

Inserting the above estimations for J_l $(1 \le l \le 13)$ into (2.26) and then choosing ε_0, ϵ and η so small, and integrating (2.26) over $[0,T]$ and using (2.7), Cauchy-Schwarz inequality with $0 < \eta < 1$, we get (2.24). This completes the proof of Lemma 2.2.

Lemma 2.3 *Assume the conditions in Proposition* 2.1 *hold, then we have the following* *energy estimate for* $t \in [0,T]$,

$$\|\partial_x\psi\|^2 + \int_0^t \|\partial_x^2\psi\|^2 \mathrm{d}\tau \le C\|[\zeta_0, m_0]\|^2 + C\|[\varphi_0,\psi_0]\|_{H^1}^2 + C\varepsilon_0 \int_0^t \|\partial_x^2 m\|^2 \mathrm{d}\tau + C\epsilon^{\frac{1}{5}}. \quad (2.27)$$

Proof Multiplying $(1.5)_2$ by $-\frac{\partial_x^2\psi}{\rho}$ and then integrating the resulting equation over $\mathbb{R}$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (\partial_x\psi)^2 \mathrm{d}x + \int_{\mathbb{R}} \frac{\mu}{\rho}(\partial_x^2\psi)^2 \mathrm{d}x \\
&= \int_{\mathbb{R}} u\partial_x\psi\partial_x^2\psi \mathrm{d}x + \int_{\mathbb{R}} \frac{\partial_x^2\psi}{\rho}\partial_x[p(\rho,\theta) - p(\rho^r,\theta^r)]\mathrm{d}x - \int_{\mathbb{R}} \mu\frac{\partial_x^2\psi}{\rho}\partial_x^2 u^r \mathrm{d}x + \int_{\mathbb{R}} \psi\partial_x^2\psi\partial_x u^r \mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \frac{\varphi}{\rho\rho^r}\partial_x^2\psi\partial_x p(\rho^r,\theta^r)\mathrm{d}x + \int_{\mathbb{R}} \frac{\delta}{2\rho}\partial_x^2\psi\partial_x(\theta(\partial_x m)^2)\mathrm{d}x \\
&= \sum_{l=14}^{19} J_l, \qquad (2.28)
\end{aligned}$$

where J_l $(14 \le l \le 19)$ denote the corresponding terms on the left-hand side of (2.28).

We now turn to estimate J_l $(14 \le l \le 19)$ term by term. By applying Hölder's inequality, Cauchy-Schwarz inequality with $0 < \eta < 1$, Sobolev inequality (2.6), Lemma 1.2, (2.3), (2.5) and integrating by parts, it is direct to derive the following estimates:

$$\begin{aligned}
&J_{14} + J_{16} + J_{17} + J_{18} \\
&\le C\int_{\mathbb{R}} |u\partial_x\psi\partial_x^2\psi| + |\partial_x^2\psi\partial_x^2 u^r| + |\psi\partial_x u^r\partial_x^2\psi| + |\varphi\partial_x\rho^r\partial_x^2\psi| + |\varphi\partial_x\theta^r\partial_x^2\psi|\mathrm{d}x \\
&\le \eta\|\partial_x^2\psi\|^2 + C_\eta(\|\partial_x\psi\|^2 + \|\partial_x^2 u^r\|^2) + C\|\partial_x[u^r,\rho^r]\|_{L^\infty}\|[\varphi,\psi]\|\|\partial_x^2\psi\| \\
&\le \eta\|\partial_x^2\psi\|^2 + C_\eta\|\partial_x\psi\|^2 + C\epsilon(1+t)^{-2} + C\epsilon^{\frac{1}{4}}(1+t)^{-\frac{3}{4}}\|[\varphi,\psi]\|\|\partial_x^2\psi\| \\
&\le \eta\|\partial_x^2\psi\|^2 + C_\eta\|\partial_x\psi\|^2 + C\epsilon(1+t)^{-2} + C\epsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}, \\
J_{15} &\le C\int_{\mathbb{R}} |\partial_x^2\psi\partial_x\rho\zeta| + |\partial_x^2\psi\partial_x\zeta| + |\partial_x^2\psi\partial_x\varphi| + |\partial_x^2\psi\varphi\partial_x\theta^r|\mathrm{d}x \\
&\le C\|\zeta\|_{L^\infty}\|\partial_x^2\psi\|\|\partial_x\varphi\| + C\|\partial_x[\rho^r,\theta^r]\|_{L^\infty}\|[\varphi,\zeta]\|\|\partial_x^2\psi\| + \eta\|\partial_x^2\psi\|^2 \\
&\quad + C_\eta(\|\partial_x\zeta\|^2 + \|\partial_x\varphi\|^2) \\
&\le C\varepsilon_0(\|\partial_x\varphi\|^2 + \|\partial_x^2\psi\|^2) + C\epsilon^{\frac{1}{4}}(1+t)^{-\frac{3}{4}}\|[\varphi,\zeta]\|\|\partial_x^2\psi\| \\
&\quad + \eta\|\partial_x^2\psi\|^2 + C_\eta(\|\partial_x\zeta\|^2 + \|\partial_x\varphi\|^2) \\
&\le (C\varepsilon_0 + C_\eta)\|\partial_x\varphi\|^2 + (C\varepsilon_0 + \eta)\|\partial_x^2\psi\|^2 + C_\eta\|\partial_x\zeta\|^2 + C\epsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}, \\
J_{19} &\le C\int_{\mathbb{R}} |\partial_x^2\psi\partial_x\theta(\partial_x m)^2| + |\partial_x^2\psi\partial_x m\partial_x^2 m|\mathrm{d}x \\
&\le C\|\partial_x m\|_{L^\infty}^2\|\partial_x\zeta\|\|\partial_x^2\psi\| + C\|\partial_x\theta^r\|_{L^\infty}\|\partial_x m\|_{L^\infty}\|\partial_x^2\psi\|\|\partial_x m\| + C\|\partial_x m\|_{L^\infty}\|\partial_x^2 m\|\|\partial_x^2\psi\| \\
&\le C\varepsilon_0(\|\partial_x\zeta\|^2 + \|\partial_x m\|^2 + \|\partial_x^2\psi\|^2 + \|\partial_x^2 m\|^2).
\end{aligned}$$

Inserting the above estimations for J_l $(14 \le l \le 19)$ into (2.28) and then choosing ε_0, ϵ and η so small, and integrating (2.28) over $[0,T]$, using (2.7) and (2.24), we get (2.27).

Lemma 2.4 *Assume the conditions in Proposition* 2.1 *hold, then we have the following energy estimate for* $t \in [0,T]$,

$$\|\partial_x\zeta\|^2 + \int_0^t \|\partial_x^2\zeta\|^2 \mathrm{d}\tau \le C\|[\varphi_0,\psi_0,\zeta_0]\|_{H^1}^2 + C\|m_0\|^2 + C\varepsilon_0 \int_0^t \|\partial_x^2 m\|^2 \mathrm{d}\tau + C\epsilon^{\frac{1}{5}}. \tag{2.29}$$

Proof Multiplying $(1.5)_3$ by $-\frac{\partial_x^2\zeta}{\rho}$ and then integrating the resulting equation over $\mathbb{R}$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{R}{\gamma-1}(\partial_x\zeta)^2\mathrm{d}x + \int_{\mathbb{R}} \frac{\beta}{\rho}(\partial_x^2\zeta)^2\mathrm{d}x \\
&= -\int_{\mathbb{R}} \frac{R}{2(\gamma-1)}\partial_x u(\partial_x\zeta)^2\mathrm{d}x + \int_{\mathbb{R}} R\theta\partial_x^2\zeta\partial_x\psi\mathrm{d}x - \int_{\mathbb{R}} \frac{\beta}{\rho}\partial_x^2\zeta\partial_x^2\theta^r\mathrm{d}x - \int_{\mathbb{R}} \frac{\mu}{\rho}\partial_x^2\zeta(\partial_x u)^2\mathrm{d}x \\
&\quad + \int_{\mathbb{R}} R\zeta\partial_x u^r\partial_x^2\zeta\mathrm{d}x + \int_{\mathbb{R}} \frac{R}{\gamma-1}\psi\partial_x\theta^r\partial_x^2\zeta\mathrm{d}x + \int_{\mathbb{R}} \frac{\delta}{2\rho}\theta\partial_x u(\partial_x m)^2\partial_x^2\zeta\mathrm{d}x \\
&= \sum_{l=20}^{26} J_l,
\end{aligned} \tag{2.30}$$

where J_l $(20 \le l \le 26)$ denote the corresponding terms on the left-hand side of (2.30).

We now turn to estimate J_l $(20 \le l \le 26)$ term by term. By applying Hölder's inequality, Cauchy-Schwarz inequality with $0 < \eta < 1$, Sobolev inequality (2.6), Lemma 1.2, (2.3), (2.5) and integrating by parts, it is direct to derive the following estimates:

$$\begin{aligned}
J_{20} &\le \int_{\mathbb{R}} |\partial_x\psi(\partial_x\zeta)^2| + |\partial_x u^r(\partial_x\zeta)^2|\mathrm{d}x \\
&\le C\|\partial_x\psi\|_{L^\infty}\|\partial_x\zeta\|^2 + C\|\partial_x u^r\|_{L^\infty}\|\partial_x\zeta\|^2 \\
&\le C(\|\partial_x\psi\| + \|\partial_x^2\psi\|)\|\partial_x\zeta\|^2 + C\|\partial_x u^r\|_{L^\infty}\|\partial_x\zeta\|^2 \\
&\le C(\varepsilon_0+\epsilon)\|\partial_x\zeta\|^2 + C\varepsilon_0\|\partial_x^2\psi\|^2, \\
J_{21} &\le \int_{\mathbb{R}} |\theta\partial_x^2\zeta\partial_x\psi|\mathrm{d}x \\
&\le C\|\theta\|_{L^\infty}\|\partial_x^2\zeta\|\|\partial_x\psi\| \\
&\le C\varepsilon_0(\|\partial_x^2\zeta\|^2 + \|\partial_x\psi\|^2), \\
J_{22} &\le C\int_{\mathbb{R}} |\partial_x^2\zeta\partial_x^2\theta^r|\mathrm{d}x \\
&\le \eta\|\partial_x^2\zeta\|^2 + C_\eta\epsilon(1+t)^{-2}, \\
J_{23} &\le C\int_{\mathbb{R}} |\partial_x^2\zeta(\partial_x u)^2|\mathrm{d}x \\
&\le C\|\partial_x\psi\|_{L^\infty}\|\partial_x^2\zeta\|\|\partial_x\psi\| + C\|\partial_x u^r\|_{L^\infty}\|\partial_x^2\zeta\|\|\partial_x u^r\| \\
&\le C(\|\partial_x\psi\| + \|\partial_x^2\psi\|)\|\partial_x^2\zeta\|\|\partial_x\psi\| + C\epsilon^{\frac{1}{2}}(1+t)^{-1}\|\partial_x^2\zeta\| \\
&\le C(\varepsilon_0+\epsilon^{\frac{1}{2}})\|\partial_x^2\zeta\|^2 + C\varepsilon_0(\|\partial_x^2\psi\|^2 + \|\partial_x\psi\|^2) + C\epsilon^{\frac{1}{2}}(1+t)^{-2}, \\
J_{24}+J_{25} &\le C\int_{\mathbb{R}} |\zeta\partial_x u^r\partial_x^2\zeta| + |\psi\partial_x\theta^r\partial_x^2\zeta|\mathrm{d}x \\
&\le C\|\partial_x[u^r,\theta^r]\|_{L^\infty}\|[\zeta,\psi]\|\|\partial_x^2\zeta\| \\
&\le C\epsilon^{\frac{1}{4}}(1+t)^{-\frac{3}{4}}\|\partial_x^2\zeta\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \eta\|\partial_x^2\zeta\|^2+\epsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}},\\
J_{26}&\leq C\int_{\mathbb{R}}|\partial_x u(\partial_x m)^2\partial_x^2\zeta|\mathrm{d}x\\
&\leq C\|\partial_x m\|_{L^\infty}^2\|\partial_x\psi\|\|\partial_x^2\zeta\|+C\|\partial_x u^r\|_{L^\infty}\|\partial_x m\|_{L^\infty}\|\partial_x m\|\|\partial_x^2\zeta\|\\
&\leq C\varepsilon_0(\|\partial_x\psi\|^2+\|\partial_x^2\zeta\|^2+\|\partial_x m\|^2).
\end{aligned}
$$

Inserting the above estimations for J_l $(20\leq l\leq 26)$ into (2.35) and then choosing ε_0, ϵ and η so small, and integrating (2.35) over $[0,T]$, using (2.7), (2.24) and (2.27), we get (2.29).

Lemma 2.5 *Assume the conditions in Proposition* 2.1 *hold, then we have the following energy estimate for* $t\in[0,T]$,

$$
\|\partial_x m\|^2+\int_0^t\|\partial_x^2 m\|^2\mathrm{d}\tau\leq C\|[\varphi_0,\psi_0,\zeta_0,m_0]\|_{H^1}^2+C\epsilon^{\frac{1}{5}}. \tag{2.31}
$$

Proof Multiplying $(1.5)_3$ by $-\frac{\partial_x^2 m}{\rho}$ and then integrating the resulting equation over $\mathbb{R}$, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\partial_x m)^2\mathrm{d}x+\int_{\mathbb{R}}\frac{\delta\theta}{a\rho}(\partial_x^2 m)^2\mathrm{d}x\\
&=-\int_{\mathbb{R}}\frac{\partial_x u}{2}(\partial_x m)^2\mathrm{d}x+\int_{\mathbb{R}}\frac{\theta}{a\delta}(m^3+3m^2+2m)\partial_x^2 m\mathrm{d}x\\
&=\sum_{l=27}^{28}J_l,
\end{aligned}\tag{2.32}
$$

where J_l $(27\leq l\leq 28)$ denote the corresponding terms on the left-hand side of (2.32).

We now turn to estimate J_l $(27\leq l\leq 28)$ term by term. By applying Hölder's inequality, Cauchy-Schwarz inequality with $0<\eta<1$, Sobolev inequality (2.6), Lemma 1.2, (2.3), (2.5) and integrating by parts, it is direct to derive the following estimates:

$$
\begin{aligned}
J_{27}&\leq C\int_{\mathbb{R}}|\partial_x\psi(\partial_x m)^2|+|\partial_x u^r(\partial_x m)^2|\mathrm{d}x\\
&\leq C\|\partial_x m\|_{L^\infty}\|\partial_x\psi\|\|\partial_x m\|+C\|\partial_x u^r\|_{L^\infty}\|\partial_x m\|^2\\
&\leq C(\varepsilon_0+\epsilon)\|\partial_x m\|^2+C\varepsilon_0\|\partial_x\psi\|^2,\\
J_{28}&\leq C\int_{\mathbb{R}}|m\partial_x\zeta\partial_x m|+|m\partial_x\theta^r\partial_x m|+|\partial_x m|^2\mathrm{d}x\\
&\leq C\|m\|_{L^\infty}\|\partial_x\zeta\|\|\partial_x m\|+C\|\partial_x\theta^r\|_{L^\infty}\|m\|\|\partial_x m\|+C\|\partial_x m\|^2\\
&\leq C\varepsilon_0(\|\partial_x\zeta\|^2+\|\partial_x m\|^2)+C\epsilon^{\frac{1}{4}}(1+t)^{-\frac{3}{4}}\|m\|\|\partial_x m\|+C\|\partial_x m\|^2\\
&\leq (C\varepsilon_0+C+\eta)\|\partial_x m\|^2+C\varepsilon_0\|\partial_x\zeta\|^2+C\epsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}.
\end{aligned}
$$

Inserting the above estimations for J_l $(27\leq l\leq 28)$ into (2.35) and then choosing ε_0, ϵ and η so small, and integrating (2.35) over $[0,T]$, using (2.7), (2.24) and (2.27), we get (2.31).

Lemma 2.6 *Assume the conditions in Proposition* 2.1 *hold, then we have the following energy estimate for* $t\in[0,T]$,

$$
\|\partial_x^2 m\|^2+\int_0^t\|\partial_x^3 m\|^2\mathrm{d}\tau\leq C\|[\varphi_0,\psi_0,\zeta_0]\|_{H^1}^2+C\|m_0\|_{H^2}^2+C\epsilon^{\frac{1}{5}}. \tag{2.33}
$$

Proof Differentiate $(1.5)_4$ with respect to x, we deduce that

$$\begin{aligned}
&\rho(\partial_{xt}m+u\partial_x^2m+\partial_xu\partial_xm)+\partial_x\rho(\partial_tm+u\partial_xm)\\
=&\frac{\delta}{a}\theta\partial_x^3m+\frac{\delta}{a}\partial_x\theta\partial_x^2m-\frac{\rho\theta}{a\delta}(3m^2+6m+2)\partial_xm\\
&-\frac{\theta}{a\delta}\partial_x\rho(m^3+3m^2+2m)-\frac{\rho}{a\delta}\partial_x\theta(m^3+3m^2+2m).
\end{aligned}\tag{2.34}$$

Then, multiplying (2.34) by $-\frac{\partial_x^3m}{\rho}$, and integrating the resulting equation over $\mathbb{R}$ and using integrating by parts, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\rho^2(\partial_x^2m)^2\mathrm{d}x+\int_{\mathbb{R}}\frac{\delta\theta}{a\rho}(\partial_x^3m)^2\mathrm{d}x\\
=&\int_{\mathbb{R}}u\partial_x^2m\partial_x^3m\mathrm{d}x+\int_{\mathbb{R}}\partial_xu\partial_xm\partial_x^3m\mathrm{d}x+\int_{\mathbb{R}}\frac{\partial_x\rho}{\rho}\partial_x^3m(\partial_tm+u\partial_xm)\mathrm{d}x-\int_{\mathbb{R}}\frac{\delta}{a\rho}\partial_x\theta\partial_x^2m\partial_x^3m\mathrm{d}x\\
&+\int_{\mathbb{R}}\frac{\theta}{a\delta}(3m^2+6m+2)\partial_xm\partial_x^3m\mathrm{d}x+\int_{\mathbb{R}}\frac{\theta}{a\delta\rho}\partial_x\rho(m^3+3m^2+2m)\partial_x^3m\mathrm{d}x\\
&+\int_{\mathbb{R}}\frac{\partial_x\theta}{a\delta}(m^3+3m^2+2m)\partial_x^3m\mathrm{d}x\\
=&\sum_{l=29}^{35}J_l.
\end{aligned}\tag{2.35}$$

We now turn to estimate J_l $(29\le l\le 35)$ term by term. By applying Hölder's inequality, Cauchy-Schwarz inequality with $0<\eta<1$, Sobolev inequality (2.6), Lemma 1.2, (2.3), (2.5), $(1.5)_4$, it is direct to derive the following estimates:

$$\begin{aligned}
J_{29}+J_{33}\le&\, C\int_{\mathbb{R}}|u\partial_x^2m\partial_x^3m|+|\partial_xm\partial_x^3m|\mathrm{d}x\\
\le&\,\eta\|\partial_x^3m\|^2+C_\eta(\|\partial_xm\|^2+\|\partial_x^2m\|^2),\\
J_{30}\le&\, C\int_{\mathbb{R}}|\partial_x\psi\partial_xm\partial_x^3m|+|\partial_xu^r\partial_xm\partial_x^3m|\mathrm{d}x\\
\le&\, C\|\partial_xm\|_{L^\infty}\|\partial_x\psi\|\|\partial_x^3m\|+\|\partial_xu^r\|_{L^\infty}\|\partial_xm\|\|\partial_x^3m\|\\
\le&\, C(\varepsilon_0+\epsilon)\|\partial_x^3m\|^2+C\varepsilon_0\|\partial_x\psi\|^2+C\epsilon\|\partial_xm\|^2,\\
J_{31}=&\int_{\mathbb{R}}\frac{\partial_x\rho}{\rho}\partial_x^3m\Big[\frac{\delta\theta}{a\rho}\partial_x^2m-\frac{\theta}{a\delta}(m^3+3m^2+2m)\Big]\mathrm{d}x\\
\le&\int_{\mathbb{R}}|\partial_x\rho\partial_x^3m\partial_x^2m|+|m\partial_x\rho\partial_x^3m|\mathrm{d}x\\
\le&\, C\|\partial_x^2m\|_{L^\infty}\|\partial_x\varphi\|\|\partial_x^3m\|+C\|m\|_{L^\infty}\|\partial_x\varphi\|\|\partial_x^3m\|+C\|\partial_x\rho^r\|_{L^\infty}\|\partial_x^2m\|\|\partial_x^3m\|\\
&+C\|\partial_x\rho^r\|_{L^\infty}\|\partial_x^3m\|\|m\|\\
\le&\, C(\|\partial_x^2m\|+\|\partial_x^3m\|)\|\partial_x\varphi\|\|\partial_x^3m\|+C\varepsilon_0(\|\partial_x\varphi\|^2+\|\partial_x^3m\|^2)\\
&+C\epsilon(\|\partial_x^3m\|^2+\|\partial_x^2m\|^2)+C\epsilon^{\frac14}(1+t)^{-\frac34}\|\partial_x^3m\|\|m\|\\
\le&\, C(\varepsilon_0+\epsilon+\eta)\|\partial_x^3m\|^2+C(\varepsilon_0+\epsilon)\|\partial_x^2m\|^2+C\varepsilon_0\|\partial_x\varphi\|^2+C\epsilon^{\frac12}(1+t)^{-\frac32},\\
J_{32}\le&\, C\int_{\mathbb{R}}|\partial_x\zeta\partial_x^2m\partial_x^3m|+|\partial_x\theta^r\partial_x^2m\partial_x^3m|\mathrm{d}x\\
\le&\, C\|\partial_x^2m\|_{L^\infty}\|\partial_x\zeta\|\|\partial_x^3m\|+C\|\partial_x\theta^r\|_{L^\infty}\|\partial_x^2m\|\|\partial_x^3m\|
\end{aligned}$$

$$
\begin{aligned}
&\le C(\|\partial_x^2 m\| + \|\partial_x^3 m\|)\|\partial_x\zeta\|\|\partial_x^3 m\| + C\|\partial_x^2 m\|\|\partial_x^3 m\| \\
&\le C(\varepsilon_0+\epsilon)(\|\partial_x^2 m\|^2 + \|\partial_x^3 m\|^2),\\
J_{34}+J_{35} &\le C\int_{\mathbb{R}} |m\partial_x\varphi\partial_x^3 m| + |m\partial_x\rho^r\partial_x^3 m| + |m\partial_x\zeta\partial_x^3 m| + |m\partial_x\theta^r\partial_x^3 m|\mathrm{d}x\\
&\le C\|m\|_{L^\infty}\|\partial_x[\varphi,\zeta]\|\|\partial_x^3 m\| + C\|\partial_x[\rho^r,\theta^r]\|_{L^\infty}\|m\|\|\partial_x^3 m\|\\
&\le C\varepsilon_0(\|\partial_x\varphi\|^2 + \|\partial_x\zeta\|^2 + \|\partial_x^3 m\|^2) + C\epsilon^{\frac14}(1+t)^{-\frac34}\|m\|\|\partial_x^3 m\|\\
&\le C(\varepsilon_0+\eta)\|\partial_x^3 m\|^2 + C\varepsilon_0(\|\partial_x\varphi\|^2 + \|\partial_x\zeta\|^2) + C\epsilon^{\frac12}(1+t)^{-\frac32}.
\end{aligned}
$$

Inserting the above estimations for J_l ($29 \le l \le 35$) into (2.35) and then choosing ε_0, ϵ and η so small, and integrating (2.35) over [0,T], using (2.7), (2.24), (2.27), (2.29) and (2.31), we get (2.33).

Proof of Proposition 2.1 Combinations of the estimates (2.7), (2.24), (2.27), (2.29), (2.31), (2.33) and taking ε_0 and ϵ sufficiently small, we can get the desired estimate (2.4). Thus the proof of Proposition 2.1 is completed.

3 Global Existence and Large Time Behavior

We are now in a position to complete the following.

Proof of Theorem1.1 By the a priori estimates (2.4), there exists a positive constant C_0 such that

$$
\|[\varphi,\psi,\zeta]\|_{H^1}^2 + \|m\|_{H^2}^2 \le C_0(\|[\varphi_0,\psi_0,\zeta_0]\|_{H^1}^2 + \|m_0\|_{H^2}^2 + \epsilon^{\frac15}). \tag{3.1}
$$

By letting $\epsilon > 0$ be small enough, the global existence of the solution to Cauchy problem (2.1)–(2.2) then follows from the standard continuation argument based on the local existence and the a priori estimates (2.4). Moreover, (3.1) implies (1.21). For the large time behavior in (1.22), one can verify that

$$
\lim_{t\to+\infty}\|\partial_x[\varphi,\psi,\zeta,m](t)\|_{L^2}^2 = 0. \tag{3.2}
$$

To prove (3.2), we get from $(2.1)_1$, (2.4), (2.25), (2.28) and (2.35) that

$$
\begin{aligned}
&\int_0^{+\infty}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x[\varphi,\psi,\zeta,m]\|^2\Big|\mathrm{d}t\\
&= 2\int_0^{+\infty}\Big|\int_{\mathbb{R}}\partial_t\partial_x\varphi\partial_x\varphi\mathrm{d}x\Big|\mathrm{d}t + \int_0^{+\infty}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x[\psi,\zeta,m]\|^2\Big|\mathrm{d}t\\
&\le C + C\int_0^{+\infty}\|\partial_x[\varphi,\psi,\zeta,\partial_x[\psi,\zeta,m,\partial_x m]]\|^2\mathrm{d}t < +\infty.
\end{aligned} \tag{3.3}
$$

Consequently,(3.3) together with (2.4) gives (3.2). Then (1.22) follows from (3.2), Lemma 1.2 (iii) and Sobolev's inequality. This ends the proof of Theorem 1.1.

Acknowledgement The author would like to thank Professor Changjiang Zhu for his continuous help and encouragements.

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